THE LIMIT OF $\mathbb{F}_p$-BETTI NUMBERS OF A TOWER OF FINITE COVERS WITH AMENABLE FUNDAMENTAL GROUPS

PETER LINNELL, WOLFGANG LÜCK, AND ROMAN SAUER

Abstract. We prove an analogue of the Approximation Theorem of L^2-Betti numbers by Betti numbers for arbitrary coefficient fields and virtually torsionfree amenable groups. The limit of Betti numbers is identified as the dimension of some module over the Ore localization of the group ring.

0. Introduction

A *residual chain* of a group G is a sequence $G = G_0 \supset G_1 \supset G_2 \supset \cdots$ of normal subgroups of finite index such that $\bigcap_{i\geq 0} G_i = \{e\}$. The n-th L^2-Betti number of any finite free G-CW complex X is the limit of the n-th Betti numbers of $G_i\backslash X$ normalized by the index $[G : G_i]$ for $i \to \infty$ [11]. If we instead consider Betti numbers $b_n(G_i\backslash X; k)$ with respect to a field of characteristic $p > 0$, the questions whether the limit exists, what it is, and whether it is independent of the residual chain are completely open for arbitrary residually finite G.

For $G = \mathbb{Z}^k$ and every field k Elek showed that $\lim_{i\to\infty} b_n(G_i\backslash X; k)$ exists and expresses it in terms of the entropy of G-actions on the Pontrjagin duals of finitely generated kG-modules [4] – his techniques play an important role in this paper (see Section 1.3). It was observed in [1, Theorem 17] that the mere convergence of the right hand side of (i) in Theorem 0.2 for every amenable G and every field k follows from a general convergence principle for subadditive functions on amenable groups [9] and a theorem by Weiss [18].

The main purpose of this paper is to determine the limit $\lim_{i\to\infty} b_n(G_i\backslash X; k)$ in algebraic terms for a large class of amenable groups including virtual torsionfree elementary amenable groups. This makes the limit computable by homological techniques; see e.g., the spectral sequence argument of Example 6.3.

More precisely, the limit will be expressed in terms of the *Ore dimension*. The group ring kG of a torsionfree amenable group satisfying the zero-divisor conjecture fulfills the Ore condition with respect to the subset $S = kG - \{0\}$ [12, Example 8.16 on page 324]; we will review the Ore localization in Subsection 1.1. The Ore localization $S^{-1}kG$ is a skew field containing kG. Therefore the following definition makes sense:

Definition 0.1 (Ore dimension)**.** Let G be a torsionfree amenable group such that kG contains no zero-divisors. The *Ore dimension* of a kG-module M is defined by

$$\dim_{kG}^{\mathrm{Ore}}(M) = \dim_{S^{-1}kG}\big(S^{-1}kG \otimes_{kG} M\big).$$

Date: March 1, 2010.

2000 *Mathematics Subject Classification.* 16U20, 55P99.

Key words and phrases. Amenability, Ore localization, Betti numbers.

The authors thank the HIM at Bonn for its hospitality during the Trimester program “Rigidity” in the fall 2009 when this paper was written. This work was financially supported by the Leibniz-Preis of the second author.

The following theorem is our main result; we will prove a more general version, including virtually torsionfree groups, in Section 5.

Theorem 0.2. *Let k be a field. Let G be a torsionfree amenable group for which kG has no zero-divisors*[1]*. Let $(G_n)_{n\geq 0}$ be a residual chain of G. Then:*

(i) Consider a finitely presented kG-module M. Then

$$\dim_{kG}^{\mathrm{Ore}}(M) = \lim_{n\to\infty} \frac{\dim_k\bigl(k \otimes_{kG_n} M\bigr)}{[G:G_n]};$$

(ii) Consider a finite free kG-chain complex C_*. Then we get for all $i \geq 0$

$$\dim_{kG}^{\mathrm{Ore}}\bigl(H_i(C_*)\bigr) = \lim_{n\to\infty} \frac{\dim_k\bigl(H_i(k \otimes_{kG_n} C_*)\bigr)}{[G:G_n]};$$

(iii) Let X be a finite free G-CW-complex. Then we get for all $i \geq 0$

$$\dim_{kG}^{\mathrm{Ore}}\bigl(H_i(X)\bigr) = \lim_{n\to\infty} \frac{\dim_k\bigl(H_i(G_n\backslash X; k)\bigr)}{[G:G_n]}.$$

Remark 0.3 (Fields of characteristic zero)**.** Let G be a group with a residual chain $(G_n)_{n\geq 0}$, and let M be a finitely presented kG-module. Then the Approximation Theorem for L^2-Betti numbers says that

$$\dim_{\mathcal{N}(G)}\bigl(\mathcal{N}(G) \otimes_{kG} M\bigr) = \lim_{n\to\infty} \frac{\dim_k\bigl(k \otimes_{kG_n} M\bigr)}{[G:G_n]} \tag{0.4}$$

provided k is an algebraic number field. Here $\mathcal{N}(G)$ is the group von Neumann algebra, and $\dim_{\mathcal{N}(G)}$ is the von Neumann dimension. See [11] for $k = \mathbb{Q}$ and [3] for the general case.

Let k be a field of characteristic zero and let $u = \sum_{g\in G} x_g \cdot g \in kG$ be an element. Let F be the finitely generated field extension of $\mathbb{Q}$ given by $F = \mathbb{Q}(x_g \mid g \in G) \subset k$. Then u is already an element in FG. The field F embeds into $\mathbb{C}$: since F is finitely generated, it is a finite algebraic extension of a transcendental extension F' of $\mathbb{Q}$ [8, Theorem 1.1 on p. 356], and F' has finite transcendence degree over $\mathbb{Q}$. Since the transcendence degree of $\mathbb{C}$ over $\mathbb{Q}$ is infinite, there exists an embedding $F' \hookrightarrow \mathbb{C}$ induced by an injection of a transcendence basis of $F'/\mathbb{Q}$ into a transcendence basis $\mathbb{C}/\mathbb{Q}$, which extends to $F \hookrightarrow \mathbb{C}$ because $\mathbb{C}$ is algebraically closed. This reduces the case of fields of characteristic zero to the case $k = \mathbb{C}$. In [6] Elek proved (0.4) for amenable G and $k = \mathbb{C}$ (see also [13]).

Moreover, if G is a torsionfree amenable group such that $\mathbb{C}G$ contains no zero-divisors and k is a field of characteristic zero, then

$$\dim_{\mathcal{N}(G)}\bigl(\mathcal{N}(G) \otimes_{kG} M\bigr) = \dim_{kG}^{\mathrm{Ore}}(M).$$

This follows from [12, Theorem 6.37 on page 259, Theorem 8.29 on page 330, Lemma 10.16 on page 376, and Lemma 10.39 on page 388]. In particular, Theorem 0.2 follows for k of characteristic zero. So the interesting new case is the one of a field of prime characteristic.

