

SURVEY ON ASPHERICAL MANIFOLDS

WOLFGANG LÜCK

ABSTRACT. This is a survey on known results and open problems about closed aspherical manifolds, i.e., connected closed manifolds whose universal coverings are contractible. Many examples come from certain kinds of non-positive curvature conditions. The property aspherical, which is a purely homotopy theoretical condition, implies many striking results about the geometry and analysis of the manifold or its universal covering, and the ring theoretic properties and the K - and L -theory of the group ring associated to its fundamental group. The Borel Conjecture predicts that closed aspherical manifolds are topologically rigid. The article contains new results about product decompositions of closed aspherical manifolds and an announcement of a result joint with Arthur Bartels and Shmuel Weinberger about hyperbolic groups with spheres of dimension ≥ 6 as boundary. At the end we describe (winking) our universe of closed manifolds.

0. INTRODUCTION

A space X is called *aspherical* if it is path connected and all its higher homotopy groups vanish, i.e., $\pi_n(X)$ is trivial for $n \geq 2$. This survey article is devoted to aspherical closed manifolds. These are very interesting objects for many reasons. Often interesting geometric constructions or examples lead to aspherical closed manifolds. The study of the question which groups occur as fundamental groups of closed aspherical manifolds is intriguing. The condition aspherical is of purely homotopy theoretical nature. Nevertheless there are some interesting questions and conjectures about curvature properties of a closed aspherical Riemann manifold and about the spectrum of the Laplace operator on its universal covering. The Borel Conjecture predicts that aspherical closed topological manifolds are topologically rigid and that aspherical compact Poincaré complexes are homotopy equivalent to closed manifolds. We discuss the status of some of these questions and conjectures. Examples of exotic aspherical closed manifolds come from hyperbolization techniques and we list certain examples. At the end we describe (winking) our universe of closed manifolds.

The results about product decompositions of closed aspherical manifolds in Section 6 are new and Section 8 contains an announcement of a result joint with Arthur Bartels and Shmuel Weinberger about hyperbolic groups with spheres of dimension ≥ 6 as boundary.

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1. HOMOTOPY THEORY OF ASPHERICAL MANIFOLDS

From the homotopy theory point of view an aspherical CW -complex is completely determined by its fundamental group. Namely

Theorem 1.1 (Homotopy classification of aspherical spaces).

- (i) *Two aspherical CW -complexes are homotopy equivalent if and only if their fundamental groups are isomorphic;*
- (ii) *Let X and Y be connected CW -complexes. Suppose that Y is aspherical. Then we obtain a bijection*

$$[X, Y] \xrightarrow{\cong} [\Pi(X), \Pi(Y)], \quad [f] \mapsto [\Pi(f)],$$

where $[X, Y]$ is the set of homotopy classes of maps from X to Y , $\Pi(X)$, $\Pi(Y)$ are the fundamental groupoids, $[\Pi(X), \Pi(Y)]$ is the set of natural equivalence classes of functors from $\Pi(X)$ to $\Pi(Y)$ and $\Pi(f): \Pi(X) \rightarrow \Pi(Y)$ is the functor induced by $f: X \rightarrow Y$.

Proof. (ii) One easily checks that the map is well-defined. For the proof of surjectivity and injectivity one constructs the desired preimage or the desired homotopy inductively over the skeletons of the source.

(i) This follows directly from assertion (ii). □

The description using fundamental groupoids is elegant and base point free, but a reader may prefer its more concrete interpretation in terms of fundamental groups, which we will give next: Choose base points $x \in X$ and $y \in Y$. Let $\text{hom}(\pi_1(X, x), \pi_1(Y, y))$ be the set of group homomorphisms from $\pi_1(X, x)$ to $\pi_1(Y, y)$. The group $\text{Inn}(\pi_1(Y, y))$ of inner automorphisms of $\pi_1(Y, y)$ acts on

$\text{hom}(\pi_1(X, x), \pi_1(Y, y))$ from the left by composition. We leave it to the reader to check that we obtain a bijection

$$\text{Inn}(\pi_1(Y, y)) \backslash \text{hom}(\pi_1(X, x), \pi_1(Y, y)) \xrightarrow{\cong} [\Pi(X), \Pi(Y)],$$

under which the bijection appearing in Lemma 1.1 (ii) sends $[f]$ to the class of $\pi_1(f, x)$ for any choice of representative of f with $f(x) = y$. In the sequel we will often ignore base points especially when dealing with the fundamental group.

Lemma 1.2. *A CW-complex X is aspherical if and only if it is connected and its universal covering \tilde{X} is contractible.*

Proof. The projection $p: \tilde{X} \rightarrow X$ induces isomorphisms on the homotopy groups π_n for $n \geq 2$ and a connected CW-complex is contractible if and only if all its homotopy groups are trivial (see [99, Theorem IV.7.17 on page 182]). \square

An aspherical CW-complex X with fundamental group π is the same as an Eilenberg Mac-Lane space $K(\pi, 1)$ of type $(\pi, 1)$ and the same as the classifying space $B\pi$ for the group π .

2. EXAMPLES OF ASPHERICAL MANIFOLDS

In this section we give examples and constructions of aspherical closed manifolds.

2.1. Non-positive curvature. Let M be a closed smooth manifold. Suppose that it possesses a Riemannian metric whose sectional curvature is non-positive, i.e., is ≤ 0 everywhere. Then the universal covering \tilde{M} inherits a complete Riemannian metric whose sectional curvature is non-positive. Since \tilde{M} is simply-connected and has non-positive sectional curvature, the Hadamard-Cartan Theorem (see [45, 3.87 on page 134]) implies that \tilde{M} is diffeomorphic to \mathbb{R}^n and hence contractible. We conclude that \tilde{M} and hence M is aspherical.

2.2. Low-dimensions. A connected closed 1-dimensional manifold is homeomorphic to S^1 and hence aspherical.

Let M be a connected closed 2-dimensional manifold. Then M is either aspherical or homeomorphic to S^2 or $\mathbb{R}P^2$. The following statements are equivalent: i.) M is aspherical. ii.) M admits a Riemannian metric which is *flat*, i.e., with sectional curvature constant 0, or which is *hyperbolic*, i.e., with sectional curvature constant -1 . iii.) The universal covering of M is homeomorphic to \mathbb{R}^2 .

A connected closed 3-manifold M is called *prime* if for any decomposition as a connected sum $M \cong M_0 \sharp M_1$ one of the summands M_0 or M_1 is homeomorphic to S^3 . It is called *irreducible* if any embedded sphere S^2 bounds a disk D^3 . Every irreducible closed 3-manifold is prime. A prime closed 3-manifold is either irreducible or an S^2 -bundle over S^1 (see [53, Lemma 3.13 on page 28]). A closed orientable 3-manifold is aspherical if and only if it is irreducible and has infinite fundamental group. A closed 3-manifold is aspherical if and only if it is irreducible and its fundamental group is infinite and contains no element of order 2. This follows from the Sphere Theorem [53, Theorem 4.3 on page 40].

Thurston's Geometrization Conjecture implies that a closed 3-manifold is aspherical if and only if its universal covering is homeomorphic to \mathbb{R}^3 . This follows from [53, Theorem 13.4 on page 142] and the fact that the 3-dimensional geometries which have compact quotients and whose underlying topological spaces are contractible have as underlying smooth manifold \mathbb{R}^3 (see [89]).

A proof of Thurston's Geometrization Conjecture is given in [74] following ideas of Perelman.

There are examples of closed orientable 3-manifolds that are aspherical but do not support a Riemannian metric with non-positive sectional curvature (see [66]).

For more information about 3-manifolds we refer for instance to [53, 89].

2.3. Torsionfree discrete subgroups of almost connected Lie groups. Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup. Let $G \subseteq L$ be a discrete torsionfree subgroup. Then $M = G \backslash L / K$ is a closed aspherical manifold with fundamental group G since its universal covering L/K is diffeomorphic to \mathbb{R}^n for appropriate n (see [52, Theorem 1. in Chapter VI]).

