Inheritance of Isomorphism Conjectures under colimits

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Outline

- We define the notion of an equivariant homology theory.
- We explain the notion of a classifying G-space of a family of subgroups.
- We explain what an Isomorphism Conjecture is.
- We give some applications of the Farrell-Jones Conjecture.
- We prove inheritance properties under colimits.
- We explain consequences of these inheritance properties.
- Convention: group will always mean discrete group.

Equivariant homology theories

Definition (*G*-homology theory)

A *G-homology theory* $\mathcal{H}_*$ is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups together with natural transformations

$$\partial_n(X, A)\colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Definition (Equivariant homology theory)

An *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A) there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \quad \to \quad \mathcal{H}_n^G(\mathrm{ind}_\alpha(X, A))$$

satisfying:

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then ind_α is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

Example (Equivariant homology theories)

- Given a $\mathcal{K}_*$ non-equivariant homology theory, put

$$\begin{aligned} \mathcal{H}_*^G(X) &:= \mathcal{K}_*(X/G); \\ \mathcal{H}_*^G(X) &:= \mathcal{K}_*(EG \times_G X) \quad \text{Borel homology}. \end{aligned}$$

- Equivariant bordism $\Omega_*^?(X)$;
- Equivariant topological K-homology $K_*^?(X)$ in the sense of Kasparov.

Definition (Spectrum)

A *spectrum*

$$\mathbf{E} = \{(E(n), \sigma(n)) \mid n \in \mathbb{Z}\}$$

is a sequence of pointed spaces $\{E(n) \mid n \in \mathbb{Z}\}$ together with pointed maps called *structure maps*

$$\sigma(n) \colon E(n) \wedge S^1 \longrightarrow E(n+1).$$

A *map of spectra*

$$\mathbf{f} \colon \mathbf{E} \to \mathbf{E}'$$

is a sequence of maps $f(n) \colon E(n) \to E'(n)$ which are compatible with the structure maps $\sigma(n)$, i.e., $f(n+1) \circ \sigma(n) \ = \ \sigma'(n) \circ (f(n) \wedge \mathrm{id}_{S^1})$ holds for all $n \in \mathbb{Z}$.

- Given a spectrum $\mathbf{E}$, a classical construction in algebraic topology assigns to it a homology theory $H_*(-, \mathbf{E})$ with the property

$$H_n(\mathrm{pt}; \mathbf{E}) = \pi_n(\mathbf{E}).$$

  Put

$$H_n(X; \mathbf{E}) := \pi_n\left(X_+ \wedge \mathbf{E}\right).$$

- The basic example of a spectrum is the sphere spectrum $\mathbf{S}$. Its n-th space is S^n and its n-th structure map is the standard homeomorphism $S^n \wedge S^1 \xrightarrow{\cong} S^{n+1}$. Its associated homology theory is stable homotopy $\pi_*^s(-) = H_*(-; \mathbf{S})$.
- This construction can be extended to the equivariant setting as follows.

Theorem (L.-Reich (2005))

Given a functor $\mathbf{E}\colon \mathrm{Groupoids} \to \mathrm{Spectra}$ *sending equivalences to weak equivalences, there exists an equivariant homology theory* $\mathcal{H}_*^?(-;\mathbf{E})$ *satisfying*

$$\mathcal{H}_n^H(pt) \cong \mathcal{H}_n^G(G/H) \cong \pi_n(\mathbf{E}(H)).$$

Theorem (Equivariant homology theories associated to K and L-theory, Davis-L. (1998))

Let R be a ring (with involution). There exist covariant functors

$$\begin{aligned} \mathbf{K}_R, \mathbf{L}_R^{\langle -\infty \rangle}, \mathbf{K}_{l^1}^{\mathrm{top}} \colon \mathrm{Groupoids} &\to \mathrm{Spectra}; \\ \mathbf{K}^{\mathrm{top}} \colon \mathrm{Groupoids}^{\mathrm{inj}} &\to \mathrm{Spectra}, \end{aligned}$$

with the following properties:

