Reidemeister torsion and the K-theory of von Neumann algebras by Wolfgang Lück and Mel Rothenberg

0. Introduction

The purpose of this paper is to introduce and study a new topological type invariant for cocompact properly discontinous actions of discrete groups of isometries on Riemannian manifolds. This work is inspired in part by the work of Carey and Mathai [4] and our invariant is a generalization of theirs. It is however much more refined and has the advantage of encompasing other powerful invariants such as the Alexander polynomial and the equivariant Reidemeister torsions studied in Lott-Rothenberg [18] and Lück [20], which are themselves useful generalizations of the classical notions.

Let M be a Riemannian manifold and Γ a discrete properly discontinuous group of isometries with $\Gamma\backslash M$ compact. Properly discontinuous means that for each pair of points (x,y) in M there are neighborhoods U_x and U_y such that $\{\gamma \in \Gamma \mid \gamma U_x \cap U_y \neq \emptyset\}$ is finite. An important case to keep in mind is the following. Let $p : M \longrightarrow N$ be a locally isometric Galois covering over a compact Riemannian manifold N with group of deck transformations π. One may choose Γ to be π. More generally, one can lift an action of a finite group G on N by isometries to an action of a group Γ, such that Γ is an extension of π and G and the Γ-action extends the π-action and covers the G-action.

Let $\mathcal{A}$ be a finite von Neumann algebra and V be a finitely generated Hilbert module over $\mathcal{A}$. Consider a unitary representation $\mu : \Gamma \longrightarrow Iso_{\mathcal{A}}(V)^{op}$. Let $K_1^w(\mathcal{A})$ be the K-theory of weak automorphisms of finitely generated Hilbert $\mathcal{A}$-modules with the involution given by taking adjoints. We define the *Reidemeister von Neumann torsion*

$$\rho(M;V) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

in 5.7. Some of the main properties of this invariant and the relevant K-theory are listed below :

- The Reidemeister von Neumann torsion $\rho(M)$ is an invariant of the Γ-isometry class of M. If the l^2-homology $H_*(M;V)$ vanishes, $\rho(M;V)$ depends only on the simple Γ-homotopy type of M (see theorem 3.11). We remark that in a lot of interesting cases

$H_*(M;V)$ indeed vanishes, e.g., the universal covering of a closed hyperbolic manifold M, the universal covering of a compact manifold M admitting a fixed point free S^1-action such that the inclusion of one (and hence all orbits) induces an injection on the fundamental groups and the universal covering of a prime Haken 3-manifold provided in all cases that V is $l^2(\pi_1(M))$. The first example follows from Dodziuk [11] and [12], the second statement is proved in theorem 3.20 and the last statement will appear in a forthcoming preprint. Roughly speaking, the image of Whitehead torsion under a change of rings homomorphism is the difference of Reidemeister von Neumann torsion if $H_*(M;V)$ vanishes.

- Let $M \cup_X N$ be obtained from M and N by glueing along a common union of components of the boundary. Then $\rho(M \cup_X N;V) - \rho(M;V) - \rho(N;V) + \rho(X;V)$ is given by the long Mayer Vietoris homology sequence, more precisely, by its torsion (see theorem 3.14) and is in particular zero if all l^2-homology groups vanish.

- There is a product formula $\rho(M \times N; V \otimes W) = \chi^{\mathcal{A}}(M) \cdot [V] \otimes \rho(N;W) + \chi^{\mathcal{B}}(N) \cdot \rho(M;V) \otimes [W]$, where the integers $\chi^{\mathcal{A}}(M)$ and $\chi^{\mathcal{B}}(N)$ are Euler characteristic type invariants (see theorem 3.16).

- The Reidemeister von Neumann torsion is compatible with restriction to subgroups of finite index (see lemma 3.17).

- The Reidemeister von Neumann torsion $\rho(M)$ satisfies Poincaré duality (see theorem 5.13). If Γ acts freely this means $\rho(M;V) = (-1)^{1+dim(M)} \cdot \rho(M, \partial M; V)$.

- Let $N(\Gamma)$ be the von Neumann algebra of the group Γ. The K-group $K_1^w(N(\Gamma))^{\mathbf{Z}/2}$ is the group of almost everywhere invertible functions from the r-dimensional torus T^r to $\mathbf{R}$ if Γ is $\mathbf{Z}^r$, and reduces to the ordinary K-group of the complex group ring $K_1(\mathbf{C}[\Gamma])^{\mathbf{Z}/2}$, if Γ is finite (see theorem 2.5).

We illustrate these computational tools by proving the following corollary 3.21. Let G be a connected compact Lie group and $p : X \longrightarrow Y$ be a G-principal bundle of finite CW-complexes such that the image of $\pi_1(G) \longrightarrow \pi_1(X)$ is infinite. Then $\widetilde{X}$ is $l^2(\pi_1(X))$-acyclic, where $\widetilde{X}$ denotes the universal covering of X. If G is S^1, then $\rho(\widetilde{X}; l^2(\pi_1(X)))$ is given by $\chi(Y) \cdot [[(w-1) : l^2(\pi_1(X)) \longrightarrow l^2(\pi_1(X))]]$ in $K_1^w(N(\pi_1(X)))^{\mathbf{Z}/2}$ where $N(\pi_1(X))$ is the von Neumann algebra of the fundamental group and $w \in \pi_1(X)$ is given by an S^1-orbit. If G is not S^1, then $\rho(X; l^2(\pi_1(X)))$ vanishes. Our computations apply to Seifert 3-manifolds (see remark 3.22).

For a finite-dimensional representation V the equivariant Reidemeister von Neumann torsion reduces to the PL-torsion invariants defined in Lott-Rothenberg [18] and Lück [20]. In particular one obtains for free actions the PL-torsion of Ray and Singer [23]. If ∂M

is empty, its logarithm is the analytic torsion as shown independently by Cheeger [5] and Müller [22]. This result is extended to the equivariant case and the case with boundary in Lott-Rothenberg [18] and Lück [20]. We will relate Reidemeister von Neumann torsion to the Alexander polynomial of a link (see example 4.7) and to the Lefschetz zeta function of an endomorphism of a finite CW-complex (see example 4.8).

The complex group ring $\mathbf{C}[\Gamma]$ of a group Γ is semisimple if and only if Γ is finite. The semisimplicity is crucial for the definition of classical Reidemeister torsion. If one completes $\mathbf{C}[\Gamma]$ to $l^2(\Gamma)$, one is lead to the theory of Hilbert $N(\Gamma)$-modules as established in Atiyah [1], Cheeger and Gromov [6], [7], [8] and Dixmier [10]. The proof that $\mathbf{C}[\Gamma]$ is semisimple for finite Γ is based on the existence of a Hilbert structure. Hence one obtains semisimplicity also for infinite Γ for Hilbert $N(\Gamma)$-modules. Now our definition of Reidemeister von Neumann torsion follows the standard pattern. The main technical difficulty comes from the fact that one has to define homology as the quotient of the kernel and the closure of the image of the relevant differentials, so that the vanishing of homology does not imply contractibility. In particular one has to deal in the K-theory instead of isomorphisms with weak isomorphisms, i.e. morphisms with trivial kernel and dense image. Therefore a lot of the material for Hilbert $\mathcal{A}$-modules of section 6 and 7 is essentially standard, but the proofs are different and harder than in the case of modules over the complex numbers.

Here is a short survey of the construction of Reidemeister von Neumann torsion. The relevant K-group $K_1^w(\mathcal{A})$ has weak automorphisms $f : M \longrightarrow M$ of finitely generated Hilbert $\mathcal{A}$-modules as generators $[f]$ and the relations are $[g \circ f] = [g] + [f]$, $[id] = 0$ and $[f] + [h] = [g]$, if there is a exact sequence $0 \longrightarrow (M, f) \longrightarrow (N, g) \longrightarrow (P, h) \longrightarrow 0$. Let M be a compact smooth manifold with fundamental group π and $\mu : \pi \longrightarrow Iso_{\mathcal{A}}(V)^{op}$ be a unitary representation. We give the definition of $\rho(M; V)$ in the case that $H_*(M; V)$ is trivial (for all $* \geq 0$). Let $C(M; V)$ be the Hilbert $\mathcal{A}$-chain complex $V \otimes_{\mathbf{Z}[\Gamma]} C(M)$ and $\Delta_p : C_p(M; V) \longrightarrow C_p(M; V)$ be the associated Laplace operator $d_{p+1} \circ d_{p+1}^* + d_p^* \circ d_p$ where d is the differential of $C(M; V)$. As $H_p(M; V)$ vanishes, Δ_p is a weak automorphism of finitely generated Hilbert $\mathcal{A}$-modules by the Hodge decomposition theorem. The Reidemeister von Neumann torsion $\rho(M; V) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$ is given by $\sum_{p \geq 0} (-1)^p \cdot p \cdot [\Delta_p]$. We will introduce another definition using weak chain contractions which has some advantages for technical conceptual reasons and is closer to the classical definitions (see definition 7.10). Moreover, it allows in the acyclic case the definition of a refined invariant, the acyclic Reidemeister von Neumann torsion (see 3.24). Both definitions will be identified in lemma 7.12. If $H(M; V)$ is not trivial, one needs a Riemannian metric on M. We remark that $H_*(M; l^2(\Gamma))$ has a good chance to be trivial (for all $* \geq 0$), whereas $H_0(M; \mathbf{C})$ never vanishes for non-empty M.

In this article we develop the foundations of Reidemeister von Neumann torsion in the sense that we give its definition, collect its main properties, give computational tools, relate it to known invariants and analyse some examples. The following problems seem to be the natural continuation of these investigations and their relevance and meaning is evident from the discussion above. We will deal with them in forthcoming papers where the results of this article will be applied to these problems.

- Compute the K-groups $K_1^w(N(\Gamma))$ also for non-abelian groups and find detecting homomorphisms into known groups.
- What is the analytic interpretation of the Reidemeister von Neumann torsion in terms of the spectral theory of the Laplace operator?
- Compute Reidemeister von Neumann torsion for certain classes of manifolds (crystallographic manifolds, hyperbolic manifolds, 3-manifolds) and investigate how sharp it is.

If one wants to get a quick survey about the results, one may skip the first two sections and read sections 3 and 4. To get a sufficient impression from section 1 and 2 it suffices to consider von Neumann algebras of a group as explained in example 1.9 and read in section 2 until Theorem 2.5 only assuming $\mathcal{A} = N(\mathbf{Z}^r)$ and $X = T^r$. The material simplifies considerably if one assumes that the action of Γ is free what is true in a lot of interesting examples. Under this assumption one does not need the material about G-CW-complexes and permutation modules in the beginning of section 3 and may start with definition 3.6. In section 5 one can skip the equivariant triangulation theorem as it follows from the non-equivariant one in the case of a free action. Moreover, the definition of Poincaré von Neumann torsion becomes irrelevant in view of theorem 5.13 and the upshot of the discussion about Poincaré duality is that Reidemeister von Neumann torsion satisfies $\rho(M;V) = (-1)^{1+dim(M)} \cdot \rho(M,\partial M;V)$.

The paper is organized as follows :

1. Preliminaries about von Neumann algebras and Hilbert modules

In this section we collect the basic properties about von Neumann algebras and Hilbert modules.

Let H be a Hilbert space with inner product $\langle \xi, \eta \rangle$ and induced norm $|\xi|$. A Hilbert space is always to be understood as a seperable complex Hilbert space. Let $B(H)$ be the C^*-algebra of bounded linear operators $H \longrightarrow H$. Recall that the *operator norm* is given by $\| f \| := sup\{ |f(\xi)| \mid \xi \in H, |\xi| = 1\}$ and the involution $* : B(H) \longrightarrow B(H)$ maps f to its adjoint f^*. The *norm topology* is the vector space topology induced by the norm $\| f \|$. The *weak* resp. *strong topology* on $B(H)$ is the topology induced by the family of seminorms $\{p_{\xi,\eta} \mid \xi, \eta \in H\}$ resp. $\{p_\xi \mid \xi \in H\}$ defined by $p_{\xi,\eta}(f) := \langle f(\xi), \eta \rangle$ resp. by $p_\xi(f) := |f(\xi)|$. A *subalgebra* A of $B(H)$ is a subset closed under addition, multiplication with scalars and multiplication and contains the unit of $B(H)$. It is called *selfadjoint* , if it is closed under the involution.

Definition 1.1 *A* von Neumann algebra (in H) *is a selfadjoint subalgebra $\mathcal{A}$ of $B(H)$ which is closed in the weak topology.* ■

The *commutant* of a subset M of $B(H)$ is $M' := \{f \in B(H) \mid fg = gf$ for all $g \in M\}$. Obviously $M \subset M''$ holds. The following theorem is due to von Neumann (see e.g. Sunder [28] page 12 for a proof).

Theorem 1.2 (Double commutant theorem)
Let A be a selfadjoint subalgebra of $B(H)$. Then the following assertions are equivalent :

1.) $A = A''$

2.) A is weakly closed.

3.) A is strongly closed. ■

Given a von Neumann algebra $\mathcal{A} \subset B(H)$, let $\mathcal{A}^+$ be the cone of positive elements. Recall that $f : H \longrightarrow H$ is called *positive,* if f is selfadjoint and $\langle f(\xi), \xi \rangle \geq 0$ holds for all $\xi \in H$. A map $tr : \mathcal{A}^+ \longrightarrow [0, \infty]$ is a *trace* , if for $a, b \in \mathcal{A}^+$ and $\lambda \in [0, \infty[$ the following holds :

$$tr(a) + tr(b) = tr(a) + tr(b) \qquad tr(\lambda a) = \lambda tr(a) \qquad tr(aa^*) = tr(a^*a)$$

A trace tr is *finite* , if $tr(a) < \infty$ holds for all $a \in \mathcal{A}^+$. It is *faithful* , if $tr(a) = 0$ implies $a = 0$ for all $a \in \mathcal{A}^+$. We call tr *normal* if $tr(a) = sup\{tr(a_i) \mid i \in I\}$ holds for any monotone increasing net $\{a_i \mid i \in I\}$ in $\mathcal{A}^+$ with a as supremum. Let $\mathcal{M}$ be the ideal in $\mathcal{A}$ given by finite sums of products of elements $a \in \mathcal{A}$ satisfying $tr(aa^*) < \infty$. Then tr extends uniquely to a $\mathbf{C}$-linear form also denoted $tr : \mathcal{M} \longrightarrow \mathbf{C}$ satisfying $tr(ab) = tr(ba)$ for all $a, b \in \mathcal{M}$. In particular a finite trace extends uniquely to a $\mathbf{C}$-linear form $tr : \mathcal{A} \longrightarrow \mathbf{C}$.

Definition 1.3 A von Neumann algebra $\mathcal{A}$ is called *finite* , if it possesses a finite, normal and faithful trace. ■

In the sequel any von Neumann algebra $\mathcal{A}$ is assumed to be finite and comes with a finite normal and faithful trace $tr : \mathcal{A} \longrightarrow \mathbf{C}$. Define a pre-Hilbert structure on $\mathcal{A}$ by $\langle a, b\rangle = tr(b^*a)$. Let $l^2(\mathcal{A})$ be the Hilbert completion of $(\mathcal{A}, \langle\ ,\ \rangle)$. Denote by $|\, a\,|$ the induced norm on $l^2(\mathcal{A})$. Given $a \in \mathcal{A}$, we obtain a linear operator $l(a) : \mathcal{A} \longrightarrow \mathcal{A}$ sending b to ab. This operator is bounded and has the operator norm $\|\, l(a)\,\| = |\, a\,|$. Hence it extends uniquely to a bounded operator $\overline{l(a)} : l^2(\mathcal{A}) \longrightarrow l^2(\mathcal{A})$ satisfying $\|\, \overline{l(a)}\,\| = |\, a\,|$. Thus we obtain a left $\mathcal{A}$-module structure on $l^2(A)$. Analogously we get $r(a) : \mathcal{A} \longrightarrow \mathcal{A}$ sending b to ba and an induced operator $\overline{r(a)} : l^2(\mathcal{A}) \longrightarrow l^2(\mathcal{A})$. Notice that $l(a)$ and $r(b)$ and hence $\overline{l(a)}$ and $\overline{r(b)}$ commute for $a, b \in \mathcal{A}$. In particular we obtain the *right regular representation* $\nu_r : \mathcal{A}^{op} \longrightarrow B_{\mathcal{A}}(l^2(\mathcal{A}))$ from the opposite algebra $\mathcal{A}^{op}$ of $\mathcal{A}$ into the subalgebra $B_{\mathcal{A}}(l^2(\mathcal{A}))$ of linear bounded $\mathcal{A}$-operators of $B(l^2(\mathcal{A}))$. Recall that the opposite algebra $\mathcal{A}^{op}$ is obtained from $\mathcal{A}$ by reversing the multiplication, i.e. ab in $\mathcal{A}^{op}$ is given by ba in $\mathcal{A}$. The following result is fundamental for the theory of Hilbert modules over a finite von Neumann algebra (see Dixmier [10], page 80 theorem 1, page 99 theorem 2).

Theorem 1.4 *Let $\mathcal{A}$ be a finite von Neumann algebra. Then the right regular representation*

$$\nu = \nu_r : \mathcal{A}^{op} \longrightarrow B_{\mathcal{A}}(l^2(\mathcal{A}))$$

is a bijection ■

Next we introduce the category of Hilbert $\mathcal{A}$-modules over a finite von Neumann algebra $\mathcal{A}$. A *Hilbert $\mathcal{A}$-module* M is a Hilbert space M together with a continuous left $\mathcal{A}$-module structure such that there exists an isometric linear embedding onto a closed subspace of $l^2(A) \otimes H$ for some Hilbert space H. The embedding is not part of the structure. A morphism between Hilbert $\mathcal{A}$-modules $f : M \longrightarrow N$ is a bounded linear operator compatible with the $\mathcal{A}$-module structures. We get from the theorem 1.4 above a bijection of $\mathbf{C}$-algebras.

1.5 $\Omega : \hom_{\mathcal{A}}(\oplus_{i=1}^n l^2(\mathcal{A}), \oplus_{i=1}^n l^2(\mathcal{A})) \longrightarrow M(n, n, \mathcal{A})^{op}$

Hence we may think of $\oplus_{i=1}^n l^2(\mathcal{A})$ as the free Hilbert $\mathcal{A}$-module of rank n. A Hilbert $\mathcal{A}$-module M is called *finitely generated*, if there is an epimorphism of Hilbert $\mathcal{A}$-modules from $\oplus_{i=1}^n l^2(A)$ onto M for some integer $n \geq 0$.

Lemma 1.6 *Any finitely generated Hilbert $\mathcal{A}$-module M is projective in the following sense: there is a finitely generated Hilbert $\mathcal{A}$-module N such that there exists an isometric isomorphism of Hilbert $\mathcal{A}$-modules from $M \oplus N$ to $\oplus_{i=1}^n l^2(\mathcal{A})$ for appropiate n.*

Proof : By definition there is an epimorphism of Hilbert $\mathcal{A}$-modules f from $\oplus_{i=1}^n l^2(\mathcal{A})$ to M for appropiate n. It induces a bijective morphism of Hilbert $\mathcal{A}$-modules between $ker(f)^{\perp}$ and M. This is an isomorphism of Hilbert $\mathcal{A}$-modules by the open mapping theorem. The unitary part in its polar decomposition is an isometric isomorphism of Hilbert $\mathcal{A}$-modules from $ker(f)^{\perp}$ to M. Now the claim follows from the orthogonal decomposition of $\oplus_{i=1}^n l^2(\mathcal{A})$ into $ker(f) \oplus ker(f)^{\perp}$. ■

This lemma 1.6 will enables us to carry over the notion of Reidemeister torsion invariants for finite transformation groups to infinite transformation groups. The construction of Reidemeister torsion for finite groups G is based on the algebraic fact that the group ring $\mathbf{C}[G]$ is semi-simple if and only if G is finite. Hence the construction does not go through in the infinite case. However, if one extends the group ring to its Hilbert completion and hence deals with the von Neumann algebra of the group, one gets semi-simplicity again. Recall that the proof of semisimplicity of $\mathbf{C}[G]$ for finite G is based on the fact that $\mathbf{C}[G]$ is a Hilbert space for finite G.

Let $f : M \longrightarrow N$ be a morphism of Hilbert $\mathcal{A}$-modules . Its kernel in the categorial sense in just the ordinary kernel $ker(f)$, whereas the cokernel in the categorial sense is given by $N/clos(im(f))$. We have to divide out the closure of the image and not the image itself, since Hilbert $\mathcal{A}$-modules are required to be complete. Finite coproducts and products are given by finite direct sums. The zero object is given by $\{0\}$. However, the category of Hilbert $\mathcal{A}$-modules is not abelian, since it is neither true that any epimorphism is a cokernel nor that any monomorphism is a kernel. This will force us to deal with two different K-theories and to modify the usual definitions of torsion invariants.

A Hilbert $\mathcal{A}$-chain complex $C = (C_*, c_*)$ is a sequence of Hilbert $\mathcal{A}$-modules

$$\ldots \xrightarrow{c_{n+1}} C_n \xrightarrow{c_n} C_{n-1} \xrightarrow{c_{n-1}} \ldots$$

such that $c_{n+1} \circ c_n = 0$ holds for $n \in \mathbf{Z}$. We call C *bounded* if there is $N \in \mathbf{Z}$ such that C_n is zero for $\mid n \mid > N$. If C is bounded and C_n is finitely generated for all $n \in \mathbf{Z}$, we say that C is *finite*. The homology of C is the $\mathbf{Z}$-graded Hilbert $\mathcal{A}$-module $H(C)$ given by

$H_n(C) = ker(c_n)/clos(im(c_{n+1}))$. We call C *weakly acyclic* if $H_n(C)$ is zero for all $n \in \mathbf{Z}$. Notice that weakly acyclic does not imply acyclic. We call $0 \longrightarrow M \longrightarrow N \longrightarrow P \longrightarrow 0$ *weakly exact*, if it is weakly acyclic as a Hilbert $\mathcal{A}$-chain complex . A morphism $f : M \longrightarrow N$ is a *weak isomorphism* if its kernel is trivial and its image is dense, or, equivalently, if $0 \longrightarrow M \xrightarrow{f} N \longrightarrow 0 \longrightarrow 0$ is weakly exact. This is equivalent to the condition that f and its adjoint f^* are injective because of $ker(f^*)^{\perp} = clos(im(f))$. The following lemma is a direct consequence of the polar decomposition theorem.