2.4. Hyperbolization. A very important construction of aspherical manifolds comes from the *hyperbolization technique* due to Gromov [49]. It turns a cell complex into a non-positively curved (and hence aspherical) polyhedron. The rough idea is to define this procedure for simplices such that it is natural under inclusions of simplices and then define the hyperbolization of a simplicial complex by gluing the results for the simplices together as described by the combinatorics of the simplicial complex. The goal is to achieve that the result shares some of the properties of the simplicial complexes one has started with, but additionally to produce a non-positively curved and hence aspherical polyhedron. Since this construction preserves local structures, it turns manifolds into manifolds.

We briefly explain what the *orientable hyperbolization procedure* gives. Further expositions of this construction can be found in [19, 22, 24, 25]. We start with a finite-dimensional simplicial complex Σ and assign to it a cubical cell complex $h(\Sigma)$ and a natural map $c: h(\Sigma) \rightarrow \Sigma$ with the following properties:

- (i) $h(\Sigma)$ is non-positively curved and in particular aspherical;
- (ii) The natural map $c: h(\Sigma) \rightarrow \Sigma$ induces a surjection on the integral homology;
- (iii) $\pi_1(f): \pi_1(h(\Sigma)) \rightarrow \pi_1(\Sigma)$ is surjective;
- (iv) If Σ is an orientable manifold, then
 - (a) $h(\Sigma)$ is a manifold;
 - (b) The natural map $c: h(\Sigma) \rightarrow \Sigma$ has degree one;
 - (c) There is a stable isomorphism between the tangent bundle $Th(\Sigma)$ and the pullback $c^*T\Sigma$;

Remark 2.1 (Characteristic numbers and aspherical manifolds). Suppose that M is a closed manifold. Then the pullback of the characteristic classes of M under the natural map $c: h(M) \rightarrow M$ yield the characteristic classes of $h(M)$, and M and $h(M)$ have the same characteristic numbers. This shows that the condition aspherical does not impose any restrictions on the characteristic numbers of a manifold.

Remark 2.2 (Bordism and aspherical manifolds). The conditions above say that c is a normal map in the sense of surgery. One can show that c is normally bordant to the identity map on M . In particular M and $h(M)$ are oriented bordant.

Consider a bordism theory Ω_* for PL-manifolds or smooth manifolds which is given by imposing conditions on the stable tangent bundle. Examples are unoriented bordism, oriented bordism, framed bordism. Then any bordism class can be represented by an aspherical closed manifold. If two closed aspherical manifolds represent the same bordism class, then one can find an aspherical bordism between them. See [22, Remarks 15.1] and [25, Theorem B].

2.5. Exotic aspherical manifolds. The following result is taken from Davis-Januszkiewicz [25, Theorem 5a.1].

Theorem 2.3. *There is a closed aspherical 4-manifold N with the following properties:*

- (i) N is not homotopy equivalent to a PL-manifold;
- (ii) N is not triangulable, i.e., not homeomorphic to a simplicial complex;
- (iii) The universal covering \tilde{N} is not homeomorphic to \mathbb{R}^4 ;
- (iv) N is homotopy equivalent to a piecewise flat, non-positively curved polyhedron.

The next result is due to Davis-Januszkiewicz [25, Theorem 5a.4].

Theorem 2.4 (Non-PL-example). *For every $n \geq 4$ there exists a closed aspherical n -manifold which is not homotopy equivalent to a PL-manifold*

The proof of the following theorem can be found in [23], [25, Theorem 5b.1].

Theorem 2.5 (Exotic universal covering). *For each $n \geq 4$ there exists a closed aspherical n -dimensional manifold such that its universal covering is not homeomorphic to \mathbb{R}^n .*

By the Hadamard-Cartan Theorem (see [45, 3.87 on page 134]) the manifold appearing in Theorem 2.5 above cannot be homeomorphic to a smooth manifold with Riemannian metric with non-positive sectional curvature.

The following theorem is proved in [25, Theorem 5c.1 and Remark on page 386] by considering the ideal boundary, which is a quasiisometry invariant in the negatively curved case.

Theorem 2.6 (Exotic example with hyperbolic fundamental group). *For every $n \geq 5$ there exists an aspherical closed smooth n -dimensional manifold N which is homeomorphic to a strictly negatively curved polyhedron and has in particular a hyperbolic fundamental group such that the universal covering is homeomorphic to \mathbb{R}^n but N is not homeomorphic to a smooth manifold with Riemannian metric with negative sectional curvature.*

The next results are due to Belegarde [8, Corollary 5.1], Mess [71] and Weinberger (see [22, Section 13]).

Theorem 2.7 (Exotic fundamental groups).

- (i) *For every $n \geq 4$ there is a closed aspherical manifold of dimension n whose fundamental group contains an infinite divisible abelian group;*
- (ii) *For every $n \geq 4$ there is a closed aspherical manifold of dimension n whose fundamental group has an unsolvable word problem and whose simplicial volume is non-zero.*

Notice that a finitely presented group with unsolvable word problem is not a CAT(0)-group, not hyperbolic, not automatic, not asynchronously automatic, not residually finite and not linear over any commutative ring (see [8, Remark 5.2]).

The proof of Theorem 2.7 is based on the *reflection group trick* as it appears for instance in [22, Sections 8,10 and 13]. It can be summarized as follows.

Theorem 2.8 (Reflection group trick). *Let G be a group which possesses a finite model for BG . Then there is a closed aspherical manifold M and a map $i: BG \rightarrow M$ and $r: M \rightarrow BG$ such that $r \circ i = \text{id}_{BG}$.*

Remark 2.9 (Reflection group trick and various conjectures). Another interesting immediate consequence of the reflection group trick is (see also [22, Sections 11]) that many well-known conjectures about groups hold for every group which possesses a finite model for BG if and only if it holds for the fundamental group of every closed aspherical manifold. This applies for instance to the Kaplansky Conjecture, Unit Conjecture, Zero-divisor-conjecture, Baum-Connes Conjecture, Farrell-Jones Conjecture for algebraic K -theory for regular R , Farrell-Jones Conjecture for algebraic L -theory, the vanishing of $\tilde{K}_0(\mathbb{Z}G)$ and of $\text{Wh}(G) = 0$, For information about

these conjectures and their links we refer for instance to [6],[68] and [70]. Further similar consequences of the reflection group trick can be found in Belegradek [8].

3. NON-ASPHERICAL CLOSED MANIFOLDS

A closed manifold of dimension ≥ 1 with finite fundamental group is never aspherical. So prominent non-aspherical manifolds are spheres, lens spaces, real projective spaces and complex projective spaces.

Lemma 3.1. *The fundamental group of an aspherical finite-dimensional CW-complex X is torsionfree.*

Proof. Let $C \subseteq \pi_1(X)$ be a finite cyclic subgroup of $\pi_1(X)$. We have to show that C is trivial. Since X is aspherical, $C \backslash \tilde{X}$ is a finite-dimensional model for BC . Hence $H_k(BC) = 0$ for large k . This implies that C is trivial. \square

Lemma 3.2. *If M is a connected sum $M_1 \sharp M_2$ of two closed manifolds M_1 and M_2 of dimension $n \geq 3$ which are not homotopy equivalent to a sphere, then M is not aspherical.*