---

[1] This assumption is satisfied if G is torsionfree elementary amenable. See Remark 1.1

1. Review of Ore localization and Elek's dimension function

1.1. **Ore localization.**

We review the Ore localization of rings. For proofs and more information the reader is referred to [17]. Consider a torsionfree group G and a field k. Let S be the set of non-zero-divisors of kG. This is a multiplicatively closed subset of kG and contains the unit element of kG. Suppose that kG satisfies the *Kaplansky Conjecture* or *zero-divisor conjecture*, i.e., $S = kG - \{0\}$. Further assume that S satisfies the *left Ore condition*, i.e., for $r \in kG$ and $s \in S$ there exists $r' \in kG$ and $s' \in S$ with $s'r = r's$. Then we can consider the *Ore localization* $S^{-1}kG$. Recall that every element in $S^{-1}kG$ is of the form $s^{-1} \cdot r$ for $r \in kG$ and $s \in S$ and $s_0^{-1} \cdot r_0 = s_1^{-1} \cdot r_1$ holds if and only there exists $u_0, u_1 \in R$ satisfying $u_0 r_0 = u_1 r_1$ and $u_0 s_0 = u_1 s_1$. Addition is given on representatives by $s_0^{-1} r_0 + s_1^{-1} r_1 = t^{-1}(c_0 r_0 + c_1 r_1)$ for $t = c_0 s_0 = c_1 s_1$. Multiplication is given on representatives by $s_0^{-1} r_0 \cdot s_1^{-1} r_1 = (ts_0)^{-1} c r_1$, where $cs_1 = tr_0$. The zero element is $e^{-1} \cdot 0$ and the unit element is $e^{-1} \cdot e$. The Ore localization $S^{-1}kG$ is a skew field and the canonical map $kG \to S^{-1}kG$ sending r to $e^{-1} \cdot r$ is injective. The functor $S^{-1}kG \otimes_{kG} -$ is exact.

Remark 1.1 (The Ore condition for group rings)**.** If a torsionfree amenable group G satisfies the Kaplansky Conjecture, i.e., kG contains no zero-divisor, then for $S = kG - \{0\}$ the Ore localization $S^{-1}kG$ exists and is a skew field [12, Example 8.16 on page 324]. Every torsionfree elementary amenable group satisfies the assumptions above for all fields k [7, Theorem 1.2; 10, Theorem 2.3]. If the group G contains the free group of rank two as subgroup, then the Ore condition is never satisfied for kG [10, Proposition 2.2].

From the previous remark and the discussion above we obtain:

Theorem 1.2. *Let G be a torsionfree amenable group such that kG contains no zero-divisors. Then the Ore dimension $\dim_{kG}^{\mathrm{Ore}}$ has the following properties:*

(i) $\dim_{kG}^{\mathrm{Ore}}(kG) = 1$*;*
(ii) For any short exact sequence of kG-modules $0 \to M_0 \to M_1 \to M_2 \to 0$ we get

$$\dim_{kG}^{\mathrm{Ore}}(M_1) = \dim_{kG}^{\mathrm{Ore}}(M_0) + \dim_{kG}^{\mathrm{Ore}}(M_2).$$

1.2. **Crossed products, Goldie rings, and the generalized Ore localization.**

Throughout, let G be a group, let k be a skew field.

Let R be a ring. The notion of crossed product generalizes the one of group ring. A *crossed product* $R * G = R *_{c,\tau} G$ is determined by maps $c \colon G \to \mathrm{aut}(R)$ and $\tau \colon G \times G \to R^{\times}$ such that, roughly speaking, c is a homomorphism up to the 2-cocycle τ. We refer to the survey [12, 10.3.2 on p. 398] for details. If G is an extension of H by Q, then the group ring kG is isomorphic to a crossed product $kH * Q$. Some results in this paper are formulated for crossed products, although we only need the case of group rings for Theorem 0.2. So the reader may think of group rings most of the time. However, crossed products show up naturally, e.g., in proving that the virtual Ore dimension (5.1) is well defined.

We recall the following definition.

Definition 1.3. A ring R is *left Goldie* if there exists $d \in \mathbb{N}$ such that every direct sum of nonzero left ideals of R has at most d summands and the left annihilators $a(x) = \{r \in R;\ rx = 0\}$, $x \in R$, satisfy the maximum condition for ascending chains. A ring R is *prime* if for any two ideals A, B in R, $AB = 0$ implies $A = 0$ or $B = 0$.

The subgroup of G generated by its finite normal subgroups will be denoted by $\Delta^+(G)$. Then $\Delta^+(G)$ is also the set of elements of finite order which have only finitely many conjugates. We need the following three results:

Lemma 1.4 ([15, Corollary 5 of Lecture 4])**.** *If $\Delta^+(G) = 1$, then $k * G$ is prime.*

Theorem 1.5 ([14, Theorem 4.10 on p. 456])**.** *The set of non-zero-divisors in a prime left Goldie ring satisfies the Ore condition. The Ore localization $S^{-1}R$ is isomorphic to $\mathrm{M}_d(D)$ for some $d \in \mathbb{N}$ and some skew field D.*

Theorem 1.6. *If G is amenable and $k * G$ is a domain, then $k * G$ is a prime left Goldie ring. If G is an elementary amenable group such that the orders of the finite subgroups are bounded, then $k * G$ is left Goldie.*

Proof. If G is amenable and $k * G$ is a domain, then $k * G$ satisfies the Ore condition [3, Theorem 6.3], thus its Ore localization with respect to $S = k * G - \{0\}$ is a skew field. By [14, Theorem 4.10 on p. 456] $k * G$ is a prime left Goldie ring. The second assertion is taken from [7, Proposition 4.2]. $\square$

Next we extend the definition of Ore dimension to prime left Goldie rings. Let R be such a ring. The functor $S^{-1}R \otimes_R -$ will still be exact [17, Proposition II.1.4 on page 51]. If M is a left R-module, then $S^{-1}R \otimes_R M$ will be a direct sum of n irreducible $S^{-1}R$-modules for some non-negative integer n, and then the *(generalized) Ore dimension* of M is defined as

$$\dim_R^{\mathrm{Ore}}(M) = \frac{n}{d}.$$

Since $S^{-1}R \cong \mathrm{M}_d(D)$ (Theorem 1.5) and $\mathrm{M}_d(D)$ decomposes into d copies of the irreducible module D^d, we have $\dim_R^{\mathrm{Ore}}(R) = 1$.

1.3. **Elek's dimension function.**

Throughout this subsection let G be a finitely generated amenable group. We review Elek's definition [5] of a dimension function $\dim_{kG}^{\mathrm{Elek}}$ for finitely generated kG-modules.

