manuscripta math. 58, 67 - 75 (1987)

manuscripta
mathematica


# EQUIVARIANT EILENBERG MACLANE SPACES $K(\mathcal{G},\mu,1)$ FOR POSSIBLY NON-CONNECTED OR EMPTY FIXED POINT SETS

Wolfgang Lück

Equivariant Eilenberg-MacLane spaces are constructed in [1, p. II.13], [3, p. 277], [8, p. 45], however, only for non-empty connected H-fixed point sets for all $H \subset G$ and in the pointed category. This is a reasonable assumption in equivariant homotopy theory (equivariant Posnikov-systems, homology, obstruction theory) but too restrictive for the study of equivariant manifolds. Therefore we develope a treatment of equivariant Eilenberg-MacLane spaces of type one in full generality. They are used, for example, in equivariant L-theory as reference spaces (see [5]) or in [4].

A groupoid is a small category such that all morphisms are isomorphisms. The fundamental groupoid $\pi(X)$ of a space has as objects points x in X. A morphism $y \rightarrow x$ is a homotopy class of paths from x to y. The objects in the orbit category $\mathfrak{G}(G)$ of a compact Lie group G are homogenous spaces G/H for closed H, morphisms are G-maps.

<u>DEFINITION 1.:</u> <u>The fundamental</u> $\mathfrak{G}(G)$-<u>groupoid of a</u> G-<u>space</u> X <u>is the contravariant functor</u>

$\Pi^G X : \mathcal{O}(G) \to \{\text{groupoids}\}, G/H \mapsto \Pi(X^H) = \Pi(HOM_G(G/H,X))$. □

An $\mathcal{O}(G)$-groupoid is a contravariant functor $\mathcal{G} : \mathcal{O}(G) \to \{\text{groupoids}\}$, an $\mathcal{O}(G)$-functor $F : \mathcal{G}_0 \to \mathcal{G}_1$ is a natural transformation. If $\hat{I}$ is the category of two objects 0 and 1 and two morphisms $0 \to 1$ and $1 \to 0$ beside the identities, an $\mathcal{O}(G)$-transformation $\varphi : F_0 \to F_1$ between $\mathcal{O}(G)$-functors $F_i : \mathcal{G}_0 \to \mathcal{G}_1$ is an $\mathcal{O}(G)$-functor $\varphi : \mathcal{G}_0 \times \hat{I} \to \mathcal{G}_1$ with $\varphi|\mathcal{G}_0 \times \{i\} = F_i$. If such a $\varphi$ exists we call $F_0$ and $F_1$ homotopic. Let $[\mathcal{G}_0, \mathcal{G}_1]^{\mathcal{O}(G)}$ be the set of homotopy classes of $\mathcal{O}(G)$-functors $\mathcal{G}_0 \to \mathcal{G}_1$. Obviously a G-map $f : X \to Y$ induces an $\mathcal{O}(G)$-functor $\Pi^G f : \Pi^G X \to \Pi^G Y$ and a G-homotopy $h : X \times I \to Y$ an $\mathcal{O}(G)$-transformation $\Pi^G h_0 \to \Pi^G h_1$. If $[X,Y]^G$ is the set of G-homotopy classes of G-maps $X \to Y$ we get

$$[\Pi^G ?] : [X,Y]^G \to [\Pi^G X, \Pi^G Y]^{\mathcal{O}(G)}$$

An $\mathcal{O}(G)$-functor $F : \mathcal{G}_0 \to \mathcal{G}_1$ is an $\mathcal{O}(G)$-homotopy equivalence if there exists $F' : \mathcal{G}_1 \to \mathcal{G}_0$ with both composites homotopic to the identity. We call F a weak $\mathcal{O}(G)$-homotopy equivalence if $F(G/H) : \mathcal{G}_0(G/H) \to \mathcal{G}_1(G/H)$ is an equivalence of categories for each G/H, that is, there exists a functor in the other direction with both composites naturally equivalent to the identity or, equivalently, F(G/H) induces a bijection $\overline{\mathcal{G}_0(G/H)} \to \overline{\mathcal{G}_1(G/H)}$ between the set of isomorphism classes of objects and a bijection $\mathrm{Aut}(x) \to \mathrm{Aut}(F(x))$ for each $x \in \mathcal{G}_0(G/H)$ (see [6], p. 98). A G-map $f : X \to Y$ induces a weak $\mathcal{O}(G)$-homotopy equivalence $\Pi^G f$ if and only if

$\pi_o(f^H)$ and $\pi_1(f^H,x)$ are bijective for all $x \in X^H$ and closed $H \subset G$.