Proof. We proceed by contradiction. Suppose that M is aspherical. The obvious map $f: M_1 \sharp M_2 \rightarrow M_1 \vee M_2$ given by collapsing S^{n-1} to a point is $(n-1)$ -connected, where n is the dimension of M_1 and M_2 . Let $p: \widetilde{M_1 \vee M_2} \rightarrow M_1 \vee M_2$ be the universal covering. By the Seifert-van Kampen Theorem the fundamental group of $\pi_1(M_1 \vee M_2)$ is $\pi_1(M_1) * \pi_1(M_2)$ and the inclusion of $M_k \rightarrow M_1 \vee M_2$ induces injections on the fundamental groups for $k = 1, 2$. We conclude that $p^{-1}(M_k) = \pi_1(M_1 \vee M_2) \times_{\pi_1(M_k)} \widetilde{M}_k$ for $k = 1, 2$. Since $n \geq 3$, the map f induces an isomorphism on the fundamental groups and an $(n-1)$ -connected map $\tilde{f}: \widetilde{M_1 \sharp M_2} \rightarrow \widetilde{M_1 \vee M_2}$. Since $\widetilde{M_1 \sharp M_2}$ is contractible, $H_m(\widetilde{M_1 \vee M_2}) = 0$ for $1 \leq m \leq n-1$. Since $p^{-1}(M_1) \cup p^{-1}(M_2) = \widetilde{M_1 \vee M_2}$ and $p^{-1}(M_1) \cap p^{-1}(M_2) = p^{-1}(\{\bullet\}) = \pi_1(M_1 \vee M_2)$, we conclude $H_m(p^{-1}(M_k)) = 0$ for $1 \leq m \leq n-1$ from the Mayer-Vietoris sequence. This implies $H_m(\widetilde{M}_k) = 0$ for $1 \leq m \leq n-1$ since $p^{-1}(M_k)$ is a disjoint union of copies of \widetilde{M}_k .

Suppose that $\pi_1(M_k)$ is finite. Since $\pi_1(M_1 \sharp M_2)$ is torsionfree by Lemma 3.1, $\pi_1(M_k)$ must be trivial and $M_k = \widetilde{M}_k$. Since M_k is simply connected and $H_m(M_k) = 0$ for $1 \leq m \leq n-1$, M_k is homotopy equivalent to S^n . Since we assume that M_k is not homotopy equivalent to a sphere, $\pi_1(M_k)$ is infinite. This implies that the manifold \widetilde{M}_k is non-compact and hence $H_n(\widetilde{M}_k) = 0$. Since \widetilde{M}_k is n -dimensional, we conclude $H_m(\widetilde{M}_k) = 0$ for $m \geq 1$. Since \widetilde{M}_k is simply connected, all homotopy groups of \widetilde{M}_k vanish by the Hurewicz Theorem [99, Corollary IV.7.8 on page 180]. We conclude from Lemma 1.2 that M_1 and M_2 are aspherical. Using the Mayer-Vietoris argument above one shows analogously that $M_1 \vee M_2$ is aspherical. Since M is by assumption aspherical, $M_1 \sharp M_2$ and $M_1 \vee M_2$ are homotopy equivalent by Lemma 1.1 (i). Since they have different Euler characteristics, namely $\chi(M_1 \sharp M_2) = \chi(M_1) + \chi(M_2) - (1 + (-1)^n)$ and $\chi(M_1 \vee M_2) = \chi(M_1) + \chi(M_2) - 1$, we get a contradiction. \square

4. THE BOREL CONJECTURE

In this section we deal with

Conjecture 4.1 (Borel Conjecture for a group G). *If M and N are closed aspherical manifolds of dimensions ≥ 5 with $\pi_1(M) \cong \pi_1(N) \cong G$, then M and N are homeomorphic and any homotopy equivalence $M \rightarrow N$ is homotopic to a homeomorphism.*

Definition 4.2 (Topologically rigid). We call a closed manifold N *topologically rigid* if any homotopy equivalence $M \rightarrow N$ with a closed manifold M as source is homotopic to a homeomorphism.

If the Borel Conjecture holds for all finitely presented groups, then every closed aspherical manifold is topologically rigid.

The main tool to attack the Borel Conjecture is surgery theory and the Farrell-Jones Conjecture. We consider the following special version of the Farrell-Jones Conjecture.

Conjecture 4.3 (Farrell-Jones Conjecture for torsionfree groups and regular rings). *Let G be a torsionfree group and let R be a regular ring, e.g., a principal ideal domain, a field, or \mathbb{Z} . Then*

- (i) $K_n(RG) = 0$ for $n \leq -1$;
- (ii) The change of rings homomorphism $K_0(R) \rightarrow K_0(RG)$ is bijective. (This implies in the case $R = \mathbb{Z}$ that the reduced projective class group $\tilde{K}_0(\mathbb{Z}G)$ vanishes;
- (iii) The obvious map $K_1(R) \times G/[G, G] \rightarrow K_1(RG)$ is surjective. (This implies in the case $R = \mathbb{Z}$ that the Whitehead group $\text{Wh}(G)$ vanishes);
- (iv) For any orientation homomorphism $w: G \rightarrow \{\pm 1\}$ the w -twisted L -theoretic assembly map

$$H_n(BG; {}^w \mathbf{L}^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG, w)$$

is bijective.

Lemma 4.4. *Suppose that the torsionfree group G satisfies the version of the Farrell-Jones Conjecture stated in Conjecture 4.3 for $R = \mathbb{Z}$.*

Then the Borel Conjecture is true for closed aspherical manifolds of dimension ≥ 5 with G as fundamental group. It is true for closed aspherical manifolds of dimension 4 with G as fundamental group if G is good in the sense of Freedman (see [42], [43]).

Sketch of the proof. We treat the orientable case only. The *topological structure set* $\mathcal{S}^{\text{top}}(M)$ of a closed topological manifold M is the set of equivalence classes of homotopy equivalences $M' \rightarrow M$ with a topological closed manifold as source and M as target under the equivalence relation, for which $f_0: M_0 \rightarrow M$ and $f_1: M_1 \rightarrow M$ are equivalent if there is a homeomorphism $g: M_0 \rightarrow M_1$ such that $f_1 \circ g$ and f_0 are homotopic. The Borel Conjecture 4.1 for a group G is equivalent to the statement that for every closed aspherical manifold M with $G \cong \pi_1(M)$ its topological structure set $\mathcal{S}^{\text{top}}(M)$ consists of a single element, namely, the class of $\text{id}: M \rightarrow M$.

The *surgery sequence* of a closed orientable topological manifold M of dimension $n \geq 5$ is the exact sequence

$$\begin{aligned} \dots \rightarrow \mathcal{N}_{n+1}(M \times [0, 1], M \times \{0, 1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial} \mathcal{S}^{\text{top}}(M) \\ \xrightarrow{\eta} \mathcal{N}_n(M) \xrightarrow{\sigma} L_n^s(\mathbb{Z}\pi_1(M)), \end{aligned}$$

which extends infinitely to the left. It is the basic tool for the classification of topological manifolds. (There is also a smooth version of it.) The map σ appearing in the sequence sends a normal map of degree one to its surgery obstruction. This map can be identified with the version of the L -theory assembly map where one works with the 1-connected cover $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$ of $\mathbf{L}^s(\mathbb{Z})$. The map $H_k(M; \mathbf{L}^s(\mathbb{Z})\langle 1 \rangle) \rightarrow H_k(M; \mathbf{L}^s(\mathbb{Z}))$ is injective for $k = n$ and an isomorphism for $k > n$. Because of the K -theoretic assumptions we can replace the s -decoration with the $\langle -\infty \rangle$ -decoration. Therefore the Farrell-Jones Conjecture implies that the maps

$\sigma: \mathcal{N}_n(M) \rightarrow L_n^s(\mathbb{Z}\pi_1(M))$ and $\mathcal{N}_{n+1}(M \times [0, 1], M \times \{0, 1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M))$ are injective respectively bijective and thus by the surgery sequence that $\mathcal{S}^{\text{top}}(M)$ is a point and hence the Borel Conjecture 4.1 holds for M . More details can be found e.g., in [39, pages 17,18,28], [87, Chapter 18]. \square

Remark 4.5 (The Borel Conjecture in low dimensions). The Borel Conjecture is true in dimension ≤ 2 by the classification of closed manifolds of dimension 2. It is true in dimension 3 if Thurston's Geometrization Conjecture is true. This follows from results of Waldhausen (see Hempel [53, Lemma 10.1 and Corollary 13.7]) and Turaev (see [93]) as explained for instance in [65, Section 5]. A proof of Thurston's Geometrization Conjecture is given in [74] following ideas of Perelman.