Fix a finite set of generators and equip G with the associated word metric d_G. A *Følner sequence* $(F_n)_{n\geq 0}$ is a sequence of finite subsets of G such that for any fixed $R > 0$ we have

$$\lim_{n\to\infty} \frac{|\partial_R F_n|}{|F_n|} = 0,$$

where $\partial_R F_n = \{g \in G \mid d(g, F_k) \leq R \text{ and } d(g, G \setminus F_k) \leq R\}$.

Let k be an arbitrary skew field endowed with the discrete topology and let $\mathbb{N}$ denote the positive integers $\{1, 2, \dots\}$. Let $n \in \mathbb{N}$. We equip the space of functions $\mathrm{map}(G, k^n) = \prod_{g\in G} k^n$ with the product topology, which is the same as the topology of pointwise convergence. The natural right G-action on $\mathrm{map}(G, k^n)$ is defined by

$$(\phi g)(x) = \phi(xg^{-1}) \text{ for } g, x \in G, \phi \in \mathrm{map}(G, k^n).$$

Also $\mathrm{map}(G, k^n)$ is a right k-vector space by defining $(\phi k)(x) = \phi(x)k$. For any subset $S \subset G$ and any subset $W \subset \mathrm{map}(G, k^n)$ let

$$W|_S = \{f : S \to k^n \mid \exists g \in W \text{ with } g|_S = f\}.$$

A right k-linear subspace $V \subset \mathrm{map}(G, k^n)$ is called *invariant* if V is closed and invariant under the right G-action.

Elek defines the *average dimension* $\dim_G^{\mathcal{A}}(V)$ of an invariant subspace V by choosing a Følner sequence $(F_n)_{n\in\mathbb{N}}$ of G and setting

$$\dim_G^{\mathcal{A}}(V) = \limsup_{n\to\infty} \frac{\dim_k\bigl(V|_{F_n}\bigr)}{|F_n|}. \tag{1.7}$$

Theorem 1.8 ([5, Prop. 7.2 and Prop. 9.2])**.** *The sequence in* (1.7) *converges and its limit* $\dim_G^{\mathcal{A}}(V)$ *is independent of the choice of the Følner sequence.*

Remark 1.9. Elek actually defines $\dim_G^{\mathcal{A}}(V)$ using Følner exhaustions, i.e. increasing Følner sequences $(F_{n\in\mathbb{N}})$ with $\bigcup_{n\in\mathbb{N}} F_n = G$. This makes no difference since the existence of the limit of $(\dim_k\bigl(V|_{F_n}\bigr)/|F_n|)_{n\in\mathbb{N}}$ for arbitrary Følner sequences (and thus its independence of the choice) follows from [9, Theorem 6.1].

Let M be a finitely generated left kG-module. The k-dual $M^* = \hom_k(M,k)$ (where M and k are viewed as left k-modules, and $(\phi a)m = \phi(am)$ for $\phi \in M^*$, $a \in k$ and $m \in M$) carries the natural right G-action $(\phi g)(m) = \phi(gm)$. The dual of the free left kG-module kG^n is canonically isomorphic to $\operatorname{map}(G,k^n)$. Any left kG-surjection $f\colon kG^n \twoheadrightarrow M$ induces a right kG-injection $f^*\colon M^* \to \operatorname{map}(G,k^n)$ such that $\operatorname{im}(f^*)$ is a G-invariant k-subspace.

Definition 1.10 (Elek's dimension function)**.** Let M be a finitely generated left kG-module. Its *dimension in the sense of Elek* is defined by choosing a left kG-surjection $f\colon kG^n \twoheadrightarrow M$ and setting

$$\dim_{kG}^{\mathrm{Elek}}(M) = \dim_G^{\mathcal{A}}\bigl(\operatorname{im}(f^*)\bigr). \tag{1.11}$$

Theorem 1.12 (Main properties of Elek's dimension function)**.** *Let G be a finitely generated amenable group. The definition* (1.11) *of* $\dim_{kG}^{\mathrm{Elek}}(M)$ *is independent of the choice of the surjection f, and* $\dim_{kG}^{\mathrm{Elek}}$ *has the following properties:*

- *(i)* $\dim_{kG}^{\mathrm{Elek}}(kG) = 1$*;*
- *(ii) For any short exact sequence of finitely generated kG-modules* $0 \to M_0 \to M_1 \to M_2 \to 0$ *we get*
$$\dim_{kG}^{\mathrm{Elek}}(M_1) = \dim_{kG}^{\mathrm{Elek}}(M_0) + \dim_{kG}^{\mathrm{Elek}}(M_2);$$
- *(iii) If the finitely generated kG-module M satisfies* $\dim_{kG}^{\mathrm{Elek}}(M) = 0$*, then every quotient module Q of M satisfies* $\dim_{kG}^{\mathrm{Elek}}(Q) = 0$.

Proof. The first two assertions are proved in [5, Theorem 1]. Notice that the third condition does not necessarily follow from additivity since the kernel of the epimorphism $M \to Q$ may not be finitely generated. But the third statement is a direct consequence of the definition of Elek's dimension. □

Remark 1.13 (The dual of finitely generated kG-modules)**.** Identify the left kG-module kG^n with the finitely supported functions in $\operatorname{map}(G,k^n)$. Here we view $\operatorname{map}(G,k^n)$ as a left k-vector space by $(af)(g) = af(g)$, and the left G-action is given by $(hf)(g) = f(h^{-1}g)$ for $h, g \in G$ and $a \in k$. Let

$$\langle \_, \_ \rangle\colon kG^n \times \operatorname{map}(G,k^n)$$

be the canonical pairing (evaluation) of kG^n and its dual $\operatorname{map}(G,k^n)$. If we view an element $f \in kG^n$ as a finitely supported function $G \to k^n$ (in $\operatorname{map}(G,k^n)$), then the pairing of

$f \in kG^n$ with $l \in \operatorname{map}(G,k^n)$ is given by

$$\langle f, l\rangle = \sum_{g\in G} (f(g), l(g)),$$

where $(\_,\_)$ denotes the standard inner product in k^n. For a subset $W \subset kG^n$ let

$$W^{\perp} = \bigl\{f \in \operatorname{map}(G,k^n) \mid \langle x, f\rangle = 0 \ \forall x \in W\bigr\}.$$

If M is a finitely generated kG-module and $f\colon kG^n \twoheadrightarrow M$ is a left kG-surjection, then $f^*\colon M^* \hookrightarrow \operatorname{map}(G,k^n)$ is a right kG-injection and

$$\operatorname{im}(f^*) = \ker(f)^{\perp} \subseteq \operatorname{map}(G,k^n).$$

2. Approximation for finitely presented kG-modules for Elek's dimension function

The main result of this section is:

Theorem 2.1. *Let G be a finitely generated amenable group. Consider a sequence of normal subgroups of finite index*

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

such that $\bigcap_{n\geq 0} G_n = \{1\}$. Then every finitely presented kG-module M satisfies

$$\dim_{kG}^{\mathrm{Elek}}(M) = \lim_{n\to\infty} \frac{\dim_k\bigl(k \otimes_{kG_n} M\bigr)}{[G:G_n]}.$$

Its proof needs some preparation.