Lemma 1.7 *Given a weak isomorphism $f : M \longrightarrow N$ of Hilbert $\mathcal{A}$-modules , there are a positive selfadjoint weak automorphism of Hilbert $\mathcal{A}$-modules $h : M \longrightarrow M$ and an isometric isomorphism of Hilbert $\mathcal{A}$-modules $g : M \longrightarrow N$ satisfying $f = g \circ h$.* ■

Lemma 1.8 *An endomorphism $f : M \longrightarrow M$ of a finitely generated Hilbert $\mathcal{A}$-module is a weak isomorphism, if and only if f is injective.*

Proof : This follows from dimension theory of von Neumann algebras (see Cheeger-Gromov [8] section 1). Namely, for any morphism $g : M \longrightarrow N$ of finitely generated Hilbert $\mathcal{A}$-modules we have :

$$dim(M) + dim(clos(im(g))^{\perp}) = dim(N) + dim(ker(g))$$

and $dim(M)$ is zero if and only if M is zero. We will give a direct proof of this fact in the abelian case later. ■

Example 1.9 Our main example will be the finite von Neumann algebra $N(\Gamma)$ of a countable discrete group Γ. Let $\mathbf{C}[\Gamma]$ be the complex group ring. It becomes a pre-Hilbert space by the inner product

$$\langle \sum_{\gamma\in\Gamma} \lambda_\gamma \cdot \gamma, \sum_{\gamma\in\Gamma} \mu_\gamma \cdot \gamma \rangle \ = \sum_{\gamma\in\Gamma} \lambda_\gamma \cdot \overline{\mu_\gamma}$$

The Hilbert completion of $\mathbf{C}[\Gamma]$ is denoted by $l^2(\Gamma)$ and consists of square-summable sums $\sum_{\gamma\in\Gamma} \lambda_\gamma \cdot g$. An element $\gamma \in \Gamma$ determines an operator $\overline{l(\gamma)} : l^2(\Gamma) \longrightarrow l^2(\Gamma)$ whose restriction to the group ring is given by left multiplication with γ. We obtain the left regular representation

$$\nu_l : \mathbf{C}[\Gamma] \longrightarrow B(l^2(\Gamma), l^2(\Gamma))$$

and the von Neumann algebra $N(\Gamma)$ is the closure of its image in the weak topology. The trace $tr : \mathbf{C}[\Gamma] \longrightarrow \mathbf{C}$ sending $\sum_{\gamma\in\Gamma} \lambda_\gamma \cdot \gamma$ to λ_e extends to the so called natural trace on $N(\Gamma)$. Notice that with respect to this trace $l^2(N(\Gamma))$ can be identified with $l^2(\Gamma)$. One can view $N(\Gamma)$ as the von Neumann algebra associated to the Hilbert algebra $\mathbf{C}[\Gamma]$ (see Dixmier [10] III.7.6.)

In particular we get using Fourier transforms for $\Gamma = \mathbf{Z}^r$ that $l^2(\mathbf{Z}^r)$ is the Hilbert space $L^2(T^r)$ of square-integrable functions on the r-dimensional torus T^r with values in $\mathbf{C} \cup \{\infty\}$ and $N(\mathbf{Z}^r)$ is the space $L^\infty(T^r)$ of almost everywhere bounded measurable functions on T^r with values in $\mathbf{C} \cup \{\infty\}$ and the regular representation is given by the pointwise multiplication of functions. An example of a weak $N(\mathbf{Z})$-automorphism which is not an automorphism is $l^2(\mathbf{Z}) \longrightarrow l^2(\mathbf{Z})$ given by multiplication with $(z-1)$ for $z \in \mathbf{Z}$ a generator.

We will deal with finite von Neumann algebras only and do not try to give the most general version of our constructions, as in the applications we will use von Neumann algebras of groups and these are always finite.

2. Algebraic K-theory of von Neumann algebras

In this section we define the algebraic K-groups of a finite von Neumann algebra $\mathcal{A}$ which will be the value groups for our torsion invariants. If $\mathcal{A}$ is an abelian von Neumann algebra, we compute them by a determinant. This computation will be crucial for applications in topology.

Let $\mathcal{A}$ be a finite von Neumann algebra. Define $K_0(\mathcal{A})$ to be the abelian group generated by isomorphism classes of finitely generated Hilbert $\mathcal{A}$-modules satisfying the relation $[M \oplus N] = [M] + [N]$. This can be identified with the Grothendieck group of the abelian semi-group of isomorphism classes of finitely generated Hilbert $\mathcal{A}$-modules with the addition given by $\oplus$. Let $K_1(\mathcal{A})$ resp. $K_1^w(\mathcal{A})$ be the abelian group generated by conjugation classes of automorphisms, resp. weak automorphisms of finitely generated Hilbert $\mathcal{A}$-modules satisfying the following relations

- $[f] + [g] = [h]$, if there is an exact sequence of automorphisms
$$0 \longrightarrow (M, f) \xrightarrow{i} (N, g) \xrightarrow{p} (P, h) \longrightarrow 0$$
- $[g \circ f] = [f] + [g]$, if f and g are automorphisms resp. weak automorphisms of the same finitely generated Hilbert $\mathcal{A}$-module
- $[id : M \longrightarrow M] = 0$

Remark 2.1 The group $K_1(\mathcal{A})$ is the abelianization of the general linear group $GL(\mathcal{A})$. If one wants to define also higher K-groups, one can use Waldhausen's construction (see [30]) applied to the following category with weak isomorphisms and cofibrations. The underlying category is the category of finitely generated Hilbert $\mathcal{A}$-modules. Cofibrations are split injections of finitely generated Hilbert $\mathcal{A}$-modules and weak isomorphisms are isomorphisms resp. weak isomorphisms of finitely generated Hilbert $\mathcal{A}$-modules. The resulting K-groups in dimension 1 can be identified with the groups defined above. ∎

Remark 2.2 The first relation still holds, if we substitute exact sequence by weak exact sequence. The following relations are sometimes useful. We get $[f \circ g] = [g \circ f]$ for weak isomorphisms $f : M \longrightarrow N$ and $g : N \longrightarrow P$ and $[f] = [g]$ for weak isomorphisms $u : M \longrightarrow N$, $f : M \longrightarrow M$ and $g : N \longrightarrow N$ satisfying $u \circ f = g \circ u$. ∎

Taking the adjoint induces involutions $*$ on $K_1(\mathcal{A})$ and $K_1^w(\mathcal{A})$. The forgetful functor induces a homomorphism

2.3 $F : K_1(\mathcal{A}) \longrightarrow K_1^w(\mathcal{A})$

compatible with the involutions. We will be interested in the fixed point sets $K_1(\mathcal{A})^{\mathbf{Z}/2}$ and $K_1^w(\mathcal{A})^{\mathbf{Z}/2}$.

We want to compute these K-groups for abelian von Neumann algebras. Let X be a compact second-countable space together with a positive finite measure ν. Let $L^\infty(X;\nu)$ be the set of almost everywhere bounded measurable functions from X to $\mathbf{C} \cup \{\infty\}$. In the sequel we abbreviate $\{x \in X \mid |f(x)| \leq K\}$ by $\{\,|f| \leq K\}$. A measurable function is almost everywhere bounded, if there is a real number K such that $X - \{\,|f| \leq K\}$ is contained in a zero set. Define $\|f\|_\infty$ to be the infimum over such numbers K. Denote by $L^2(X;\nu)$ the (seperable complex) Hilbert space of square integrable functions on X. Given $f \in L^\infty(X;\nu)$, we define an operator $m_f : L^2(X;\nu) \longrightarrow L^2(X;\nu)$ sending g to $f \cdot g$. Let $\mathcal{A}$ be the subset of $B(L^2(X;\nu))$ consisting of all operators m_f for $f \in L^\infty(X;\nu)$. Then $\mathcal{A}$ is an abelian finite von Neumann algebra and $m : L^\infty(X;\nu) \longrightarrow \mathcal{A}$ is an isometric isomorphism of normed algebras. It is compatible with the involutions $*$ on $L^\infty(X;\nu)$ given by complex conjugation and the involution $*$ given on $\mathcal{A}$ by taking the adjoint. The abelian von Neumann algebra $L^\infty(X;\nu)$ is finite, namely, a finite faithful and normal trace is given by $tr : L^\infty(X;\nu)^+ \longrightarrow \mathbf{R}^{\geq 0}$ sending f to $\int_X f d\nu$. In the sequel we will use this trace. Notice that then $l^2(\mathcal{A})$ can be identified with $L^2(X;\nu)$. Any abelian von Neumann algebra $\mathcal{A} \subset B(H)$ for H a seperable complex Hilbert space is of this type (see Dixmier [10], I.7.3).

Consider the finite abelian von Neumann algebra $\mathcal{A} = L^\infty(X;\nu)$. Let $Inv(X;\nu)$ be the multiplicative group of almost everywhere invertible measurable functions from X to $\mathbf{C} \cup \{\infty\}$. Almost everywhere invertible means that the preimage of 0 and the preimage of ∞ are zero-sets. Any element $h \in Inv(X;\nu)$ can be written as a quotient of almost everywhere bounded invertible functions $h = \frac{h_0}{h_1}$. E.g., put $h_1(x) = 1$, if $h(x) = 0$, $h_1(x) = h(x)/\ |h(x)|$, if $0 < |\,h(x)\,| \leq 1$ and $h_1(x) = h(x)^{-1}$ otherwise, and define $h_0 = h \cdot h_1$. Let f be a almost everywhere bounded function. If f is almost everywhere invertible, then the associated operators m_f and $(m_f)^* = m_{f^*}$ are injective and hence weak isomorphisms because of $clos(im(m_f)) = ker((m_f)^*)^\perp$. Hence m_f is a weak isomorphism if and only if f is almost everywhere invertible. Therefore we can define homomorphisms :

2.4
$$i : L^\infty(X;\nu)^\times \longrightarrow K_1(\mathcal{A}) \qquad f \mapsto [m_f]$$
$$i : Inv(X;\nu) \longrightarrow K_1^w(\mathcal{A}) \qquad \frac{h_0}{h_1} \mapsto [m_{h_0}] - [m_{h_1}]$$

where $L^\infty(X;\nu)^\times$ denotes the abelian group of units, i.e. function which are almost everywhere bounded from above and below.

Consider an endomorphism $f : M \longrightarrow M$ of a finitely generated Hilbert $\mathcal{A}$-module. Because of lemma 1.6 there is a finitely generated Hilbert $\mathcal{A}$-module N, a non-negative integer n and an isomorphism of Hilbert $\mathcal{A}$-modules $\phi : M \oplus N \longrightarrow \oplus_{i=1}^{n} l^2(\mathcal{A})$. Define the endomorphism $g : \oplus_{i=1}^{n} l^2(\mathcal{A}) \longrightarrow \oplus_{i=1}^{n} l^2(\mathcal{A})$ to be $\phi \circ (f \oplus id_N) \circ \phi^{-1}$. Let $\Omega(g)$ be the (n,n)-matrix over $\mathcal{A}$ defined for g in 1.5. Define *the determinant* $det(f) \in \mathcal{A}$ to be the ordinary determinant $det(\Omega(g))$ of a quadratic matrix over a commutative ring such as $\mathcal{A}$. We leave it to the reader to check that this is independent of the choices of N, n and ϕ. We will see that this determinant inherits from the ordinary determinant for commutative rings all the expected properties. The main result of this section is:

Theorem 2.5 *Let $\mathcal{A} = L^\infty(X;\nu)$ be a finite abelian von Neumann algebra. Then the determinant induces isomorphisms :*

$$det : K_1(\mathcal{A}) \longrightarrow L^\infty(X;\nu)^\times$$
$$det : K_1^w(\mathcal{A}) \longrightarrow Inv(X;\nu)$$

These maps are compatible with the involutions given on the K-groups by taking the adjoint and on the targets by complex conjugation. The inverse maps of these isomorphisms are given by the maps of 2.4. ■

The following Lemma 2.6 and lemma 2.7 imply the main theorem 2.5 of this section. One may say in view of lemma 2.6 that its proof is based on a kind of Euclidean algorithm based on the support of functions in $L^\infty(X;\nu)$. A block matrix G is a matrix whose first row or first column consists of zero entries except the $(1,1)$-entry.

Lemma 2.6 *Let G be a (n,n)-matrix over $\mathcal{A}$ such that the associated homomorphism of Hilbert $\mathcal{A}$-modules is an isomorphism resp. weak isomorphism. Then there are block (n,n)-matrices S_0, S_1, $\dots S_n$ with the following properties :*

1.) S_i is an isomorphism for $2 \le i \le n-1$.

2.) S_0 and S_1 are isomorphisms resp. weak isomorphisms.

3.) $S_0 = S_1 \cdot S_2 \cdot \dots S_n \cdot G$.

Proof : Denote by $\| G \|_\infty$ the supremum of all $\| g_{i,j} \|_\infty$. Recall that $\| g_{i,j} \|_\infty$ is the infimum over all real numbers K for which $\{ \, | g_{i,j} | > K \}$ is a zero-set. We have :

$$\| G \cdot G' \|_\infty \le n \cdot \; \| G \|_\infty \; \cdot \; \| G' \|_\infty$$

Now fix $\epsilon \geq 0$. Given $2 \leq i \leq n$, denote by Y the set $\{ |g_{1,1}| \leq \epsilon\} \cap \{ |g_{i,1}| > \epsilon\}$. Denote by χ_Y the characteristic function of Y. Let S_i be the matrix having 1 as entries on the diagonal, $2 \cdot \chi_Y$ as (1,i)-entry and zero as other entries. Put $G' = S_i \cdot G$. Since the sum of a number of norm $\geq 2 \cdot \epsilon$ and a number of norm $\leq \epsilon$ has norm $\geq \epsilon$ by the triangle inequality, we have by construction :

$$
\begin{array}{ll}
\{ |g'_{1,1}| \leq \epsilon\} \subset \{ |g'_{i,1}| \leq \epsilon\} & \\
\{ |g'_{1,1}| \leq \epsilon\} \subset \{ |g_{1,1}| \leq \epsilon\} & \\
\{ |g'_{j,1}| \leq \epsilon\} = \{ |g_{j,1}| \leq \epsilon\} & \text{for } j \neq 1 \\
\text{ffi } \| G' \|_\infty \leq \| S \|_\infty \cdot \| G \|_\infty \cdot n \leq max\{1, 2 \cdot \| G \|_\infty \} \cdot \| G \|_\infty \cdot n &
\end{array}
$$

By iterating this process for $i = n, n-1, \ldots 2$ we obtain a matrix $\widetilde{G}$ and block matrices S_i with the following properties :

$$
\begin{array}{ll}
S_i \text{ is an isomorphism} & \\
\widetilde{G} = S_2 \cdot S_3 \ldots S_n \cdot G & \\
\{ |\widetilde{g}_{1,1}| \leq \epsilon\} \subset \{ |\widetilde{g}_{i,1}| \leq \epsilon\} & \text{for } 1 \leq i \leq n \\
\| \widetilde{G} \|_\infty \leq max\{1, \| G \|_\infty\}^{2 \cdot n} \cdot (2n)^{2 \cdot n - 1} &
\end{array}
$$

Next we finish the proof in the case, where G is a weak isomorphism. Then $\widetilde{G}$ is also a weak isomorphism. Choose ϵ to be zero. Notice that $\widetilde{G}$ maps $(\chi_{\{ |\widetilde{g}_{1,1}| \leq 0\}}, 0, \ldots 0)$ to zero. Hence $\{ | \widetilde{g}_{1,1} | \leq 0\}$ is a zero set. This shows that $\widetilde{g}_{1,1}$ is a weak isomorphism. Let S_1 be the block matrix having $\widetilde{g}_{1,1}$ as (i,i)-entry and $-\widetilde{g}_{i,1}$ as $(i,1)$-entry for $2 \leq i \leq n$, 1 as $(1,1)$-entry and zero as other entries. Then $S_0 := S_1 \cdot \widetilde{G}$ is a block matrix and S_0 and S_1 are weak isomorphisms by the implication b.) $\Rightarrow$ a.) in claim 5.) of lemma 2.7.

Finally, we deal with the case where G and hence $\widetilde{G}$ are isomorphisms. Then there are elements $h_i \in \mathcal{A}$ such that $\sum_{i=1}^{n} h_i \cdot g_{i,1} = 1$ holds. Choose $\epsilon > 0$ such that $\epsilon \cdot \sum_{i=1}^{n} \| h_i \|_\infty$ is smaller than 1. Then $\{ |\widetilde{g}_{1,1}| \leq \epsilon\}$ must be a zero-set. Therefore $g_{1,1}$ is a unit. Let S_1 be the block matrix having 1 on each diagonal entry, $-g_{i,1} \cdot g_{1,1}^{-1}$ as $(i,1)$-entry for $2 \leq i \leq n$ and zero as other entries. Then $S_0 = S_1 \cdot G$ is a block matrix and S_0 and S_1 are isomorphisms by the implication b.) $\Rightarrow$ a.) in claim 4.) of lemma 2.7. ■

Lemma 2.7 *1.) Given to endomorphisms f and g of the same finitely generated Hilbert $\mathcal{A}$-module, we get :*

$$det(f \circ g) = det(f) \cdot det(g)$$

2.) Let $f : M \longrightarrow M$ and $g : N \longrightarrow N$ be endomorphisms of finitely generated Hilbert $\mathcal{A}$-modules and $u : M \longrightarrow N$ be a weak isomorphism such that $u \circ f = g \circ u$ holds. Then *we have :*

$$det(f) = det(g)$$

3.) If $0 \longrightarrow (M,f) \xrightarrow{i} (N,g) \xrightarrow{p} (P,h) \longrightarrow 0$ *is a weakly exact sequence of endomorphisms of finitely generated Hilbert* $\mathcal{A}$*-modules, we have :*

$$det(f) \cdot det(h) = det(g)$$

4.) The following assertions are equivalent for an endomorphism $f : M \longrightarrow M$ *of a finitely generated Hilbert* $\mathcal{A}$*-module* M *:*

a.) f *is an isomorphism.*

b.) $det(f) : l^2(\mathcal{A}) \longrightarrow l^2(\mathcal{A})$ *is an isomorphism.*

c.) $det(f) \in \mathcal{A}$ *is a unit.*

5.) The following assertions are equivalent for an endomorphism $f : M \longrightarrow M$ *of a finitely generated Hilbert* $\mathcal{A}$*-module* M *:*

a.) f *is a weak isomorphism.*

b.) $m_{det(f)} : l^2(\mathcal{A}) \longrightarrow l^2(\mathcal{A})$ *is a weak isomorphism.*

c.) $det(f) \in \mathcal{A} = L^\infty(X;\nu)$ *is almost everywhere invertible.*

d.) f *is injective.*

e.) f *has dense image*

Proof : 1.) follows directly from the corresponding property of the determinant for commutative rings.

4.) Obviously $det(f_1 \oplus f_2) = det(f_1) \cdot det(f_2)$ holds. Hence we may suppose that $f$is an endomorphisms of $\oplus_{i=1}^n l^2(\mathcal{A}) = \oplus_{i=1}^n L^2(X;\nu)$. Choose a matrix

$$G = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{n\Upsilon,1} & g_{n,2} & \cdots & g_{n,n} \end{pmatrix}$$

with entries in $\mathcal{A} = L^\infty(X;\nu)$ such that f and G correspond to one another under Ω (see 1.5). By Cramer's rule there exists a n-n-matrix G' with entries in $\mathcal{A} = L^\infty(X;\nu)$ satisfying $G' \cdot G = G \cdot G' = det(g) \cdot id$. Now assertion 4.) follows.

5.) We have shown c.) $\Leftrightarrow$ b.) already before 2.4. We derive b.) $\Rightarrow$ a.) from Cramer's rule. Obviously a.) implies d.) and e.).

Next we prove a.) $\Rightarrow$ c.), by induction over the size of the matrix G associated to f. The induction begin $n = 1$ is trivial, the induction step from $n-1$ to $n \geq 2$ done as follows. Let f be a weak isomorphism. Because of lemma 2.6 we can assume without of loss of generality that G is given by a block matrix of the following shape

$$G = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ 0 & & & \\ \vdots & & \widehat{G} & \\ 0 & & & \end{pmatrix}$$

where $\widehat{G}$ is a $(n-1)$-$(n-1)$-matrix. Then we get:

$$det(G) = det(\widehat{G}) \cdot g_{1,1}$$

As G defines a weak isomorphism, the morphism given by $g_{1,1}$ is injective and the one given by $\widehat{G}$ has dense image. By induction hypothesis $det(\widehat{G})$ and $g_{1,1}$ are almost everywhere invertible and hence the same is true for $det(G)$. This shows a.) $\Rightarrow$ c.).

Next we show d.) $\Rightarrow$ c.). If f is injective, f^*f is injective because of the following calculation for $x \in ker(f^*f)$

$$0 = \langle f^*f(x), x\rangle = \langle f(x), f(x)\rangle = |\, f(x)\,|^2 \Rightarrow \; f(x) = 0 \; \Rightarrow \; x = 0$$

As f^*f is selfadjoint, f^*f is a weak isomorphism so that $det(f^*f) = |\, det(f)\,|^2$ and hence $det(f)$ are almost everywhere invertible by a.) $\Rightarrow$ c.). This finishes the proof of d.) $\Rightarrow$ c.). If f has dense image, f^* is injective and hence $det(f) = det(f^*)^*$ is almost everywhere invertible. This shows e.) $\Rightarrow$ c.) and hence 5.) is true.

2.) follows from 1. and 5. Namely, because $det(u)$ is almost everywhere invertible, $det(f) \cdot det(u) = det(u) \cdot det(g)$ implies $det(g) = det(f)$.