Remark 4.6 (Topological rigidity for non-aspherical manifolds). Topological rigidity phenomena do hold also for some non-aspherical closed manifolds. For instance the sphere S^n is topologically rigid by the Poincaré Conjecture. The Poincaré Conjecture is known to be true in all dimensions. This follows in high dimensions from the h -cobordism theorem, in dimension four from the work of Freedman [42], in dimension three from the work of Perelman as explained in [62, 73] and in dimension two from the classification of surfaces.

Many more examples of classes of manifolds which are topologically rigid are given and analyzed in Kreck-Lück [65]. For instance the connected sum of closed manifolds of dimension ≥ 5 which are topologically rigid and whose fundamental groups do not contain elements of order two, is again topologically rigid and the connected sum of two manifolds is in general not aspherical (see Lemma 3.2). The product $S^k \times S^n$ is topologically rigid if and only if k and n are odd. An integral homology sphere of dimension $n \geq 5$ is topologically rigid if and only if the inclusion $\mathbb{Z} \rightarrow \mathbb{Z}[\pi_1(M)]$ induces an isomorphism of simple L -groups $L_{n+1}^s(\mathbb{Z}) \rightarrow L_{n+1}^s(\mathbb{Z}[\pi_1(M)])$.

Remark 4.7 (The Borel Conjecture does not hold in the smooth category). The Borel Conjecture 4.1 is false in the smooth category, i.e., if one replaces topological manifold by smooth manifold and homeomorphism by diffeomorphism. The torus T^n for $n \geq 5$ is an example (see [97, 15A]). Other counterexample involving negatively curved manifolds are constructed by Farrell-Jones [31, Theorem 0.1].

Remark 4.8 (The Borel Conjecture versus Mostow rigidity). The examples of Farrell-Jones [31, Theorem 0.1] give actually more. Namely, it yields for given $\epsilon > 0$ a closed Riemannian manifold M_0 whose sectional curvature lies in the interval $[1 - \epsilon, -1 + \epsilon]$ and a closed hyperbolic manifold M_1 such that M_0 and M_1 are homeomorphic but not diffeomorphic. The idea of the construction is essentially to take the connected sum of M_1 with exotic spheres. Notice that by definition M_0 were hyperbolic if we would take $\epsilon = 0$. Hence this example is remarkable in view of *Mostow rigidity*, which predicts for two closed hyperbolic manifolds N_0 and N_1 that they are isometrically diffeomorphic if and only if $\pi_1(N_0) \cong \pi_1(N_1)$ and any homotopy equivalence $N_0 \rightarrow N_1$ is homotopic to an isometric diffeomorphism.

One may view the Borel Conjecture as the topological version of Mostow rigidity. The conclusion in the Borel Conjecture is weaker, one gets only homeomorphisms and not isometric diffeomorphisms, but the assumption is also weaker, since there are many more aspherical closed topological manifolds than hyperbolic closed manifolds.

Remark 4.9 (The work of Farrell-Jones). Farrell-Jones have made deep contributions to the Borel Conjecture. They have proved it in dimension ≥ 5 for non-positively curved closed Riemannian manifolds, for compact complete affine flat

manifolds and for closed aspherical manifolds whose fundamental group is isomorphic to the fundamental group of a complete non-positively curved Riemannian manifold which is A-regular (see [32, 33, 35, 36]).

The following result is due to Bartels and Lück [4].

Theorem 4.10. *Let \mathcal{C} be the smallest class of groups satisfying:*

- *Every hyperbolic group belongs to \mathcal{C} ;*
- *Every group that acts properly, isometrically and cocompactly on a complete proper CAT(0)-space belongs to \mathcal{C} ;*
- *If G_1 and G_2 belong to \mathcal{C} , then both $G_1 * G_2$ and $G_1 \times G_2$ belong to \mathcal{C} ;*
- *If H is a subgroup of G and $G \in \mathcal{C}$, then $H \in \mathcal{C}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{C}$ for every $i \in I$. Then the directed colimit $\text{colim}_{i \in I} G_i$ belongs to \mathcal{C} .*

Then every group G in \mathcal{C} satisfies the version of the Farrell-Jones Conjecture stated in Conjecture 4.3.

Remark 4.11 (Exotic closed aspherical manifolds). Theorem 4.10 implies that the exotic aspherical manifolds mentioned in Subsection 2.5 satisfy the Borel Conjecture in dimension ≥ 5 since their universal coverings are CAT(0)-spaces.

Remark 4.12 (Directed colimits of hyperbolic groups). There are also a variety of interesting groups such as *lacunary groups* in the sense of Olshanskii-Osin-Sapir [80] or *groups with expanders* as they appear in the counterexample to the *Baum-Connes Conjecture with coefficients* due to Higson-Lafforgue-Skandalis [54] and which have been constructed by Arzhantseva-Delzant [2, Theorem 7.11 and Theorem 7.12]. Since these arise as colimits of directed systems of hyperbolic groups, they do satisfy the Farrell-Jones Conjecture and the Borel Conjecture in dimension ≥ 5 by Theorem 4.10.

The *Bost Conjecture* has also been proved for colimits of hyperbolic groups by Bartels-Echterhoff-Lück [3].

The original source for the (Fibered) Farrell-Jones Conjecture is the paper by Farrell-Jones [34, 1.6 on page 257 and 1.7 on page 262]. The C^* -analogue of the Farrell-Jones Conjecture is the Baum-Connes Conjecture whose formulation can be found in [7, Conjecture 3.15 on page 254]. For more information about the Baum-Connes Conjecture and the Farrell-Jones Conjecture and literature about them we refer for instance to the survey article [70].

5. POINCARÉ DUALITY GROUPS

The following definition is due to Johnson-Wall [59].

Definition 5.1 (Poincaré duality group). A group G is called a *Poincaré duality group of dimension n* if the following conditions holds:

- (i) The group G is of type FP, i.e., the trivial $\mathbb{Z}G$ -module \mathbb{Z} possesses a finite-dimensional projective $\mathbb{Z}G$ -resolution by finitely generated projective $\mathbb{Z}G$ -modules;
- (ii) We get an isomorphism of abelian groups

$$H^i(G; \mathbb{Z}G) \cong \begin{cases} \{0\} & \text{for } i \neq n; \\ \mathbb{Z} & \text{for } i = n. \end{cases}$$

The next definition is due to Wall [96]. Recall that a CW -complex X is called *finitely dominated* if there exists a finite CW -complex Y and maps $i: X \rightarrow Y$ and $r: Y \rightarrow X$ with $r \circ i \simeq \text{id}_X$.

Definition 5.2 (Poincaré complex). Let X be a finitely dominated connected CW -complex with fundamental group π .

It is called a *Poincaré complex of dimension n* if there exists an orientation homomorphism $w: \pi \rightarrow \{\pm 1\}$ and an element

$$[X] \in H_n^\pi(\tilde{X}; {}^w\mathbb{Z}) = H_n(C_*(\tilde{X}) \otimes_{\mathbb{Z}\pi} {}^w\mathbb{Z})$$

in the n -th π -equivariant homology of its universal covering \tilde{X} with coefficients in the $\mathbb{Z}G$ -module ${}^w\mathbb{Z}$, such that the up to $\mathbb{Z}\pi$ -chain homotopy equivalence unique $\mathbb{Z}\pi$ -chain map

$$-\cap [X]: C^{n-*}(\tilde{X}) = \text{hom}_{\mathbb{Z}\pi}(C_{n-*}(\tilde{X}), \mathbb{Z}\pi) \rightarrow C_*(\tilde{X})$$

is a $\mathbb{Z}\pi$ -chain homotopy equivalence. Here ${}^w\mathbb{Z}$ is the $\mathbb{Z}G$ -module, whose underlying abelian group is \mathbb{Z} and on which $g \in \pi$ acts by multiplication with $w(g)$.

If in addition X is a finite CW -complex, we call X a *finite Poincaré duality complex of dimension n* .