Throughout, let G be a finitely generated amenable group. For any subset $S \subset G$ let $k[S]$ be the k-subspace of kG generated by $S \subset kG$. Let $j[S]\colon k[S] \to k[G]$ be the inclusion and $\operatorname{pr}[S]\colon kG \to k[S]$ be the projection given by

$$\operatorname{pr}[S](g) = \begin{cases} g & \text{if } g \in S;\\ 0 & \text{if } g \in G\backslash S.\end{cases}$$

Theorem 2.2. *Let G be a finitely generated amenable group. Let M be a finitely presented left kG-module M with a presentation $kG^r \xrightarrow{f} kG^s \xrightarrow{p} M \to 0$. For every subset $S \subset G$ we define*

$$M[S] = \operatorname{coker}\Bigl(\operatorname{pr}[S]\circ f \circ j[S]\colon k[S]^r \to k[S]^s\Bigr).$$

Let $(F_n)_{n\geq 0}$ be a Følner sequence of G. Then

$$\dim_{kG}^{\mathrm{Elek}}(M) = \lim_{n\to\infty} \frac{\dim_k\left(M[F_n]\right)}{|F_n|}.$$

Proof. The map f is given by right multiplication with a matrix $A \in M_{r,s}(kG)$. Viewing A as a map $G \to k^{r\times s}$ it is clear what we mean by the support $\operatorname{supp}(A)$ of A. Let $R > 0$ be the diameter of $\operatorname{supp}(A) \cup \operatorname{supp}(A)^{-1}$. Since

$$\lim_{n\to\infty} \frac{\bigl|\partial_R F_n\bigr|}{\bigl|F_n\bigr|} = 0,$$

it is enough to show that for every $n \geq 1$

$$\bigl|\dim_k(M[F_n]) - \dim_k\bigl(\operatorname{im}(p^*)|_{F_n}\bigr)\bigr| \leq s \cdot \bigl|\partial_R F_n\bigr|. \tag{2.3}$$

For the definition of inner products $(\_,\_)$ and $\langle\_,\_\rangle$ we refer to Remark 1.13. Define the following k-linear subspaces of $\operatorname{map}(F_n, k^s)$:

$$\begin{aligned} W_n &= \bigl\{\phi\colon F_n \to k^s \mid \langle \operatorname{pr}_n \circ f \circ j_n(x), \phi\rangle = 0 \ \forall x \in k[F_n]^r\bigr\};\\ V_n &= \bigl\{\phi\colon F_n \to k^s \mid \exists\bar\phi : G \to k^s \text{ satisfying } \bar\phi|_{F_n} = \phi, \langle f(y), \bar\phi\rangle = 0 \ \forall y \in kG^r\bigr\};\\ Z_n &= \bigl\{\phi\colon F_n \to k^s \mid \phi|_{\partial_R F_n} = 0\bigr\}. \end{aligned}$$

Since $\dim_k(M[F_n]) = \dim_k(W_n)$ and $\dim_k\bigl(\operatorname{im}(p^*)|_{F_n}\bigr) = \dim_k(V_n)$, the desired estimate (2.3) is equivalent to

$$\bigl|\dim_k(W_n) - \dim_k(V_n)\bigr| \le s \cdot \bigl|\partial_R F_n\bigr|. \tag{2.4}$$

By additivity of $\dim_k$ we obtain that

$$\begin{aligned} \dim_k(W_n \cap Z_n) &\ge \dim_k(W_n) - \dim_k(\operatorname{map}(F_n, k^s)) + \dim_k(Z_n)\\ &\ge \dim_k(W_n) - s\cdot|F_n| + s\cdot(|F_n| - |\partial_R F_n|)\\ &= \dim_k(W_n) - s\cdot|\partial_R F_n|. \end{aligned}$$

Similarly, we get

$$\dim_k(V_n \cap Z_n) \ge \dim_k(V_n) - s\cdot|\partial_R F_n|.$$

To prove (2.4) it hence suffices to show that

$$W_n \cap Z_n \subset V_n; \tag{2.5}$$

$$V_n \cap Z_n \subset W_n. \tag{2.6}$$

Let $\phi \in W_n \cap Z_n$. Extend ϕ by zero to a function $\bar\phi : G \to k^s$. Let $y \in kG^r$. Then we can decompose y as $y = y_0 + y_1$ with $\operatorname{supp}(y_0) \subset F_n$ and $\operatorname{supp}(y_1) \subset G\backslash F_n$. By definition of the radius R it is clear that $\operatorname{supp}(f(y_1)) \subset G\backslash F_n \cup \partial_R F_n$. Because of $\phi \in Z_n$ we have $\langle f(y_1), \bar\phi\rangle = 0$. The fact that $\phi \in W_n$ implies that

$$\langle f(y_0), \bar\phi\rangle = \langle \operatorname{pr}_n \circ f \circ j_n(y_0), \phi\rangle = 0.$$

So we obtain that $\langle f(y), \bar\phi\rangle = 0$, meaning that $\phi \in V_n$. The proof of (2.6) is similar. □

The following theorem is due to Weiss. Its proof can be found in [2, Proposition 5.5].

Theorem 2.7 (Weiss)**.** *Let G be a countable amenable group. Let $G_n \subset G$, $n \ge 1$, be a sequence of normal subgroups of finite index with $\bigcap_{n\ge1} G_n = \{1\}$. Then there exists, for every $R \ge 1$ and every $\epsilon > 0$, an integer $M = M(R,\epsilon) \ge 1$ such that for $n \ge M$ there is a fundamental domain $Q_n \subset G$ of the coset space G/G_n such that*

$$\frac{\bigl|\partial_R Q_n\bigr|}{\bigl|Q_n\bigr|} < \epsilon.$$

Now we are ready to prove Theorem 2.1

Proof of Theorem 2.1. According to Theorem 2.7 let $(Q_n)_{n\ge0}$ be a Følner sequence of G such that $Q_n \subset G$ is a fundamental domain for G/G_n. Choose a finite presentation of M:

$$kG^r \xrightarrow{f} kG^s \to M \to 0.$$

Let $f_n = k[G/G_n] \otimes_{kG} f$. By right-exactness of tensor products we have the exact sequence

$$k[G/G_n]^r \xrightarrow{f_n} k[G/G_n]^s \to k[G/G_n] \otimes_{kG} M \to 0.$$

The natural map $Q_n \subset G \to G/G_n$ induces an isomorphism $j_n\colon k[Q_n] \to k[G/G_n]$ of k-vector spaces. The map f is given by right multiplication $f = R_A$ with a matrix $A \in M_{r,s}(kG)$. Viewing A as a map $G \to k^{r\times s}$ let $\operatorname{supp}(A)$ be the support of A. Let $R > 0$ be the diameter of $\operatorname{supp}(A) \cup \operatorname{supp}(A)^{-1}$ (with respect to the fixed word metric on G). Then f restricts to a map

$$f|_{Q_n\setminus\partial_R Q_n}\colon k[Q_n \setminus \partial_R Q_n]^r \to k[Q_n]^s.$$