- *They send equivalences to weak equivalences;*
- *For every group G and all $n \in \mathbb{Z}$ we have:*

$$\begin{aligned} \pi_n(\mathbf{K}_R(G)) &\cong K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle -\infty \rangle}(G)) &\cong L_n^{\langle -\infty \rangle}(RG); \\ \pi_n(\mathbf{K}^{\mathrm{top}}(G)) &\cong K_n(C_r^*(G)); \\ \pi_n(\mathbf{K}_{l^1}^{\mathrm{top}}(G)) &\cong K_n(l^1(G)). \end{aligned}$$

Example (Equivariant homology theories associated to K and L-theory)

We get equivariant homology theories:

$$H_*^?(-;\mathbf{K}_R);$$
$$H_*^?(-;\mathbf{L}_R^{\langle -\infty\rangle});$$
$$H_*^?(-;\mathbf{K}^{\mathrm{top}});$$
$$H_*^?(-;\mathbf{K}_{l^1}^{\mathrm{top}}),$$

satisfying for $H \subseteq G$:

$$\begin{array}{lllll}
H_n^G(G/H;\mathbf{K}_R) & \cong & H_n^H(\mathrm{pt};\mathbf{K}_R) & \cong & K_n(RH); \\
H_n^G(G/H;\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & H_n^H(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & L_n^{\langle -\infty\rangle}(RH); \\
H_n^G(G/H;\mathbf{K}^{\mathrm{top}}) & \cong & H_n^H(\mathrm{pt};\mathbf{K}^{\mathrm{top}}) & \cong & K_n(C_r^*(H)); \\
H_n^G(G/H;\mathbf{K}_{l^1}^{\mathrm{top}}) & \cong & H_n^H(\mathrm{pt};\mathbf{K}_{l^1}^{\mathrm{top}}) & \cong & K_n(l^1(H)).
\end{array}$$

Classifying spaces for families of subgroups

Definition (*G-CW*-complex)

A *G-CW-complex* X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration, and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

Example (Simplicial actions)

Let X be a simplicial complex. Suppose that G acts simplicially on X. Then G acts simplicially also on the barycentric subdivision X', and the G-space X' inherits the structure of a G-CW-complex.

Example (Smooth actions)

If G acts properly and smoothly on a smooth manifold M, then M inherits the structure of G-CW-complex.

Definition (Family of subgroups)

A *family $\mathcal{F}$ of subgroups* of G is a set of subgroups of G which is closed under conjugation and taking subgroups.

Examples for $\mathcal{F}$ are:

$$\begin{array}{lcl} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{FIN} & = & \{\text{finite subgroups}\}; \\ \mathcal{VCYC} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

Definition (Classifying G-space for a family of subgroups, tom Dieck(1974))

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-space for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.

We abbreviate $\underline{E}G := E_{\mathcal{FIN}}(G)$ and call it the *universal G-space for proper G-actions*.
We also write $EG = E_{\mathcal{TR}}(G)$.

- If $\mathcal{F} \subseteq \mathcal{G}$ are families of subgroups of G, there is up to G-homotopy precisely one G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$.

Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- *There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;*
- *Two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;*
- *A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is contractible.*

- If $\mathcal{F} \subseteq \mathcal{G}$ are families of subgroups of G, then $E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G)$ is a model for $E_{\mathcal{F}}(G)$.

The spaces $\underline{E}G$ are interesting in their own right and have often very nice geometric models which are rather small. For instance

- Rips complex for word hyperbolic groups;
- Teichmüller space for mapping class groups;
- Outer space for the group of outer automorphisms of free groups;
- L/K for an almost connected Lie group L, a maximal compact subgroup $K \subseteq L$ and $G \subseteq L$ a discrete subgroup;
- CAT(0)-spaces with proper isometric G-actions, e.g., simply connected Riemannian manifolds with non-positive sectional curvature or trees.

