The Farrell-Jones Conjecture for algebraic K-theory holds for word-hyperbolic groups and arbitrary coefficients.

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Outline

- We explain our main Theorem that the *Farrell-Jones Conjecture for algebraic K-theory* is true for every word-hyperbolic group G and every coefficient ring R.
- It predicts the structure of the algebraic K-groups $K_n(RG)$.
- We discuss new applications focussing on
    - Vanishing of the reduced projective class group and the Whitehead group of torsionfree groups;
    - Conjectures generalizing Moody's Induction Theorem;
    - Bass Conjecture;
    - Kaplanky Conjecture
    - Algebraic versus homotopy K-theory, Nil-groups;
    - L^2-invariants;
- We make a few comments about the proof.

Conjecture

The Farrell-Jones Conjecture for algebraic K-theory with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- R is any (associative) ring (with unit) and G is discrete;
- $K_n(RG)$ is the algebraic K-theory of the group ring RG;
- $\mathcal{VC}$yc is the family of virtually cyclic subgroups;
- Given a family of subgroups $\mathcal{F}$, let $E_{\mathcal{F}}(G)$ be the classifying space associated to it;
- $H_*^G(-;\mathbf{K}_R)$ is the G-homology theory with the property that for every subgroup $H \subseteq G$

$$H_n^G(G/H; \mathbf{K}_R) = K_n(RH).$$

The Farrell-Jones Conjecture gives a way to compute $K_n(RG)$ in terms of $K_m(RV)$ for all virtually cyclic subgroups $V \subseteq G$ and all $m \leq n$.

It is analogous to the Baum-Connes Conjecture.

Conjecture

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{F}in}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(pt, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Here $H_*^G(-;\mathbf{K}^{\mathrm{top}})$ is the G-homology theory with the property that for every subgroup $H \subseteq G$

$$H_n^G(G/H; \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(H)).$$

Theorem (Bartels-L.-Reich (2006))

The (Fibered) Farrell-Jones Conjecture for algebraic K-theory with (G-twisted) coefficients in any ring R is true for word-hyperbolic groups G.

We emphasize that this result holds for all rings R and not only for $R = \mathbb{Z}$.

Corollary

If G is a torsionfree word-hyperbolic group and R any ring, then we get an isomorphism

$$H_n(BG;\mathbf{K}(R)) \oplus \left(\bigoplus_{\substack{(C), C \subseteq G, C \neq 1 \\ C \text{ maximal cyclic}}} NK_n(R) \right) \xrightarrow{\cong} K_n(RG).$$

We are not (yet?) able to prove the L-theory version. The L-theory version implies the Novikov Conjecture.
If one knows the K- and L-theory version for a group G in the case $R = \mathbb{Z}$, one gets the Borel Conjecture in dimension ≥ 5

Conjecture

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

Let $\mathcal{FJ}(R)$ be the class of groups which satisfy the Fibered Farrell-Jones Conjecture for algebraic K-theory with coefficients in R.

Theorem (Bartels-L.-Reich (2006))

1. *Every word-hyperbolic group and every virtually nilpotent group belongs to $\mathcal{FJ}(R)$;*
2. *If G_1 and G_2 belong to $\mathcal{FJ}(R)$, then $G_1 \times G_2$ belongs to $\mathcal{FJ}(R)$;*
3. *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}(R)$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}(R)$;*
4. *If H is a subgroup of G and $G \in \mathcal{FJ}(R)$, then $H \in \mathcal{FJ}(R)$.*

In order to illustrate the depth of the Farrell-Jones Conjecture, we present some conclusions which are interesting in their own right.

Corollary

Let R be a regular ring. Suppose that G is torsionfree and $G \in \mathcal{FJ}(R)$. Then

1. $K_n(RG) = 0$ *for* $n \leq -1$;
2. *The change of rings map* $K_0(R) \to K_0(RG)$ *is bijective. In particular* $\widetilde{K}_0(RG)$ *is trivial if and only if* $\widetilde{K}_0(R)$ *is trivial;*
3. *The Whitehead group* $\operatorname{Wh}^R(G)$ *is trivial.*

The idea of the proof is to study

$$H_n(BG; \mathbf{K}(R)) = H_n^G(E_{\mathcal{TR}}(G); \mathbf{K}_R) \to H_n^G(E_{\mathcal{VC}\text{yc}}(G); \mathbf{K}_R) \to K_n(RG).$$

In particular we get for a torsionfree group $G \in \mathcal{FJ}(\mathbb{Z})$

- $K_n(\mathbb{Z}G) = 0$ for $n \leq -1$;
- $\widetilde{K}_0(\mathbb{Z}G) = 0$;
- $\mathrm{Wh}(G) = 0$;
- Every finitely dominated CW-complex X with $G = \pi_1(X)$ is homotopy equivalent to a finite CW-complex;
- Every compact h-cobordism $W = (W; M_0, M_1)$ of dimension ≥ 6 with $\pi_1(W) \cong G$ is trivial, i.e., diffeomorphic to $M_0 \times [0,1]$ relative M_0. (For $G = \{1\}$ this implies the Poincaré Conjecture in dimensions ≥ 5.)