3.) Let $0 \longrightarrow (ker(p), g_1) \longrightarrow N \longrightarrow (ker(p)^{\perp}, g_2) \longrightarrow 0$ be the exact sequence of endomorphisms induced from the given weakly exact sequence. We derive from the corresponding statement for determinants of commutative rings that $det(g_1) \cdot det(g_2) = det(g)$ holds. We conclude $det(f) = det(g_1)$ and $det(h) = det(g_2)$ from assertion 2.). ∎

Next we give the proof of theorem 2.5. Obviously det is a well-defined homomorphism and $det \circ i = id$ if i is the map defined in 2.4. It remains to show surjectivity of i. We prove inductively over the size of a quadratic matrix G representing an element $[G]$ in the K_1-group that $[G]$ is given by a sum of elements represented by $(1,1)$-matrices. Because of lemma 2.6 we may assume that G is a block matrix. But then $[G]$ is the sum of elements given by matrices of smaller size. ∎

We now drop the assumption that $\mathcal{A}$ is abelian. There is a well-defined notion of a tensor product $\mathcal{A} \otimes \mathcal{B}$ of von Neumann algebras. Let $\mathcal{A}$ and $\mathcal{B}$ be finite von Neumann algebras with given finite normal faithful traces. Then $\mathcal{A} \otimes \mathcal{B}$ is a finite von Neumann algebra and inherits a finite normal faithful trace. There is a natural isomorphism $l^2(\mathcal{A}) \otimes l^2(\mathcal{B}) \longrightarrow l^2(\mathcal{A} \otimes \mathcal{B})$. If M resp. N is a Hilbert $\mathcal{A}$ resp. $\mathcal{B}$- module, then the tensor product of Hilbert spaces $M \otimes N$ comes with a canonical Hilbert $\mathcal{A} \otimes \mathcal{B}$-structure. If M and N are finitely generated, then also $M \otimes N$. This tensor product of Hilbert modules over finite von Neumann algebras is functorial. If $\mathcal{A} = N(A)$ and $\mathcal{B} = N(B)$ for countable discrete groups A and B, then $\mathcal{A} \otimes \mathcal{B}$ can be identified with $N(A \times B)$. We obtain a pairings

2.8 $\otimes : K_0(\mathcal{A}) \otimes K_1(\mathcal{B}) \longrightarrow K_1(\mathcal{A} \otimes \mathcal{B})$
$\otimes : K_0(\mathcal{A}) \otimes K_1^w(\mathcal{B}) \longrightarrow K_1^w(\mathcal{A} \otimes \mathcal{B})$
$\otimes : K_0(\mathcal{A}) \otimes K_1(\mathcal{B})^{\mathbf{Z}/2} \longrightarrow K_1(\mathcal{A} \otimes \mathcal{B})^{\mathbf{Z}/2}$
$\otimes : K_0(\mathcal{A}) \otimes K_1^w(\mathcal{B})^{\mathbf{Z}/2} \longrightarrow K_1^w(\mathcal{A} \otimes \mathcal{B})^{\mathbf{Z}/2}$

sending $[M] \otimes [f : N \longrightarrow N]$ to $[id_M \otimes f : M \otimes N \longrightarrow M \otimes N]$.

Proposition 2.9 *Let G be a finite group. Then $K_0(N(G))$ is the complex representation ring $Rep_{\mathbf{C}}(G)$ and the pairing 2.8 induces isomorphisms*

$$\begin{aligned} Rep_{\mathbf{C}}(G) \otimes K_1(\mathcal{A}) &\longrightarrow K_1(\mathbf{C}[G] \otimes \mathcal{A}) \\ Rep_{\mathbf{C}}(G) \otimes K_1^w(\mathcal{A}) &\longrightarrow K_1^w(\mathbf{C}[G] \otimes \mathcal{A}) \end{aligned}$$

Proof : We construct the inverse isomorphism. Let $f : M \longrightarrow M$ be a (weak) automorphism of a finitely generated Hilbert $\mathbf{C}[G] \otimes \mathcal{A}$- module. Let I be a complete set of representatives of the isomorphism classes of irreducible unitary G-representations. Let $\bar{V}$ be the dual unitary G-representation of V. Equip $(\bar{V} \otimes M)^G$ with the induced Hilbert $\mathcal{A}$-structure. The inverse map sends $[f]$ to the sum $\sum_{V \in I} [V] \otimes [(id_{\bar{V}} \otimes f)^G]$. This is an inverse as there is a natural isometric $\mathbf{C}[G] \otimes \mathcal{A}$-isomorphism from $\sum_{V \in I} V \otimes (\bar{V} \otimes M)^G$ to M sending $v \otimes w \otimes m \in V \otimes (\bar{V} \otimes M)^G$ to $\langle v, w \rangle \cdot m$. ■

Corollary 2.10 *Let G be finite group and r be a non-negative integer. Then there are isomorphisms :*

$$\begin{array}{lcl} K_1(N(G \times \mathbf{Z}^r)) & \longrightarrow & Rep_{\mathbf{C}}(G) \otimes L^\infty(T^r, \mathbf{C} \cup \{\infty\})^\times \\ K_1(N(G \times \mathbf{Z}^r))^{\mathbf{Z}/2} & \longrightarrow & Rep_{\mathbf{C}}(G) \otimes L^\infty(T^r, \mathbf{R} \cup \{\infty\})^\times \\ K_1^w(N(G \times \mathbf{Z}^r)) & \longrightarrow & Rep_{\mathbf{C}}(G) \otimes Inv(T^r, \mathbf{C} \cup \{\infty\}) \\ K_1^w(N(G \times \mathbf{Z}^r))^{\mathbf{Z}/2} & \longrightarrow & Rep_{\mathbf{C}}(G) \otimes Inv(T^r, \mathbf{R} \cup \{\infty\}) \end{array}$$ ■

3. Torsion invariants for Γ-CW-complexes

In this section we want to introduce Reidemeister von Neumann torsion for finite proper Γ-CW-complexes for a discrete group Γ. We state basic properties like sum and product formulas and relate Reidemeister von Neumann torsion to Whitehead torsion. As an illlustration we compute the Reidemeister von Neumann torsion of spaces carrying appropiate torus actions. Most of the technical proofs are deferred to sections 6 and 7. The definition of Reidemeister von Neumann torsion in this section has the advantage that it is easy to state. For technical purposes, however, we will introduce a different definition in section 7 which will be shown to be equivalent to the one in this section. If one is only interested in the case where Γ acts freely, one may skip the first part and start directly with definition 3.6.

Let Γ be a discrete group. A Γ-*CW-complex* X is a Γ-space X together with a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset X_n \subset \ldots \subset X$ such that X has the weak topology with respect to the filtration $\{X_n \mid n \geq -1\}$ and for each $n \geq 0$ there exists a Γ-push out

3.1
$$\begin{tikzcd}
\coprod_{i\in I_n} \Gamma/\Gamma_i \times S^{n-1} \arrow[r, "\coprod_{i\in I_n} q_i^n"] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i\in I_n} \Gamma/\Gamma_i \times D^n \arrow[r, "\coprod_{i\in I_n} Q_i^n"] & X_n
\end{tikzcd}$$

A Γ-CW-complex X is *finite* , if the set $\coprod_{n\geq 0} I_n$ is finite and X is *proper,* if the map $\Gamma \times X \longrightarrow X \times X$ sending (γ, x) to $(x, \gamma \cdot x)$ is proper. Since we are working in the category of compactly generated spaces, a map is proper if and only if preimages of compact sets are compact. A Γ-CW-complex X is finite if and only if $\Gamma\backslash X$ is compact. It is proper if and only if the isotropy subgroup $\Gamma_x = \{\gamma \in \Gamma \mid \gamma x = x\}$ is finite for all $x \in X$ (see Lück [19], theorem 1.23). Since Γ is discrete, the Γ-action is proper if and only it is properly discontinuos in the sense of the introduction. Notice that the Γ-push outs appearing in the definition of a Γ-CW-complex are not part of the structure. The universal covering of a compact CW-complex X is a proper free $\pi_1(X)$-CW-complex. More generally we have the following examples.

Example 3.2 Let G be a finite group and X a compact G-CW-complex. Denote by $p : \widetilde{X} \longrightarrow X$ the universal covering of X and identify $\pi = \pi_1(X, x)$ with the group of deck transformations. Let Γ be the discrete group

3.3 $\Gamma := \{(\widetilde{f}, g) \mid f : \widetilde{X} \to \widetilde{X}, g \in G, p \circ \widetilde{f} = l(g) \circ p\}$

where $l(g) : X \to X$ denotes multiplication with g. There is an obvious exact sequence $0 \longrightarrow \pi \xrightarrow{i} \Gamma \xrightarrow{q} G \longrightarrow 0$ and an operation of Γ on $\widetilde{X}$ making the following diagram commute

$$\begin{tikzcd}
\pi \times \widetilde{X} \arrow[r] \arrow[d, "i \times id"] & \widetilde{X} \arrow[d, "id"] \\
\Gamma \times \widetilde{X} \arrow[r] \arrow[d, "q \times p"] & \widetilde{X} \arrow[d, "p"] \\
G \times X \arrow[r] & X
\end{tikzcd}$$

The G-CW-structure on X induces a finite proper Γ-CW-structure on $\widetilde{X}$. ∎

Next we prepare the definition of the cellular Hilbert $\mathcal{A}$-chain complex. Let P be a finitely generated projective $\mathbf{C}[\Gamma]$-module. Consider left Γ-sets T and S. We call two $\mathbf{C}[\Gamma]$-isomorphisms $\alpha : P \longrightarrow \mathbf{C}[T]$ and $\beta : P \longrightarrow \mathbf{C}[S]$ *equivalent,* if there is a bijective Γ-map $f : S \longrightarrow T$ and a Γ-invariant map $\epsilon : S \longrightarrow \{\pm 1\}$ such that $\beta \circ \alpha^{-1} : \mathbf{C}[T] \longrightarrow \mathbf{C}[S]$ sends $\sum_{t \in T} \lambda_t \cdot t$ to $\sum_{s \in S} \epsilon(s) \cdot \lambda_{f(s)} \cdot s$. A *permutation* $\mathbf{C}[\Gamma]$- *module* is a $\mathbf{C}[\Gamma]$-module together with a choice of equivalence classes of $\mathbf{C}[\Gamma]$-isomorphisms $\alpha : P \longrightarrow \mathbf{C}[T]$ for some Γ-set T. Notice for a Γ-set T that the $\mathbf{C}[\Gamma]$-module $\mathbf{C}[T]$ is finitely generated and projective if and only if $\Gamma \backslash T$ is finite and the isotropy groups of elements in T under the Γ-action are all finite.

Let X be a finite proper Γ-CW-complex. Then the cellular $\mathbf{C}[\Gamma]$-chain complex $C_*(\widetilde{X})$ inherits from the Γ-CW-structure the structure of a finite permutation $\mathbf{C}[\Gamma]$-chain complex for all $n \geq 0$. Namely, an explicit choice of Γ-push outs 3.1 together with a choice of generators of the homology groups $H_n(D^n, S^{n-1}; \mathbf{Z})$ determine a $\mathbf{C}[\Gamma]$-isomorphism $C_n(\widetilde{X}) \longrightarrow \mathbf{C}[\coprod_{i \in I_n} \Gamma/H_i]$. The associated structure of a permutation $\mathbf{C}[\Gamma]$-module depends only on the Γ-CW-structure of X (see Lück [19] lemma 13.2).

Let $\mathcal{A}$ be a finite von Neumann algebra with a finite normal faithful trace. Let V be a finitely generated Hilbert $\mathcal{A}$-module. A *unitary representation of* Γ *in* $Iso_{\mathcal{A}}(V)^{op}$

3.4 $\mu : \Gamma \longrightarrow Iso_{\mathcal{A}}(V)^{op}$

is a group homomorphism from Γ into the opposite group of the group of isometric $\mathcal{A}$-automorphisms of V. The associated algebra homomorphism is denoted in the same way by $\mu : \mathbf{C}[\Gamma] \longrightarrow End_{\mathcal{A}}(V)^{op}$. Notice that Γ acts from the left on $\mathbf{C}[T]$ and from the right on V. Hence the tensor product $V \otimes_{\mathbf{C}[\Gamma]} \mathbf{C}[T]$ is defined. If we consider $Hom_{\mathbf{C}[\Gamma]}(\mathbf{C}[T], V)$ we use on V the induced left module structure given by $\gamma \cdot v := v \cdot g^{-1}$ for $\gamma \in \Gamma$ and $v \in V$.

Lemma 3.5 *Let P and Q be a finitely generated projective permutation $\mathbf{C}[\Gamma]$-modules and $f : P \longrightarrow Q$ a $\mathbf{C}[\Gamma]$-homomorphism. Then :*

1.) *There are preferred structures of a finitely generated Hilbert $\mathcal{A}$-module on $V \otimes_{\mathbf{C}[\Gamma]} P$ and $Hom_{\mathbf{C}[\Gamma]}(P, V)$.*

2.) *The induced maps $id_V \otimes_{\mathbf{C}[\Gamma]} f$ and $Hom_{\mathbf{C}[\Gamma]}(f, id_V)$ are homomorphisms of Hilbert $\mathcal{A}$-modules.*

3.) *There is an isometric isomorphism $D(P) : V \otimes_{\mathbf{C}[\Gamma]} P \longrightarrow Hom_{\mathbf{C}[\Gamma]}(P, V)$. It is natural, i.e., $Hom_{\mathbf{C}[\Gamma]}(f, id_V) \circ D(Q)$ and $D(P) \circ (id_V \otimes_{\mathbf{C}[\Gamma]} f)^*$ agree.*

Proof : Consider a Γ-set T such that $\Gamma\backslash T$ is finite and the isotropy group Γ_t of any $t \in T$ under the Γ-action is finite. Now $V \otimes_{\mathbf{C}[\Gamma]} \mathbf{C}[T]$ and $Hom_{\mathbf{C}[\Gamma]}(\mathbf{C}[T], V)$ inherit $\mathcal{A}$-module structures from V. As V is a unitary representation, there is a Hilbert structure on $V \otimes_{\mathbf{C}[\Gamma]} \mathbf{C}[T]$ given by

$$\langle m \otimes_{\mathbf{C}[\Gamma]} t, n \otimes_{\mathbf{C}[\Gamma]} s \rangle \ := \frac{1}{|\Gamma_t|} \cdot \sum_{\{\gamma \in \Gamma | \gamma \cdot t = s\}} \langle \gamma \cdot m, n \rangle$$

and on $Hom_{\mathbf{C}[\Gamma]}(\mathbf{C}[T], V)$ given by

$$\langle \phi, \psi \rangle \ = \sum_{\Gamma t \in \Gamma \backslash T} \langle \phi(t), \psi(t) \rangle$$

These data determine the preferred structure of a finitely generated Hilbert $\mathcal{A}$-module. One easily checks that the maps induced by f are continuous.

The isometric Hilbert $\mathcal{A}$-isomorphism $D(\mathbf{C}[T]) : V \otimes_{\mathbf{C}[\Gamma]} \mathbf{C}[T] \longrightarrow Hom_{\mathbf{C}[\Gamma]}(\mathbf{C}[T], V)$ sends $v \otimes_{\mathbf{C}[\Gamma]} t$ to the $\mathbf{C}[\Gamma]$-map from $\mathbf{C}[T]$ to V which assigns $| \Gamma_t |^{-1} \cdot \sum_{\{\gamma \in \Gamma | \gamma \cdot t = s\}} v \cdot \gamma^{-1}$ to $s \in T$. ■

Notation 3.6 Given a pair (X, X_1) of proper Γ-CW-complexes, let $C_*(X, X_1; V)$ be the Hilbert $\mathcal{A}$-chain complex $V \otimes_{\mathbf{C}[\Gamma]} C_*(X, X_1)$. Denote by $H_*(X, X_1; V)$ its homology. Define the Hilbert $\mathcal{A}$-cochain complex $C^*(X, X_1; V)$ by $Hom_{\mathbf{C}[\Gamma]}(C_*(X, X_1), V)$. Denote its cohomology by $H^*(X, X_1; V)$. ∎

For the definition of Reidemeister von Neumann torsion we need the following result. Let C be a finite Hilbert $\mathcal{A}$-chain complex. Its Laplace operator $\Delta_n : C_n \longrightarrow C_n$ in dimension n is given by $c_{n+1} \circ c_{n+1}^* + c_n^* \circ c_n$. Let $\mathcal{H}_n(C)$ be the kernel of Δ_n.

Theorem 3.7 (Hodge decomposition theorem) *Let C be a finite Hilbert $\mathcal{A}$- chain complex. Then:*

1.) $C_n = \mathcal{H}_n(C) \oplus clos(im(c_n^*)) \oplus clos(im(c_{n+1}))$

2.) *We have $H_n^{harm}(C) \subset ker(c_n)$. The natural projection induces an isometric isomorphism :*
$$\mathcal{H}_n(C) \longrightarrow H_n(C)$$

Proof : As Δ_n is selfadjoint, we have :

$$\mathcal{H}_n(C) \oplus clos(im(\Delta_n)) = C_n$$

Since $\langle c_n^*(x), c_{n+1}(y)\rangle = \langle x, c_n c_{n+1}(y)\rangle$ holds, the spaces $clos(im(c_n^*))$ and $clos(im(c_{n+1}))$ are orthogonal. For $x \in \mathcal{H}_n(C)$ we compute :

$$0 = \langle x, \Delta_n(x)\rangle = \langle c_n(x), c_n(x)\rangle + \langle c_{n+1}^*(x), c_{n+1}^*(x)\rangle$$

As x lies in $ker(c_n) = clos(im(c_n^*))^{\perp}$ and in $ker(c_{n+1}^*) = clos(im(c_{n+1}))^{\perp}$, we get

$$clos(im(c_n^*))^{\perp} \cap clos(im(c_{n+1}))^{\perp} = H_n^{harm}(C)$$

This implies :

$$clos(im(c_n^*)) \oplus clos(im(c_{n+1})) = \mathcal{H}_n(C)^{\perp} = clos(im(\Delta_n)) \qquad ∎$$

Let C be a finite Hilbert $\mathcal{A}$-chain complex. Let C' be the orthogonal complement of $\mathcal{H}(C)$ in C. We have $\Delta' \oplus 0 = \Delta$ if Δ' resp. Δ are the Laplace operators of C' and C. Then the morphism Δ' is a selfadjoint weak automorphism and defines an element $[\Delta_n']$ in $K_1^w(\mathcal{A})^{\mathbf{Z}/2}$.

Definition 3.8 *Define the Reidemister torsion of C by:*

$$\rho(C) = -\sum_n (-1)^n \cdot n \cdot [\Delta'_n] \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

Definition 3.9 *Let (X, X_1) be a pair of finite proper Γ-CW-complexes. Consider a finite von Neumann algebra $\mathcal{A}$ with finite normal and faithful trace. Let V be a finitely generated Hilbert $\mathcal{A}$-module and $\mu : \Gamma \longrightarrow Iso_{\mathcal{A}}(V)^{op}$ be a unitary representation of Γ. Define* Reidemeister-von Neumann torsion *of X*

$$\rho(X, X_1; V) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

by the Reidemeister von Neumann torsion $\rho(C_*(X, X_1; V))$ of $C_*(X, X_1; V)$ (see definition 7.10). ■

As it stands, the definition above makes only sense for connected X, but can easily be generalized to arbitrary X. For a component C of X let Γ_C be its isotropy group under the Γ-action on $\pi_0(X)$. Suppose we have assigned to any component a representation $\mu_C : \Gamma_C \longrightarrow Iso_{\mathcal{A}}(V_C)^{op}$ such that V_C and $V_{\gamma C}$ agree for any $\gamma \in \Gamma$ and $\mu_{\gamma C} \circ c_\gamma = \mu_C$ holds for the homomorphism $c_\gamma : \Gamma_C \longrightarrow \Gamma_{\gamma C}$ sending δ to $\gamma\delta\gamma^{-1}$. We define $\rho(X, X_1; V)$ to be the sum $\sum_{\Gamma C \in \Gamma \backslash \pi_0(X)} \rho(C, C \cap X_1; V_C)$.

We prefer chain complexes instead of cochain complexes in this definition, although later we have to deal with deRham cohomology and hence with cochain complexes. In many cases of interest the spaces are acyclic, so that no cohomology is involved, and it is more convenient to deal with the cellular chain complex instead of the cochain complex since this is done in related classical cases like Whitehead and Reidemeister torsion. In principal there is no difference because of the following lemma. Recall that the dual cochain complex $(C_*)^*$ of a Hilbert $\mathcal{A}$-chain complex C_* has the same underlying chain modules and the codifferentials are the adjoint of the differentials.

Lemma 3.10 *There are a natural isometric Hilbert $\mathcal{A}$-isomorphisms*

$$(C_*(X, X_1; V))^* \longrightarrow C^*(X, X_1; V) \qquad \textit{and} \qquad (H_*(X, X_1; V))^* \longrightarrow H^*(X, X_1; V)$$

Proof : The first identification comes from lemma 3.5. Let C be a finite Hilbert $\mathcal{A}$-chain complex.We obtain from the Hodge decomposition theorem 3.7 a natural identification $H(C^*) = \mathcal{H}(C^*) = \mathcal{H}(C) = H(C)$ and thus the second identification from the first. ■

If $(f, f_1) : (X, X_1) \longrightarrow (Y, Y_1)$ is a Γ-homotopy equivalence of pairs of finite proper Γ-CW-complexes, its equivariant Whitehead torsion

$$\tau^\Gamma(f, f_1) \in Wh(\mathbf{Z}\Pi/(\Gamma, Y))$$

is defined in Lück [19], page 68, page 284 (see also Illman [15] and Dovermann-Rothenberg [13] in the case of a finite group Γ). This is an important invariant, e.g. for the classification of smooth G-manifolds. The following result reflects the general idea that Whitehead torsion is the difference of Reidemeister torsion. The map (f, f_1) induces a weak isomorphism on homology

$$H_n(f, f_1; V) : H_n(X, X_1; V) \longrightarrow H_n(Y, Y_1; V)$$

It defines an element $[[H_n(f, f_1; V)]]$ in $K_1^w(\mathcal{A})^{\mathbf{Z}/2}$ by the class of the weak automorphism $H_n(f, f_1; V)^* \circ H_n(f, f_1; V)$ of the finitely generated Hilbert $\mathcal{A}$-module $H_n(X, X_1; V)$ (see notation 7.2).

Theorem 3.11 *There is a natural homomorphism*

$$\Phi = \Phi(Y, V) : Wh(\mathbf{Z}\Pi/(\Gamma, Y)) \longrightarrow K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

such that

$$\Phi(\tau^\Gamma(f, f_1)) - \sum_n (-1)^n \cdot [[H_n(f, f_1; V)]] = \rho(Y, Y_1; V) - \rho(X, X_1; V) \qquad \blacksquare$$

We do not give the precise definition of Φ here, its construction is obvious, if one is familar with the language of modules over the fundamental category as developed in Lück [19]. In the case, where Γ acts freely, $Wh(\mathbf{Z}\Pi/(\Gamma, Y))$ reduces to the ordinary Whitehead group $Wh(\Gamma)$ and Φ sends an element in $Wh(\Gamma)$ represented by the automorphism g of $\oplus_m \mathbf{Z}[\pi]$ to $[[id_V \otimes_{\mathbf{Z}[\Gamma]} g]]$. The following conclusion is of particular interest:

Corollary 3.12 *If (X, X_1) is V-acyclic, i.e. $H(X, X_1; V)$ is trivial, then $\rho(X, X_1; V)$ depends only the simple Γ-homotopy type of (X, X_1).*

The next two results are the basic tools for computations. Consider the cellular Γ-push out of finite proper Γ-CW-complexes, where i_1 is an inclusion of finite proper Γ-CW-complexes.