Conjecture (Isomorphism Conjecture)

Let $\mathcal{H}^?_*$ be an equivariant homology theory. It satisfies the Isomorphism Conjecture for the group G and the family $\mathcal{F}$ if the projection $E_{\mathcal{F}}(G) \to pt$ induces for all $n \in \mathbb{Z}$ a bijection

$$\mathcal{H}^G_n(E_{\mathcal{F}}(G)) \to \mathcal{H}^G_n(pt).$$

- The point is to find an as small as possible family $\mathcal{F}$.
- The Isomorphism Conjecture is always true for $\mathcal{F} = \mathcal{ALL}$ since it becomes a trivial statement because of $E_{\mathcal{ALL}}(G) = \text{pt}$.
- The philosophy is to be able to compute the functor of interest for G by knowing it on the values of elements in $\mathcal{F}$.

Example (Farrell-Jones Conjecture)

The Farrell-Jones Conjecture for K-theory or L-theory respectively with coefficients in R is the Isomorphism Conjecture for $\mathcal{H}^?_* = H_*(-;\mathbf{K}_R)$ or $\mathcal{H}^?_* = H_*(-;\mathbf{L}_R^{\langle -\infty\rangle})$ respectively and $\mathcal{F} = \mathcal{VCYC}$.
In other words, it predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G),\mathbf{K}_R) \to H_n^G(\mathrm{pt},\mathbf{K}_R) = K_n(RG)$$

or

$$H_n^G(E_{\mathcal{VCYC}}(G),\mathbf{L}_R^{\langle -\infty\rangle}) \to H_n^G(\mathrm{pt},\mathbf{L}_R^{\langle -\infty\rangle}) = L_n^{\langle -\infty\rangle}(RG)$$

respectively is bijective for all $n \in \mathbb{Z}$.

Example (Baum-Connes Conjecture)

The Baum-Connes Conjecture is the Isomorphism Conjecture for $\mathcal{H}^?_* = K^?_* = H^?_*(-;\mathbf{K}^{\mathrm{top}})$ and $\mathcal{F} = \mathcal{FIN}$.
In other words it predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(\mathrm{pt}, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Example (Bost Conjecture)

The Bost Conjecture is the Isomorphisms Conjecture for $\mathcal{H}^?_* = K^?_* = H^?_*(-;\mathbf{K}^{\mathrm{top}}_{l^1})$ and $\mathcal{F} = \mathcal{FIN}$.
In other words it predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}_{l^1}) \to H_n^G(\mathrm{pt}, \mathbf{K}^{\mathrm{top}}_{l^1}) = K_n(l^1(G))$$

is bijective for all $n \in \mathbb{Z}$.

Applications

Definition (Whitehead group)

The *Whitehead group* of a group G is defined to be

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}.$$

Definition (h-cobordism)

An *h-cobordism* over a closed manifold M_0 is a compact manifold W whose boundary is the disjoint union $M_0 \amalg M_1$ such that both inclusions $M_0 \to W$ and $M_1 \to W$ are homotopy equivalences.

Theorem (*s*-Cobordism Theorem, Barden, Mazur, Stallings, Kirby-Siebenmann)

Let M_0 be a closed (smooth) manifold of dimension $n \geq 5$. Let $(W; M_0, M_1)$ be an h-cobordism over M_0.
Then W is homeomorphic (diffeomorpic) to $M_0 \times [0,1]$ relative M_0 if and only if its Whitehead torsion

$$\tau(W, M_0) \in \mathrm{Wh}(\pi_1(M_0))$$

vanishes.

- The s-cobordism theorem is a key ingredient in the surgery program for the classification of closed manifolds due to Browder, Novikov, Sullivan and Wall.
- If $\mathrm{Wh}(G)$ vanishes, every h-cobordism $(W; M_0, M_1)$ of dimension ≥ 6 with $G \cong \pi_1(W)$ is trivial and in particular $M_0 \cong M_1$.
- The K-theoretic Farrell-Jones Conjecture implies for a torsionfree group G that $\mathrm{Wh}(G)$ is trivial.
- The Poincaré Conjecture in dimension ≥ 5 is a consequence of the s-cobordism theorem since $\mathrm{Wh}(\{1\})$ vanishes.