Theorem

1. *Let R be a regular ring with $\mathbb{Q} \subseteq R$. Suppose $G \in \mathcal{FJ}(R)$. Then the map given by induction from finite subgroups of G*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(RH) \to K_0(RG)$$

is bijective;

2. *Let F be a field of characteristic p for a prime number p. Suppose that $G \in \mathcal{FJ}(F)$. Then the map*

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

is bijective.

Conjecture

Let R be a commutative integral domain and let G be a group. Let $g \in G$ be an element in G. Suppose that either the order $|g|$ is infinite or that the order $|g|$ is finite and not invertible in R. Then the Bass Conjecture predicts that for every finitely generated projective RG-module P the value of its Hattori-Stallings rank $\mathrm{HS}_{RG}(P)$ at (g) is trivial.

Theorem

Let G be a group. Suppose that

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}in}(G)} K_0(FH) \otimes_{\mathbb{Z}} \mathbb{Q} \to K_0(FG) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is surjective for all fields F of prime characteristic. (This is true if $G \in \mathcal{FJ}(F)$ for every field F of prime characteristic).
Then the Bass Conjecture is satisfied for every integral domain R.

Conjecture

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

The Kaplansky Conjecture is related to the vanishing of $\widetilde{K}_0(RG)$.

Lemma

Let F be a field and let G be a group with $G \in \mathcal{FJ}(F)$. Suppose that F has characteristic zero and G is torsionfree or that F has characteristic p, all finite subgroups of G are p-groups and G is residually amenable.
Then 0 and 1 are the only idempotents in FG.

Conjecture

Let R be a regular ring with $\mathbb{Q} \subseteq R$. Then we get for all groups G and all $n \in \mathbb{Z}$ that

$$NK_n(RG) = 0$$

and that the canonical map from algebraic to homotopy K-theory

$$K_n(RG) \to KH_n(RG)$$

is bijective.

Theorem

Let R be a regular ring with $\mathbb{Q} \subseteq R$. If $G \in \mathcal{FJ}(R)$, then the conjecture above is true.

Conjecture

If X and Y are det-L^2-acyclic finite G-CW-complexes, which are G-homotopy equivalent, then their L^2-torsion agree:

$$\rho^{(2)}(X; \mathcal{N}(G)) = \rho^{(2)}(Y; \mathcal{N}(G)).$$

- The L^2-torsion of closed Riemannian manifold M is defined in terms of the heat kernel on the universal covering. If M is hyperbolic and has odd dimension, its L^2-torsion is up to dimension constant its volume.
- The conjecture above allows to extend the notion of a volume to word-hyperbolic groups whose L^2-Betti numbers all vanish.

Theorem

Suppose that $G \in \mathcal{FJ}(\mathbb{Z})$. Then G satisfies the Conjecture above.

- Deninger can define a p-adic Fuglede-Kadison determinant for a group G and relate it to p-adic entropy provided that $\mathrm{Wh}^{\mathbb{F}_p}(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ is trivial.
- The surjectivity of the map

$$\operatorname*{colim}_{\mathrm{Or}_{\mathcal{F}\mathrm{in}}(G)} K_0(\mathbb{C}H) \to K_0(\mathbb{C}G)$$

plays a role in a program to prove the Atiyah Conjecture which predicts for a closed Riemannian manifold with torsionfree fundamental group that the L^2-Betti numbers of its universal covering are all integers.

- There is no group known for which the Farrell-Jones Conjecture, the Fibered Farrell-Jones Conjecture or the Baum-Connes Conjecture is false.
- However, Higson, Lafforgue and Skandalis have constructed counterexamples to the Baum-Connes-Conjecture with coefficients. They describe precisely what properties a group Γ must have so that it does *not* satisfy the Baum-Connes Conjecture with coefficients. Gromov outlines the construction of such a group Γ as a colimit over a directed system of groups $\{G_i \mid i \in I\}$ such that each G_i is word-hyperbolic.
- Our main result implies that the Fibered Farrell-Jones Conjecture for algebraic K-theory with twisted coefficients in any ring does hold for Γ.

Here are the basic steps of the proof of the main Theorem.
Step 1: Interprete the assembly map as a forget control map.

Step 2: Show for a finitely generated group G that $G \in \mathcal{FJ}(R)$ holds for all rings R if one can construct the following geometric data:

- A G-space X, such that the underlying space X is the realization of an abstract simplicial complex;
- A G-space $\overline{X}$, which contains X as an open G-subspace. The underlying space of $\overline{X}$ should be compact, metrizable and contractible,

such that the following assumptions are satisfied:

- Z-set-condition
  There exists a homotopy $H\colon \overline{X} \times [0,1] \to \overline{X}$, such that $H_0 = \mathrm{id}_{\overline{X}}$ and $H_t(\overline{X}) \subset X$ for every $t > 0$;
- Long thin covers
  There exists an $N \in \mathbb{N}$ that only depends on the G-space $\overline{X}$, such that for every $\beta \geq 1$ there exists an $\mathcal{VC}$yc-covering $\mathcal{U}(\beta)$ of $G \times \overline{X}$ with the following two properties:
  - For every $g \in G$ and $x \in \overline{X}$ there exists a $U \in \mathcal{U}(\beta)$ such that $\{g\}^\beta \times \{x\} \subset U$. Here g^β denotes the β-ball around g in G with respect to the word metric;
  - The dimension of the covering $\mathcal{U}(\beta)$ is smaller than or equal to N.

Step 3: Prove the existence of the geometric data above.