An $\mathfrak{O}(G)$-homotopy equivalence is a weak $\mathfrak{O}(G)$-homotopy equivalence, the converse is false. Namely, let G be $\mathbb{Z}/p \times \mathbb{Z}/q$ and $\mathcal{G}$ the $\mathfrak{O}(G)$-groupoid with $\mathcal{G}(G/L)$ the trivial groupoid $\{*\}$ for $L \neq G$ and $\mathcal{G}(G/G) = \emptyset$. If X is a G-space with simply connected $X, X^{\mathbb{Z}/p}$ and $X^{\mathbb{Z}/q}$ and with $X^G = \emptyset$ the obvious projection $\Pi^G X \to \mathcal{G}$ is a weak $\mathfrak{O}(G)$-homotopy equivalence. It cannot be an $\mathfrak{O}(G)$-homotopy equivalence since any $\mathfrak{O}(G)$-functor $F : \mathcal{G} \to \Pi^G X$ must send the object in $\mathcal{G}(G)$ to a point in $X^{\mathbb{Z}/p} \cap X^{\mathbb{Z}/q} = X^G$, a contradiction.

DEFINITION 2: *A G-CW-complex Y together with an $\mathfrak{O}(G)$-functor $\mu : \Pi^G Y \to \mathcal{G}$ is an equivariant Eilenberg-MacLane space $K(\mathcal{G},\mu,1)$ of type $(\mathcal{G},1)$ if the map $[X,Y]^G \to [\Pi^G X,\mathcal{G}]^{\mathfrak{O}(G)}, [f] \mapsto [\mu \circ \Pi^G f]$ is bijective for all G-CW-complexes X.* □

THEOREM 3:

a) *A G-CW-complex Y together with an $\mathfrak{O}(G)$-functor $\mu : \Pi^G Y \to \mathcal{G}$ is a $K(\mathcal{G},\mu,1)$ if and only if $\mu$ is a weak* $\mathfrak{O}(G)$-equivalence and $\pi_n(Y^H,y) = 0$ for all closed $H \subset G$, $y \in Y^H$, $n \geq 2$.

b) There is a $K(\mathcal{G},\mu,1)$ for any $\mathfrak{O}(G)$-groupoid $\mathcal{G}$. Any two of them are G-homotopic. □

COROLLARY 4:

a) An $\mathfrak{G}(G)$-functor $F : \mathcal{G}_0 \to \mathcal{G}_1$ is a weak $\mathfrak{G}(G)$-equivalence if and only if $F_* : [\Pi^G X, \mathcal{G}_0]^{\mathfrak{G}(G)} \to [\Pi^G X, \mathcal{G}_1]^{\mathfrak{G}(G)}$ is bijective for all G-CW-complexes X.

b) Each weak $\mathfrak{G}(G)$-homotopy equivalence $\Pi^G X \to \Pi^G Y$ for X and Y G-CW-complexes is an $\mathfrak{G}(G)$-homotopy equivalence.

PROOF: Because of theorem 3 it suffices to prove for a G-map $f : Y \to Z$ between G-spaces that $f^H : Y^H \to Z^H$ is a (non-equivariant) weak homotopy equivalence for all closed $H \subset G$ if and only if $f_* : [X,Y]^G \to [X,Z]^G$ is bijective for all G-CW-complexes X. As in [9], p. 220 this follows from elementary obstruction theory. Now a) implies b). □

The homotopy colimit of $\Pi^G X$ is the fundamental group category used in [2] to introduce equivariant finiteness obstruction, Whitehead torsion and obstruction theory.

If $\Gamma_i$ is a group and $\hat{\Gamma}_i$ the groupoid with one object and elements of $\Gamma$ as morphisms the set $[\hat{\Gamma}_1, \hat{\Gamma}_2]$ of natural equivalence classes of functors $\Gamma_1 \to \Gamma_2$ can be identified with $HOM(\Gamma_1, \Gamma_2)/INN(\Gamma_2, \Gamma_2)$. We rediscover the (non-equivariant) statement that (free) homotopy classes of maps $K(\Gamma_1, 1) \to K(\Gamma_2, 1)$ correspond bijectively to $HOM(\Gamma_1, \Gamma_2)/INN(\Gamma_2, \Gamma_2)$ (see [9, p. 226]).