3.13
$$\begin{tikzcd}
X_0 \arrow[r, "i_2"] \arrow[d, "i_1"'] \arrow[dr, "j_0"] & X_2 \arrow[d, "j_2"] \\
X_1 \arrow[r, "j_1"] & X
\end{tikzcd}$$

We get an exact sequence of finite Hilbert $\mathcal{A}$-chain complexes

$$\{0\} \longrightarrow C(X_0; j_0^*V) \xrightarrow{i_{1*} \oplus i_{2*}} C(X_1; j_1^*V) \oplus C_*(X_2; j_2^*V) \xrightarrow{j_{1*} - j_{2*}} C_*(X; V) \longrightarrow \{0\}$$

Denote by $LHS(X; X_1, X_2, X_0; V)$ the long weakly exact homology sequence of the sequence above (see 6.1). If $k : Y_1 \longrightarrow Y$ is the inclusion of a pair of proper finite G-CW-complexes, let $LHS(Y, Y_1; V)$ be the weakly exact long homology sequence of the pair. We derive from theorem 7.16.

Theorem 3.14 (Sum formula) *We have :*

$$\rho(X; V) = \rho(X_1; j_1^*V) + \rho(X_2, j_2^*V) - \rho(X_0; j_0^*V) + \rho(LHS(X; X_1, X_2, X_0; V))$$
$$\rho(Y, Y_1; V) = \rho(Y; V) - \rho(Y_1; k^*V) + \rho(LHS(Y, Y_1; V)) \qquad \blacksquare$$

Let Γ_1 and Γ_2 be discrete groups. Let (X, X_1) resp. (Y, Y_1) be pairs of a finite proper Γ_1- resp. Γ_2-CW-complexes. Then the product space $(X, X_1) \times (Y, Y_1)$ inherits the structure of a finite proper $\Gamma_1 \times \Gamma_2$-CW-complex. Consider finite von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ with finite normal and faithful traces. Let $\Gamma_1 \longrightarrow Iso_{\mathcal{A}}(V)^{op}$ and $\Gamma_2 \longrightarrow Iso_{\mathcal{B}}(W)^{op}$ be unitary representations. Define

3.15 $\chi^{\mathcal{A}}(X, X_1; V) \in K_0(\mathcal{A})$

by $\chi^{\mathcal{A}}(C(X, X_1; V)) = \sum_n (-1)^n \cdot [C_n(X, X_1; V)] = \sum_n (-1)^n \cdot [H_n(X, X_1; V)]$ (see 7.14). If Γ_1 acts freely on X, then $\chi^{\mathcal{A}}(X; V)$ reduces to $\chi(\Gamma_1\backslash X, \Gamma_1\backslash X_1) \cdot [V]$, where the integer $\chi(\Gamma_1\backslash X, \Gamma_1\backslash X_1)$ is the ordinary Euler characteristic. Because there is an isometric $\mathcal{A} \otimes \mathcal{B}$-isomorphism from $C_*(X, X_1; V) \otimes C_*(Y, Y_1; W)$ to $C_*((X; X_1) \times (Y, Y_1); V \otimes W)$, we derive from lemma 7.13 and lemma 7.15.

Theorem 3.16 (Product formula) *We get under the assumptions above:*

1.) $\rho((X,X_1)\times(Y,Y_1);V\otimes W) = \chi^{\mathcal{A}}(X,X_1;V)\otimes\rho(Y,Y_1;W)+\rho(X,X_1;V)\otimes\chi^{\mathcal{B}}(Y,Y_1;W)$

2.) $H_*(X,X_1;V)\otimes H_*(Y,Y_1;W)$ *and* $H_*((X;X_1)\times(Y,Y_1);V\otimes W)$ *are isometrically* $\mathcal{A}\otimes\mathcal{B}$*-isomorphic.* ■

Let $i:\Gamma_0\longrightarrow\Gamma$ be the canonical inclusion of a subgroup Γ_0 of Γ of finite index. Let (X,X_1) be a pair of finite Γ-CW-complexes. The restriction $(res(X),res(X_1))$ to Γ_0 inherits the strucure of a pair of finite Γ_0-CW-complexes. Let $\mu:\Gamma_0\longrightarrow Iso_{\mathcal{A}}(V)^{op}$ be a unitary representation and $i_*\mu:\Gamma\longrightarrow Iso_{\mathcal{A}}(V\otimes_{\mathbf{C}[\Gamma_0]}\mathbf{C}[\Gamma])$ be the induced one.

Lemma 3.17 (Restriction formula) *We obtain under the conditions above :*

1.) $H_*(res(X),res(X_1);V)=H_*(X,X_1;i_*V)$ *and*

$$\rho(res(X),res(X_1);V)=\rho(X,X_1;i_*V)$$

2.) *There is a natural restriction homomorphism* $res:K_1^w(N(\Gamma))\longrightarrow K_1^w(N(\Gamma_0))$ *which sends* $\rho(X,X_1;l^2(\Gamma))$ *to* $\rho(res(X),res(X_1);l^2(\Gamma_0))$.

Proof : The first assertion follows from the existence of an isometric Hilbert $\mathcal{A}$-chain isomorphism $C(res(X),res(X_1);V)\longrightarrow C(X,X_1;i_*V)$. Any finitely generated Hilbert $N(\Gamma)$-module V can be viewed by restriction as a finitely generated Hilbert $N(\Gamma_0)$-module denoted by $res(V)$ and similar for morphisms. This induces the restriction homomorphism $res:K_1^w(N(\Gamma))\longrightarrow K_1^w(N(\Gamma_0))$ Obviously it sends $\rho(X,X_1;l^2(\Gamma))$ to $\rho(X,X_1;res(l^2(\Gamma)))$. One easily checks that the representations of Γ given by $res(l^2(\Gamma))$ and $i_*l^2(\Gamma_0)$ are conjugate by an isometric isomorphism of Hilbert $N(\Gamma_0)$-modules. Hence $\rho(X,X_1;res(l^2(\Gamma)))$ and $\rho(X,X_1;i_*l^2(\Gamma_0))$ agree and the second assertion follows from the first. ■

For the remainder of the section we use

Notation 3.18 If X is a finite CW-complex and $\mu:\pi_1(X)\longrightarrow Iso_{\mathcal{A}}(V)^{op}$ is a unitary representation of its fundamental group, we write $\rho(X;V)$ instead of $\rho(\widetilde{X};V)$ and $C_*(X;V)$ for $C_*(\widetilde{X};V)$ ■

We want to illustrate these computational tools by determing the torsion for spaces carrying a torus action. As a preliminary we need the following lemma.

Lemma 3.19 *Let T be the n-dimensional torus for $n \geq 1$ and $\mu : \pi_1(T) \longrightarrow Iso_{\mathcal{A}}(V)$ a unitary representation. Suppose the existence of $w \in \pi_1(T)$ such that $w \neq 0$ and the endomorphism $(\mu(w) - 1) : V \longrightarrow V$ is a weak isomorphism. Then:*

1.) T is weakly V-acyclic, i.e. $H_*(T; V) = \{0\}$.

2.) $\rho(T; V) = 0$ for $n \geq 2$.

3.) If $n = 1$ and $w_1 \in \pi_1(T)$ is a generator we get:

$$\rho(T; V) = [[(\mu(w_1) - 1) : V \longrightarrow V]]$$

Proof : Choose a decomposition $T = S^1 \times T'$ where T' is the $n-1$-dimensional torus such that $w = w_1^m$ for some m and w_1 given by $S^1 \times \{*\}$. Since $(\mu(w) - 1)$ is a weak isomorphism by assumption and $(\mu(w_1) - 1) \circ \left(\sum_{j=0}^{m-1} \mu(w_1^j)\right) = \left(\sum_{j=0}^{m-1} \mu(w_1^j)\right) \circ (\mu(w_1) - 1) = (\mu(w) - 1)$ is true, $(\mu(w_1) - 1)$ is a weak isomorphism. Because $C(\widetilde{T}) = C(\widetilde{S^1}) \otimes_{\mathbf{Z}} C(\widetilde{T'})$ is the mapping cone of $id \otimes (w_1 - 1) : \mathbf{Z}[\mathbf{Z}] \otimes_{\mathbf{Z}} C(\widetilde{T'}) \longrightarrow \mathbf{Z}[\mathbf{Z}] \otimes_{\mathbf{Z}} C(\widetilde{T'})$. Therefore $C(T; V)$ is the mapping cone of

$$id_V \otimes_{\mathbf{Z}[\pi_1(T)]} (id \otimes (w_1 - 1)) : V \otimes_{\mathbf{Z}[\pi_1(T)]} \left(\mathbf{Z}[\mathbf{Z}] \otimes_{\mathbf{Z}} C(\widetilde{T'})\right) \longrightarrow V \otimes_{\mathbf{Z}[\pi_1(T)]} \left(\mathbf{Z}[\mathbf{Z}] \otimes_{\mathbf{Z}} C(\widetilde{T'})\right)$$

In dimension r this chain map is up to conjugation with an isometric isomorphism of Hilbert $\mathcal{A}$-modules given by the weak automorphism $(\mu(w_1) - 1) : \oplus_{b(r)} V \longrightarrow \oplus_{b(r)} V$, where $b(r)$ is the rank of $C_r(\widetilde{T'})$ over $\mathbf{Z}[\pi_1(T')]$. We conclude from the long weakly exact homology sequence (see theorem 6.1 and corollary 6.2) that $C(T; V)$ is weakly V-acyclic. We derive from the sum formula for Reidemeister von Neumann torsion in theorem 7.16:

$$\rho(T; V) = \sum_{r \geq 0} (-1)^r \cdot b(r) \cdot [[(\mu(w_1) - 1) : V \longrightarrow V]] = \chi(T') \cdot [[(\mu(w_1) - 1) : V \longrightarrow V]]$$

Now the claim follows since $\chi(T')$ is zero for $n \geq 2$ and one for $n = 1$. ■

Let X be a finite T-CW-complex for T the n-dimensional torus. (The definition of Γ-CW-complex above carries over to Γ a Lie group). Consider a unitary representation $\mu : \pi_1(X) \longrightarrow Iso_{\mathcal{A}}(V)^{op}$. The space X itself does not carry a canonical CW-complex structure, but has a well-defined simple structure in the sense of Lück [19], page 74 and section 7. Hence $\rho(X; V)$ is well-defined, provided that X is V-acyclic (compare corollary 3.12). Denote by r the rank of the abelian group $im(\pi_1(T) \longrightarrow \pi_1(X))$. Let X^H be $\{x \in X \mid H \subset T_x\}$ and define $X^{>H}$ by $\{x \in X \mid H \subset T_x, H \neq T_x\}$, where $T_x \subset T$ denotes the isotropy group at $x \in X$. Let I be the set of subgroups H of T of dimension $n-1$. Consider $H \in I$ and a component C of X^H. Given a base point in C, the orbit through X is a circle and thus defines after choosing an orientation $w(H,C) \in \pi_1(X)$. Make the assumption that the endomorphism $\mu(w(H,C)-1) : V \longrightarrow V$ is a weak isomorphism. Hence it defines a class $[[(\mu(w(H,C)-1) : V \longrightarrow V]]$ in $K_1^w(\mathcal{A})^{\mathbf{Z}/2}$. Notice that the assumption as well as the class $[[(\mu(w(H,C)-1)]]$ are independent of the choice of base point and choice of orientation of the orbit. Namely, given a weak automorphism f and an isomorphism g with the same source then $[[f]] = [[f^*]] = [[g \circ f \circ g^{-1}]]$ and changing the base point corresponds to conjugating with g and changing the orientation corresponds to taking the adjoint of f.

Theorem 3.20 *Let T be the n-dimensional torus for $n \geq 1$ and $\mu : \pi_1(T) \longrightarrow Iso_{\mathcal{A}}(V)^{op}$ be a unitary representation. Let X be a finite T-CW-complex. Assume that there is an element w in $im(\pi_1(T) \longrightarrow \pi_1(X))$ such that $w \neq 0$ and $(\mu(w)-1) : V \longrightarrow V$ is a weak isomorphism. (This assumption is fullfilled for $\mathcal{A} = N(\pi_1(X))$ and V the right regular representation $l^2(\pi_1(X))$, provided that w has infinite order). Then :*

1.) X is weakly V-acyclic.

2.) If $r \geq 2$, then $\rho(X;V)$ vanishes.

3.) $\rho(X;V) = \sum_{H\in I} \sum_{C\in\pi_0(X^H)} \chi((C, C\cap X^{>H})/T) \cdot [[(\mu(w(H,C)-1) : V \longrightarrow V]]$

Proof : We show for any T-CW-subcomplex $k : Y \longrightarrow X$

$$\rho(Y;k^*V) = \sum_{H\in I} \sum_{C\in\pi_0(X^H)} \chi((k^{-1}(C), k^{-1}(C)\cap X^{>H})/T) \cdot [[(\mu(w(H,C)-1) : V \longrightarrow V]]$$

The induction begin $Y = \emptyset$ is trivial. In the induction step we assume that Y is obtained from Z be attaching an equivariant cell $T/H \times D^k$. Let $j_1 : T/H \times D^k \longrightarrow X$, $j_0 : T/H \times S^{k-1} \longrightarrow X$ and $j : Z \longrightarrow X$ be the obvious maps induced by k and denote by $l : T/H \longrightarrow T/H \times D^k$ the inclusion. We have $j_1^*V \simeq l^*j_1^*V \otimes \mathbf{C}$ and $j_0^*V \simeq l^*j_0^*V \otimes \mathbf{C}$ where $\mathbf{C}$ denotes the trivial representation. We derive from the product formula 3.16:

$H_*(T/H \times D^k; j_1^*V) = H_k(T/H; l^*j_1^*V) \otimes H_k(D^k; \mathbf{C})$
$H_*(T/H \times S^{k-1}; j_1^*V) = H_k(T/H; l^*j_1^*V) \otimes H_k(S^{k-1}; \mathbf{C})$
$\rho(T/H \times D^k; j_1^*V) = \rho(T/H; l^*j_1^*V)$
$\rho(T/H \times S^{k-1}; j_0^*V) = (1+(-1)^k) \cdot \rho(T/H, l^*j_1^*V)$

We obtain from lemma 3.19 that $H_k(T/H; l^*j_1^*V)$ vanishes and $\rho(T/H; l^*j_1^*V)$ is zero, if H does not belong to I and is $[[(\mu(w(H,C)-1) : V \longrightarrow V]]$ otherwise where C is a component of X^H which meets the new cell $T/H \times D^k$. Since Z is j^*V acyclic by induction hypothesis, we get from the long weakly exact Mayer Vietoris sequence 6.1 that Y is k^*V-acyclic. One checks directly that the difference

$$\sum_{H\in I}\sum_{C\in\pi_0(X^H)} \chi((k^{-1}(C), k^{-1}(C)\cap X^{>H})/T)\cdot[[(\mu(w(H,C)-1) : V \longrightarrow V]] -$$
$$\sum_{H\in I}\sum_{C\in\pi_0(X^H)} \chi((j^{-1}(C), j^{-1}(C)\cap X^{>H})/T)\cdot[[(\mu(w(H,C)-1) : V \longrightarrow V]]$$

is $(-1)^k \cdot [[(\mu(w(H,C)-1) : V \longrightarrow V]]$, if H is in I and C is a component of X^H which meets the new cell $G/H \times T$, and is zero if H is not in I. Now the claim follows from the sum formula 3.14. ■

Corollary 3.21 *Let $p : X \longrightarrow Y$ be a G-principal bundle over a finite CW-complex Y for G a connected compact Lie group such that the image of $\pi_1(G) \longrightarrow \pi_1(X)$ is not finite. Then we have :*

1.) *X is $l^2(\pi_1(X))$-acyclic.*

2.) *$\rho(X; l^2(\pi_1(X)))$ vanishes, if G is not S^1.*

3.) *Let G be S^1 and $w \in \pi_1(X)$ the image of the generator of $\pi_1(S^1)$. Then :*

$$\rho(X; l^2(\pi_1(X))) = \chi(Y)\cdot[[(w-1) : l^2(\pi_1(X)) \longrightarrow l^2(\pi_1(X))]] \qquad ■$$

Proof : We restrict the G-action to the action of the maximal torus T and apply theorem 3.20. Notice that $SO(3)$ and $SU(2)$ are the only Lie groups of dimension bigger than one whose maximal torus has dimension one (see Bröcker-tom Dieck [2], page 185) and these Lie groups have finite fundamental groups. ■

Remark 3.22 The computations above apply in particular to orientable compact 3-manifolds admitting a fixed point free S^1-action since for such manifolds $\pi_1(S^1) \longrightarrow \pi_1(M)$ is injective. Any Seifert fibre 3-manifold is covered by such a manifold. Hence any Seifert fibre 3-manifold is L^2-acyclic by corollary 3.20 and lemma 3.17. ■

Let (X, X_1) be a pair of finite CW-complexes and $\mu : \pi_1(X) \longrightarrow Iso_A^{op}$ be a unitary representation. Under the assumption that $H_*(X, X_1; V)$ vanishes, we can improve our invariant, namely, we can drop the symmetrization process.

The Hilbert dimension as defined in Cheeger-Gromov [8], section 1 induces a homomorphism

3.23 $dim : K_0(\mathcal{A}) \longrightarrow \mathbf{R}$

Then the Euler characteristic $\chi(X, X_1) \in \mathbf{Z}$ is trivial, since $\chi^{\mathcal{A}}(X, X_1; V) = \chi(X, X_1) \cdot [V] \in K_0(\mathcal{A})$ vanishes and $dim(V) = 0$ implies $V = \{0\}$. Choose a cellular bases for the finite free v$\mathbf{Z}[\pi]$-chain complex $C(\widetilde{X}, \widetilde{X_1})$. As $\chi(X, X_1)$ vanishes, the bases yield an $\mathbf{Z}[\pi]$- isomorphism $\Phi : C(\widetilde{X}, \widetilde{X_1})_{ev} \longrightarrow C(\widetilde{X}, \widetilde{X_1})_{odd}$. It induces an isomorphism of finitely generated Hilbert $\mathcal{A}$-modules $\Phi_V : C(X, X_1; V)_{ev} \longrightarrow C(X, X_1; V)_{odd}$. As $H_*(X, X_1; V)$ is trivial, we can choose a weak chain contraction (γ, u) for $C(X, X_1; V)$ (see definition 6.4). Let $U_V \subset K_1^w(\mathcal{A})$ be the subgroup of *trivial units*, element of the shape $[\pm\mu(w) : V \longrightarrow V]$ for $w \in \pi_1(X)$. Define the *acyclic Reidemeister von Neumann torsion*

3.24 $\rho_a(X, X_1; V) \in K_1^w(\mathcal{A})/U_V$

by $\rho_a(X, X_1; V) = [\Phi_V \circ (uc + \gamma)_{odd}] - [u_{odd}]$. We have to divide out U_V, as the cellular bases is not quite unique. One easily checks

3.25 $\rho(X, X_1; V) = \rho_a(X, X_1; V) + *\rho_a(X, X_1; V)$

If we have a push out of finite CW-complexes as indicated in 3.13 and X_i is j_i^*V-acyclic for $i = 0, 1, 2$, then X is V-acyclic and we get :

3.26 $\rho_a(X; V) = \rho_a(X_1; j_1^*V) + \rho_a(X_2; j_2^*V) - \rho_a(X_0; j_0^*V)$

Let (Y, Y_1) be a pair of finite CW-complexes and $\nu : \pi_1(Y) \longrightarrow Iso_{\mathcal{B}}(W)$ be a unitary representation. Then $(X, X_1) \times (Y, Y_1)$ is $V \otimes W$ acyclic and we get using the pairing 2.8

3.27 $\rho_a((X, X_1) \times (Y, Y_1), V \otimes W) = \chi(Y, Y_1) \cdot (\rho_a(X, X_1; V) \otimes [W])$

Theorem 3.11, lemma 3.17, theorem 3.20 and corollary 3.21 have obvious analogues for acyclic Reidemeister von Neumann torsion.

4. Free abelian fundamental groups

In this section we investigate the case of a free abelian fundamental group which is completely understood. Then our torsion invariant is related to the Alexander torsion.