Conjecture (Kaplansky Conjecture)

The Kaplansky Conjecture says for a torsionfree group G and an integral domain R that 0 and 1 are the only idempotents in RG.

Theorem (The Baum-Connes Conjecture and the Kaplansky Conjecture)

If the torsionfree group G satisfies the Baum-Connes Conjecture, then the Kaplansky Conjecture is true for $C_r^*(G)$ and hence for $\mathbb{C}G$.

Theorem (The Farrell-Jones Conjecture and the Kaplansky Conjecture, Bartels-L.-Reich (2007))

] Let F be a skew-field and let G be a group satisfying the K-theoretic Farrell-Jones Conjecture with coefficients in F. Suppose that one of the following conditions is satisfied:

- *F is commutative and has characteristic zero and G is torsionfree.*
- *G is torsionfree and sofic, e.g., residually amenable.*
- *The characteristic of F is p, all finite subgroups of G are p-groups and G is sofic.*

Then 0 and 1 are the only idempotents in FG.

Conjecture (Borel Conjecture)

The Borel Conjecture for G predicts for two closed aspherical manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ that any homotopy equivalence $M \to N$ is homotopic to a homeomorphism and in particular that M and N are homeomorphic.

- The Borel Conjecture can be viewed as the topological version of Mostow rigidity. A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of Farrell-Jones(1989).
- There are also non-aspherical manifolds which are topological rigid in the sense of the Borel Conjecture (see Kreck-L. (2005)).

Theorem (The Farrell-Jones Conjecture and the Borel Conjecture)

If the K- and L-theoretic Farrell-Jones Conjecture hold for G in the case $R = \mathbb{Z}$, then the Borel Conjecture is true in dimension ≥ 5 and in dimension 4 if G is good in the sense of Freedman.

- Thurston's Geometrization Conjecture implies the Borel Conjecture in dimension 3.
- The Borel Conjecture in dimension 1 and 2 is obviously true.

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\mathrm{sign}_x(M,f) := \langle \mathcal{L}(M) \cup f^*x, [M]\rangle$$

is an oriented homotopy invariant of (M,f), i.e., for every orientation preserving homotopy equivalence of closed oriented manifolds $g\colon M_0 \to M_1$ and homotopy equivalence $f_i\colon M_0 \to M_1$ with $f_1 \circ g \simeq f_2$ we have

$$\mathrm{sign}_x(M_0,f_0) = \mathrm{sign}_x(M_1,f_1).$$

Theorem (The Farrell-Jones, the Baum-Connes and the Novikov Conjecture)

Suppose that one of the following assembly maps

$$\begin{array}{rcl} H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{-\infty}) & \to & H_n^G(pt, \mathbf{L}_R^{-\infty}) = L_n^{-\infty}(RG); \\ K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{FIN}}(G), \mathbf{K}^{\mathrm{top}}) & \to & H_n^G(pt, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G)), \end{array}$$

is rationally injective.
Then the Novikov Conjecture holds for the group G.

Inheritance properties

- Fix an equivariant homology theory $\mathcal{H}_*^?$.

Theorem (Transitivity Principle)

Suppose $\mathcal{F} \subseteq \mathcal{G}$ are two families of subgroups of G. Assume that for every element $H \in \mathcal{G}$ the group H satisfies the Isomorphism Conjecture for $\mathcal{F}|_H = \{K \subseteq H \mid K \in \mathcal{F}\}$.
Then the map

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$$

is bijective for all $n \in \mathbb{Z}$.
Moreover, $(G, \mathcal{G})$ satisfies the Isomorphism Conjecture if and only if $(G, \mathcal{F})$ satisfies the Isomorphism Conjecture.

Sketch of proof.

