Introduction to L^2-invariants

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Some appetizers

Theorem

Let G be a group with finite classifying space BG. Suppose that G contains a normal infinite solvable subgroup. Then

$$\chi(BG) = 0.$$

Theorem

Let M be a closed hyperbolic manifold of even dimension $n = 2k$. Then

$$(-1)^k \cdot \chi(M) > 0,$$

and every S^1-action on M is trivial.

Theorem

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that* G *is finitely presented and* H *is finitely generated. Then:*

1. $\operatorname{defi}(G) \leq 1$*;*
2. *Let* M *be a closed oriented* 4*-manifold with* G *as fundamental group. Then*

$$|\operatorname{sign}(M)| \leq \chi(M).$$

Conjecture (Zero-divisor Conjecture)

Let F be a field of characteristic zero and G be a torsionfree group. Then the group ring FG has no non-trivial zero-divisors.

Theorem

Let M be a closed Kähler manifold. Suppose that it admits some Riemannian metric with negative sectional curvature.
Then M is a projective algebraic variety.

- The point is that the statements of these theorems have nothing to do with L^2-invariants, but their proofs have. This list can be extended considerably.

Basic motivation

- Given an invariant for finite *CW*-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group π into account.

- Examples:

Classical notion	Generalized version
Homology with coefficients in $\mathbb{Z}$	Homology with coefficients in representations of π
Euler characteristic $\in \mathbb{Z}$	Walls finiteness obstruction in $K_0(\mathbb{Z}\pi)$
Signature $\in \mathbb{Z}$	Surgery invariants in $L_*(\mathbb{Z}\pi)$
—	torsion invariants

- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X;\mathbb{C})).$$

- Here are two naive attempts which fail:
  - $\dim_{\mathbb{C}}(H_n(\widetilde{X};\mathbb{C}))$
  - $\dim_{\mathbb{C}\pi}(H_n(\widetilde{X};\mathbb{C}))$,
    where $\dim_{\mathbb{C}\pi}(M)$ for a $\mathbb{C}\pi$-module could be chosen for instance as $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}\pi} M)$.
- The problem is that $\mathbb{C}\pi$ is in general not Noetherian and $\dim_{\mathbb{C}\pi}(M)$ is in general not additive under exact sequences.
- We will use the following successful approach which is essentially due to Atiyah and motivated by L^2-index theory.

Group von Neumann algebras

- Given a ring R and a group G, denote by RG the group ring.
- Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients r_g are non-zero.
- Addition is given by adding the coefficients.
- Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of $\cdot$).
- In general RG is a very complicated ring.

- Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

Definition

Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$. The von Neumann trace is defined by

$$\operatorname{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

Example (Finite G)

If G is finite, then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$. The trace $\operatorname{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient λ_e.

Example ($G = \mathbb{Z}^n$)

Let G be $\mathbb{Z}^n$. Let $L^2(T^n)$ be the Hilbert space of L^2-integrable functions $T^n \to \mathbb{C}$. Fourier transform yields an isometric $\mathbb{Z}^n$-equivariant isomorphism

$$L^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n).$$

Let $L^\infty(T^n)$ be the Banach space of essentially bounded measurable functions $f\colon T^n \to \mathbb{C}$. We obtain an isomorphism

$$L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n), \quad f \mapsto M_f$$

where $M_f\colon L^2(T^n) \to L^2(T^n)$ is the bounded $\mathbb{Z}^n$-operator $g \mapsto g \cdot f$.

Under this identification the trace becomes

$$\operatorname{tr}_{\mathcal{N}(\mathbb{Z}^n)}\colon L^\infty(T^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu.$$

von Neumann dimension

Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module V is a Hilbert space V together with a linear isometric G-action such that there exists an isometric linear G-embedding of V into $L^2(G)^n$ for some $n \geq 0$.
A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f\colon V \to W$ is a bounded G-equivariant operator.