A topological space X is called an *absolute neighborhood retract* or briefly ANR if for every normal space Z , every closed subset $Y \subseteq Z$ and every (continuous) map $f: Y \rightarrow X$ there exists an open neighborhood U of Y in Z together with an extension $F: U \rightarrow X$ of f to U . A *compact n -dimensional homology ANR-manifold* X is a compact absolute neighborhood retract such that it has a countable basis for its topology, has finite topological dimension and for every $x \in X$ the abelian group $H_i(X, X - \{x\})$ is trivial for $i \neq n$ and infinite cyclic for $i = n$. A closed n -dimensional topological manifold is an example of a compact n -dimensional homology ANR-manifold (see [21, Corollary 1A in V.26 page 191]).

Theorem 5.3 (Homology ANR-manifolds and finite Poincaré complexes). *Let M be a closed topological manifold, or more generally, a compact homology ANR-manifold of dimension n . Then M is homotopy equivalent to a finite n -dimensional Poincaré complex.*

Proof. A closed topological manifold, and more generally a compact ANR, has the homotopy type of a finite CW -complex (see [61, Theorem 2.2]. [98]). The usual proof of Poincaré duality for closed manifolds carries over to homology manifolds. \square

Theorem 5.4 (Poincaré duality groups). *Let G be a group and $n \geq 1$ be an integer. Then:*

- (i) *The following assertions are equivalent:*
 - (a) *G is finitely presented and a Poincaré duality group of dimension n ;*
 - (b) *There exists an n -dimensional aspherical Poincaré complex with G as fundamental group;*
- (ii) *Suppose that $\tilde{K}_0(\mathbb{Z}G) = 0$. Then the following assertions are equivalent:*
 - (a) *G is finitely presented and a Poincaré duality group of dimension n ;*
 - (b) *There exists a finite n -dimensional aspherical Poincaré complex with G as fundamental group;*
- (iii) *A group G is a Poincaré duality group of dimension 1 if and only if $G \cong \mathbb{Z}$;*
- (iv) *A group G is a Poincaré duality group of dimension 2 if and only if G is isomorphic to the fundamental group of a closed aspherical surface;*

Proof. (i) Every finitely dominated CW -complex has a finitely presented fundamental group since every finite CW -complex has a finitely presented group and a group which is a retract of a finitely presented group is again finitely presented [94, Lemma 1.3]. If there exists a CW -model for BG of dimension n , then the cohomological dimension of G satisfies $\text{cd}(G) \leq n$ and the converse is true provided that

$n \geq 3$ (see [14, Theorem 7.1 in Chapter VIII.7 on page 205], [29], [94], [95]). This implies that the implication (i)b \implies (i)a holds for all $n \geq 1$ and that the implication (i)a \implies (i)b holds for $n \geq 3$. For more details we refer to [59, Theorem 1]. The remaining part to show the implication (i)a \implies (i)b for $n = 1, 2$ follows from assertions (iii) and (iv).

(ii) This follows in dimension $n \geq 3$ from assertion (i) and Wall's results about the finiteness obstruction which decides whether a finitely dominated CW -complex is homotopy equivalent to a finite CW -complex and takes values in $\tilde{K}_0(\mathbb{Z}\pi)$ (see [37, 72, 94, 95]). The implication (ii)b \implies (ii)a holds for all $n \geq 1$. The remaining part to show the implication (ii)a \implies (ii)b holds follows from assertions (iii) and (iv).

(iii) Since $S^1 = B\mathbb{Z}$ is a 1-dimensional closed manifold, \mathbb{Z} is a finite Poincaré duality group of dimension 1 by Theorem 5.3. We conclude from the (easy) implication (i)b \implies (i)a appearing in assertion (i) that \mathbb{Z} is a Poincaré duality group of dimension 1. Suppose that G is a Poincaré duality group of dimension 1. Since the cohomological dimension of G is 1, it has to be a free group (see [91, 92]). Since the homology group of a group of type FP is finitely generated, G is isomorphic to a finitely generated free group F_r of rank r . Since $H^1(BF_r) \cong \mathbb{Z}^r$ and $H_0(BF_r) \cong \mathbb{Z}$, Poincaré duality can only hold for $r = 1$, i.e., G is \mathbb{Z} .

(iv) This is proved in [27, Theorem 2]. See also [10, 11, 26, 28]. \square

Conjecture 5.5 (Aspherical Poincaré complexes). *Every finite Poincaré complex is homotopy equivalent to a closed manifold.*

Conjecture 5.6 (Poincaré duality groups). *A finitely presented group is a n -dimensional Poincaré duality group if and only if it is the fundamental group of a closed n -dimensional topological manifold.*

Because of Theorem 5.3 and Theorem 5.4 (i), Conjecture 5.5 and Conjecture 5.6 are equivalent.

The *disjoint disk property* says that for any $\epsilon > 0$ and maps $f, g: D^2 \rightarrow M$ there are maps $f', g': D^2 \rightarrow M$ so that the distance between f and f' and the distance between g and g' are bounded by ϵ and $f'(D^2) \cap g'(D^2) = \emptyset$.

Lemma 5.7. *Suppose that the torsionfree group G and the ring $R = \mathbb{Z}$ satisfy the version of the Farrell-Jones Conjecture stated in Theorem 4.3. Let X be a Poincaré complex of dimension ≥ 6 with $\pi_1(X) \cong G$. Then X is homotopy equivalent to a compact homology ANR-manifold satisfying the disjoint disk property.*

Proof. See [87, Remark 25.13 on page 297], [15, Main Theorem on page 439 and Section 8] and [16, Theorem A and Theorem B]. \square

Remark 5.8 (Compact homology ANR-manifolds versus closed topological manifolds). In the following all manifolds have dimension ≥ 6 . One would prefer if in the conclusion of Lemma 5.7 one could replace “compact homology ANR-manifold” by “closed topological manifold”. The problem is that in the geometric exact surgery sequence one has to work with the 1-connective cover $\mathbf{L}\langle 1 \rangle$ of the L -theory spectrum \mathbf{L} , whereas in the assembly map appearing in the Farrell-Jones setting one uses the L -theory spectrum \mathbf{L} . The L -theory spectrum \mathbf{L} is 4-periodic, i.e., $\pi_n(\mathbf{L}) \cong \pi_{n+4}(\mathbf{L})$ for $n \in \mathbb{Z}$. The 1-connective cover $\mathbf{L}\langle 1 \rangle$ comes with a map of spectra $\mathbf{f}: \mathbf{L}\langle 1 \rangle \rightarrow \mathbf{L}$ such that $\pi_n(\mathbf{f})$ is an isomorphism for $n \geq 1$ and $\pi_n(\mathbf{L}\langle 1 \rangle) = 0$ for $n \leq 0$. Since $\pi_0(\mathbf{L}) \cong \mathbb{Z}$, one misses a part involving $L_0(\mathbb{Z})$ of the so called *total surgery obstruction* due to Ranicki, i.e., the obstruction for a finite Poincaré complex to be homotopy equivalent to a closed topological manifold, if one deals with the periodic L -theory spectrum \mathbf{L} and picks up only the obstruction for a finite Poincaré

complex to be homotopy equivalent to a compact homology ANR-manifold, the so called *four-periodic total surgery obstruction*. The difference of these two obstructions is related to the *resolution obstruction* of Quinn which takes values in $L_0(\mathbb{Z})$. Any element of $L_0(\mathbb{Z})$ can be realized by an appropriate compact homology ANR-manifold as its *resolution obstruction*. There are compact homology ANR-manifolds that are not homotopy equivalent to closed manifolds. But no example of an aspherical compact homology ANR-manifold that is not homotopy equivalent to a closed topological manifold is known. For an aspherical compact homology ANR-manifold M , the total surgery obstruction and the resolution obstruction carry the same information. So we could replace in the conclusion of Lemma 5.7 “compact homology ANR-manifold” by “closed topological manifold” if and only if every aspherical compact homology ANR-manifold with the disjoint disk property admits a resolution.