- For a G-CW-complex X with isotropy group in $\mathcal{G}$ consider the natural map induced by the projection

$$s_*^G(X)\colon \mathcal{H}_*^G(X \times E_{\mathcal{F}}(G)) \to \mathcal{H}_*^G(X).$$

- This a natural transformation of G-homology theories defined for G-CW-complexes with isotropy groups in $\mathcal{G}$.
- In order to show that it is a natural equivalence it suffices to show that $s_n^G(G/H)$ is an isomorphism for all $H \in \mathcal{G}$ and $n \in \mathbb{Z}$.

□

Sketch of proof (continued).

- The G-space $G/H \times E_{\mathcal{F}}(G)$ is G-homeomorphic to $G \times_H \operatorname{res}_G^H E_{\mathcal{F}}(G)$ and $\operatorname{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}|_H}(H)$.
- Hence by the induction structure $s_n^G(G/H)$ can be identified with the assembly map

$$\mathcal{H}_*^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_*^H(\mathrm{pt}),$$

which is bijective by assumption.
- Now apply this to $X = E_{\mathcal{G}}(G)$ and observe that $E_{\mathcal{G}}(G) \times E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}}(G)$.

□

Example (Baum-Connes Conjecture and $\mathcal{VCYC}$)

- Consider the Baum-Connes setting, i.e., take $\mathcal{H}_*^? = K_*^?$.
- Consider the families $\mathcal{FIN} \subseteq \mathcal{VCYC}$.
- For every virtually cyclic group V the Baum-Connes Conjecture is true, i.e.,

$$K_n^V(E_{\mathcal{FIN}}(V)) \to K_n(C_r^*(V))$$

  is bijective for $n \in \mathbb{Z}$.
- Hence by the Transitivity principle the following map is bijective for all groups G and all $n \in \mathbb{Z}$

$$K_n^G(\underline{E}G) = K_n^G(E_{\mathcal{FIN}}(G)) \to K_n^G(E_{\mathcal{VCYC}}(G)).$$

- This explains why in the Baum-Connes setting it is enough to deal with $\mathcal{FIN}$ instead of $\mathcal{VCYC}$.
- This is not true in the Farrell-Jones setting and causes many extra difficulties there (NIL and UNIL-phenomena).
- This difference is illustrated by the following isomorphisms due to Pimsner-Voiculescu and Bass-Heller-Swan:

$$\begin{aligned} K_n(C_r^*(\mathbb{Z})) &\cong K_n(\mathbb{C}) \oplus K_{n-1}(\mathbb{C}); \\ K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R). \end{aligned}$$

- Consider a directed system of groups $\{G_i \mid i \in I\}$ with structure maps $\psi_i \colon G_i \to G$ for $i \in I$. Put $G = \operatorname{colim}_{i \in I} G_i$.
- Let X be a G-CW-complex.
- We have the canonical G-map

$$ad \colon (\psi_i)_* \psi_i^* X = G \times_{G_i} X \to X, \quad (g, x) \mapsto gx.$$

- Define a homomorphism

$$t_n^G(X) \colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^* X) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

by the colimit of the system of maps indexed by $i \in I$

$$\mathcal{H}_n^{G_i}(\psi_i^* X) \xrightarrow{\operatorname{ind}_{\psi_i}} \mathcal{H}_n^G((\psi_i)_* \psi_i^* X) \xrightarrow{\mathcal{H}_n^G(ad)} \mathcal{H}_n^G(X).$$

Definition (Strongly continuous equivariant homology theory)

An equivariant homology theory $\mathcal{H}^?_*$ is called *strongly continuous* if for every group G and every directed system of groups $\{G_i \mid i \in I\}$ with $G = \operatorname{colim}_{i \in I} G_i$ the map

$$t_n^G(\mathrm{pt})\colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\mathrm{pt}) \to \mathcal{H}_n^G(\mathrm{pt})$$

is an isomorphism for every $n \in \mathbb{Z}$.