Let (X, X_1) be a pair of finite CW-complexes and $\phi : \pi_1(X) \longrightarrow \mathbf{Z}^r$ be an epimorphism into the free abelian group of rank r. Denote by $\widehat{X} \longrightarrow X$ the corresponding covering. As $\mathbf{Z}[\mathbf{Z}^r]$ is an integral domain, we may consider its quotient field $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$. We assume (X, X_1) is *ϕ-acyclic*,i.e., $C(\widehat{X}, \widehat{X_1})_{(0)} = C(\widehat{X}, \widehat{X_1}) \otimes_{\mathbf{Z}[\mathbf{Z}^r]} \mathbf{Z}[\mathbf{Z}^r]_{(0)}$ is $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$-acyclic. We can choose a $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$-chain contraction γ of $C(\widehat{X}, \widehat{X_1})_{(0)}$,i.e. a map γ of degree 1 satisfying $c \circ \gamma + \gamma \circ c = id$ where c is the differential. We get an isomorphism of finitely generated free $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$-modules $(c_{(0)} + \gamma)_{odd} : C(\widehat{X}, \widehat{X_1})_{(0)odd} \longrightarrow C(\widehat{X}, \widehat{X_1})_{(0)ev}$. Fix a $\mathbf{Z}[\mathbf{Z}^r]$-bases of $C(\widehat{X}, \widehat{X_1})$. Let A be the invertible $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$-matrix of $(c_{(0)} + \gamma)_{odd}$ with respect to the induced $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$-bases. Define the *Alexander torsion*

4.1 $a(X, X_1; \phi) \in \mathbf{Z}[\mathbf{Z}^r]^*_{(0)} / \mathbf{Z}[\mathbf{Z}^r]^*$

by the determinant of A. Since we have divided out the units of $\mathbf{Z}[\mathbf{Z}^r]$, this is well-defined and we get for a homotopy equivalence of pairs of connected finite CW-complexes $f : (X', X'_1) \longrightarrow (X, X_1)$:

4.2 $a(X', X'_1; \phi \circ f_*) = a(X, X_1; \phi)$

Suppose we have a push out of finite CW-complexes as indicated in 3.13 and X_i is $\phi \circ j_{i*}$-acyclic for $i = 0, 1, 2$, then X is ϕ-acyclic and we obtain :

4.3 $a(X; \phi) = a(X_1, \phi \circ j_1) + a(X_2, \phi \circ j_2) - a(X_0, \phi \circ j_0)$

Let (X, X_1) and (Y, Y_1) be pairs of finite CW-complexes and $\phi : \pi_1(X) \longrightarrow \mathbf{Z}^r$ and $\psi : \pi_1(Y) \longrightarrow \mathbf{Z}^s$ be epimorphisms such that (X, X_1) is ϕ-acyclic. Then $(X, X_1) \times (Y, Y_1)$ is $\phi \times \psi$-acyclic. Let $i : \mathbf{Z}[\mathbf{Z}^r]^*_{(0)} / \mathbf{Z}[\mathbf{Z}^r]^* \longrightarrow \mathbf{Z}[\mathbf{Z}^{r+s}]^*_{(0)} / \mathbf{Z}[\mathbf{Z}^{r+s}]^*$ be the inclusion. The product formula for Alexander torsion is given by:

4.4 $a((X, X_1) \times (Y, Y_1); \phi \times \psi) = \chi(Y, Y_1) \cdot i(a(X, X_1; \phi))$

Let $\nu : \mathbf{Z}^r \longrightarrow L^2(T^r)$ be the right regular representation of $\mathbf{Z}^r$ where $L^2(T^r)$ is the Hilbert space of L^2-integrable functions from the r-dimensional torus into $\mathbf{C} \cup \{\infty\}$. Recall that $N(\mathbf{Z}^r)$ is $L^\infty(T^r)$ and $L^2(T^r)$ can be identified with $l^2(N(\mathbf{Z}^r)) = l^2(\mathbf{Z}^r)$. We get from ϕ a unitary representation $\overline{\phi} : \pi_1(X) \longrightarrow Iso_{N(\mathbf{Z}^r)}(L^2(T^r))$. Suppose that (X, X_1) is $\overline{\phi}$-acyclic. Then the acyclic Reidemeister von Neumann torsion is defined (see 3.24). By theorem 2.5 it takes values in :

$$\rho_a(X, X_1; \overline{\phi}) \in Inv(T^r)/U'$$

where U' is the subgroup of elements of the form

$$T^r \longrightarrow \mathbf{C} \qquad\qquad (z_1, z_2 \ldots z_r) \mapsto \pm z_1^{a_1} \cdot z_2^{a_2} \cdot \ldots \cdot z_r^{a_r}$$

for integers $a_1, a_2, \ldots a_r$. There is an obvious injection

4.5 $i : \mathbf{Z}[\mathbf{Z}^r]^*_{(0)}/\mathbf{Z}[\mathbf{Z}^r]^* \longrightarrow Inv(T^r)/U'$

Theorem 4.6 *We get under the conditions above :*

1. *1.) (X, X_1) is ϕ-acyclic if and only if (X, X_1) is $\overline{\phi}$-acyclic.*
2. *2.) Suppose that (X, X_1) is ϕ-acyclic. Then :*

$$i(a(X, X_1; \phi)) = \rho_a(X, X_1; \overline{\Phi})$$

Proof : Suppose that (X, X_1) is ϕ-acyclic. Let γ be a chain contraction of the $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$-chain complex $C(\widehat{X}, \widehat{X_1})_{(0)}$. Choose an element $u \in \mathbf{Z}[\mathbf{Z}]^r$ with $u \neq 0$ and and a $\mathbf{Z}[\mathbf{Z}^r]$-chain homotopy $\widehat{\gamma} : C(\widehat{X}, \widehat{X_1})_* \longrightarrow C(\widehat{X}, \widehat{X_1})_{*+1}$ satisfying $u \cdot id \circ \gamma = \widehat{\gamma}_{(0)}$. Then we have:

$$a(X, X_1; \phi) = det\left((c_{(0)} + \gamma)_{odd}\right) = det\left(u_{odd} \circ (c_{(0)} + \gamma)_{odd}\right)/det(u_{odd}) = det((uc + \widehat{\gamma})_{odd})/det(u_{odd})$$

Tensoring $\widehat{\gamma}$ and $u \cdot id$ with the regular representation yields a weak chain contraction of $C(X, X_1; \overline{\phi})$ because of $u \cdot id$ is a weak isomorphism by lemma 2.7. Therefore (X, X_1) is $\overline{\phi}$-acyclic and $i(a(X, X_1; \phi))$ is $\rho_a(X, X_1; \overline{\phi})$. It remains to show that $\overline{\phi}$-acyclic implies ϕ-acyclic. For $n \geq 0$ choose a finitely generated free $\mathbf{Z}[\mathbf{Z}^r]$-module C_n together with a $\mathbf{Z}[\mathbf{Z}^r]$-map $j_n : C_n \longrightarrow H_n(C(\widehat{X}, \widehat{X_1}; \phi))$ such that $(j_n)_{(0)}$ is an isomorphism. Lift j_n to a $\mathbf{Z}[\mathbf{Z}^r]$-homomorphism $i_n : C_n \longrightarrow ker(d_n)$. We have construced a finite free $\mathbf{Z}[\mathbf{Z}^r]$-chain complex C with trivial differentials and a chain map $i : C \longrightarrow C(\widehat{X})$ such that $H(i)_{(0)}$ is an isomorphism. Then $Cone(i)$ is $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$-acyclic. By the argument above and the long weakly exact homology sequence 6.1 the induced $N(\mathbf{Z}^r)$-chain map from $C \otimes_{\mathbf{Z}[\mathbf{Z}^r]} L^2(T^r)$ to $C(X, \overline{\phi})$ is a weak homology equivalence. But C is $\mathbf{Z}[\mathbf{Z}^r]_{(0)}$-acyclic, if and only if $C \otimes_{\mathbf{Z}[\mathbf{Z}^r]} L^2(T^r)$ is weakly acyclic, as C has trivial differentials. Thus $\overline{\phi}$-acyclic implies ϕ-acyclic. ∎

Example 4.7 (Alexander polynomial) Let L be an oriented link in an oriented homology 3-sphere M. Denote by $M(L)$ the complement of a tubular neighbourhood of the link. If r is the number of components, the *Hopf homomorphism*

$$\phi : \pi_1(M(L)) \longrightarrow \mathbf{Z}^r$$

maps a loop w to the r-tuple of integers given by the linking numbers of w and the various components of the link in M. Then the Alexander torsion of $M(K)$ defined in 4.1 is just the *Alexander polynomial* of the link L provided that $M(L)$ is ϕ-acyclic. Notice that the Alexander polynomial is defined to be zero if $M(L)$ is not ϕ-acyclic (see Burde-Zieschang [3] or Turaev [29]). Hence we see by theorem 4.6 that the acyclic Reidemeister von Neumann torsion of the link complement contains the same information as the Alexander polynomial. This identification is very similar to the one in Milnor [21].

The acyclic Reidemeister von Neumann torsion provides the possibility of extending the definition of Alexander polynomial to other coverings of the link complement than abelian coverings. We will investigate this at a different place. ■

Example 4.8 (Lefschetz zeta function of selfmaps)
Given a selfmap $f : X \longrightarrow X$ of a space of the homotopy type of a finite CW-complex, define the *Lefschetz zeta function* of f by the formal power series with leading coefficient zero

4.9 $L(f) = \sum_{n \geq 1} \frac{\lambda(f^n)}{n} t^n$

where $\lambda(f^n)$ is the Lefschetz index of the self map f^n of X. Denote by T_f the mapping torus of f. This space is obtained from $X \times [0,1]$ by the identification $(x,1) \sim (f(x),0)$. There is a natural map $T_f \longrightarrow S^1$ inducing a homomorphism $\phi : \pi_1(T_f) \longrightarrow \mathbf{Z}$. The mapping torus is ϕ-acyclic and the Alexander torsion $a(T_f;\phi)$ can be computed by

$$a(T_f;\phi) = \prod_{n \geq 0} det(z - H_n(f))^{(-1)^n}$$

Given an endomorphism $g : \mathbf{C}^r \longrightarrow \mathbf{C}^r$ of $\mathbf{C}$-modules, we obtain :

$$det(id - t \cdot g) = exp(\sum_{n \geq 1} \frac{tr(g^n)}{n} \cdot t^n)^{-1}$$

Hence $exp(L(f))$ is a rational function $\frac{p(t)}{q(t)}$ in t and $\frac{q(z^{-1})}{p(z^{-1})} \cdot z^{\chi(X)}$ represents $a(T_f,\phi)$. This observation is due to Milnor [21]. Theorem 4.6 says that the acyclic Reidemeister von Neumann torsion of the mapping torus and the Lefschetz Zeta function determine one another.
■

5. Torsion invariants for Riemannian Γ-manifolds

In this section we define Reidemeister von Neumann torsion for Riemannian manifolds with a cocompact properly discontinuous action of a dicrete group Γ by isometries. We investigate Poincaré duality and relate Reidemeister von Neumann torsion to classical Reidemeister torsion and to analytic torsion.

The definition of Reidemeister von Neumann torsion for Riemannian manifolds needs some preparation. We need two ingredients, an equivariant triangulation theorem and a deRham isomorphism. Before we establish these and give the final definition, we investigate an example to illustrate why the Riemannian metric comes in if the manifold is not weakly acyclic. The naive approach to choose a triangulation and to define the Reidemeister von Neumann torsion of a manifold by the Reidemeister von Neumann torsion of the triangulation does only work if the manifold is weakly acyclic. The reason is the appearance of the term depending on the map induced on homology in the formula in theorem 3.11 where Whitehead torsion and Reidemeister von Neumann torsion are compared. If one has two triangulations, they differ by a simple Γ-homotopy equivalence f. Simple means that its equivaraint Whitehead torsion $\tau^\Gamma(f)$ is zero. However, the map induced on homology represents in general a non-trivial class in the K-theory unless there is no homology. This will force us to "fix" the homology or cohomology of the given triangulation by comparing it with the space of harmonic l^2-forms by the deRham isomorphism.

Example 5.1 Consider the 1-dimensional manifold D^1 given by the unit intervall. In this example Γ is trivial. Consider for $n \geq 1$ the triangulation of D^1 with $\{\{\frac{k}{n}\} \mid 0 \leq k \leq n\}$ as set of 0-simplices and $\{[\frac{k}{n}, \frac{k+1}{n}] \mid 0 \leq k < n\}$ as set of 1-simplices. Let $D^1(n)$ be the CW-complex given by this explicit triangulation. We want to compute its Reidemeister von Neumann torsion introduced in 3.9 for $\mathcal{A} = \mathbf{C}$ and $V = \mathbf{C}$.

The cellular $\mathbf{C}$-chain complex $C = C(D^1(n))$ looks like :

$$A = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix} : \oplus_n \mathbf{C} \longrightarrow \oplus_{n+1} \mathbf{C}$$

Let N be the element $(1, 1, \dots, 1) \in C_0$. Its norm with respect to the standard Hilbert structure is $(n+1)^{1/2}$ and it is orthogonal to the image of the differential. Hence we obtain an identification of Hilbert spaces by :

$$\mathbf{C} \longrightarrow H_0(C) \qquad \lambda \mapsto \lambda \cdot (n+1)^{-1/2} \cdot N$$

Let $i_0 : \mathbf{C} \longrightarrow C_0$ be the $\mathbf{C}$-map sending λ to $\lambda \cdot (n+1)^{-1/2} \cdot N$. We get with the identification above a $\mathbf{C}$-chain map $i : H(C) \longrightarrow C$ satisfying $H(i) = id$. The Reidemeister von Neumann torsion of $D^1(n)$

$$\rho(D^1(n)) \in (\mathbf{C}^*)^{\mathbf{Z}/2}$$

is given by $\rho(D^1(n)) = det(B(n+1,(n+1)^{-1/2})) \cdot \overline{det(B(n+1,(n+1)^{-1/2}))}$, if we denote by $B(n+1,\lambda)$ the following $(n+1)$-$(n+1)$-matrix :

$$B(n+1,\lambda) \;=\; \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \lambda \\ -1 & 1 & 0 & \ldots & 0 & \lambda \\ 0 & -1 & 1 & \ldots & 0 & \lambda \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & \lambda \end{pmatrix}$$

Developing the determinant after the first row gives :

$$det(B(n+1,\lambda)) = 1 \cdot det(B(n,\lambda)) + (-1)^n \cdot \lambda \cdot (-1)^n = det(B(n,\lambda)) + \lambda$$

By induction over n we get $det(B(n+1,\lambda)) = (n+1) \cdot \lambda$. This shows :

$$\rho(D^1(n)) = (n+1)$$

In particular we see that $\rho(D^1(n))$ does depend on the triangulation. ■

Next we deal with the equivariant triangulation theorem. Let Γ be a discrete group acting cocompact and properly discontinuosly on a smooth manifold M. Recall that properly discontinuous means that for each pair of points (x,y) in M there are neighborhoods U_x and U_y such that $\{\gamma \in \Gamma \mid \gamma U_x \cap U_y \neq \emptyset\}$ is finite, and the action is cocompact if the quotient $\Gamma \backslash M$ is compact. Our assumptions imply that each point x in $int(M)$ resp. ∂M has a neighborhood U_x such that U_x is Γ_x-invariant and Γ_x-diffeomorphic to the unit disk of an orthogonal Γ_x-representation resp. half unit disk in the direct sum of an orthogonal Γ-representation and the trivial representation $\mathbf{R}$ and $\{\gamma \in \Gamma \mid \gamma U_x \cap U_x \neq \emptyset\} = \Gamma_x$ where Γ_x is the isotropy subgroup of the point x. Such a neighborhood will be called a *nice smooth neighborhood.* If we are working in the PL-category, it will be called an *nice PL-neighborhood.* Because $\Gamma \backslash M$ is compact, there is a finite set of points $\{x_1, x_2, \ldots x_m\}$ in M with smooth nice neighborhoods $\{U_{x_1}, U_{x_2}, \ldots U_{x_m}\}$ such that $M = \Gamma\,(U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_m})$. Such a family will be called an Γ-*equivariant covering basis* for M. A *nice simplicial* Γ-*complex* is a locally finite simplicial complex on which Γ acts simplicially, cocompact and properly discontinuously such that the regular neighborhood of each point is a nice PL-neighborhood.

Definition 5.2 *A smooth* Γ*-equivariant triangulation of* M *is a pair of nice simplicial* Γ*-complexes* $(K, \partial K)$ *and a* Γ*-equivariant homeomorphism* $(f, \partial f) : (K, \partial K) \longrightarrow (M, \partial M)$ *of pairs such that* h *is smooth on each simplex.* ■

Theorem 5.3 *There is a smooth Γ-equivariant triangulation of M. Given two such smooth Γ-equivariant triangulations $(f_i, \partial f_i) : (K_i, \partial K_i) \longrightarrow (M, \partial M)$ for $i = 1, 2$, the composition $(h_2^{-1} \circ h_1, \partial h_2^{-1} \circ \partial h_1)$ is Γ-isotopic to a Γ-equivariant PL-homeomorphism of pairs. In particular its equivariant Whitehead torsion vanishes.* ■

Illman [16] gives a proof for finite Γ and it generalizes easily to our situation. Notice that the theorem follows from Illman's provided that Γ contains a normal subgroup π of finite index acting freely. Namely, one applies Illman's result to the action of Γ/π on $\pi\backslash M$ and lifts the triangulation to M.

Here is the basic idea. For simplicity assume $\partial M = \emptyset$. Choose a Γ-equivariant covering basis for M. Then $K_1 = \Gamma \times_{\Gamma_{x_2}} U_{x_2}$ has a structure of a nice simplicial Γ-complex. Using the basic affine approximation method of Whitehead move U_{x_2} slightly such that the intersection $U_{x_2} \cap \Gamma \times_{\Gamma_{x_1}} K_1$ is a Γ-subcomplex of both. Then $K_2 = K_1 \cup \Gamma \times_{\Gamma_{x_1}} U_{x_1}$ has a structure of a nice simplicial Γ-complex. Continueing in this way yields eventually $K = K_m$. The uniqueness result follows in a similar way directly from Whitehead's argument.

Next we need a generalized deRham isomorphism. To state this we need some preparation. Let M be a Riemannian manifold of dimension m whose boundary ∂M is the disjoint union $\partial_0 M \coprod \partial_1 M$. Let Γ be a discrete group acting by isometries and cocompact, properly discontinuous on M. Let $\mathcal{A}$ be a finite von Neumann algebra and V be a finitely generated Hilbert module over $\mathcal{A}$. Consider a unitary representation $\mu : \Gamma \longrightarrow Iso_{\mathcal{A}}(V)^{op}$. We will assume that our manifold M is orientable. We can do this without loss of generality since otherwise one can pass to the orientation covering of M. We define the *orientation homomorphism*

5.4 $w = \Gamma \longrightarrow \{\pm 1\}$

by sending $\gamma \in \Gamma$ to $+1$ resp. ± 1 if γ acts orientation preserving resp. reversing.

The fibre over $x \in M$ of the Hilbert bundle $Alt^p(TM, V)$ consists of alterating p-forms on T_xM with coefficients in V. The Hilbert structure on the fibres comes from the one on V and the Riemannian metric. Recall that the space $\Lambda^p(M; V)$ of p-forms on M with coefficients in V is the space of smooth sections in $Alt^p(TM, V)$. Since Γ acts on M by isometries, Γ acts by isomorphisms of Hilbert bundles on $Alt^p(TM, V)$. This induces a action of Γ on $\Lambda^p(M; V)$. Let $\Lambda^p_\Gamma(M; V)$ be the Γ-fixed point set of $\Lambda^p(M; V)$.

Denote by ${}^wV$ the w-twisted representation given by $w(\gamma) \cdot v\gamma$ for $\gamma \in \Gamma$ and $v \in V$. Let $*\Lambda^p(M; V) \longrightarrow \Lambda^{m-p}(M; {}^w V)$ be the *Hodge star operator*, $d : \Lambda^p(M; V) \longrightarrow \Lambda^{p+1}(M; V)$

be the differential. We have the product $\wedge : \Lambda^p(M;V) \otimes \Lambda^q(M;V) \otimes \Lambda^{p+q}(M;^w \mathbf{C})$ and the *volume form* $dM \in \Lambda^n(M;^w \mathbf{C})$. The standard definitions for forms with coefficients in $\mathbf{C}$ carry directly over to the case of coefficients in V. Notice that d is compatible with the Γ-action as well as $*$ and dM is Γ-invariant.

We obtain for ω and η in $\Lambda^p(TM,V)$ a function $\langle \omega, \eta \rangle_x$ on M from the Hilbert bundles structure on $Alt^p(TM,V)$. One easily checks from the definitions $\omega \wedge (*\eta) = < \omega, \eta >_x \cdot dM$. Let $\mathcal{F}$ be a fundamental domain for the Γ-action on M. Given ω and η in $\Lambda^p_\Gamma(M;V)$, define their inner product

$$\langle \omega, \eta \rangle = \int_{\mathcal{F}} \langle \omega, \eta \rangle_x dM = \int_{\mathcal{F}} \omega \wedge (*\eta)$$

This is independent of the choice of fundamental domain, since ω and η are Γ-invariant. The adjoint $\delta^p : \Lambda^p(M;V) \longrightarrow \Lambda^{p-1}(M;V)$ of the differential d^{p-1} is $(-1)^{mp+p+1} * d^{m-p} *$. Define the *Laplace operator* $\Delta^p : \Lambda^p(M;V) \longrightarrow \Lambda^p(M;V)$ by $d^{p-1}\delta^p + \delta^{p+1}d^p$.

Given a p-form $\omega \in \Lambda^p(M;V)$, let ω_{tan} be the p-form on ∂M coming from restriction with $Ti : T\partial M \longrightarrow TM$ for the inclusion i. Let ω_{nor} be the $(p-1)$-form $*_{\partial M}(*_M\omega)_{tan}$. We say ω satisfies *Dirichlet boundary conditions* on $\partial_0 M$ if $\omega_{tan} = 0$ and $(\delta\omega)_{tan} = 0$ on $\partial_0 M$ holds, and *Neumann boundary conditions* on $\partial_1 M$, if $\omega_{nor} = 0$ and $(d\omega)_{nor} = 0$ on M_2 holds. Let $\Lambda^p(M, \partial_0 M; V)$ be the subspace of $\Lambda^p(M;V)$ consisting of p-forms satisfying Dirichlet boundary conditions on $\partial_0 M$ and Neumann boundary conditions on $\partial_1 M$. The *space of harmonic p-forms* $\mathcal{H}^p(M, \partial_0 M; V)$ is the subspace of $\Lambda^p(M, \partial_0 M; V)$ given by p-forms in the kernel of the Laplace operator. Define the space of invariant harmonic p-forms $\mathcal{H}^p_\Gamma(M, \partial_0 M; V)$ to be the fixed point set $(\mathcal{H}^p(M, \partial_0 M; V))^\Gamma$. It inherits a pre Hilbert structure from $\Lambda^p_\Gamma(M;V)$

Let $(f; f_0, f_1) : (K; K_0, K_1) \longrightarrow (M; \partial_0 M, \partial_1 M)$ be a smooth Γ-equivariant triangulation. Consider a Γ-invariant harmonic p-form ω in $\mathcal{H}^p_\Gamma(M, \partial_0 M; V)$. Let $\sigma \subset K$ be a p-simplex. The integral $\int_\sigma f^*\omega$ is an element in V. Define the de Rham homomorphism

$$A^p(f, f_0; V) : \mathcal{H}^p_\Gamma(M, \partial_0 M; V) \longrightarrow H^p(K, K_0; f^*V)$$

by sending ω to the class represented by $\widetilde{\sigma} \mapsto \int_\sigma f^*\omega$.