We refer for instance to [15, 38, 85, 86, 87] for more information about this topic.

Question 5.9 (Vanishing of the resolution obstruction in the aspherical case). Is every aspherical compact homology ANR-manifold homotopy equivalent to a closed manifold?

6. PRODUCT DECOMPOSITIONS

In this section we show that, roughly speaking, a closed aspherical manifold M is a product $M_1 \times M_2$ if and only if its fundamental group is a product $\pi_1(M) = G_1 \times G_2$ and that such a decomposition is unique up to homeomorphism.

Theorem 6.1 (Product decomposition). *Let M be a closed aspherical manifold of dimension n with fundamental group $G = \pi_1(M)$. Suppose we have a product decomposition*

$$p_1 \times p_2: G \xrightarrow{\cong} G_1 \times G_2.$$

Suppose that G , G_1 and G_2 satisfy the version of the Farrell-Jones Conjecture stated in Theorem 4.3 in the case $R = \mathbb{Z}$.

Then G , G_1 and G_2 are Poincaré duality groups whose cohomological dimensions satisfy

$$n = \text{cd}(G) = \text{cd}(G_1) + \text{cd}(G_2).$$

Suppose in the sequel:

- *the cohomological dimension $\text{cd}(G_i)$ is different from 3, 4 and 5 for $i = 1, 2$.*
- *$n \geq 5$ or $n \leq 2$ or ($n = 4$ and G is good in the sense of Freedmann);*

Then:

- (i) *There are topological closed aspherical manifolds M_1 and M_2 together with isomorphisms*

$$v_i: \pi_1(M_i) \xrightarrow{\cong} G_i$$

and maps

$$f_i: M \rightarrow M_i$$

for $i = 1, 2$ such that

$$f = f_1 \times f_2: M \rightarrow M_1 \times M_2$$

is a homeomorphism and $v_i \circ \pi_1(f_i) = p_i$ (up to inner automorphisms) for $i = 1, 2$;

- (ii) *Suppose we have another such choice of topological closed aspherical manifolds M'_1 and M'_2 together with isomorphisms*

$$v'_i: \pi_1(M'_i) \xrightarrow{\cong} G_i$$

and maps

$$f'_i: M \rightarrow M'_i$$

for $i = 1, 2$ such that the map $f' = f'_1 \times f'_2$ is a homotopy equivalence and $v'_i \circ \pi_1(f'_i) = p_i$ (up to inner automorphisms) for $i = 1, 2$. Then there are for $i = 1, 2$ homeomorphisms $h_i: M_i \rightarrow M'_i$ such that $h_i \circ f_i \simeq f'_i$ and $v_i \circ \pi_1(h_i) = v'_i$ holds for $i = 1, 2$.

Proof. In the sequel we identify $G = G_1 \times G_2$ by $p_1 \times p_2$. Since the closed manifold M is a model for BG and $\text{cd}(G) = n$, we can choose BG to be an n -dimensional finite Poincaré complex in the sense of Definition 5.2 by Theorem 5.3.

From $BG = B(G_1 \times G_2) \simeq BG_1 \times BG_2$ we conclude that there are finitely dominated CW -models for BG_i for $i = 1, 2$. Since $\tilde{K}_0(\mathbb{Z}G_i)$ vanishes for $i = 0, 1$ by assumption, we conclude from the theory of the finiteness obstruction due to Wall [94, 95] that there are finite models for BG_i of dimension $\max\{\text{cd}(G_i), 3\}$. We conclude from [47], [84] that BG_1 and BG_2 are Poincaré complexes. One easily checks using the Künneth formula that

$$n = \text{cd}(G) = \text{cd}(G_1) + \text{cd}(G_2).$$

If $\text{cd}(G_i) = 1$, then BG_i is homotopy equivalent to a manifold, namely S^1 , by Theorem 5.4 (iii). If $\text{cd}(G_i) = 2$, then BG_i is homotopy equivalent to a manifold by Theorem 5.4 (iv). Hence it suffices to show for $i = 1, 2$ that BG_i is homotopy equivalent to a closed aspherical manifold, provided that $\text{cd}(G_i) \geq 6$.

Since by assumption G_i satisfies the version of the Farrell-Jones Conjecture stated in Theorem 4.3 in the case $R = \mathbb{Z}$, there exists a compact homology ANR-manifold M_i that satisfies the disjoint disk property and is homotopy equivalent to BG_i (see Lemma 5.7). Hence it remains to show that Quinn's resolution obstruction $I(M_i) \in (1 + 8 \cdot \mathbb{Z})$ is 1 (see [86, Theorem 1.1]). Since this obstruction is multiplicative (see [86, Theorem 1.1]), we get $I(M_1 \times M_2) = I(M_1) \cdot I(M_2)$. In general the resolution obstruction is not a homotopy invariant, but it is known to be a homotopy invariant for aspherical compact ANR-manifolds if the fundamental group satisfies the Novikov Conjecture 7.2 (see [15, Proposition on page 437]). Since G_i satisfies the version of the Farrell-Jones Conjecture stated in Theorem 4.3 in the case $R = \mathbb{Z}$, it satisfies the Novikov Conjecture by Lemma 4.4 and Remark 7.4. Hence $I(M_1 \times M_2) = I(M)$. Since $I(M)$ is a closed manifold, we have $I(M) = 1$. Hence $I(M_i) = 1$ and M_i is homotopy equivalent to a closed manifold. This finishes the proof of assertion (i).

Assertion (ii) follows from Lemma 4.4. \square

Remark 6.2 (Product decompositions and non-positive sectional curvature). The following result has been proved by Gromoll-Wolf [48, Theorem 2]. Let M be a closed Riemannian manifold with non-positive sectional curvature. Suppose that we are given a splitting of its fundamental group $\pi_1(M) = G_1 \times G_2$ and that the center of $\pi_1(M)$ is trivial. Then this splitting comes from an isometric product decomposition of closed Riemannian manifolds of non-positive sectional curvature $M = M_1 \times M_2$.

7. NOVIKOV CONJECTURE

Let G be a group and let $u: M \rightarrow BG$ be a map from a closed oriented smooth manifold M to BG . Let

$$\mathcal{L}(M) \in \bigoplus_{k \in \mathbb{Z}, k \geq 0} H^{4k}(M; \mathbb{Q})$$

be the L -class of M . Its k -th entry $\mathcal{L}(M)_k \in H^{4k}(M; \mathbb{Q})$ is a certain homogeneous polynomial of degree k in the rational Pontrjagin classes $p_i(M; \mathbb{Q}) \in H^{4i}(M; \mathbb{Q})$ for

$i = 1, 2, \dots, k$ such that the coefficient s_k of the monomial $p_k(M; \mathbb{Q})$ is different from zero. The L -class $\mathcal{L}(M)$ is determined by all the rational Pontrjagin classes and vice versa. The L -class depends on the tangent bundle and thus on the differentiable structure of M . For $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ define the *higher signature of M associated to x and u* to be the integer

$$(7.1) \quad \text{sign}_x(M, u) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle.$$

We say that sign_x for $x \in H^*(BG; \mathbb{Q})$ is *homotopy invariant* if for two closed oriented smooth manifolds M and N with reference maps $u: M \rightarrow BG$ and $v: N \rightarrow BG$ we have

$$\text{sign}_x(M, u) = \text{sign}_x(N, v),$$

whenever there is an orientation preserving homotopy equivalence $f: M \rightarrow N$ such that $v \circ f$ and u are homotopic. If $x = 1 \in H^0(BG)$, then the higher signature $\text{sign}_x(M, u)$ is by the Hirzebruch signature formula (see [56, 57]) the signature of M itself and hence an invariant of the oriented homotopy type. This is one motivation for the following conjecture.

Conjecture 7.2 (Novikov Conjecture). *Let G be a group. Then sign_x is homotopy invariant for all $x \in \prod_{k \in \mathbb{Z}, k \geq 0} H^k(BG; \mathbb{Q})$.*

This conjecture appears for the first time in the paper by Novikov [78, §11]. A survey about its history can be found in [39]. More information can be found for instance in [39, 40, 64].