Lemma

Consider a directed system of groups $\{G_i \mid i \in I\}$ *with* $G = \operatorname{colim}_{i \in I} G_i$*. Let* X *be a* G*-CW-complex. Suppose that* $\mathcal{H}^?_*$ *is strongly continuous. Then the homomorphism*

$$t_n^G(X)\colon \operatorname*{colim}_{i \in I} \mathcal{H}_n^{G_i}(\psi_i^* X) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

is bijective for every $n \in \mathbb{Z}$*.*

Idea of proof.

- Show that t_*^G is a transformation of G-homology theories.
- Prove that the strong continuity implies that $t_n^G(G/H)$ is bijective for all $n \in \mathbb{Z}$ and $H \subseteq G$.
- Then a general comparison theorem gives the result.

□

- Let $\phi\colon K \to G$ be a group homomorphism and let $\mathcal{F}$ be a family of subgroups of G.
  Define the family $\phi^*\mathcal{F}$ of subgroups of K by

$$\phi^*\mathcal{F} := \{L \subseteq K \mid \phi(L) \in \mathcal{F}\}.$$

- Basic property: $\phi^* E_{\mathcal{F}}(G) = E_{\phi * \mathcal{F}}(K)$.

Lemma

Let $\mathcal{F}$ be a family of subgroups of G. Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$ and structure maps $\psi_i\colon G_i \to G$. Suppose that $\mathcal{H}^?_*$ is strongly continuous and for every $i \in I$ the Isomorphism Conjecture holds for G_i and $\psi_i^*\mathcal{F}$.
Then the Isomorphism Conjecture holds for G and $\mathcal{F}$.

Proof.

This follows from the following commutative square, whose horizontal arrows are bijective because of the last lemma, and the identification $\psi_i^* E_{\mathcal{F}}(G) = E_{\psi_i^*\mathcal{F}}(G_i)$

$$\begin{array}{ccc}
\operatorname{colim}_{i\in I} \mathcal{H}_n^{G_i}(E_{\psi_i^*\mathcal{F}}(G_i)) & \xrightarrow[\cong]{t_n^G(E_{\mathcal{F}}(G))} & \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \\
\downarrow & & \downarrow \\
\operatorname{colim}_{i\in I} \mathcal{H}_n^{G_i}(\mathrm{pt}) & \xrightarrow[\cong]{t_n^G(\mathrm{pt})} & \mathcal{H}_n^G(\mathrm{pt})
\end{array}$$

□

- Fix a class of groups $\mathcal{C}$ closed under isomorphisms, taking subgroups and taking quotients, e.g., the class of finite groups or the class of virtually cyclic groups.
- For a group G let $\mathcal{C}(G)$ be the family of subgroups of G which belong to $\mathcal{C}$.

Theorem (Inheritance under colimits for Isomorphism Conjectures)

Let $\{G_i \mid i \in I\}$ be a directed system of groups with $G = \operatorname{colim}_{i \in I} G_i$. Suppose that $\mathcal{H}^?_*$ is strongly continuous and that the Isomorphism Conjecture is true for $(H, \mathcal{C}(H))$ for every $i \in I$ and every subgroup $H \subseteq G_i$.
Then for every subgroup $K \subseteq G$ the Isomorphism Conjecture is true for K and $\mathcal{C}(K)$.

Proof.

- If G is the colimit of the directed system $\{G_i \mid i \in I\}$, then the subgroup $K \subseteq G$ is the colimit of the directed system $\{\psi_i^{-1}(K) \mid i \in I\}$. Hence we can assume $G = K$ without loss of generality.
- Since $\mathcal{C}$ is closed under quotients by assumption, we have $\mathcal{C}(G_i) \subseteq \psi_i^* \mathcal{C}(G)$ for every $i \in I$. Hence we can consider for any $i \in I$ the composition

$$H_n^{G_i}(E_{\mathcal{C}(G_i)}(G_i)) \to H_n^{G_i}(E_{\psi_i^* \mathcal{C}(G)}(G_i)) \to H_n^{G_i}(\mathrm{pt}).$$

- By the last lemma it suffices to show that the second map is bijective.
- By assumption the composition of the two maps is bijective. Hence it remains to show that the first map is bijective.