Hence there is precisely one k-linear map g for which the following diagram of k-vector spaces commutes:

$$\begin{tikzcd}
k[G/G_n]^r \arrow[r, "f_n"] & k[G/G_n]^s \arrow[r] & k[G/G_n] \otimes_{kG} M \arrow[r] & 0 \\
k[Q_n \setminus \partial_R Q_n]^r \arrow[u, "j_n|_{Q_n\setminus\partial_R Q_n}"] \arrow[r, "f|_{Q_n\setminus\partial_R Q_n}"] & k[Q_n]^s \arrow[u, "j_n"', "\cong"] \arrow[r, "\mathrm{pr}"] & \operatorname{coker}(f|_{Q_n\setminus\partial_R Q_n}) \arrow[u, two heads, "g"'] \arrow[r] & 0
\end{tikzcd} \tag{2.8}$$

One easily verifies that g is surjective and that

$$\ker(g) \subset \operatorname{im}\bigl(\mathrm{pr} \circ j_n^{-1} \circ f_n\colon k[G/G_n]^r \to \operatorname{coker}(f|_{Q_n\setminus\partial_R Q_n})\bigr).$$

The map $\mathrm{pr} \circ j_n^{-1} \circ f_n$ descends to a map

$$\mathrm{pr} \circ j_n^{-1} \circ f_n\colon \ \operatorname{coker}(j_n|_{Q_n\setminus\partial_R Q_n}) \to \operatorname{coker}(f|_{Q_n\setminus\partial_R Q_n}).$$

Note that

$$\dim_k\bigl(\operatorname{coker}(j_n|_{Q_n\setminus\partial_R Q_n})\bigr) = r \cdot |\partial_R Q_n|.$$

Thus,

$$\dim_k(\operatorname{coker}(f|_{Q_n\setminus\partial_R Q_n})) - \dim_k\left(k[G/G_n] \otimes_{kG} M\right) = \dim_k \ker(g) \leq r \cdot \bigl|\partial_R Q_n\bigr|.$$

By replacing the upper row in diagram (2.8) by

$$k[Q_n]^r \xrightarrow{\mathrm{pr}[Q_n]\circ f\circ j[Q_n]} k[Q_n]^s \to M[Q_n] \to 0$$

and essentially running the same argument as before we obtain that

$$\dim_k(\operatorname{coker}(f|_{Q_n\setminus\partial_R Q_n})) - \dim_k(\operatorname{coker}(M[Q_n])) \leq r \cdot \bigl|\partial_R Q_n\bigr|.$$

Since

$$\frac{|\partial_R Q_n|}{[G/G_n]} = \frac{|\partial_R Q_n|}{|Q_n|} \xrightarrow{n\to\infty} 0$$

we get that

$$\lim_{n\to\infty} \frac{\dim_k(k[G/G_n] \otimes_{kG} M)}{[G : G_n]}$$

exists if and only if

$$\lim_{n\to\infty} \frac{\dim_k(M[Q_n])}{|Q_n|}$$

exists, and in this case they are equal. Now the assertion follows from Theorem 2.2. □

3. Comparing dimensions

The main result of this section is:

Theorem 3.1 (Comparing dimensions)**.** *Let G be a group, let k be a skew field, and let $k*G$ be a crossed product which is prime left Goldie. Let* dim *be any dimension function which assigns to a finitely generated left $k*G$-module a nonnegative real number and satisfies*

(i) $\dim(k*G) = 1$.

(ii) *For every short exact sequence $0 \to M_0 \to M_1 \to M_2 \to 0$ of finitely generated left $k*G$-modules, we get*
$$\dim(M_1) = \dim(M_0) + \dim(M_2).$$

(iii) *If the finitely generated left $k*G$-module M satisfies $\dim(M) = 0$, then every quotient module Q of M satisfies $\dim(Q) = 0$.*

Then for every finitely presented left $k*G$-module M, we get $\dim(M) = \dim^{\mathrm{Ore}}_{k*G}(M)$.

Proof. Let S denote the non-zero-divisors of $k*G$. We have to show that for all $r, s \in \mathbb{N}$ and every $r \times s$ matrix A with entries in $k*G$

$$\dim^{\mathrm{Ore}}_{k*G}\bigl(\operatorname{coker}\bigl(r_A \colon S^{-1}k*G^r \to S^{-1}k*G^s\bigr)\bigr) = \dim\bigl(\operatorname{coker}\bigl(r_A \colon k*G^r \to k*G^s\bigr)\bigr), \tag{3.2}$$

where r_A denotes the module homomorphism given by right multiplication with A. First note that we may assume that $r = s$. Indeed if $r < s$, replace A with the $s \times s$ matrix which is A for the first r rows, and has 0's on the bottom $s - r$ rows. On the other hand if $r > s$, replace A with the $r \times r$ matrix B with entries (b_{ij}) which is A for the first s columns, and has $b_{ij} = \delta_{ij}$ if $i > s$, where δ_{ij} is the Kronecker delta.

We will often use the obvious long exact sequence associated to homomorphisms $f \colon M_0 \to M_1$ and $g \colon M_1 \to M_2$

$$0 \to \ker(f) \to \ker(g \circ f) \to \ker(g) \to \operatorname{coker}(f) \to \operatorname{coker}(g \circ f) \to \operatorname{coker}(g) \to 0. \tag{3.3}$$

We now assume that A is an $r \times r$ matrix. Note that equation (3.2) is true if A is invertible over $S^{-1}k*G$; this is because then $\ker r_A = 0$ (whether A is considered as a matrix over $k*G$ or $S^{-1}k*G$).

Next observe that if $U \in \mathrm{M}_r(k*G)$ which is invertible over $\mathrm{M}_r(S^{-1}k*G)$, then equation (3.2) holds for A if and only if it holds for AU, and also if and only if it hold for UA. This follows from (3.3), $\ker U = 0$, $\dim(\operatorname{coker} U) = \dim^{\mathrm{Ore}}_{k*G}(\operatorname{coker} U) = 0$, and in the second case we use the third property of dim.