We mention the following deep result due to Novikov [75, 76, 77].

Theorem 7.3 (Topological invariance of rational Pontrjagin classes). *The rational Pontrjagin classes $p_k(M, \mathbb{Q}) \in H^{4k}(M; \mathbb{Q})$ are topological invariants, i.e. for a homeomorphism $f: M \rightarrow N$ of closed smooth manifolds we have*

$$H_{4k}(f; \mathbb{Q})(p_k(M; \mathbb{Q})) = p_k(N; \mathbb{Q})$$

for all $k \geq 0$ and in particular $H_*(f; \mathbb{Q})(\mathcal{L}(M)) = \mathcal{L}(N)$.

The rational Pontrjagin classes are not homotopy invariants and the integral Pontrjagin classes $p_k(M)$ are not homeomorphism invariants (see for instance [64, Example 1.6 and Theorem 4.8]).

Remark 7.4 (The Novikov Conjecture and aspherical manifolds). Let $f: M \rightarrow N$ be a homotopy equivalence of closed aspherical manifolds. Suppose that the Borel Conjecture 4.1 is true for $G = \pi_1(N)$. This implies that f is homotopic to a homeomorphism and hence by Theorem 7.3

$$f_*(\mathcal{L}(M)) = \mathcal{L}(N).$$

But this is equivalent to the conclusion of the Novikov Conjecture in the case $N = BG$.

Conjecture 7.5. *A closed aspherical smooth manifold does not admit a Riemannian metric of positive scalar curvature.*

Proposition 7.6. *Suppose that the strong Novikov Conjecture is true for the group G , i.e., the assembly map*

$$K_n(BG) \rightarrow K_n(C_r^*(G))$$

is rationally injective for all $n \in \mathbb{Z}$. Let M be a closed aspherical smooth manifold whose fundamental group is isomorphic to G .

Then M carries no Riemannian metric of positive scalar curvature.

Proof. See [88, Theorem 3.5]. □

Proposition 7.7. *Let G be a group. Suppose that the assembly map*

$$K_n(BG) \rightarrow K_n(C_r^*(G))$$

is rationally injective for all $n \in \mathbb{Z}$. Let M be a closed aspherical smooth manifold whose fundamental group is isomorphic to G .

Then M satisfies the Zero-in-the-Spectrum Conjecture 9.5

Proof. See [67, Corollary 4]. □

We refer to [70, Section 5.1.3] for a discussion about the large class of groups for which the assembly map $K_n(BG) \rightarrow K_n(C_r^*(G))$ is known to be injective or rationally injective.

8. BOUNDARIES OF HYPERBOLIC GROUPS

We announce the following two theorems joint with Arthur Bartels and Shmuel Weinberger. For the notion of the boundary of a hyperbolic group and its main properties we refer for instance to [60].

Theorem 8.1. *Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6 . Then:*

- (i) *The following statements are equivalent:*
 - (a) *The boundary ∂G is homeomorphic to S^{n-1} ;*
 - (b) *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering \widetilde{M} is homeomorphic to \mathbb{R}^n and the compactification of \widetilde{M} by ∂G is homeomorphic to D^n ;*
- (ii) *The aspherical manifold M appearing in the assertion above is unique up to homeomorphism.*

The proof depends strongly on the surgery theory for compact homology ANR-manifolds due to Bryant-Ferry-Mio-Weinberger [15] and the validity of the K - and L -theoretic Farrell-Jones Conjecture for hyperbolic groups due to Bartels-Reich-Lück [5] and Bartels-Lück [4]. It seems likely that this result holds also if $n = 5$. Our methods can be extended to this case if the surgery theory from [15] can be extended to the case of 5-dimensional compact homology ANR-manifolds.

We do not get information in dimensions $n \leq 4$ for the usual problems about surgery. For instance, our methods give no information in the case, where the boundary is homeomorphic to S^3 , since virtually cyclic groups are the only hyperbolic groups which are known to be good in the sense of Friedman [43]. In the case $n = 3$ there is the conjecture of Cannon [17] that a group G acts properly, isometrically and cocompactly on the 3-dimensional hyperbolic plane \mathbb{H}^3 if and only if it is a hyperbolic group whose boundary is homeomorphic to S^2 . Provided that the infinite hyperbolic group G occurs as the fundamental group of a closed irreducible 3-manifold, Bestvina-Mess [9, Theorem 4.1] have shown that its universal covering is homeomorphic to \mathbb{R}^3 and its compactification by ∂G is homeomorphic to D^3 , and the Geometrization Conjecture of Thurston implies that M is hyperbolic and G satisfies Cannon's conjecture. The problem is solved in the case $n = 2$, namely, for a hyperbolic group G its boundary ∂G is homeomorphic to S^1 if and only if G is a Fuchsian group (see [18, 41, 44]).

For every $n \geq 5$ there exists a strictly negatively curved polyhedron of dimension n whose fundamental group G is hyperbolic, which is homeomorphic to a closed aspherical smooth manifold and whose universal covering is homeomorphic to \mathbb{R}^n , but the boundary ∂G is not homeomorphic to S^{n-1} , see [25, Theorem 5c.1 on page 384 and Remark on page 386]. Thus the condition that ∂G is a sphere for a torsion-free hyperbolic group is (in high dimensions) not equivalent to the existence of an aspherical manifold whose fundamental group is G .

Theorem 8.2. *Let G be a torsion-free hyperbolic group and let n be an integer ≥ 6 . Then*

- (i) *The following statements are equivalent:*
 - (a) *The boundary ∂G has the integral Čech cohomology of S^{n-1} ;*
 - (b) *G is a Poincaré duality group of dimension n ;*
 - (c) *There exists a compact homology ANR-manifold M homotopy equivalent to BG . In particular, M is aspherical and $\pi_1(M) \cong G$;*
- (ii) *If the statements in assertion (i) hold, then the compact homology ANR-manifold M appearing there is unique up to s -cobordism of compact ANR-homology manifolds.*

The discussion of compact homology ANR-manifolds versus closed topological manifolds of Remark 5.8 and Question 5.9 are relevant for Theorem 8.2 as well.

In general the boundary of a hyperbolic group is not locally a Euclidean space but has a fractal behavior. If the boundary ∂G of an infinite hyperbolic group G contains an open subset homeomorphic to Euclidean n -space, then it is homeomorphic to S^n . This is proved in [60, Theorem 4.4], where more information about the boundaries of hyperbolic groups can be found.

9. L^2 -INVARIANTS

Next we mention some prominent conjectures about aspherical manifolds and L^2 -invariants. For more information about these conjectures and their status we refer to [68] and [69].

9.1. The Hopf and the Singer Conjecture.

Conjecture 9.1 (Hopf Conjecture). *If M is an aspherical closed manifold of even dimension, then*

$$(-1)^{\dim(M)/2} \cdot \chi(M) \geq 0.$$

If M is a closed Riemannian manifold of even dimension with sectional curvature $\sec(M)$, then

$$\begin{array}{lll} (-1)^{\dim(M)/2} \cdot \chi(M) > 0 & \text{if} & \sec(M) < 0; \\ (-1)^{\dim(M)/2} \cdot \chi(M) \geq 0 & \text{if} & \sec(M) \leq 0; \\ \chi(M) = 0 & \text{if} & \sec(M) = 0; \\ \chi(M) \geq 0 & \text{if} & \sec(M) \geq 0; \\ \chi(M) > 0 & \text{if} & \sec(M) > 0. \end{array}$$

Conjecture 9.2 (Singer Conjecture). *If M is an aspherical closed manifold, then*

$$b_p^{(2)}(\widetilde{M}) = 0 \quad \text{if } 2p \neq \dim(M).$$

If M is a closed connected Riemannian manifold with negative sectional curvature, then

$$b_p^{(2)}(\widetilde{M}) \begin{cases} = 0 & \text{if } 2p \neq \dim(M); \\ > 0 & \text{if } 2p = \dim(M). \end{cases}$$