□

Proof (continued).

- By the Transitivity Principle this follows from the assumption that the Isomorphism Conjecture holds for every subgroup $H \subseteq G_i$ and in particular for any $H \in \psi_i^*\mathcal{C}(G)$ for $\mathcal{C}(G_i)|_H = \mathcal{C}(H)$.

□

- Notice that it is very convenient for the proof to allow arbitrary families of subgroups and to have the definition of $\mathcal{H}_*^G(X)$ at hand for arbitrary (not necessarily proper) G-CW-complexes X.

Lemma

The homology theories

$$H^?_*(-;\mathbf{K}_R);$$
$$H^?_*(-;\mathbf{L}_R^{\langle -\infty\rangle});$$
$$H^?_*(-;\mathbf{K}^{\mathrm{top}}_{l^1}),$$

are strongly continuous.

- For instance one has to show that the canonical map induced by the various structure maps $G_i \to G$ induces an isomorphism

$$\operatorname*{colim}_{i\in I} K_n\big(l^1(G_i)\big) \xrightarrow{\cong} K_n\big(l^1\big(\operatorname*{colim}_{i\in I} G_i\big)\big).$$

- This statement does not make sense for the reduced group C^*-algebra since it is not functorial under arbitrary group homomorphisms.
- For instance, $C_r^*(\mathbb{Z} * \mathbb{Z})$ is a simple C^*-algebra and hence no epimorphism $C_r^*(\mathbb{Z} * \mathbb{Z}) \to C_r^*(\{1\})$ exists.

- Let $\{G_i \mid i \in I\}$ be a directed system of groups with (not necessarily injective) structure maps $\phi_{i,j} \colon G_i \to G_j$. Let $G = \operatorname{colim}_{i \in I} G_i$ be its colimit.
- Next we pass to twisted coefficients: Let R be a ring (with involution) and let A be a C^*-algebra, both with structure preserving G-action.
- Given $i \in I$ and a subgroup $H \subseteq G_i$, we let H act on R and A by restriction with the group homomorphism $(\psi_i)|_H \colon H \to G$.
- The following result follows for untwisted coefficients from the previous result. In the twisted case one has to modify the setting by considering everything over a fixed reference group.

Theorem (Inheritance under colimits for the Farrell-Jones and the Bost Conjecture, Bartels-Echterhoff-Lück (2007))

In the situation above we get:

- *Suppose that the assembly map*

  $$H_n^H(E_{\mathcal{VCYC}}(H);\mathbf{K}_R) \to H_n^H(pt;\mathbf{K}_R) = K_n(R \rtimes H)$$

  is bijective for all $n \in \mathbb{Z}$, all $i \in I$ and all subgroups $H \subseteq G_i$. Then for every subgroup K of G the assembly map

  $$H_n^K(E_{\mathcal{VCYC}}(K);\mathbf{K}_R) \to H_n^K(pt;\mathbf{K}_R) = K_n(R \rtimes K)$$

  is bijective for all $n \in \mathbb{Z}$.
- *The corresponding version is true for the assembly maps in the L-theory setting and for the Bost Conjecture.*

Theorem (Bartels-L.-Reich (2007))

Let G be a subgroup of a finite product of hyperbolic groups. Let R be a ring with structure preserving G-action.
Then the K-theoretic Farrell-Jones Conjecture holds for G and R, i.e., the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(K); \mathbf{K}_R) \to H_n^G(pt; \mathbf{K}_R) = K_n(R \rtimes G)$$

is bijective for all $n \in \mathbb{Z}$.