We may write $S^{-1}k*G = \mathrm{M}_d(D)$ for some $d \in \mathbb{N}$ and some skew field D. By applying the Morita equivalence from $\mathrm{M}_d(D)$ to D and back and doing Gaussian elimination over D we see that there are invertible matrices $U, V \in \mathrm{M}_{rd}(S^{-1}k*G)$ such that $U \operatorname{diag}(A, \dots, A) V = J$, where there are d A's and J is a matrix of the form $\operatorname{diag}(1, \dots, 1, 0, \dots, 0)$. Now choose $u, v \in S$ such that $uU, vV \in \mathrm{M}_{rd}(k*G)$. Then $(uU)\operatorname{diag}(A, \dots, A)(Vv) = uJv$, and the result follows. □

Theorem 3.4 (Comparing Elek's dimension and the Ore dimension)**.** *Let G be a finitely generated group and let k be a skew field. Suppose that kG is a prime left Goldie ring. Then for any finitely presented left kG-module M*
$$\dim^{\mathrm{Elek}}_{kG}(M) = \dim^{\mathrm{Ore}}_{kG}(M).$$

Proof. This follows from Theorem 3.1 and Theorem 1.12. □

4. Proof of the main theorem

Proof of Theorem 0.2. (i) In the first step we reduce the claim to the case, where G is finitely generated. Consider a finitely presented left kG-module M. Choose a matrix $A \in M_{r,s}(kG)$ such that M is kG-isomorphic to the cokernel of $r_A \colon kG^r \to kG^s$. Since A is a finite matrix and each element in kG has finite support, we can find a finitely generated subgroup $H \subseteq G$ such that $A \in M_{r,s}(kH)$. Both kG and kH are prime left Goldie by Lemma 1.4 and Theorem 1.6. Consider the finitely presented kH-module $N := \operatorname{coker}\bigl(r_A \colon kH^r \to kH^s\bigr)$. Then $M = kG \otimes_{kH} N$. We can also consider the Ore localization $T^{-1}kH$ for T the set of non-zero-divisors of kH. Put $H_n = H \cap G_n$. We obtain a residual chain $(H_n)_{n \geq 0}$ of H and have:

$$\begin{aligned}
\dim_{kG}^{\mathrm{Ore}}(M) &= \dim_{S^{-1}kG}\bigl(S^{-1}kG \otimes_{kG} M\bigr)\\
&= \dim_{S^{-1}kG}\bigl(S^{-1}kG \otimes_{kG} kG \otimes_{kH} N\bigr)\\
&= \dim_{S^{-1}kG}\bigl(S^{-1}kG \otimes_{T^{-1}kH} T^{-1}kH \otimes_{kH} N\bigr)\\
&= \dim_{T^{-1}kH}\bigl(T^{-1}kH \otimes_{kH} N\bigr)\\
&= \dim_{kH}^{\mathrm{Ore}}(N).
\end{aligned}$$

We compute

$$\begin{aligned}
\frac{\dim_k\bigl(k \otimes_{kG_n} M\bigr)}{[G:G_n]} &= \frac{\dim_k\bigl(k[G/G_n] \otimes_{kG} M\bigr)}{[G:G_n]}\\
&= \frac{\dim_k\bigl(k[G/G_n] \otimes_{kG} kG \otimes_{kH} N\bigr)}{[G:G_n]}\\
&= \frac{\dim_k\bigl(k[G/G_n] \otimes_{k[H/H_n]} k[H/H_n] \otimes_{kH} N\bigr)}{[G:G_n]}\\
&= \frac{[G/G_n : H/H_n] \cdot \dim_k\bigl(k[H/H_n] \otimes_{kH} N\bigr)}{[G:G_n]}\\
&= \frac{[G/G_n : H/H_n] \cdot \dim_k\bigl(k \otimes_{kH_n} N\bigr)}{[G/G_n : H/H_n] \cdot [H:H_n]}\\
&= \frac{\dim_k\bigl(k \otimes_{kH_n} N\bigr)}{[H:H_n]}.
\end{aligned}$$

Therefore the claim holds for M over kG if it holds for N over kH. Hence we can assume without loss of generality that G is finitely generated.

Now apply Theorem 2.1 and Theorem 3.4.

(ii) We obtain from additivity, the exactness of the functor $S^{-1}kG \otimes_{kG} \_$ and the right exactness of the functor $k \otimes_{kG} \_$ that

$$\begin{aligned}
\dim_{kG}^{\mathrm{Ore}}\bigl(H_i(C_*)\bigr) &= \dim_{kG}^{\mathrm{Ore}}\bigl(\operatorname{coker}(c_{i+1})\bigr) + \dim_{kG}^{\mathrm{Ore}}\bigl(\operatorname{coker}(c_i)\bigr) - \dim_{kG}^{\mathrm{Ore}}\bigl(C_{i-1}\bigr),\\
\dim_k\bigl(H_i(k \otimes_{kG_n} C_*)\bigr) &= \dim_k\bigl(k \otimes_{kG_n} \operatorname{coker}(c_{i+1})\bigr) + \dim_k\bigl(k \otimes_{kG_n} \operatorname{coker}(c_i)\bigr)\\
&\quad - \dim_k\bigl(k \otimes_{kG_n} C_{i-1}\bigr).
\end{aligned}$$

Hence the claim follows from assertion (i) applied to the finitely presented kG-modules $\operatorname{coker}(c_{i+1})$, $\operatorname{coker}(c_i)$ and C_{i-1}.

(iii) This follows from assertion (ii) applied to the cellular chain complex of X. □

5. Extension to the virtually torsionfree case

Next we explain how Theorem 0.2 can be extended to the virtually torsionfree case.

For the remainder of this section let k be a skew field, let G be an amenable group which possesses a subgroup H of finite index with $\Delta^+(H) = 1$, and let $k * G$ be a crossed product such that $k*H$ is a left Goldie ring. We define the *virtual Ore dimension* of a $k*G$-module M by

$$\operatorname{vdim}_{k*G}^{\mathrm{Ore}}(M) = \frac{\dim_{k*H}^{\mathrm{Ore}}\bigl(\operatorname{res}_{k*G}^{k*H} M\bigr)}{[G:H]}, \tag{5.1}$$

where $\operatorname{res}_{k*G}^{k*H} M$ is the $k * H$-module obtained from the $k * G$-module M by restricting the G-action to H.

We have to show that this is independent of the choice of H. Since every subgroup of finite index contains a normal subgroup of finite index, it is enough to show that if K is a normal subgroup of finite index in H and $K \le H \le G$ with H torsion free, then for every $k * H$-module N,

$$\frac{\dim_{k*K}^{\mathrm{Ore}}\bigl(\operatorname{res}_{k*H}^{k*K} N\bigr)}{[H:K]} = \dim_{k*H}^{\mathrm{Ore}}(N). \tag{5.2}$$

Let T denote the set of non-zero-divisors of $k * H$ and write $S = (k * K) \cap T$. Note that $\Delta^+(K) = 1$, so $k * K$ is still a prime left Goldie ring and hence the ring $S^{-1}k * K$ exists. Then $S^{-1}k * H \cong (S^{-1}k * K) * [H/K]$ and there is a natural ring monomorphism $\theta\colon S^{-1}k * H \hookrightarrow T^{-1}k * H$. Since $S^{-1}k * K$ is a matrix ring over a skew field by Theorem 1.5, we see that $(S^{-1}k*K)[H/K]$ is an Artinian ring, because H/K is finite. But every element of T is a non-zero-divisor in $(S^{-1}k * K)[H/K]$, and since every non-zero-divisor in an Artinian ring is invertible (compare [16, Exercise 22 of Chapter 15 on p. 16]), we see that every element of T is invertible in $S^{-1}k * H$ and we conclude that θ is onto and hence is an isomorphism. We deduce that $\dim_{S^{-1}k*K}(T^{-1}k * H) = [H : K]$ and that the natural map $S^{-1}N \to T^{-1}N$ induced by $s^{-1}n \mapsto s^{-1}n$ is an isomorphism. This proves (5.2).