We end with the proof of theorem 3.

a) We start with the "if"-statement. We only show that $\mu_* \circ \Pi^G$: $[X,Y]^G \to [\Pi^G X, \mathcal{Y}]^{\mathfrak{G}(G)}$ is surjective because the easier proof of injectivity is similar. Given $\psi : \Pi^G X \to \mathcal{Y}$, we have to construct $f : X \to Y$ and an $\mathfrak{G}(G)$-equivalence $\varphi$ between $\mu \circ \Pi^G f$ and $\psi$. We define f inductively over the skeletons of X as $f_r : X_r \to Y$.

We fix for each zero-cell a characteristic map $p : G/H \to X$. Since $\overline{\mu(G/H)} : \overline{\Pi(Y^H)} \to \overline{\mathcal{Y}}(G/H)$ is bijective we can choose a point y in $Y^H$ and an isomorphism $u : \mu(G/H)(y) \to \psi(G/H)(p(eH))$. Define $f_o : X_o \to Y$ such that $f_o \circ p(eH)$ is y.

Now we also fix a characteristic map $q : G/K \times I \to X_1$ for each one-cell. With the choices above there is a unique zero-cell with characteristic map $p_i : G/H_i \to X_o$ and a unique G-map $\sigma_i : G/K \to G/H_i$ with $q|G/K \times \{i\} = p_i \circ \sigma_i$ for $i = 0,1$. Let $u_i : \mu(G/H_i)(f_o \circ p_i(eH_i)) \to \psi(G/H_i)(p_i(eH))$ be the isomorphism choosen above. Now interprete $q\; eK \times I$ as a morphism in $\Pi(X^K)$. Since $\mu$ is a weak $\mathfrak{G}(G)$-homotopy equivalence there is exactly one morphism $w: f_o \circ q(eK \times 0) \to f_o \circ q(eK \times 1)$ in $\Pi(Y^K)$ making the following diagram commutative:

$$
\begin{array}{ccc}
\mu(G/K)(f_o \circ q(eK \times 0)) & \xrightarrow{\mu(G/K)(w)} & \mu(G/K)(f_o \circ q(eK \times 1)) \\
\| & & \| \\
\sigma_o^*\mu(G/H_o)(f_o \circ p_o(eH_o)) & & \sigma_1^*\mu(G/H_1)(f_o \circ p_1(eH_1)) \\
\sigma_o^* u_o \downarrow & & \downarrow \sigma_1^* u_1 \\
\sigma_o^*\psi(G/H_o)(p_o(eH_o)) & & \sigma_1^*\psi(G/H_1)(p_1(eH_1)) \\
\| & & \| \\
\psi(G/K)(q(eK \times 0) & \xrightarrow[\psi(G/K)(q|eK \times I)]{} & \psi(G/K)(q(eK \times 1))
\end{array}
$$

Use w to extend $f_0$ over the one-cell q. Thus we get $f_1 : X_1 \to Y$.

Each path in $X_1^H$ between two points in $X_0^H$ is given up to homotopy by a sequence of oriented one-cells. Therefore $f_1$ has the following property.

Let $p_i : G/H \to X$ be any **zero-cell**, $\sigma_i : G/L \to G/H_i$ any G-map for i = 0,1 and v any path in $X_1^L$ from $\sigma_1^* p_1(eH_1)$ to $\sigma_0^* p_0(eH_0)$. Denote by $\nu$ the composition $\mu \circ \Pi^G f_1$. Then the following diagram commutes

$$\begin{array}{ccc}
\nu(G/L)(\sigma_0^* p_0(eH_0)) & \xrightarrow{\nu(G/L)(v)} & \nu(G/L)(\sigma_1^* p_1(eH_1)) \\
\| & & \| \\
\sigma_0^* \nu(G/H_0)(p_0(eH_0)) & & \sigma_1^* \nu(G/H_1)(p_1(eH_1)) \\
\sigma_0^* u_0 \downarrow & & \downarrow \sigma_1^* u_1 \\
\sigma_0^* \psi(G/H_0)(p_0(eH_0)) & & \sigma_1^* \psi(G/H_1)(p_1(eH_1)) \\
\| & & \| \\
\psi(G/L)(p_0(eH_0)) & \xrightarrow[\psi(G/L)(v)]{} & \psi(G/L)(p_1(eH_1))
\end{array}$$

Let $r : G/L \times S^1 \to X_1$ be the attaching map of a two-cell. We can assume without loss of generality that there is a zero-cell $p : G/H \to X_0$ and a G-map $\sigma : G/L \to G/H$ with $p \circ \sigma = r | G/L \times *$ for $*$ a base point in $S^1$. Let $v : r(eL \times *) \to r(eL \times *)$ be the morphism in $\Pi(X^L)$ given by $r | eL \times S^1$. Since $r | eL \times *$ is nullhomotopic in $X^L$ this morphism v and hence $\psi(G/L)(v)$ are the identity. Because of the diagram above $\nu(G/L)(v) = \mu(G/L) \circ \Pi(f_1^L)(v)$ is also the identity.