Definition (von Neumann dimension)

Let V be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a G-equivariant projection $p\colon L^2(G)^n \to L^2(G)^n$ with $\operatorname{im}(p) \cong_{\mathcal{N}(G)} V$.
Define the von Neumann dimension of V by

$$\dim_{\mathcal{N}(G)}(V) := \operatorname{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \quad \in \mathbb{R}^{\geq 0}.$$

Example (Finite G)

For finite G a finitely generated Hilbert $\mathcal{N}(G)$-module V is the same as a unitary finite dimensional G-representation and

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example ($G = \mathbb{Z}^n$)

Let G be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} \colon L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with χ_X. Its image V is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \operatorname{vol}(X).$$

In particular each $r \in \mathbb{R}^{\geq 0}$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$.

Theorem (Main properties of the von Neumann dimension)

1. *Faithfulness*

   We have for a finitely generated Hilbert $\mathcal{N}(G)$-module V

   $$V = 0 \Longleftrightarrow \dim_{\mathcal{N}(G)}(V) = 0;$$

2. *Additivity*

   If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules, then

   $$\dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V);$$

3. *Cofinality*

   Let $\{V_i \mid i \in I\}$ be a directed system of Hilbert $\mathcal{N}(G)$- submodules of V, directed by inclusion. Then

   $$\dim_{\mathcal{N}(G)}\left(\overline{\bigcup_{i \in I} V_i}\right) = \sup\{\dim_{\mathcal{N}(G)}(V_i) \mid i \in I\}.$$

L^2-homology and L^2-Betti numbers

Definition (L^2-homology and L^2-Betti numbers)

Let X be a connected CW-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.

Define its cellular L^2-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

Define its n-th L^2-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Define its n-th L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)}\left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

Theorem (Main properties of L^2-Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

  $$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{Y});$$

- *Euler-Poincaré formula*

  We have

  $$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X});$$

- *Poincaré duality*

  Let M be a closed manifold of dimension d. Then

  $$b_n^{(2)}(\widetilde{M}) = b_{d-n}^{(2)}(\widetilde{M});$$

Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

- *Finite coverings*

  If $X \to Y$ is a finite covering with d sheets, then

$$b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

Example (Finite π)

If π is finite then

$$b_n^{(2)}(\widetilde{X}) = \frac{b_n(\widetilde{X})}{|\pi|}.$$

Example ($\pi = \mathbb{Z}^d$)

Let X be a connected CW-complex of finite type with fundamental group $\mathbb{Z}^d$. Let $\mathbb{C}[\mathbb{Z}^d]^{(0)}$ be the quotient field of the commutative integral domain $\mathbb{C}[\mathbb{Z}^d]$. Then

$$b_n^{(2)}(\widetilde{X}) = \dim_{\mathbb{C}[\mathbb{Z}^d]^{(0)}} \left(\mathbb{C}[\mathbb{Z}^d]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^d]} H_n(\widetilde{X})\right)$$

Obviously this implies

$$b_n^{(2)}(\widetilde{X}) \in \mathbb{Z}.$$

Some computations and results

Theorem (S^1-actions on aspherical manifolds, Lück)

Let M be an aspherical closed manifold with non-trivial S^1-action. Then we get for $n \geq 0$

$$\begin{aligned} b_n^{(2)}(\widetilde{M}) &= 0; \\ \chi(M) &= 0. \end{aligned}$$

Theorem (mapping tori, Lück)

Let $f\colon X \to X$ be a cellular selfhomotopy equivalence of a connected CW-complex X of finite type. Let T_f be the mapping torus. Then

$$b_n^{(2)}(\widetilde{T_f}) = 0 \quad \text{for } n \geq 0.$$

Theorem (L^2-Hodge - de Rham Theorem, Dodziuk)