Theorem 5.3 (Extension to the virtually torsionfree case)**.** *Let G be an amenable group which possesses a subgroup $E \subseteq G$ of finite index such that kE is left Goldie and $\Delta^+(E) = 1$, and let k be a skew field. Then assertions (i), (ii) and (iii) of Theorem 0.2 remain true, provided we replace $\dim_{kG}^{\mathrm{Ore}}$ by $\operatorname{vdim}_{kG}^{\mathrm{Ore}}$ everywhere.*

Proof. It suffices to prove the claim for assertion (i) since the proof in Theorem 0.2 that it implies the other two assertions applies also in this more general situation. Let $(G_n)_{n\ge 0}$ be a residual chain of G. To prove the result in general, we may assume that G is finitely generated. Since kE is left Goldie and $[G : E] < \infty$, the group ring kG is also left Goldie. Further, every kG_n is left Goldie. Since $\Delta^+(G)$ is finite (its order is bounded by $[G : E]$), there exists $N \in \mathbb{N}$ such that $G_N \cap \Delta^+(G) = 1$, and then $\Delta^+(G_N) = 1$ and $G_i \subseteq G_N$ for all $i \ge N$. Set $H = G_N$, so that kH is prime by Lemma 1.4. Then for a finitely presented kH-module L,

$$\dim_{kH}^{\mathrm{Ore}}(L) = \lim_{n\to\infty} \frac{\dim_k\bigl(k \otimes_{k[G_n\cap H]} L\bigr)}{[H : H \cap G_n]}$$

by Theorems 2.1 and 3.1. We have $[G:G_n] = [G:H]\cdot[H:H\cap G_n]$ for $n \geq N$. This implies for every finitely presented kG-module M

$$\begin{aligned}\operatorname{vdim}^{\mathrm{Ore}}_{kG}(M) = \frac{\dim^{\mathrm{Ore}}_{kH}\bigl(\operatorname{res}^{kH}_{kG} M\bigr)}{[G:H]} &= \lim_{n\to\infty} \frac{\dim_k\bigl(k \otimes_{k[H\cap G_n]} \operatorname{res}^{kH}_{kG} M\bigr)}{[G:H]\cdot[H:H\cap G_n]} \\ &= \lim_{n\to\infty} \frac{\dim_k\bigl(k\otimes_{kG_n} M\bigr)}{[G:G_n]}. \qquad \square\end{aligned}$$

Remark 5.4. Because of Theorem 1.6, Theorem 0.2 is true in the case k is a skew field and G is an elementary amenable group in which the orders of the finite subgroups are bounded (clearly $\Delta^+(G_n) = 1$ for sufficiently large n). In particular Theorem 0.2 is true for any virtually torsionfree elementary amenable group.

6. EXAMPLES

Remark 6.1. Let $(G_n)_{n\geq 0}$ be a residual chain of a group G. Let X be a finite free G-CW-complex. Let k be a field of characteristic $\operatorname{char}(k)$. For a prime p denote by $\mathbb{F}_p$ the field of p elements. Then we conclude from the universal coefficient theorem

$$\begin{aligned}\dim_k\bigl(H_i(G_n\backslash X;k)\bigr) &= \dim_{\mathbb{Q}}\bigl(H_i(G_n\backslash X;\mathbb{Q})\bigr) && \operatorname{char}(k)=0;\\ \dim_k\bigl(H_i(G_n\backslash X;k)\bigr) &= \dim_{\mathbb{F}_p}\bigl(H_i(G_n\backslash X;\mathbb{F}_p)\bigr) && p=\operatorname{char}(k)\neq 0;\\ \dim_{\mathbb{F}_p}\bigl(H_i(G_n\backslash X;\mathbb{F}_p)\bigr) &\geq \dim_{\mathbb{Q}}\bigl(H_i(G_n\backslash X;\mathbb{Q})\bigr).\end{aligned}$$

In particular we conclude from Remark 0.3 that

$$\liminf_{n\to\infty} \frac{\dim_k\bigl(H_i(G_n\backslash X;k)\bigr)}{[G:G_n]} \geq \lim_{n\to\infty}\frac{\dim_k\bigl(H_i(G_n\backslash X;\mathbb{Q})\bigr)}{[G:G_n]} = b_i^{(2)}(X;\mathcal{N}(G)),$$

where the latter term denotes the i-th L^2-Betti number of X. In particular we get from Theorem 0.2 for a torsionfree amenable group G with no zero-divisors in kG that

$$\begin{aligned}\dim^{\mathrm{Ore}}_{kG}\bigl(H_i(X;k)\bigr) = \lim_{n\to\infty}\frac{\dim_k(H_i(G_n\backslash X;k))}{[G:G_n]} &\geq \lim_{n\to\infty}\frac{\dim_k(H_i(G_n\backslash X;\mathbb{Q}))}{[G:G_n]}\\ &= b_i^{(2)}(X;\mathcal{N}(G))\\ &= \dim^{\mathrm{Ore}}_{\mathbb{C}G}\bigl(H_i(X;\mathbb{C})\bigr).\end{aligned}$$

This inequality is in general not an equality as the next example shows.