Theorem 5.5 *The deRham homomorphism*

$$A^p(f, f_0; V) : \mathcal{H}^p_\Gamma(M, \partial_0 M; V) \longrightarrow H^p(K, K_1; f^*V)$$

is a well-defined isomorphism of finitely generated Hilbert $\mathcal{A}$-modules. Given another equivariant triangulation $(g; g_0, g_1) : (L; L_0, L_1) \longrightarrow (M; \partial_0 M, \partial_1 M)$, we get :

$$H^p((g, g_0)^{-1} \circ (f, f_0); g^*V) \circ A^p(g, g_0; V) = A^p(f, f_0; V) \qquad \blacksquare$$

If $\mathcal{A}$ is the field of complex numbers $\mathbf{C}$ and Γ acts freely, the proof of the theorem above can be found in Ray and Singer [23]. Suppose Γ acts freely, M has no boundary, $\mathcal{A} = N(\Gamma)$ and V is the regular right representation $l^2(\Gamma)$. Let $L^2\Lambda^p_c(M)$ be the Hilbert completion of the space $\Lambda^p_c(M)$ of smooth p-forms on M with compact support. Let $L^2\Lambda^p_\Gamma(M;l^2(\Gamma))$ be the Hilbert space completion of $\Lambda^p_\Gamma(M;l^2(\Gamma))$. There is a canonical isometric isomorphism of Hilbert $N(\Gamma)$-modules

5.6 $\beta : L^2\Lambda^p(M) \longrightarrow L^2\Lambda^p_\Gamma(M;l^2(\Gamma))$

Given ω in $\Lambda^p(M)$, $\beta(\omega)$ assigns to the p-tuple of tangent vectors $(v_1, v_2, \ldots, v_p)$ of M at x the element in $l^2(\Gamma)$ given by $\sum_{\gamma\in\Gamma} \omega_{\gamma^{-1}x}\,(T_x\gamma^{-1}(v_1), T_x\gamma^{-1}(v_2), \ldots, T_x\gamma^{-1}(v_p)) \cdot \gamma$, where $T_x\gamma$ is the differential of the map given by multiplication with γ. In particular we obtain an identification of the Hilbert space $\mathcal{H}^p_\Gamma(M, \partial_0 M; V)$ of invariant harmonic p-forms with coefficients in $l^2(\Gamma)$ with the Hilbert space of smooth harmonic L^2-integrable harmonic forms on M. Under the assumptions and identifications the theorem above reduces to the result of Dodzuik [11] applied to $\Gamma\backslash M$. The proof of the theorem above is an extension of Dodzuik's argument combined with the technique or Ray and Singer [23] to handle the boundary.

Now we can define Reidemeister von Neumann torsion for $(M, \partial_0 M)$ with coefficients in V. Choose a Γ-equivariant tringulation $(f; f_0, f_1) : (K; K_0, K_1) \longrightarrow (M, \partial_0 M, \partial_1 M)$.

Definition 5.7 *Define the Reidemeister von Neumann torsion*

$$\rho(M, \partial_0 M; V) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

by $\rho(K, K_1; f^*V) - \sum_{p\geq 0}\ (-1)^p \cdot [[A^p(f, f_0; V)\ :\ \mathcal{H}^p_\Gamma(M, \partial_0 M; V)\ \longrightarrow\ H^p(K, K_0; f^*V)]]$.
■

We have to show that this is independent of the choice of equivariant triangulation. If $(g; g_0, g_1) : (L; L_0, L_1) \longrightarrow (M, \partial_0 M, \partial_1 M)$ is a second equivariant triangulation, the Whitehead torsion of $(g, g_0)^{-1} \circ (f, f_0)$ vanishes. We drive from theorem 3.11

$$\rho(L, L_0; g^*V) - \rho(K, K_0; f^*V) = -t(H((g, g_0)^{-1} \circ (f, f_0); g^*V))$$

Because of lemma 3.10 we have

$$t(H_p((g, g_0)^{-1} \circ (f, f_0); g^*V)) = t(H^p((g, g_0)^{-1} \circ (f, f_0); g^*V))$$

We derive from theorem 5.5 and lemma 7.8

$$t(H^p((g, g_0)^{-1} \circ (f, f_0); g^*V)) = t(A(f, f_0; V)) - t(A(g, g_1; V))$$

This implies :

$$\rho(K, K_0; f^*V) - t(A(f, f_0; V)) = \rho(L, L_0; g^*V) - t(A(g, g_0; V))$$

Now the claim follows as we have

$$t(A(f, f_0; V)) = \sum_{p \geq 0} (-1)^p \cdot [[A^p(f, f_0;) : \mathcal{H}^p_\Gamma(M, \partial_0 M; V) \longrightarrow H^p(K, K_0; f^*V)]]$$

Example 5.8 Equip D^1 with a Riemannian metric. Then the constant function on D^1 with value $vol(D^1)^{-1/2}$ has norm one in the space of harmonic 0-forms. Hence $\rho(D^1)$ is $vol(D^1)$ (compare with example 5.1). ∎

Example 5.9 Reidemeister von Neumann-torsion is sensitive to $\mathcal{A}$ and V. Consider S^1 with any Riemannian metric. If we use $\mathcal{A} = N(\mathbf{Z})$ and $V = l^2(\mathbf{Z})$, the Reidemeister von Neumann torsion is independent of the Riemannian metric and represented in $K_1^w(N(\mathbf{Z})) = Inv(S^1)$ by the function $(z-1)(z^{-1}-1)$ (see lemma 3.19 and theorem 2.5). If we choose $\mathcal{A} = \mathbf{C}$ and $V = \mathbf{C}$ the Reidemeister von Neumann torsion is real number in $K_1^w(\mathbf{C})^{\mathbf{Z}/2} = \mathbf{R}$ given by the square of the volume of S^1 and does depend on the Riemannian metric. ∎

Remark 5.10 The results of section 3 for CW-complexes also apply to manifolds. In the sum formula 3.14 the extra term coming from long homology sequences is to be understood with respect to the harmonic forms. The product formula 3.16 and restriction formula 3.17 hold as they stand. In theorem 3.20 and corollary 3.21 one has to substitute finite T-CW-complex by compact smooth manifold with smooth T-action. In the Lie group one has to be more careful with the triangulation theorem. However, we obtain at least a simple structure on M compatible with restriction by Lück [19], lemma 7.4.5, what suffices for theorem 3.20 as the spaces under consideration are V-acyclic. ∎

Finally we examine Poincaré duality following Lück [20] page 26ff. In view of theorem 5.3 we can identify $(M, \partial_0 M)$ with a triangulation. Let m be dimension of M. Let σ_1, σ_2, $\ldots \sigma_r$ be the m-simplices of $\Gamma \backslash M$. Choose for any $1 \leq i \leq r$ a lift $\widetilde{\sigma_i}$ in M. Then we obtain a cycle in $C_m(M, \partial M, {}^w\mathbf{C})$ by $\sum_{i=1}^r \widetilde{\sigma_i} \otimes_{\mathbf{Z}\Gamma} {}^w\mathbf{C}$. Its class in $H_m(M, \partial M; {}^w\mathbf{C})$ is independent of the choice of the lifts and called the *fundamental class* $[M]$. We have a cellular Γ-equivariant approximation of the diagonal map $M \longrightarrow M \times M$ by an equivariant version of the cellular approximation theorem. Hence the standard definitions for cup and cap products go through and we obtain a chain map of Hilbert chain complexes well defined up to homotopy:

5.11 $\cap[M] : C^{m-p}(M, \partial_1 M, {}^w V) \longrightarrow C_p(M, \partial_0 M, V)$

This map turns out to be a chain homotopy equivalence and we refer to it as the Poincaré chain homotopy equivalence. One proof of Poincaré duality for a compact manifold uses handlebody decompositions and a Mayer-Vietoris argument to reduce the claim to a handle body itself. In our case one can use an equivariant handlebody decomposition. Since all isotropy groups are finite, this reduces the claim to an equivariant handle for a finite group and in that case Poincaré duality is well-known. Now we can define:

Definition 5.12 *Define the Hilbert Poincaré torsion or briefly Poincaré torsion*

$$\rho_{pd}(M, \partial_0 M; V) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

by the torsion of $\cap[M] : C^{m-*}(M, \partial_1 M; {}^w V) \longrightarrow C_*(M, \partial_0 M; V)$ *introduced in 7.4.* ■

If Γ acts freely, then there is another proof of Poincaré duality using dual cells. This does not go through if the action is non-free, because the dual cell of an equivariant cell is not an equivariant cell in general anymore. The Hilbert Poincaré torsion measures this failure. In particular it vanishes if Γ acts freely. Analogously to Lück [20] proposition 5.24 there is a local formula for Hilbert Poincaré torsion in terms of the Euler characteristics of the components of the H-fixed points sets and their normal slices for finite subgroups H of Γ. As in Lück [20] p.26ff one shows:

Theorem 5.13 (Poincaré Duality)

1.) $\rho(M, \partial_0 M; V) + (-1)^m \cdot \rho(M, \partial_1; {}^w V) = \rho_{pd}(M, \partial_0 M; V)$

2.) $\rho_{pd}(M, \partial_0 M; V) = (-1)^m \cdot \rho_{pd}(M, \partial_1 M; V)$

3.) $\chi^{\mathcal{A}}(M, \partial_0 M; V) = (-1)^m \cdot \chi^{\mathcal{A}}(M, \partial_1 M; V)$

4.) If Γ *acts freely,* $\rho_{pd}(M, \partial_0 M; V)$ *vanishes.* ■

Suppose M is closed and orientable and Γ acts orientation preserving. Then we obtain $2 \cdot \rho(M; V) = \rho_{pd}(M; V)$, if M has even dimension, and $2 \cdot \rho_{pd}(M; V) = 0$, if M has odd dimension. In particular $2 \cdot \rho(M; V)$ vanishes, if Γ acts freely and orientation preserving and M is an orientable closed Riemannian manifold of even dimension.

Next we want to relate the Reidemeister von Neumann torsion to other well-known invariants

Example 5.14 Let X be a finite CW-complex with finite fundamental group π. Consider a finite-dimensional complex representation $\mu : \pi \longrightarrow Iso_{\mathbf{C}}(V)$. This means that we have choosen $\mathcal{A}$ to be $\mathbf{C}$. Suppose that X is V-acyclic. The acyclic Reidemeister von Neumann torsion $\rho_a(X;V)$ defined in3.24 is essentially a complex number, more precisely, it is an element in $\mathbf{C}^*/U$, where U is given by $\{det(\pm\mu(g) : V \longrightarrow V) \mid g \in \pi\}$. This is just the classical Reidemeister torsion which was used to classify Lens spaces (see Cohen [9] and Reidemeister [24]). ■

Example 5.15 Let M be a compact smooth Riemannian manifold with fundamental group π and $\mu : \pi \longrightarrow Iso_{\mathbf{R}}(V)$ be an orthogonal finite-dimensional representation. Then $V \otimes_{\mathbf{R}} \mathbf{C}$ is a unitary representation of π over the von Numann algebra $\mathcal{A} = \mathbf{C}$. The Reidemeister von Neumann torsion $\rho(M;V)$ defined in 5.7 reduces to a positive real number, because $K_1^w(\mathbf{C})$ is $\mathbf{R}^*$. This is the square of the PL-torsion $\rho_{pl}(M;V)$ introduced by Ray and Singer [23]. Ray and Singer [23] defined also the analytic counterpart $\rho_{an}(M;V)$ based on the zeta function of the Laplace operator. Cheeger [5] and Müller [22] proved independently that $ln(\rho_{pl}(M;V)) = \rho_{an}(M;V)$, provided that M is closed. If M has a boundary, a correction term proportional to the Euler characteristic of the boundary comes in (see Lück [20]). ■

Example 5.16 The notions of Ray and Singer and the results of Cheeger and Müller were extended to compact Riemannian G-manifolds by Lott and Rothenberg [18] and Lück[20] for a finite group G. Analytic proofs of deRham's theorem are established that the unit spheres of two orthogonal representations are G diffeomorphic if and only if the representations themselves are isomorphic (cf. deRham [25]). The notion of Reidemeister von Neumann torsion reduces to the notions in Lott and Rothenberg [18] and Lück [20] if $Gamma$ is finite. The meaning of equivariant Reidemeister torsion for finite transformation groups is worked out in Rothenberg [26] and Lück [19]. ■

Remark 5.17 In view of the results above one should try to give an analytic interpretation of the Reidemeister von Neumann torsion. In this context the definition of a real number, the analytic L^2- torsion by Lott [17] is very interesting. His invariant should be up to a constant the Fuglede Kadison determinant (see Fuglede and Kadison [14]) applied to the Reidemeister von Neumann torsion. The analytic L^2-torsion of Lott is only defined if all the Novikov-Shubin invariants of the manifold are positive whereas the Reidemeister von Neumann torsion is always defined. However, the vanishing of all Novikov-Shubin invariants is needed to apply the Fuglede Kadison determinant to the Reidemeister von Neumann torsion. In a forthcoming preprint the first author and Lott will prove that the Novikov-Shubin invariants are positive for a manifold satisfying the condition of theorem 3.20 and for a compact Haken 3-manifold whose boundary is empty or is a disjoint union of incompressible tori. In particlular under the conditions of theorem 3.20 the Reidemeister von Neumann torsion vanishes so that one would expect that also the analytic L^2-torsion vanishes.

We also refer in this context to Carey-Mathai [4], where a real number based on the Kadison-Fuglede determinant is assigned to a smooth closed L^2-acyclic manifold whose Novikov-Shubin invariants are positive. Their invariant when defined is just the Fuglede Kadison determinant of the Reidemeister von Neumann torsion. In the original version which appeared in Contemporary Math., vol 105 (1989) there is an error in the definition of their invariant. This has been corrected in the version we refer to. ■

6. Hilbert $\mathcal{A}$-chain complexes

In this section we generalize basic facts about long homology sequences and chain contractions for chain complexes over a ring to chain complexes over a finite von Neumann algebra $\mathcal{A}$ with a finite normal and faithful trace.

The main difficulty lies in the definition of homology as $H(C) = ker(c)/clos(im(c))$ which does not coincide with the ordinary definition as $ker(c)/im(c)$. Recall that we have to divide out the closure of the image to ensure that the homology consists again of Hilbert $\mathcal{A}$-modules. In the classical situation the constrution of torsion invariants for acyclic chain complexes is based on the existence of a chain contraction γ, i.e., a chain homotopy $\gamma : id \sim 0$. We must look for a weaker notion of chain contraction in the case of Hilbert $\mathcal{A}$-chain complexes which are weakly acyclic. Before we deal with this question, we need some preliminaries. In the sequel module resp. chain complex means Hilbert $\mathcal{A}$-module resp. Hilbert $\mathcal{A}$-chain complex unless explicitly stated differently. Recall that a chain complex is finite if C_n is finitely generated for all $n \in \mathbf{Z}$ and there is a number N such that $C_n = 0$ for $|n| \geq N$.

Theorem 6.1 (Cheeger-Gromov [6])
Let $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ be an exact sequence of finite chain complexes. Then there is a weakly exact long natural homology sequence :

$$\longrightarrow H_{n+1} \xrightarrow{\partial} H_n(C) \xrightarrow{H_n(i)} H_n(D) \xrightarrow{H_n(p)} H_n(E) \xrightarrow{\partial} \ldots$$

Proof : We firstly define ∂. Since the differential d_n of D_n maps $ker(p_n)$ to $ker(p_{n-1})$, we may write $d_n : D_n \longrightarrow D_{n-1}$ as :

$$\begin{pmatrix} d_n^1 & d_n^2 \\ 0 & d_n^3 \end{pmatrix} : ker(p_n) \oplus ker(p_n)^\perp \longrightarrow ker(p_{n-1}) \oplus ker(p_{n-1})^\perp$$

The induced morphisms $i_n : C_n \longrightarrow ker(p_n)$ and $p_n| : ker(p_n)^\perp \longrightarrow E_n$ are isomorphisms by the open mapping theorem. Consider the composition

$$\partial' = i_{n-1}^{-1} \circ d_n^2 \circ (p_n|)^{-1} : E_n \longrightarrow C_{n-1}$$

We claim $\partial'_n(ker(e_n)) \subset ker(c_{n-1})$. Namely, regard $x \in ker(e_n)$, $y \in ker(p_n)^\perp$ and $z \in C_{n-1}$ such that $p_n(y) = z$ and $d_n^2(y) = i_n(z)$ holds. We have by definition $\partial'(x) = z$ so that we have to check $c_{n-1}(z) = 0$. Since i_{n-2} is injective, it suffices to show $i_{n-2} \circ c_{n-1}(z)$ is zero. This element is just $d_{n-1} \circ d_n^2(y)$. Now $(p_{n-1}|) \circ d_n^3(y)$ is $e_n \circ p_n(0, y) = e_n(x)$ and hence vanishes. Therefore $d_n^3(y)$ is zero. This implies $d_{n-1} \circ d_n^2(y) = d_{n-1} \circ d_n(0, y) = 0$. We have shown $\partial'_n(ker(e_n)) \subset ker(c_{n-1})$. Analogously one proves $\partial'(im(e_{n+1})) \subset im(c_n)$. Hence we can define ∂ to be the map induced by ∂'.

Although the definition of the long homology sequence reduces in the case, where $\mathcal{A}$ is $\mathbf{C}[G]$ for a finite group G to the classical situation, the verification of weak exactness is much harder and not just a diagram chase. We refer to Cheeger - Gromov [6] theorem 2.1. ■

For a chain map $f : C \longrightarrow D$, denote by $Cone(f)$ its *mapping cone* given by

$$c_n \quad = \quad \begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix} : C_{n-1} \oplus D_n \longrightarrow C_{n-2} \oplus D_{n-1}$$

Recall that weak homology equivalence is a chain map inducing a weak isomorphism on the homology. A weak chain isomorphism is a chain map f such that f_n is a weak isomorphism for each $n \in \mathbf{Z}$.

Corollary 6.2

1.) *Regard the following commutative diagram of finite chain complexes with weakly exact rows.*

$$\begin{tikzcd} 0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "f"] & D \arrow[r, "p"] \arrow[d, "g"] & E \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & C' \arrow[r, "i'"] & D' \arrow[r, "p'"] & E' \arrow[r] & 0 \end{tikzcd}$$

If two of the vertical arrows are weak homology equivalences, then also the third.

2.) *Let $f : C \longrightarrow D$ and $g : D \longrightarrow E$ be chain maps of finite chain complexes. If two of the chain maps f, g and $g \circ f$ are weak homology equivalences, then also the third.*

3.) *A weak chain isomorphism of finite chain complexes is a weak homology equivalence.*

4.) *A chain map of finite chain complexs is a weak homology equivalence if and only if its mapping cone $Cone(f)$ is weakly acyclic.*

Proof : Assertion 4.) follows from the weakly exact long homology sequence of

$$0 \longrightarrow D \longrightarrow Cone(f) \longrightarrow \Sigma C \longrightarrow 0$$

introduced in theorem 6.1, where the suspension ΣC is $Cone(C \longrightarrow \{0\}$.

Next we prove assertion 1.) under the additional assumption that the rows are exact and not only weakly exact. Then we obtain an exact sequence :

$$0 \longrightarrow Cone(f) \longrightarrow Cone(g) \longrightarrow Cone(h) \longrightarrow 0$$

Now the claim follows from assertion 4.) and the weakly exact long homology sequence of theorem 6.1.

The assertion 3.) follows by induction over the dimension $n = dim(C)$. Notice that we can assume without loss of generality that C_n is zero for negative n. The induction begin $n = 0$ is trivial. The induction step follows from 1.) applied to :

$$0 \longrightarrow C\,|_n \longrightarrow C \longrightarrow (n+1)(C) \longrightarrow 0$$

where $C\,|_n$ is C restricted to dimension n and $(n+1)(C)$ is concentrated in dimension $n+1$ and given in this dimension by C_{n+1}.

We obtain assertion 2.) from assertions 1.) and 4.) by constructing a chain map $k : \Sigma^{-1}Cone(g) \longrightarrow Cone(f)$ given by

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} : D_n \oplus E_{n+1} \longrightarrow C_{n-1} \oplus D_n$$

and exact sequences as done in Lück [19] page 245 :

$$0 \longrightarrow Cone(f) \longrightarrow Cone(h) \longrightarrow Cone(g) \longrightarrow 0$$
$$0 \longrightarrow Cone(g \circ f) \longrightarrow Cone(h) \longrightarrow Cone(D) \longrightarrow 0$$

Finally we prove assertion 1.) in full generality. It suffices to show for a weakly exact sequence $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ that C, D and E are weakly acyclic, if two of them are. Consider the induced exact sequence $0 \longrightarrow ker(p) \longrightarrow D \longrightarrow ker(p)^{\perp} \longrightarrow 0$ for which assertion 1.) applies by the argument above. There are weak isomorphisms $C \longrightarrow ker(p)$ and $ker(p)^{\perp} \longrightarrow E$. Now apply assertion 2.) and 3.) ∎

The following example shows that these results are non-trivial and that the finite Hilbert $\mathcal{A}$-module structure is essential.

Example 6.3 Let l^2 be the Hilbert space $\{(a_n \in \mathbf{C})_{n\in\mathbf{N}} \mid \sum \ |a_n| \ ^2 < \infty\}$ with the inner product $\langle (a_n),(b_n)\rangle \ = \sum a_n \cdot \overline{b_n}$. Define a linear bounded operator $f : l^2 \longrightarrow l^2$ by sending (a_n) to $(1/n \cdot a_n)$. Obviously f is injective and has dense image. But f is not surjective because $u := (1/n)_n \in l^2$ cannot have a preimage. Let $pr : l^2 \longrightarrow span_{\mathbf{C}}(u)^\perp$ be the projection. Then $pr \circ f$ is injective with dense image. Hence f and $pr \circ f$ are weak isomorphisms of Hilbert spaces but pr is not. Compare this example with corollary 6.2 assertion 2.). One easily constructs out of it counterexamples to the other claims in corollary 6.2. Since any Hilbert space may be viewed as a Hilbert $\mathcal{A}$-module for the von Neumann algebra $\mathbf{C}$, the example shows that the finiteness conditions are necessary. ■

Next we introduce an appropiate notion of weak chain contraction.

Definition 6.4 *A weak chain contraction for a chain complex C is a pair (γ, u) consisting of a weak chain isomorphism $u : C \longrightarrow C$ and a chain homotopy $\gamma : u \sim 0$ satisfying $\gamma \circ u = u \circ \gamma$.* ■

Lemma 6.5 *The following assertions are equivalent for a finite chain complex C :*

1.) C is weakly acyclic.

2.) There is a weak chain contraction (γ, u).

3.) There is a weak chain contraction (γ, u) satisfying $\gamma \circ \gamma = 0$.

4.) There is a weak chain isomorphism $u : C \longrightarrow C$ and a chain homotopy $\gamma : u \sim 0$.