Theorem (Lafforgue (2002))

Let G be a subgroup of a hyperbolic group. Let A be a C^*-algebra with structure preserving G-action.
Then the Bost Conjecture holds for G and A, i.e., the assembly map

$$H_n^G(\underline{E}G; \mathbf{K}^{\text{top}}_{A,l^1}) \to H_n^G(pt; \mathbf{K}^{\text{top}}_{A,l^1}) = K_n(A \rtimes_{l^1} G)$$

is bijective for all $n \in \mathbb{Z}$.

Theorem (The Farrell-Jones and the Bost Conjecture with coefficients for colimits of hyperbolic groups, Bartels-Echterhoff-Lück (2007))

Both the K-theoretic Farrell-Jones Conjecture and the Bost Conjecture with twisted coefficients hold for a group G if G is a subgroup of a colimit of directed system of hyperbolic groups (with not necessarily injective structure maps).

- The theorem above is not true for the Baum-Connes Conjecture because of the lack of functoriality of the reduced group C^*-algebra.
- One needs for the Baum-Connes setting that all structure maps have amenable kernels.

- The groups above are certainly wild in Bridson's universe of groups.
- Many recent constructions of groups with exotic properties are given by colimits of directed systems of hyperbolic groups. Examples are.
  - groups with expanders;
  - Lacunary hyperbolic groups in the sense of Olshanskii-Osin-Sapir;
  - Tarski monsters, i.e., groups which are not virtually cyclic and whose proper subgroups are all cyclic;
  - certain infinite torsion groups.

Miscellaneous

- The Baum-Connes Conjecture and the Farrell-Jones Conjecture do not seem to be known for $SL_n(\mathbb{Z})$ for $n \geq 3$, mapping class groups and $\mathrm{Out}(F_n)$;
- Certain groups with expanders yield counterexamples to the Baum-Connes Conjecture with coefficients by a construction due to Higson-Lafforgue-Skandalis (2002).
- The K-theoretic Farrell-Jones conjecture and the Bost Conjecture are true for these groups as shown above.
- So the counterexample of Higson-Lafforgue-Skandalis (2002) shows that the map $K_n(A \rtimes_{l^1} G) \to K_n(A \rtimes_r G)$ is not bijective in general.

- It is not known whether there are counterexamples to the Farrell-Jones Conjecture or the Baum-Connes Conjecture.
- There seems to be no promising candidate of a group G which is a potential counterexample to the K- or L-theoretic Farrell-Jones Conjecture or the Bost Conjecture.
- The Baum-Connes Conjecture is the one for which it is most likely that there may exist a counterexample.
  One reason is the existence of counterexamples to the version with coefficients and that $K_n(C_r^*(G))$ has certain failures concerning functoriality which do not exists for $K_n^G(\underline{E}G)$.
  These failures are not present for $K_n(RG)$, $L^{\langle -\infty \rangle}(RG)$ and $K_n(l^1(G))$.

- Bartels and L. have a program to prove the L-theoretic Farrell-Jones Conjecture for all coefficient rings and the same class of groups for which the K-theoretic versions have been proved.
- Bartels and L. have a program to prove the Farrell-Jones Conjecture for G and all twisted coefficients if G acts properly and cocompactly on a simply connected CAT(0)-space. This would yield the same result for all subgroups of cocompact lattices in almost connected Lie groups.

$$
\begin{array}{ccc}
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathbb{Z}}^p)[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{Z}G)[1/2] \\
\downarrow\cong & & \downarrow\cong \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{\mathbb{R}}^p)[1/2] & \xrightarrow{\cong} & L_n^p(\mathbb{R}G)[1/2] \\
\downarrow\cong & & \downarrow\cong \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_{C_r^*(?;\mathbb{R})}^p)[1/2] & \xrightarrow{\cong} & L_n^p(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow\cong & & \downarrow\cong \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathbb{R}}^{\mathrm{top}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G;\mathbb{R}))[1/2] \\
\downarrow & & \downarrow \\
H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}^{\mathrm{top}})[1/2] & \xrightarrow{\cong} & K_n(C_r^*(G))[1/2]
\end{array}
$$