Example 6.2. Fix an integer $d \geq 2$ and a prime number p. Let $f_p\colon S^d \to S^d$ be a map of degree p and denote by $i\colon S^d \to S^1\vee S^d$ the obvious inclusion. Let X be the finite CW-complex obtained from $S^1 \vee S^d$ by attaching a $(d+1)$-cell with attaching map $i\circ f^d\colon S^d \to S^1\vee S^d$. Then $\pi_1(X) = \mathbb{Z}$. Let $\widetilde{X}$ be the universal covering of X which is a finite free $\mathbb{Z}$-CW-complex. Denote by X_n the covering of X associated to $n\cdot\mathbb{Z}\subseteq\mathbb{Z}$. The cellular $\mathbb{Z}C$-chain complex of $\widetilde{X}$ is concentrated in dimension $(d+1)$, d and 1 and 0, the $(d+1)$-th differential is multiplication with p and the first differential is multiplication with $(z-1)$ for a generator $z\in\mathbb{Z}$

$$0 \to \cdots \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{p} \mathbb{Z}[\mathbb{Z}] \to \cdots \to \mathbb{Z}[\mathbb{Z}] \xrightarrow{z-1} \mathbb{Z}[\mathbb{Z}].$$

If the characteristic of k is different from p, one easily checks that $H_i(C_*) = 0$ and

$$\dim^{\mathrm{Ore}}_{k\mathbb{Z}}\bigl(H_i(\widetilde{X};k)\bigr) = 0 \quad\text{for } i\in\{d,d+1\}.$$

If p is the characteristic of k, then $H_i(C_*) = k\mathbb{Z}$ and

$$\dim_{k\mathbb{Z}}^{\mathrm{Ore}}\big(H_i(\widetilde{X};k)\big) = 1 \quad \text{for } i \in \{d, d+1\}.$$

Hence $\dim_{kG}^{\mathrm{Ore}}\big(H_i(\widetilde{X};k)\big)$ does depend on k in general.

Example 6.3. Let G be a torsionfree amenable group such that kG has no zero-divisors. Let $S^1 \to X \to B$ be a fibration of connected CW-complexes such that X has fundamental group $\pi_1(X) \cong G$ and $\pi_1(S^1) \to \pi_1(X)$ is injective. Then

$$\dim_{kG}^{\mathrm{Ore}}\big(H_i(\widetilde{X};k)\big) = 0 \tag{6.4}$$

for every $i \geq 0$.

Let $S = kG - \{0\}$ and $S_0 = k\mathbb{Z} - \{0\}$. By looking at the cellular chain complex one directly sees that

$$H_i\big(\widetilde{S}^1, S_0^{-1}k\mathbb{Z}\big) = 0 \ \forall i \geq 0,$$

thus $H_i\big(\widetilde{S}^1, S^{-1}kG\big) = S^{-1}kG \otimes_{S_0^{-1}k\mathbb{Z}} H_i\big(\widetilde{S}^1, S_0^{-1}k\mathbb{Z}\big) = 0$ for every $i \geq 0$. The assertion is implied by the Hochschild-Serre spectral sequence that converges to $H_{p+q}(\widetilde{X}, S^{-1}kG)$ and has the E^2-term:

$$E^2_{pq} = H_p\big(\widetilde{B}, H_q(\widetilde{S}^1, S^{-1}kG)\big).$$

Example 6.5 (Sublinear growth of Betti numbers)**.** Let G be an infinite amenable group which possesses a subgroup H of finite index such that kH is left Goldie and $\Delta^+(H) = 1$, e.g., G is a virtually torsionfree elementary amenable group. Let k be a field. Let $(G_n)_{n\geq 0}$ be a residual chain of G. Denote by $b_i(G/G_n;K)$ the i-th Betti number of the group G/G_n with coefficients in k. Then we get for every $i \geq 0$

$$\lim_{n\to\infty} \frac{b_i(G/G_n;k)}{[G:G_n]} = 0.$$

For $i = 0$ this is obvious. For $i \geq 1$ this follows from Theorem 5.3 and $H_i(EH;k) = H_i(H;k) = 0$.

REFERENCES

[1] M. Abert, A. Jaikin-Zapirain, and N. Nikolov, *The rank gradient from a combinatorial viewpoint*, 2007. arXiv:math/0701925v2.

[2] C. Deninger and K. Schmidt, *Expansive algebraic actions of discrete residually finite amenable groups and their entropy*, Ergodic Theory Dynam. Systems **27** (2007), no. 3, 769–786. MR2322178 (2008d:37009)

[3] J. Dodziuk, P. Linnell, V. Mathai, T. Schick, and S. Yates, *Approximating L^2-invariants and the Atiyah conjecture*, Comm. Pure Appl. Math. **56** (2003), no. 7, 839–873. Dedicated to the memory of Jürgen K. Moser. MR1990479 (2004g:58040)

[4] G. Elek, *Amenable groups, topological entropy and Betti numbers*, Israel J. Math. **132** (2002), 315–335. MR1952628 (2003k:37026)

[5] ———, *The rank of finitely generated modules over group algebras*, Proc. Amer. Math. Soc. **131** (2003), no. 11, 3477–3485 (electronic). MR1991759 (2004i:43003)

[6] ———, *The strong approximation conjecture holds for amenable groups*, J. Funct. Anal. **239** (2006), no. 1, 345–355. MR2258227 (2007m:43001)

[7] P. H. Kropholler, P. A. Linnell, and J. A. Moody, *Applications of a new K-theoretic theorem to soluble group rings*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 675–684. MR89j:16016

[8] S. Lang, *Algebra*, third, Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR1878556 (2003e:00003)

[9] E. Lindenstrauss and B. Weiss, *Mean topological dimension*, Israel J. Math. **115** (2000), 1–24. MR1749670 (2000m:37018)

[10] P. A. Linnell, *Noncommutative localization in group rings*, Non-commutative localization in algebra and topology, 2006, pp. 40–59. MR2222481 (2007f:16064)

[11] W. Lück, *Approximating L^2-invariants by their finite-dimensional analogues*, Geom. Funct. Anal. **4** (1994), no. 4, 455–481. MR95g:58234

[12] ———, *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002. MR1926 649

[13] D. Pape, *A short proof of the approximation conjecture for amenable groups*, J. Funct. Anal. **255** (2008), no. 5, 1102–1106. MR2455493 (2010a:46171)

[14] D. S. Passman, *The algebraic structure of group rings*, Wiley-Interscience [John Wiley & Sons], New York, 1977. Pure and Applied Mathematics. MR81d:16001

[15] ———, *Group rings, crossed products and Galois theory*, CBMS Regional Conference Series in Mathematics, vol. 64, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986. MR840467 (87e:16033)

[16] L. H. Rowen, *Graduate algebra: noncommutative view*, Graduate Studies in Mathematics, vol. 91, American Mathematical Society, Providence, RI, 2008. MR2462400 (2009k:16001)

[17] B. Stenström, *Rings of quotients*, Springer-Verlag, New York, 1975. Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory. MR52#10782

[18] B. Weiss, *Monotileable amenable groups*, Topology, ergodic theory, real algebraic geometry, 2001, pp. 257–262. MR1819193 (2001m:22014)

Department of Mathematics, Virginia Tech, Blacksburg, VA 24061-0123,, USA
E-mail address: `plinnell@math.vt.edu`
URL: `http://www.math.vt.edu/people/plinnell/`

Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany
E-mail address: `lueck@math.uni-muenster.de`
URL: `http://www.math.uni-muenster.de/u/lueck`

Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Einsteinstr. 62, D-48149 Münster, Germany
E-mail address: `sauerr@uni-muenster.de`
URL: `http://wwwmath.uni-muenster.de/u/sauerr/`