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n_{(2)}(\widetilde{M}) = \{\widetilde{\omega} \in \Omega^n(\widetilde{M}) \mid \widetilde{\Delta}_n(\widetilde{\omega}) = 0, \; ||\widetilde{\omega}||_{L^2} < \infty\}$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules

$$\mathcal{H}^n_{(2)}(\widetilde{M}) \xrightarrow{\cong} H^n_{(2)}(\widetilde{M}).$$

Corollary (L^2-Betti numbers and heat kernels)

$$b_n^{(2)}(\widetilde{M}) = \lim_{t \to \infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x}, \tilde{x})) \, dvol \, .$$

where $e^{-t\widetilde{\Delta}_n}(\tilde{x}, \tilde{y})$ is the heat kernel on $\widetilde{M}$ and $\mathcal{F}$ is a fundamental domain for the π-action.

Theorem (hyperbolic manifolds, Dodziuk)

Let M be a hyperbolic closed Riemannian manifold of dimension d. Then:

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & \text{, if } 2n \neq d; \\ > 0 & \text{, if } 2n = d. \end{cases}$$

Corollary

Let M be a hyperbolic closed manifold of dimension d. Then

1. *If $d = 2m$ is even, then*

$$(-1)^m \cdot \chi(M) > 0;$$

2. *Every S^1-action on M is trivial S^1.*

Theorem (3-manifolds, Lott-Lück)

Let the 3-manifold M be the connected sum $M_1 \sharp \ldots \sharp M_r$ of (compact connected orientable) prime 3-manifolds M_j. Assume that $\pi_1(M)$ is infinite. Then

$$\begin{aligned}
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} - \chi(M) \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{\mid \pi_1(M_j) \mid} \\
&\quad + \left| \{ C \in \pi_0(\partial M) \mid C \cong S^2 \} \right|; \\
b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \neq 1, 2.
\end{aligned}$$

The fundamental square and the Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the Atiyah Conjecture if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

- If G is finite, its is given by

$$\begin{tikzcd} \mathbb{Z}G \arrow[r] \arrow[d] & \mathbb{C}G \arrow[d, "\mathrm{id}"] \\ \mathbb{Q}G \arrow[r] & \mathbb{C}G \end{tikzcd}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{tikzcd} \mathbb{Z}[\mathbb{Z}] \arrow[r] \arrow[d] & L^{\infty}(S^1) \arrow[d] \\ \mathbb{Q}[\mathbb{Z}]^{(0)} \arrow[r] & L(S^1) \end{tikzcd}$$

- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.
- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.
- This aspect has recently played an important role in the construction of new invariants for 3-manifolds such as the universal L^2-torsion or the L^2-polytope by Friedl-Lück.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- A torsionfree group G satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Q}G)$ the von Neumann dimension

$$\dim_{\mathcal{N}(G)}\left(\ker\left(r_A\colon L^2(G)^m \to L(G)^n\right)\right)$$

  is an integer. In this case this dimension agrees with

$$\dim_{\mathcal{D}(G)}\left(r_A\colon \mathcal{D}(G)^m \to \mathcal{D}(G)^n\right).$$

- The general version above is equivalent to the one stated before if G is finitely presented.
- An even stronger version allows $A \in M_{m,n}(\mathbb{C}G)$.

- The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero F the group ring FG has no non-trivial zero-divisors.

- There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

- However, there exist closed Riemannian manifolds whose universal coverings have an L^2-Betti number which is irrational, see Austin, Grabowski.

Theorem (Linnell)

Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions.

Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture (over $\mathbb{C}$).

- Strategy to prove the Atiyah Conjecture:
  1. Show that $K_0(\mathbb{C}) \to K_0(\mathbb{C}G)$ is surjective.
  2. Show that $K_0(\mathbb{C}G) \to K_0(\mathcal{D}(G))$ is surjective.
  3. Show that $\mathcal{D}(G)$ is semisimple.

- Notice that the Atiyah Conjecture originally was statement about an invariant extracted from the heat kernel of the universal covering, namely about the the analytic L^2-Betti number.