9.2. L^2 -torsion and aspherical manifolds.

Conjecture 9.3 (L^2 -torsion for aspherical manifolds). *If M is an aspherical closed manifold of odd dimension, then \widetilde{M} is \det - L^2 -acyclic and*

$$(-1)^{\frac{\dim(M)-1}{2}} \cdot \rho^{(2)}(\widetilde{M}) \geq 0.$$

If M is a closed connected Riemannian manifold of odd dimension with negative sectional curvature, then \widetilde{M} is \det - L^2 -acyclic and

$$(-1)^{\frac{\dim(M)-1}{2}} \cdot \rho^{(2)}(\widetilde{M}) > 0.$$

If M is an aspherical closed manifold whose fundamental group contains an amenable infinite normal subgroup, then \widetilde{M} is \det - L^2 -acyclic and

$$\rho^{(2)}(\widetilde{M}) = 0.$$

9.3. Simplicial volume and L^2 -invariants.

Conjecture 9.4 (Simplicial volume and L^2 -invariants). *Let M be an aspherical closed orientable manifold. Suppose that its simplicial volume $\|M\|$ vanishes. Then \widetilde{M} is of determinant class and*

$$\begin{aligned} b_p^{(2)}(\widetilde{M}) &= 0 & \text{for } p \geq 0; \\ \rho^{(2)}(\widetilde{M}) &= 0. \end{aligned}$$

9.4. Zero-in-the-Spectrum Conjecture.

Conjecture 9.5 (Zero-in-the-spectrum Conjecture). *Let \widetilde{M} be a complete Riemannian manifold. Suppose that \widetilde{M} is the universal covering of an aspherical closed Riemannian manifold M (with the Riemannian metric coming from M). Then for some $p \geq 0$ zero is in the Spectrum of the minimal closure*

$$(\Delta_p)_{\min}: \text{dom}((\Delta_p)_{\min}) \subset L^2\Omega^p(\widetilde{M}) \rightarrow L^2\Omega^p(\widetilde{M})$$

of the Laplacian acting on smooth p -forms on \widetilde{M} .

Remark 9.6 (Non-aspherical counterexamples to the Zero-in-the-Spectrum Conjecture). For all of the conjectures about aspherical spaces stated in this article it is obvious that they cannot be true if one drops the condition aspherical except for the zero-in-the-Spectrum Conjecture 9.5. Farber and Weinberger [30] gave the first example of a closed Riemannian manifold for which zero is not in the spectrum of the minimal closure $(\Delta_p)_{\min}: \text{dom}((\Delta_p)_{\min}) \subset L^2\Omega^p(\widetilde{M}) \rightarrow L^2\Omega^p(\widetilde{M})$ of the Laplacian acting on smooth p -forms on \widetilde{M} for each $p \geq 0$. The construction by Higson, Roe and Schick [55] yields a plenty of such counterexamples. But there are no aspherical counterexamples known.

10. THE UNIVERSE OF CLOSED MANIFOLDS

At the end we describe (winking) our universe of closed manifolds.

The idea of a random group has successfully been used to construct groups with certain properties, see for instance [2], [46], [50, 9.B on pages273ff], [51], [79],[82], [90] and [100]. In a precise statistical sense almost all finitely presented groups are hyperbolic see [81]. One can actually show that in a precise statistical sense almost all finitely presented groups are torsionfree hyperbolic and in particular have a finite model for their classifying space. In most cases it is given by the limit for $n \rightarrow \infty$ of the quotient of the number of finitely presented groups with a certain property (P) which are given by a presentation satisfying a certain condition C_n by the number of all finitely presented groups which are given by a presentation satisfying condition C_n .

It is not clear what it means in a precise sense to talk about a random closed manifold. Nevertheless, the author's intuition is that almost all closed manifolds are aspherical. (A related question would be whether a random closed smooth manifold admits a Riemannian metric with non-positive sectional curvature.) This intuition is supported by Remark 2.1. It is certainly true in dimension 2 since only finitely many closed surfaces are not aspherical. The characterization of closed 3-dimensional manifolds in Subsection 2.2 seems to fit as well. In the sequel we assume that this (vague) intuition is correct.

If we combine these considerations, we get that almost all closed manifolds are aspherical and have a hyperbolic fundamental group. Since except in dimension 4 the Borel Conjecture is known in this case by Lemma 4.4, Remark 4.5 and Theorem 4.10, we get as a consequence that almost almost all closed manifolds are aspherically and topologically rigid.

A closed manifold M is called *asymmetric* if every finite group which acts effectively on M is trivial. This is equivalent to the statement that for any choice of Riemannian metric on M the group of isometries is trivial (see [63, Introduction]). A survey on asymmetric closed manifolds can be found in [83]. The first constructions of asymmetric closed aspherical manifolds are due to Connor-Raymond-Weinberger [20]. The first simply-connected asymmetric manifold has been constructed by Kreck [63] answering a question of Raymond and Schultz [13, page 260] which was repeated by Adem and Davis [1] in their problem list. Raymond and Schultz expressed also their feeling that a random manifold should be asymmetric. Borel has shown that an aspherical closed manifold is asymmetric if its fundamental group is centerless and its outer automorphism group is torsionfree (see the manuscript “On periodic maps of certain $K(\pi, 1)$ ” in [12, pages 57–60]).

This leads to the intuitive statement:

Almost all closed manifolds are aspherical, topologically rigid and asymmetric.

In particular almost every closed manifold is determined up to homeomorphism by its fundamental group.

This is — at least on the first glance — surprising since often our favorite manifolds are not asymmetric and not determined by their fundamental group. There are prominent manifolds such as lens spaces which are homotopy equivalent but not homeomorphic. There seem to be plenty of simply connected manifolds. So why do human beings may have the feeling that the universe of closed manifolds described above is different from their expectation?

If one asks people for the most prominent closed manifold, most people name the standard sphere. It is interesting that the n -dimensional standard sphere S^n can be characterized among (simply connected) closed Riemannian manifolds of dimension n by the property that its isometry group has maximal dimension. More precisely, if M is a closed n -dimensional smooth manifold, then the dimension of its isometry group for any Riemannian metric is bounded by $n(n+1)/2$ and the maximum $n(n+1)/2$ is attained if and only if M is diffeomorphic to S^n or $\mathbb{R}P^n$; see Hsiang [58], where the Ph.D-thesis of Eisenhart is cited and the dimension of the isometry group of exotic spheres is investigated. It is likely that the human taste whether a geometric object is beautiful is closely related to the question how many symmetries it admits. In general it seems to be the case that a human being is attracted by unusual representatives among mathematical objects such as groups or closed manifolds and not by the generic ones. In group theory it is clear that random groups can have very strange properties and that these groups are to some extent scary. The analogous statement seems to hold for closed topological manifolds.

At the time of writing the author cannot really name a group which could be a potential counterexample to the Farrell-Jones Conjecture or other conjectures discussed in this article. But the author has the feeling that nevertheless the class of groups, for which we can prove the conjecture and which is for “human standards” quite large, is only a very tiny portion of the whole universe of groups and the question whether these conjectures are true for all groups is completely open.

Here is an interesting parallel to our actual universe. If you materialize at a random point in the universe it will be very cold and nothing will be there. There

is no interaction between different random points, i.e., it is rigid. A human being will not like this place, actually even worse, it cannot exist at such a random place. But there are unusual rare non-generic points in the universe, where human beings can exist such as the surface of our planet and there a lot of things and interactions are happening. And human beings tend to think that the rest of the universe looks like the place they are living in and cannot really comprehend the rest of the universe.

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WESTFÄLISCHE WILHELMS-UNIVERSITÄT MÜNSTER, MATHEMATISCHES INSTITUT, EINSTEINSTR. 62,
D-48149 MÜNSTER, GERMANY

E-mail address: lueck@math.uni-muenster.de

URL: <http://www.math.uni-muenster.de/u/lueck>