Proof : 1.) $\Rightarrow$ 3.) As C is weakly acyclic, $c_n : C_n \longrightarrow C_{n-1}$ induces a weak isomorphism $c_n|: ker(c_n)^\perp \longrightarrow ker(c_{n-1})$. There is an isomorphism $\psi_n : ker(c_n) \longrightarrow ker(c_{n+1})^\perp$ because of lemma 1.7. Define $\gamma_n : C_n \longrightarrow C_{n+1}$ by :

$$\begin{pmatrix} 0 & 0 \\ \psi_n & 0 \end{pmatrix} \ : \ ker(c_n) \oplus ker(c_n)^\perp \longrightarrow ker(c_{n+1}) \oplus ker(c_{n+1})^\perp$$

Put $u_n = c_{n+1} \circ \gamma_n + \gamma_{n-1} \circ c_n$. One easily checks that u is a chain map and $\gamma \circ \gamma = 0$. As u_n is the direct sum of weak isomorphisms, u_n itself is a weak isomorphism.

3.) $\Rightarrow$ 2.) $\Rightarrow$ 4.) are trivial

4.) $\Rightarrow$ 1.) As u is nullhomotopic, we have $H(u) = 0$. Since u is a weak chain isomorphism, u is a weak homology equivalence by lemma 6.2. Hence C is weakly acyclic. ■

Next we prove some results we will need in the construction of torsion invariants in the following section. The technical condition $\gamma \circ u = u \circ \gamma$ in the definition 6.4 of a weak chain contraction is needed for the verification of the next lemma. Its proof is a straightforward computation and is left to the reader. In the sequel we write :

6.6 $\quad C_{odd} = \oplus_{n \in \mathbf{Z}} C_{2n+1} \qquad\qquad C_{ev} = \oplus_{n \in \mathbf{Z}} C_{2n}$

Lemma 6.7 *Let (γ, u) and (δ, v) be weak chain contractions for the finite chain complex C. Define $\Theta : C_{ev} \longrightarrow C_{ev}$ by :*

$$\Theta := (v \circ u + \delta \circ \gamma) = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \dots & vu & 0 & 0 & \dots \\ \dots & \delta\gamma & vu & 0 & \dots \\ \dots & 0 & \delta\gamma & vu & \dots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then the composition :

$$\Theta' : C_{odd} \overset{(uc+\gamma)}{\longrightarrow} C_{ev} \overset{\Theta}{\longrightarrow} C_{ev} \overset{(vc+\delta)}{\longrightarrow} C_{odd}$$

is given by the triangle matrix

$$\begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \ddots \\ \dots & (v^2u^2)_{2n-1} & 0 & 0 & \dots \\ \dots & * & (v^2u^2)_{2n+1} & 0 & \dots \\ \dots & * & * & (v^2u^2)_{2n+3} & \dots \\ \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad \blacksquare$$

Lemma 6.8 *Let C be a finite chain complex. Consider $H(C)$ as a finite chain complex using the trivial differential.*

1.) There is a chain map $i : H(C) \longrightarrow C$ satisfying $H(i) = id$.

2.) Let i and j be chain maps $H(C) \longrightarrow C$ satisfying $H(i) = H(j)$. Then there is a weak chain isomorphism $f : C \longrightarrow C$ such that $f \circ i$ and $f \circ j$ are chain homotopic.

Proof : 1.) Let $s_n : H(C)_n \longrightarrow ker(c_n)$ be a section of the projection $ker(c_n) \longrightarrow H(C_n)$. Define $i_n : H(C)_n \longrightarrow C_n$ by the composition of s_n and the inclusion. Then $i : H(C) \longrightarrow C$ is a chain map satisfying $H(i) = id$.

2.) We may suppose $j = 0$, otherwise substitute i, j by $i-j$, 0. Because the map $c_n|: ker(c_n)^{\perp} \longrightarrow clos(im(c_n))$ is a weak isomorphism, we can choose an isomorphism $\psi_{n-1} : clos(im(c_n)) \longrightarrow ker(c_n)^{\perp}$ by lemma 1.7. Define $f_n : C_n \longrightarrow C_n$ to be the orthogonal sum of

$$c_{n+1}| \circ \psi_n : clos(im(c_{n+1})) \longrightarrow clos(im(c_{n+1}))$$
$$\psi_{n-1} \circ c_n| : ker(c_n)^{\perp} \longrightarrow ker(c_n)^{\perp}$$
$$id : ker(c_n) \cap clos(im(c_{n+1}))^{\perp} \longrightarrow ker(c_n) \cap clos(im(c_{n+1}))^{\perp}$$

Let $\gamma_n : H(C)_n \longrightarrow C_{n+1}$ be given by $\psi_n \circ i_n$. One easily checks that f is a weak chain isomorphism with $H(f) = id$ and γ a chain homotopy $f \sim 0 \circ i$. ■

Lemma 6.9 *Let $f : C \longrightarrow D$ be a weak chain isomorphism of finite chain complexes. Then there is weak chain isomorphism $g : D \longrightarrow C$.*

Proof : As f and the restriction $f|_n : C|_n \longrightarrow D|_n$ to dimension n are weak chain isomorphism, corollary 6.2 implies that f_n induces weak isomorphisms

$$ker(c_n) \longrightarrow ker(d_n)$$
$$ker(c_n)^{\perp} \longrightarrow ker(d_n)^{\perp}$$
$$ker(c_n) \cap clos(im(c_{n+1}))^{\perp} \longrightarrow ker(d_n) \cap clos(im(d_{n+1}))^{\perp}$$

The differential d_n induces a weak isomorphism $d_n| : ker(d_n)^{\perp} \longrightarrow clos(im(d_n))$. By lemma 1.7 we can choose isomorphisms :

$$\alpha_n : ker(d_n)^{\perp} \longrightarrow ker(c_n)^{\perp}$$
$$\beta_n : clos(im(d_n)) \longrightarrow ker(d_n)^{\perp}$$

Define weak isomorphisms :

$$u_n : ker(d_n)^{\perp} \longrightarrow ker(c_n)^{\perp}$$
$$v'_{n-1} : clos(im(d_n)) \longrightarrow clos(im(c_n))$$

by $u_n = \alpha_n \circ \beta_n \circ d_n|$ and $v'_{n-1} = c_n \circ \alpha_n \circ \beta_{n-1}$. Then $c_n \circ u_n = v'_{n-1} \circ d_n$ is valid. Extend v'_{n-1} to a weak isomorphism $v_{n-1} : ker(d_{n-1}) \longrightarrow ker(c_{n-1})$. Then $c_n \circ u_n = v_{n-1} \circ d_n$ remains true. Now we obtain a weak chain isomorphism $g : D \longrightarrow C$ by :

$$g_n = u_n \oplus v_n : ker(d_n)^{\perp} \oplus ker(d_n) \longrightarrow ker(c_n)^{\perp} \oplus ker(c_n) \qquad ■$$

Lemma 6.10 *Let $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ be a weakly exact sequence of finite chain complexes such that E is weakly acyclic. Then there is a chain map $s : E \longrightarrow D$ such that $p \circ s : E \longrightarrow E$ is a weak chain isomorphism.*

Proof : Suppose that $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ is exact. Choose for $n \in \mathbf{Z}$ a homomorphism $\sigma_n : E_n \longrightarrow D_n$ satisfying $p_n \circ s_n = id$. Choose a weak chain contraction (ϵ, w) for E. Define $s_n : E_n \longrightarrow D_n$ by $d_{n+1} \circ \sigma_{n+1} \circ \epsilon_n + \sigma_n \circ \epsilon_{n-1} \circ e_n$. Then $s : E \longrightarrow D$ is a chain map and $p \circ s = w$.

In the general case, where $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ is only weakly exact, one applies the argument above to $0 \longrightarrow ker(p) \longrightarrow D \longrightarrow ker(p)^{\perp} \longrightarrow 0$ and uses lemma 6.9 to get a weak chain isomorphism $E \longrightarrow ker(p)^{\perp}$ from the canonical weak chain isomorphism $ker(p)^{\perp} \longrightarrow E$. ■

7. Torsion invariants for Hilbert $\mathcal{A}$-chain complexes

Recall that $\mathcal{A}$ is a finite von Neumann algebra together with a finite faithful normal trace and all modules and chain complexes are understood to be Hilbert $\mathcal{A}$-modules and Hilbert $\mathcal{A}$-chain complexes. We introduce Reidemeister von Neumann torsion for chain complexes and torsion for weak homology equivalences. We verify the basic properties like sum formula, product formula, composition formula and compare these invariants.

Let C be a weakly acyclic chain complex. Because of lemma 6.5 we can choose a weak chain contraction (γ, u) of C. Recall that (γ, u) consists of a weak chain isomorphism $u : C \longrightarrow C$ and a chain homotopy $\gamma : u \sim 0$ satisfying $u \circ \gamma = \gamma \circ u$. Let (δ, v) be a second weak chain contraction for C. We have introduced in section 2 the K-groups $K_1(\mathcal{A})$ resp. $K_1^w(\mathcal{A})$ of automorphisms resp. weak automorphisms of finitely generated Hilbert $\mathcal{A}$-modules and the involution on them given by taking the adjoint.

If c denotes the differential of C, we derive $(gc + \delta)_{ev} \circ \Theta \circ (fc + \gamma)_{odd} = \Theta'$ from lemma 6.7 in the notation used there, where Θ and Θ' are triangular matrices. We also have $(uc + \gamma)_{odd} \circ u_{odd} = u_{ev} \circ (uc + \gamma)_{odd}$. Hence corollary 6.2 and the relations in the K_1-groups imply that the maps

$(uc + \gamma)_{odd} : C_{odd} \longrightarrow C_{ev}$
$(vc + \delta)_{ev} : C_{ev} \longrightarrow C_{odd}$

are weak isomorphism and satisfy :

7.1 $[[(uc + \gamma)_{odd})]] - [[u_{odd}]] = -[[(vc + \delta)_{ev}]] + [[v_{odd}]]$
$[u_{odd}] = [u_{ev}]$

where we use the following notation:

Notation 7.2 *Given a weak automorphism $g : M \longrightarrow M$ of a finitely generated Hilbert $\mathcal{A}$-module, we denote its class in $K_1^w(\mathcal{A})$ by $[g]$. If $g : M \longrightarrow N$ is a weak isomorphism of finitely generated Hilbert $\mathcal{A}$-modules, let $[[g]] \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$ be $[g^* \circ g]$* ■

Hence the following definition is independent of the choice of (u, γ).

Definition 7.3 *Let C be a finite weakly acyclic chain complex. Define the* Reidemeister von Neumann torsion *of C*

$$\rho(C) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

by $\rho(C) = [[(uc+\gamma)_{odd}]] - [[u_{odd}]]$. ■

We will later generalize this definition to finite chain complexes, dropping the assumption about weak acyclicity and identify it with the definition 3.8 we have used earlier. Recall from corollary 6.2 that the mapping cone $Cone(f)$ of a weak homology equivalence f is weakly acyclic.

Definition 7.4 *Define the* torsion *of a weak homology equivalence* $f : C \longrightarrow D$ *of finite chain complexes*

$$t(f) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

by $t(f) = \rho(Cone(f))$. ■

Notation 7.5 *Let* $0 \longrightarrow M \xrightarrow{i} N \xrightarrow{p} P \longrightarrow 0$ *be a weakly exact sequence of finitely generated modules. Define*

$$\rho(M,N,P) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

by the Reidemeister von Neumann torsion of the corresponding weakly acyclic finite chain complex having P *in dimension* 0. *Explicitely we get for any map* $s : P \longrightarrow N$ *such that* $p \circ s$ *is a weak isomorphism :*

$$\rho(M,N,P) = -[[i \oplus s : M \oplus P \longrightarrow N]] + [[p \circ s : P \longrightarrow P]]$$

Let $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ *be a weakly exact sequence of finite chain complexes. Define :*

$$\rho(C,D,E) = \sum_n (-1)^n \cdot \rho(C_n, D_n, E_n) \qquad ■$$

Lemma 7.6 *Let* $f : C \longrightarrow D$ *be a weak chain isomorphism of finite weakly acyclic chain complexes. Then*

$$\rho(D) - \rho(C) = \sum_n (-1)^n \cdot [[f_n]]$$

Proof : We use induction over the dimension n of C. The induction begin $n = 0$ is trivial, since in this case C and D are the trivial chain complexes. The case $n = 1$ is trivial as well, since $f_0 \circ c_1 = d_1 \circ f_1$ implies:

$$[[f_0]] + \rho(C) = [[f_0]] + [[c_1]] = [[d_1]] + [[f_1]] = \rho(D) + [[f_1]]$$

The induction step from $n-1 \geq 1$ to n is done as follows.

Let C' be the chain subcomplex of C concentrated in dimension n and $n-1$ satisfying $C'_n = C_n$ and $C'_{n-1} = clos(im(c_n))$. Let C'' be the quotient complex so that we have a canonical exact sequence $0 \longrightarrow C' \xrightarrow{i_C} C \xrightarrow{q_C} C'' \longrightarrow 0$. Construct an exact sequence $0 \longrightarrow D' \xrightarrow{i_D} D \xrightarrow{q_D} D'' \longrightarrow 0$ for D similiarly. Then f restricts to a weak chain isomorphism $f' : C' \longrightarrow D'$ and f and f' induce a weak chain isomorphism $f'' : C'' \longrightarrow D''$. Now one easily checks for $i \geq 0$:

$$[[f'_i]] - [[f_i]] + [[f''_i]] = 0$$

Let γ be a weak chain contraction for C. One easily constructs weak chain contractions γ' and γ'' for C' and C'' compatible with γ and the chain maps i_C and q_C. This implies:

$$\rho(C') - \rho(C) + \rho(C'')$$

Since a similiar equation holds for D, D' and D'' and the induction hypothesis applies to f' and f'', the lemma follows. ■

Lemma 7.7 (Sum formula) *Consider the commutative diagram of finite chain complexes with weakly exact rows and weak homology equivalences as vertical arrows.*

$$\begin{tikzcd}
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "f"] & D \arrow[r, "p"] \arrow[d, "g"] & E \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & C' \arrow[r, "i'"] & D' \arrow[r, "q'"] & E' \arrow[r] & 0
\end{tikzcd}$$

Then we get :

$$t(f) - t(g) + t(h) = \rho(C', D', E') - \rho(C, D, E)$$

Proof : The diagram above induces a weakly exact sequence of weakly acyclic finite chain complexes by corollary 6.2.

$$0 \longrightarrow Cone(f) \xrightarrow{j} Cone(g) \xrightarrow{q} Cone(h) \longrightarrow 0$$

By lemma 6.10 there is a chain map $s : Cone(h) \longrightarrow Cone(g)$ such that we obtain a weak chain equivalence $i \oplus s : Cone(f) \oplus Cone(h) \longrightarrow Cone(g)$. We get from lemma 7.6:

$$\rho(Cone(f) \oplus Cone(h)) - \rho(Cone(g)) = [[(i \oplus s)_{odd}]] - [[(i \oplus s)_{ev}]]$$

We derive from the definitions

$$\begin{aligned}
&\rho(Cone(f) \oplus \rho(Cone(h)) = t(f) + t(h) \\
&\rho(Cone(g)) = t(g) \\
&\rho(C', D', E') - \rho(C, D, E) = [[(i \oplus s)_{odd}]] - [[(i \oplus s)_{ev}]]
\end{aligned}$$

This finishes the proof of lemma 7.7 ■

Lemma 7.8 (Composition formula) *If $f : C \longrightarrow D$ and $g : D \longrightarrow E$ are weak homology equivalences of finite chain complexes, then :*

$$t(g \circ f) = t(g) + t(f)$$

Proof : We first prove by induction over the dimension of D that $\rho(Cone(D))$ is zero. The induction begin $dim(D) \leq 0$ is trivial. In the induction step let $D \mid_n$ be the restriction of D to dimension n and D' be the quotient of D and $D \mid_n$. There is an obvious exact sequence

$$0 \longrightarrow Cone(D \mid_n) \longrightarrow Cone(D) \longrightarrow Cone(D') \longrightarrow 0$$

Now the claim follows from the induction hypothesis applied to $Cone(D \mid_n)$ and $Cone(D')$ and lemma 7.7.

Consider the chain map $h : \Sigma^{-1}Cone(g) \longrightarrow Cone(f)$ given by

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} : D_n \oplus E_{n+1} \longrightarrow C_{n-1} \oplus D_n$$

There are exact sequences (see Lück [19] page 245 :

$$\begin{aligned}
&0 \longrightarrow Cone(f) \longrightarrow Cone(h) \longrightarrow Cone(g) \longrightarrow 0 \\
&0 \longrightarrow Cone(g \circ f) \longrightarrow Cone(h) \longrightarrow Cone(D) \longrightarrow 0
\end{aligned}$$

Since $\rho(Cone(D))$ vanishes, the claim follows from lemma 7.7 applied to these two exact sequences. ■

Lemma 7.9 (weak homotopy invariance) *Let $f, g : C \longrightarrow D$ be weak homology equivalences of finite chain complexes. Suppose the existence of weak homology equivalences of finite chain complexes $u : C' \longrightarrow C$ and $v : D \longrightarrow D'$ such that $v \circ f \circ u$ and $v \circ g \circ u$ are chain homotopic. Then :*

$$t(f) = t(g)$$

Proof : Because of lemma 7.8 we have

$$t(f) - t(g) = t(v \circ f \circ u) - t(v \circ g \circ u)$$

Hence we may suppose that there is a chain homotopy $h : f \sim g$. Consider the chain isomorphism $I : Cone(f) \longrightarrow Cone(g)$ given by :

$$\begin{pmatrix} id & o \\ h_{n-1} & id \end{pmatrix} : C_{n-1} \oplus D_n \longrightarrow C_{n-1} \oplus D_n$$

We derive from the composition formula 7.8 and sum formula 7.7 :

$$t(g) - t(f) = t(0 \longrightarrow Cone(g)) - t(0 \longrightarrow Cone(f)) = t(I) = \sum_n (-1)^n \cdot [[I_n]] = 0 \qquad \blacksquare$$

Next we define Reidemeister von Neumann torsion for finite not necessarily weakly acyclic chain complexes. We get from lemma 6.8 the existence of a chain map $i : H(C) \longrightarrow C$ satisfying $H(i) = id$. Recall that we view $H(C)$ as a chain complex using the trivial differential. The following definition makes sense because of lemma 6.8 and lemma 7.9.

Definition 7.10 *Let C be a finite chain complex. Define its* Reidemeister von Neumann torsion

$$\rho(C) \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

by $\rho(C) := t(i)$ for any chain map $i : H(C) \longrightarrow C$ satisfying $H(i) = id$. $\blacksquare$

Lemma 7.11 (Comparision formula) *Let $f : C \longrightarrow D$ be a weak homology equivalence of finite chain complexes. Then :*

$$t(f) - t(H(f)) = \rho(D) - \rho(C)$$

Proof : Choose chain maps $i_C : H(C) \longrightarrow C$ and $i_D : H(D) \longrightarrow D$ satisfying $H(i_C) = id$ and $H(i_D) = id$. There is a weak chain isomorphism $g : H(D) \longrightarrow H(C)$ because of lemma 1.7. Then $H(f \circ i_C \circ g \circ f)$ is the same as $H(i_D \circ f \circ g \circ f)$. Hence by lemma 6.8 there is a weak chain isomorphism $h : D \longrightarrow D$ such that $h \circ i_D \circ H(f) \circ g \circ H(f)$ is chain homotopic to $h \circ f \circ i_C \circ g \circ H(f)$. We get from the composition formula 7.8 and weak homotopy invariance 7.9 :

$$t(f) + \rho(C) = t(f) + t(i_C) = t(f \circ i_C) = t(i_D \circ H(f)) = t(H(f)) + t(i_D) = t(H(f)) + \rho(D)$$

■

Next we show that the definitions of Reidemeister von Neumann torsion 7.10 and 3.8 agree. The first one is more appropiate for technical purposes, whereas the second one is easier to state. Recall that $\mathcal{H}_p(C)$ is the kernel of the Laplace operator Δ_p.

Lemma 7.12 *Let C be a finite chain complex. Let C' be the orthogonal complement of $\mathcal{H}(C)$ in C. We have $\Delta' \oplus 0 = \Delta$ if Δ' resp. Δ are the Laplace operators of C' and C. Then the morphism Δ' is a selfadjoint weak automorphism and we get :*

$$\rho(C) = -\sum_n (-1)^n \cdot n \cdot [\Delta'_n] \in K_1^w(\mathcal{A})^{\mathbf{Z}/2}$$

Proof : Since $\rho(C) = \rho(C' \oplus \mathcal{H}(C)) = \rho(C') + \rho(\mathcal{H}(C))$ holds and $\rho(\mathcal{H}(C))$ vanishes, we may suppose $C = C'$, or, equivalently, that C is weakly acyclic. By the Hodge decomposition theorem 3.7 $\Delta_n : C_n \longrightarrow C_n$ is a selfadjoint weak automorphism. Let $f_n : C_n \longrightarrow C_n$ be the n-fold composition $(\Delta_n)^n = \Delta_n \circ \ldots \circ \Delta_n$. Then the following square commutes :

$$\begin{tikzcd} C_{odd} \arrow[r, "(\Delta c + c^*)_{odd}"] \arrow[d, "f_{odd}"] & C_{ev} \arrow[d, "f_{ev}"'] \\ C_{odd} \arrow[r, "(c + \Delta c^*)_{odd}"] & C_{ev} \end{tikzcd}$$

This shows :

$$\begin{aligned}
&[[(\Delta c + c^*)_{odd}]] \\
&= [(\Delta c + c^*)^*_{odd} \circ (\Delta c + c^*)_{odd}] \\
&= [(\Delta c + c^*)^*_{odd} \circ f_{ev} \circ (\Delta c + c^*)_{odd}] - [f_{ev}] \\
&= [(\Delta c + c^*)^*_{odd} \circ (c + \Delta c^*) \circ f_{odd}] - [f_{ev}] \\
&= [(c + \Delta c^*)_{ev} \circ (c + \Delta c^*)_{odd}] + [f_{odd}] - [f_{ev}]
\end{aligned}$$

As (c^*, Δ) is a weak chain contraction of C, we obtain :

$$\begin{aligned}
\rho(C) &= [[(\Delta c + c^*)_{odd}]] - [[\Delta_{odd}]] \\
&= [f_{odd}] - [f_{ev}] + [(c + \Delta c^*)_{ev} \circ (c + \Delta c^*)_{odd}] - 2 \cdot [\Delta_{odd}] \\
&= - \textstyle\sum_n (-1)^n \cdot n \cdot [\Delta_n] + [(c + \Delta c^*)_{ev} \circ (c + \Delta c^*)_{odd}] - 2 \cdot [\Delta_{odd}]
\end{aligned}$$

Hence it remains to show :

$$[(c + \Delta c^*)_{ev} \circ (c + \Delta c^*)_{odd}] = 2 \cdot [\Delta_{odd}]$$

Consider the dual chain complex C^* given by :

$$\ldots \longrightarrow C^*_{-1} \xrightarrow{c^*_0} C^*_0 \xrightarrow{c^*_1} C^*_1 \ldots$$

It has the weak chain contraction (c, Δ). Now the claim follows from lemma 6.7 and the fact that $[\Delta_{odd}] = [\Delta_{ev}]$ holds (see 7.1). ■

Let $\mathcal{A}$ resp. $\mathcal{B}$ a be finite von Neumann algebra and C resp. D a finite Hilbert $\mathcal{A}$- resp. $\mathcal{B}$-chain complex.