Now $\mu(G/L)$ induces a bijection between $\mathrm{Aut}(f_1 \circ r(eL \times *))$ in $\Pi(Y^L)$ and $\mathrm{Aut}(\mu(G/L)(f_1 \circ r(eL \times *)))$ in $\mathcal{G}(G/L)$ by assumption. Hence $f_1 \circ v$ is nullhomotopic in Y so that we can extend $f_1$ to $f_2 : X_2 \to Y$. Since $\pi_n(Y^H, y)$ vanishes for all $H \subset G$, y in $Y^H$ and $n \geq 2$ we can extend $f_2$ to $f : X \to Y$.

We next construct the $\mathfrak{G}(G)$-equivalence $\varphi : \mu \circ \Pi^G f \to \psi$. We must specify for each $L \subset G$ and x in $X^L$ an isomorphism $\varphi(G/L)(x)$ from $\mu \circ \Pi^G f(G/L)(x)$ to $\psi(G/L)(x)$ in $\mathcal{G}(G/L)$. Choose any zero cell $p : G/H \to X$, any G-map $\sigma : G/L \to G/H$ and any path w from $\sigma^*(p(eH))$ to x in $X^L$. Define $\varphi(G/L)(x)$ as the composition

$$\begin{array}{rc}
 & \mu(G/L) \circ \Pi(f^L)(x) \\
\mu(G/L) \circ \Pi(f^L)(w) & \downarrow \\
 & \mu(G/L) \circ \Pi(f^L)(\sigma^* p(eH)) \\
 & \| \\
 & \sigma^* \mu(G/H) \circ \Pi(f^H)(p(eH)) \\
\sigma^* u & \downarrow \\
 & \sigma^* \psi(G/H)(p(eH)) \\
 & \| \\
 & \psi(G/L)(\sigma^* p(eH)) \\
\psi(G/L)(w^{-1}) & \downarrow \\
 & \psi(G/L)(x)
\end{array}$$

This is independent of the choices of $p, \sigma$ and w because of the diagram above. It is left to the reader to verify that these $\varphi(G/L)(x)$ fit nicely together yielding $\varphi$. This finishes the proof of the "if"-statement.

The "only if"-statement follows from the explicit construction in the proof of b) and the if-statement.

b) Given an $\mathfrak{G}(G)$-groupoid $\mathcal{G}$ we must construct a G-CW-complex Y with a weak $\mathfrak{G}(G)$-homotopy equivalence $\Pi^G Y \to \mathcal{G}$ such that $\Pi_n(Y^H,y)$ is zero for all $H \subset G$, y in $Y^H$ and $n \geq 2$. Composing $\mathcal{G}$ with the functor "classifying space" of a category (see [7]) gives a contravariant functor $\mathfrak{G}(G) \to$ {CW-complexes}. Now Y is obtained by applying the construction C of [3].

REFERENCES

[1] Bredon, G. E.: Equivariant Cohomology Theories, LNM 34, Springer (1967)

[2] tom Dieck, T.: Transformation groups, de Gruyter (1987)

[3] Elmendorf, A. D.: Systems of fixed point sets, Trans. of the AMS 277, 275 - 284 (1983)

[4] Lück, W.: The equivariant degree, preprint, Math. Gott. Göttingen (1986)

[5] Lück, W. and Madsen, I.: Equivariant L-groups, preprint, Arhus (1987)

[6] MacLane, S.: Kategorien, Springer (1972)

[7] Segal, G. B.: Classifying spaces and spectral sequences, Publ. Math. I.H.E.S. 34, 113 - 128 (1968)

[8] Triantafillou, G.: Äquivariante rationale Homotopietheorie, Bonner Math. Schriften 110, Bonn (1978)

[9] Whitehead, G. W.: Elements of homotopy theory, grad. texts in math. 61, Springer (1978)

Wolfgang Lück
Mathematisches Institut
der Georg-August-Universität
Bunsenstraße 3 - 5

3400 Göttingen
Federal Republic of Germany

(Received October 22, 1986)