- However, the strategy described above is based on and requires K-theoretic and ring theoretic input.

Approximation

- In general there are no relations between the Betti numbers $b_n(X)$ and the L^2-Betti numbers $b_n^{(2)}(\widetilde{X})$ for a connected CW-complex X of finite type except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$

Theorem (Approximation Theorem, Lück)

Let X be a connected CW-complex of finite type. Suppose that π is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let X_i be the finite $[\pi : G_i]$-sheeted covering of X associated to G_i.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$

- Ordinary Betti numbers are not multiplicative under finite coverings, whereas the L^2-Betti numbers are. With the expression

$$\lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]},$$

we try to force the Betti numbers to be multiplicative by a limit process.

- The theorem above says that L^2-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.

- Let p be a prime and $\mathbb{F}_p$ be the field with p elements.

Conjecture (Approximation Conjecture in characteristic p)

Let X be a connected aspherical CW-complex of finite type. Suppose that π is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let X_i be the finite $[\pi : G_i]$-sheeted covering of X associated to G_i.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i; \mathbb{F}_p)}{[G : G_i]}.$$

- Schick used approximation techniques to prove the Atiyah Conjecture for matrices over $\mathbb{Q}G$ for a large class of groups.
- A lot of work has been done by Jaikin-Zapirain to extend this from $\mathbb{Q}G$ to $\mathbb{C}G$ using ring theoretic methods.

The Singer Conjecture

Conjecture (Singer Conjecture)

If M is an aspherical closed manifold, then

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \text{if } 2n \neq \dim(M).$$

If M is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2n \neq \dim(M); \\ > 0 & \text{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.
- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.

- Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{M})$$

the Singer Conjecture implies the following conjecture provided that M has non-positive sectional curvature.

Conjecture (Hopf Conjecture)

If M is a closed Riemannian manifold of even dimension with sectional curvature $\operatorname{sec}(M)$, then

$$\begin{array}{rclcrcl}
(-1)^{\dim(M)/2} \cdot \chi(M) & > & 0 & \textit{if} & \operatorname{sec}(M) & < & 0; \\
(-1)^{\dim(M)/2} \cdot \chi(M) & \geq & 0 & \textit{if} & \operatorname{sec}(M) & \leq & 0; \\
\chi(M) & = & 0 & \textit{if} & \operatorname{sec}(M) & = & 0; \\
\chi(M) & \geq & 0 & \textit{if} & \operatorname{sec}(M) & \geq & 0; \\
\chi(M) & > & 0 & \textit{if} & \operatorname{sec}(M) & > & 0.
\end{array}$$

Theorem (Gromov)

Let M be a closed Kähler manifold of complex dimension c. Suppose that it admits some Riemannian metric with negative sectional curvature. Then

$$\begin{array}{rcl} b_n^{(2)}(\widetilde{M}) & = & 0 \quad \text{if } n \neq c; \\ b_n^{(2)}(\widetilde{M}) & > & \text{if } n = c; \\ (-1)^m \cdot \chi(M) & > & 0; \end{array}$$

Moreover, M is a projective algebraic variety.

Further important problems or connections

- Conjecture about the equality of the first L^2-Betti number, cost, and rank gradients of groups, e.g., the Fixed Prize Conjecture.
- L^2-invariants and measured and geometric group theory
- Zero-in-the-spectum Conjecture.
- Determinant Conjecture.
- The Conjecture of Bergeron-Venkatesh for the growth of the torsion part of the homology and L^2-torsion.
- Conjecture about the vanishing of all L^2-invariants for closed aspherical manifolds with vanishing simplical volume.
- L^2-invariants and K-theory.
- L^2-invariants and entropy.
- L^2-invariants and graph theory.
- Twisted L^2-invariants and 3-manifolds.
- Applications to von Neumann algebras.
- and so on.