Lemma 7.13 *There is a natural isometric isomorphism of graded $\mathcal{A} \otimes \mathcal{B}$ modules*

$$H_*(C) \otimes H_*(D) \longrightarrow H_*(C \otimes D)$$

Proof : Because of the Hodge decomposition theorem 3.7 it suffices to prove that $\mathcal{H}_*(C) \otimes \mathcal{H}_*(D)$ is $\mathcal{H}_*(C \otimes D)$. This follows from the fact that the Laplace operator on $C \otimes D$ is $(\Delta_C \otimes id_D) \oplus (id_C \otimes \Delta_D)$ ■

Define the *Euler characteristic*

7.14 $\chi^{\mathcal{A}}(C) := \sum_{n \geq 0} (-1)^n \cdot [C_n] = \sum_{n \geq 0} (-1)^n \cdot [H_n(C)] \quad \in K_0(\mathcal{A})$

Lemma 7.15 (Product formula) *Let $f : C' \longrightarrow C$ resp. $g : D' \longrightarrow D$ be weak chain homology equivalence of finite $\mathcal{A}$- resp. $\mathcal{B}$-chain complexes. Then we get using the pairing 2.8*

1.) $t(f \otimes g) = \chi^{\mathcal{A}}(C) \otimes t(g) + t(f) \otimes \chi^{\mathcal{B}}(D)$

2.) $\rho(C \otimes D) = \chi^{\mathcal{A}}(C) \otimes \rho(D) + \rho(C) \otimes \chi^{\mathcal{B}}(D)$

Proof : Since $f \otimes g = (f \otimes id) \circ (id \otimes g)$ holds, the composition formula 7.8 reduces the claim 1.) to the case $g = id$. Now the claim 1.) follows from the sum formula 7.7 and claim 2.) is a consequence of claim 1.) and the Künneth formula 7.13. ■

The next result is the main theorem of this section. The remainder of the section is devoted to its proof.

Theorem 7.16 (Sum formula) *Let $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ be an exact sequence of finite chain complexes. Let $LHS(C,D,E)$ be the weakly acyclic finite chain complex given by the long homology sequence 6.1. The we have :*

$$\rho(C) - \rho(D) + \rho(E) = \rho(C,D,E) - \rho(LHS(C,D,E))$$

We have defined $\rho(C,D,E)$ in 7.5. Notice that in theorem 7.16 we demand the exactness and not only the weak exactness of $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$. This is needed to ensure that the long weakly exact homology sequence is defined (see theorem 6.1). The proof of theorem 7.16 is broken into a sequence of lemmas. We start with analysing a special case where the sequence of chain complexes is only weakly exact but nevertheless the long homology sequence exists and is weakly exact.

Lemma 7.17 *Let $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ be a weakly exact sequence of finite chain complexes. Then :*

1.) *The sequence $H(C) \xrightarrow{H(i)} H(D) \xrightarrow{H(p)} H(E)$ is weakly exact at $H(D)$.*

2.) *Suppose that $H(i) : H(C) \longrightarrow H(D)$ is injective. Define $\partial_n : H_n(E) \longrightarrow H_{n-1}(C)$ to be zero. Then we obtain a weakly exact long homology sequence*

$$\longrightarrow H_{n+1} \xrightarrow{\partial} H_n(C) \xrightarrow{H_n(i)} H_n(D) \xrightarrow{H_n(p)} H_n(E) \xrightarrow{\partial} \ldots$$

Proof : The sequence of finite chain complexes $0 \longrightarrow ker(p) \xrightarrow{j} D \xrightarrow{q} ker(p)^{\perp} \longrightarrow 0$ is exact. The chain maps i and p induce weak chain isomorphisms $\bar{i} : C \longrightarrow ker(p)$ and $\bar{p} : ker(p)^{\perp} \longrightarrow D$. Because of the weakly exact homology sequence 6.1 and corollary 6.2 we get a commutative diagram with weakly exact lower row and weak isomorphism as vertical maps.

$$\begin{array}{ccccccccc}
H(C) & \xrightarrow{H(i)} & H(D) & \xrightarrow{H(p)} & H(E) & & H(C) & \xrightarrow{H(i)} & H(D) \\
\downarrow{\scriptstyle H(\bar{i})} & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle H(\bar{p})} & & \downarrow{\scriptstyle H(\bar{i})} & & \uparrow{\scriptstyle id} \\
H(ker(p)) & \xrightarrow{H(j)} & H(D) & \xrightarrow{H(q)} & H(ker(p)^{\perp} & \xrightarrow{\partial} & H(ker(p)) & \xrightarrow{H(j)} & H(D)
\end{array}$$

One easily checks weak exactness of the upper row at $H(D)$. Suppose that $H(i)$ is injective. Then

$$H(j) \circ H(\bar{i}) = H(i) : H(C) \longrightarrow clos(im(H(i)))$$

is a weak isomorphism. Hence $H(j) : H(ker(p)) \longrightarrow clos(im(H(i))$ is a weak isomorphism by corollary 6.2 and in particular $H(j) : H(ker(p)) \longrightarrow H(D)$ is injective. By the weak exactness of the lower sequence $H(q)$ has dense image in $H(ker(p)^{\perp})$. Hence also $H(p)$ has dense image in $H(E)$. This finishes the proof of lemma 7.17. ■

Lemma 7.18 *Let $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ be a weakly exact sequence of chain complexes. Equip $ker(H(i))^{\perp}$, $H(D)$ and $clos(im(H(p)))$ with the trivial differential. Then there is a commutative diagram of chain complexes*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & ker(H(i))^{\perp} & \xrightarrow{H(i)} & H(D) & \xrightarrow{H(p)} & clos(im(H(p))) & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle i_C} & & \downarrow{\scriptstyle i_D} & & \downarrow{\scriptstyle i_E} & & \\
0 & \longrightarrow & C & \xrightarrow{i} & D & \xrightarrow{p} & E & \longrightarrow & 0
\end{array}$$

such that the rows are weakly exact and the maps i_C, i_D and i_E induce weak isomorphisms

$H(i_C) : ker(H(i))^{\perp} \longrightarrow ker(H(i))^{\perp}$
$H(i_E) : clos(im(H(p))) \longrightarrow clos(im(H(p)))$
$H(i_D) : H(D) \longrightarrow H(D)$

Proof : Choose chain maps

$$r : H(D) \longrightarrow ker(H(i))^{\perp}$$
$$s : clos(im(H(p)))^{\perp} \longrightarrow H(D)$$

such that $r \circ H(i)$ and $H(p) \circ s$ are weak chain isomorphisms. By lemma 6.8 there are chain maps

$$\widehat{i_C} : H(C) \longrightarrow C$$
$$\widehat{i_E} : H(E) \longrightarrow E$$

which induce the identity on homology. Define :

$$i_C = \widehat{i_C} \circ r \circ H(i)$$
$$i_D = i \circ \widehat{i_C} \circ r + \widehat{i_D} \circ s \circ H(p)$$
$$i_E = p \circ \widehat{i_D} \circ s$$

We compute :

$$i_D \circ H(i) = i \circ \widehat{i_C} \circ r \circ H(i) + \widehat{i_D} \circ s \circ H(p) \circ H(i) = i \circ i_C$$
$$p \circ i_D = p \circ i \circ \widehat{i_C} \circ r + p \circ \widehat{i_D} \circ s \circ H(p) = i_E \circ H(p)$$

Since $H(i_C) = r \circ H(i)$ and $H(i_E) = H(p) \circ s$ is valid, i_C and i_D induce weak isomorphisms $ker(H(i))^{\perp} \longrightarrow ker(H(i))^{\perp}$ and $clos(im(H(p))) \longrightarrow clos(im(H(p)))$ by corollary 6.2. The sequence

$$0 \longrightarrow ker(H(i))^{\perp} \xrightarrow{H(i)} H(D) \longrightarrow clos(im(H(p))) \longrightarrow 0$$

is weakly exact by lemma 7.17. By corollary 6.2 also $H(i_D) : H(D) \longrightarrow H(D)$ is a weak isomorphism. This finishes the proof of lemma 7.18. ■

Next we verify the sum formula in a special case where the sequence of chain complexes is only required to be weakly exact.

Lemma 7.19 *Let* $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ *be a weakly exact sequence of finite chain complexes such that* $H(i)$ *is injective. Then we get a weakly exact long homology sequence* $LHS(C, D, E)$ *if we put the boundary operator* ∂ *to be zero, and get :*

$$\rho(C) - \rho(D) + \rho(E) = \rho(C, D, E) - \rho(LHS(C, D, E))$$

Proof : Because of lemma 7.17 and lemma 7.18 there is a commutative diagram of chain complexes with weakly exact rows and weak homology equivalences as vertical maps

$$\begin{tikzcd}
0 \arrow[r] & H(C) \arrow[r, "H(i)"] \arrow[d, "i_C"] & H(D) \arrow[r, "H(p)"] \arrow[d, "i_D"] & H(E) \arrow[r] \arrow[d, "i_E"] & 0 \\
0 \arrow[r] & C \arrow[r, "i"] & D \arrow[r, "p"] & E \arrow[r] & 0
\end{tikzcd}$$

We get from the sum formula for torsion 7.7 :

$$t(i_C) - t(i_D) + t(i_E) = \rho(C, D, E) - \rho(H(C), H(D), H(E))$$

The comparision formula 7.11 shows:

$$\begin{aligned}
&\rho(C) - \rho(D) + \rho(E) = \\
&\ t(i_C) - t(i_D) + t(i_E) - (t(H(i_C)) - t(H(i_D)) + t(H(i_E))) \\
&+ \rho(H(C)) - \rho(H(D)) + \rho(H(E))
\end{aligned}$$

Obviously $\rho(H(C))$, $\rho(H(D))$ and $\rho(H(E))$ are zero. One easily checks :

$$\rho(H(C), H(D), H(E)) = \rho(LHS(C, D, E))$$

Hence it remains to verify :

$$t(H(i_C)) - t(H(i_D)) + t(H(i_E)) = 0$$

But this follows from the sum formula of torsion applied to

$$\begin{tikzcd}
0 \arrow[r] & H(C) \arrow[r, "H(i)"] \arrow[d, "H(i_C)"] & H(D) \arrow[r, "H(p)"] \arrow[d, "H(i_D)"] & H(E) \arrow[r] \arrow[d, "H(i_E)"] & 0 \\
0 \arrow[r] & H(C) \arrow[r, "H(i)"] & H(D) \arrow[r, "H(p)"] & H(E) \arrow[r] & 0
\end{tikzcd}$$

This finishes the proof of lemma 7.19 $\blacksquare$

Lemma 7.20 *Theorem 7.16 is true if D is weakly acyclic.*

Proof : Consider the canonical exact sequences of finite chain complexes, where Cyl is the mapping cylinder.

$$0 \longrightarrow E \xrightarrow{j} Cyl(p) \longrightarrow Cone(D) \longrightarrow 0$$
$$0 \longrightarrow \Sigma C \xrightarrow{k} Cone(p) \longrightarrow Cone(E) \longrightarrow 0$$
$$0 \longrightarrow D \longrightarrow Cyl(p) \xrightarrow{pr} Cone(p) \longrightarrow 0$$

As $Cone(D)$, $Cone(E)$ and D are weakly acyclic, we get from theorem 6.1 and lemma 7.19

$$\rho(E) - \rho(Cyl(p)) + \rho(Cone(D)) = t(H(j))$$
$$-\rho(C) - \rho(Cone(p)) + \rho(Cone(E)) = t(H(k)) - \rho(C, D, E)$$
$$\rho(D) - \rho(Cyl(p)) + \rho(Cone(p)) = -t(H(pr))$$

One checks directly $\rho(Cone(D)) = \rho(Cone(E)) = 0$. We obtain

$$\rho(C) - \rho(D) + \rho(E) = \rho(C, D, E) + t(H(pr)) + t(H(j)) - t(H(k))$$

Since $t(\partial : H(E) \longrightarrow \Sigma H(C))$ is $-\rho(H(C, D, E))$, it suffices to show

$$t(\partial) + t(H(k)) = t(H(pr)) + t(H(j))$$

We conclude from the definitions

$$H(k) \circ \partial = H(pr) \circ H(j)$$

Now an application of the composition formula 7.8 finishes the proof of lemma 7.20. ■

Now we are ready to prove theorem 7.16. Consider the commutative diagram appearing in lemma 7.18. Let D' be $clos(im(i_D))^\perp$ and $p_D : D \longrightarrow D'$ be the projection. Let $C' \subset D'$ be $clos(im(p_D \circ i))$ and $i' : C' \longrightarrow D'$ be the inclusion. Put $E' = (C')^\perp$. Let $p' : D' \longrightarrow E'$ be the projection. Then we obtain an exact sequence of finite chain complexes $0 \longrightarrow C' \longrightarrow D' \longrightarrow E' \longrightarrow 0$. As the sequence $0 \longrightarrow C \xrightarrow{i} D \xrightarrow{p} E \longrightarrow 0$ is exact, $p' \circ p_D : D \longrightarrow E'$ factorizes over $p : D \longrightarrow E$ into a map $p_E : E \longrightarrow E'$ and p_D induces $p_C; C \longrightarrow C'$. Thus we have constructed a commutative diagram of finite chain complexes with weakly exact middle column, exact lower two rows and weakly exact top row.

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & ker(H(i))^{\perp} \arrow[r, "H(i)"] \arrow[d, "i_C"] & H(D) \arrow[r, "H(p)"] \arrow[d, "i_D"] & clos(im(H(p))) \arrow[r] \arrow[d, "i_E"] & 0 \\
0 \arrow[r] & C \arrow[r, "i"] \arrow[d, "p_C"] & D \arrow[r, "p"] \arrow[d, "p_D"] & E \arrow[r] \arrow[d, "p_E"] & 0 \\
0 \arrow[r] & C' \arrow[r, "i'"] \arrow[d] & D' \arrow[r, "p'"] \arrow[d] & E' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}
$$

We claim that also the outer columns are weakly exact. We substitute in the diagram above for the upper row

$$\{0\} \longrightarrow ker(H(p)) \longrightarrow H(D) \longrightarrow ker(H(p))^{\perp} \longrightarrow \{0\}$$

in the obvious way. It suffices to consider this new diagram whose rows are exact. We may interprete it as a short exact sequence of two-dimensional chain complexes given by the columns. Since the middle row is weakly exact, image p_C is dense in C', and i_E is injective, the claim follows from the weakly exact long homology sequence of theorem 6.1.

We can apply lemma 7.19 to the columns and obtain :

$$\rho(C') - \rho(C) = \rho(ker(H(i))^{\perp}, C, C') - \rho(LHS(ker(H(i))^{\perp}, C, C'))$$
$$\rho(D') - \rho(D) = \rho(H(D), D, D') - \rho(LHS(H(D), D, D'))$$
$$\rho(E') - \rho(E) = \rho(clos(im(H(p))), E, E') - \rho(LHS(clos(im(p))), E, E')$$

Moreover, D' is weakly acyclic. Therefore we can apply lemma 7.20 to the bottom exact row and obtain :

$$\rho(C') - \rho(D') + \rho(E') = \rho(C', D', E') - \rho(LHS(C', D', E'))$$

We derive from lemma 7.19 applied to the short weakly exact sequence of weakly acyclic chain complexes given by the long homology sequences of the rows

$$\rho(LHS(ker(H(i))^{\perp}, H(D), clos(im(H(p))))) - \rho(LHS(C, D, E)) + \rho(LHS(C', D', E')) =$$
$$\rho(ker(H(i))^{\perp}; C, C') - \rho(H(D), D, D') + \rho(clos(im(H(p))), E, E')$$

Interpreting the diagram above as a short weakly exact sequence of two-dimensional weakly acyclic chain complexes given by the rows, we derive from lemma 7.19

$$\rho(ker(H(i))^{\perp}, C, C') - \rho(H(D), D, D') + \rho(clos(im(H(p))), E, E') =$$
$$\rho(ker(H(i))^{\perp}, H(D), clos(im(H(p)))) - \rho(C, D, E) + \rho(C', D', E')$$

Obviously we have :

$$\rho(ker(H(i))^{\perp}, H(D), clos(im(H(p)))) = \rho(LHS(ker(H(i))^{\perp}, H(D), clos(im(H(p)))))$$

Now theorem 7.16 follows from the equations above. $\blacksquare$

References

[1] **Atiyah, M.** : *"Elliptic operators, discrete groups and von Neumann algebras"*, Asterisque 32-33, 43 -72 (1976)

[2] **Bröcker, T. and tom Dieck, T.** : *"Representations of compact Lie groups"*, grad. texts in math. 98, Springer (1985)

[3] **Burde, G. and Zieschang, H.** : *"Knots"*, de Gruyter studies in math. 5 (1985)

[4] **Carey, A.L. and Mathai, V.** : *"L^2-torsion invariants"*, preprint (1991)

[5] **Cheeger, J.** : *"Analytic torsion and the heat equation"*, Ann. of Math. 109, 259-322 (1979)

[6] **Cheeger, J. and Gromov, M.** : *"Bounds on the von Neumann dimension of L^2-cohomology and the Gauss Bonnett theorem for open manifolds"*, J. of Diff. Geom. 21, 1 - 34 (1985)

[7] **Cheeger, J. and Gromov, M.** : *On the characteristic numbers of complete manifolds of bounded curvature and finite volume"*, Rauch Mem. vol, editors : Chavel, I. and Farkas, H., Springer, 115 - 154 (1985)

[8] **Cheeger, J. and Gromov, M.** : *"L^2-cohomology and group cohomology "*, Topology 25, 189 - 215 (1986)

[9] **Cohen, M.M.** : *"A course in simple homotopy theory"*, grad. texts in math. 10, Springer (1973)

[10] **Dixmier, J.** : *"von Neumann algebras"*, North-Holland (1981)

[11] **Dodziuk, J.** : *"DeRham-Hodge theory for L^2-cohomology of infinite coverings"*, Topology 16 , 157 - 165 (1977)

[12] **Dodziuk, J.** : *"L^2-harmonic forms on rotationally symmetric Riemannian manifolds"*, Proc AMS 77, 395 - 400 (1979)

[13] **Dovermann, K.H. and Rothenberg, M.** : *"An algebraic approach to the generalized Whitehead group"*, in "Transformation groups", Proceedings, Poznań 1985, lecture notes in mathematics vol 1217, 92-114, Springer (1986)

[14] **Fuglede, B. and Kadison, R.V.** : *"Determinant theory in finite factors"*, Ann. of math. 55, 520 -530 (1952)

[15] **Illman, S.** : *"Whitehead torsion and group actions"*, Ann. Acad. Sci. Fenn. ser. AI 588, 1-44

[16] **Illman, S.** : *"Smooth equivariant triangulations of G-manifolds for G a finite group"*, Math. Ann. 233, 199-220 (1978)

[17] **Lott, J.** : *"Heat kernels on covering spaces and topological invariants"*, preprint (1990), to appear in J. of Differential Geometry

[18] **Lott, J. and Rothenberg, M.** : *"Analytic torsion for group actions"*, preprint, to appear in J. of Differential Geometry (1989)

[19] **Lück, W.** : *"Transformation groups and algebraic K-theory"*, Springer lecture note in mathematics vol. 1408 (1989)

[20] **Lück, W.** : *"Analytic and algebraic torsion for manifolds with boundary and symmetries"*, Math. Gott. Heft 33 (1989), to appear in J. of Differential Geometry

[21] **Milnor, J.** : *"Infinite cyclic coverings"*, Proc. Conf on the topology of manifolds, 115 - 133 (1968)

[22] **Müller, W.** : *"Analytic torsion and R-torsion of Riemannian manifolds"*, Adv. in Math. 28, 233-305 (1978)

[23] **Ray, D.B. and Singer, I.** : *"R-torsion and the Laplacian on Riemannian manifolds"*, Adv. in Math. 7, 145-210 (1971)

[24] **Reidemeister, K.** : *"Homotopieringe und Linsenräume"*, Hamburger Abhandlungen 11 (1938)

[25] **deRham, G.** : *"Reidemeister's torsion invariant and rotations of S^n"*, Differential Analysis (published for the Tata Institute of Federal Research, Bombay), Oxford University Press, London, MR 32 ♯ 8355, 27-36 (1964)

[26] **Rothenberg, M** : *"Torsion invariants and finite transformation groups"*, in "Algebraic and geometric topology", part 1, Proc. Symp. Pure Math. vol 33, AMS, 267-311 (1978)

[27] **Rothenberg, M.** : *"Analytic and Combinatorial Torsion"*, Contemporary Mathematics 105 pp. 213 -245 (1988)

[28] **Sunder, V.S.** : *"An invitation to von Neumann algebras"*, Universitext, Springer (1986)

[29] **Turaev, V., G.** *"Reidemeister torsion in knot theory"*, Russian math. surveys 41 :1, 119-182 (1986)

[30] **Waldhausen, F.** : *"Algebraic K-theory of spaces"*, in "Algebraic and Geometric Topology", proc. conf. Rutgers Uni., New Brunswick 1983, lect not. in math. 1126, Springer, 318 - 419 (1983)

Current adresses :

Wolfgang Lück*
Fachbereich Mathematik
Johannes Gutenberg-Universität
Staudingerweg 9
6500 Mainz 1
Germany

Mel Rothenberg**
Department of mathematics
The University of Chicago
5734 South University Avenue
Chicago Illinois 60637
U. S. A.

* Partially supported by NSF Grant DMS - 9103327
* Partially supported by NSF Grant DMS - 8903168