

Groups of small homological dimension and the Atiyah Conjecture

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Abstract

A group has homological dimension ≤ 1 if it is locally free. We prove the converse provided that G satisfies the Atiyah Conjecture about L^2 -Betti numbers. We also show that a finitely generated elementary amenable group G of cohomological dimension ≤ 2 possesses a finite 2-dimensional model for BG and in particular that G is finitely presented and the trivial $\mathbb{Z}G$ -module \mathbb{Z} has a 2-dimensional resolution by finitely generated free $\mathbb{Z}G$ -modules.

Key words: (co-)homological dimension, von Neumann dimension, Atiyah Conjecture.

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1 Notation

Throughout this paper let G be a (discrete) group. It has *homological dimension* $\leq n$ if $H_p(G; M) = \mathrm{Tor}_p^{\mathbb{Z}G}(\mathbb{Z}, M)$ vanishes for each $\mathbb{Z}G$ -module M and each $p > n$. It has *cohomological dimension* $\leq n$ if $H^p(G; M) = \mathrm{Ext}_{\mathbb{Z}G}^p(\mathbb{Z}, M)$ vanishes for each $\mathbb{Z}G$ -module M and each $p > n$.

We call G *locally free* if each finitely generated subgroup is free. The *class of elementary amenable groups* is defined as the smallest class of groups, which contains all finite and all abelian groups and is closed under taking subgroups, taking quotient groups, extensions and directed unions. Each elementary amenable group is amenable, but the converse is not true.

2 Review of the Atiyah Conjecture

Denote by $\mathcal{N}(G)$ the group von Neumann algebra associated to G which we will view as a ring (not taking the topology into account) throughout this paper. For a $\mathcal{N}(G)$ -module M let $\dim_{\mathcal{N}(G)}(M) \in [0, \infty]$ be its dimension in the sense of [8, Theorem 6.7]. Let $\frac{1}{\mathcal{FIN}(G)}\mathbb{Z} \subseteq \mathbb{Q}$ be the additive abelian subgroup of \mathbb{Q} generated by the inverses $|H|^{-1}$ of the orders $|H|$ of finite subgroups H of G . Notice that $\frac{1}{\mathcal{FIN}(G)}\mathbb{Z}$ agrees with \mathbb{Z} if and only if G is torsion-free.

Conjecture 1 (Atiyah Conjecture). *Consider a ring A with $\mathbb{Z} \subseteq A \subseteq \mathbb{C}$. The Atiyah Conjecture for A and G says that for each finitely presented AG -module M we have*

$$\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} M) \in \frac{1}{\mathcal{FLN}(G)}\mathbb{Z}.$$

For a discussion of this conjecture and the classes of groups for which it is known we refer for instance to [8, Section 10.1]. It is not clear whether the Atiyah conjecture is subgroup closed; however in the case G is torsion-free, then it certainly is. This can be seen from [8, Theorem 6.29(2)]. We mention Linnell's result [6] that the Atiyah Conjecture is true for $A = \mathbb{C}$ and all groups G which can be written as an extension with a free group as kernel and an elementary amenable group as quotient and possess an upper bound on the orders of its finite subgroups. The Atiyah Conjecture has also been proved by Schick [9] for $A = \mathbb{Q}$ and torsion-free groups G which are residually torsion-free elementary amenable.

3 Results

Theorem 2. *A locally free group G has homological dimension ≤ 1 .*

If G is a group of homological dimension ≤ 1 and the Atiyah Conjecture 1 holds for G , then G is locally free.

Theorem 3. *Let G be an elementary amenable group of cohomological dimension ≤ 2 . Then*

1. *Suppose that G is finitely generated. Then G possesses a presentation of the form*

$$\langle x, y \mid yxy^{-1} = x^n \rangle.$$

In particular there is a finite 2-dimensional model for BG and the trivial $\mathbb{Z}G$ -module \mathbb{Z} possesses a 2-dimensional resolution by finitely generated free $\mathbb{Z}G$ -modules;

2. *Suppose that G is countable but not finitely generated. Then G is a non-cyclic subgroup of the additive group \mathbb{Q} .*

4 Proofs

Lemma 4. *Let A be a ring with $\mathbb{Z} \subset A \subset \mathbb{C}$. Let P be a projective AG -module such that for some finitely generated AG -submodule $M \subset P$ we have $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} P/M) = 0$. Then P is finitely generated.*

Proof: Choose a free AG -module F and AG -maps $i: P \rightarrow F$ and $r: F \rightarrow P$ with $r \circ i = \text{id}$. Since $M \subset P$ is finitely generated, there is a finitely generated free direct summand $F_0 \subset F$ with $i(M) \subset F_0$ and $F_1 := F/F_0$ a free AG -module.

Hence i induces a map $f: P/M \rightarrow F_1$. It suffices to show that f is trivial because then $i(P) \subset F_0$ and the restriction of r to F_0 yields an epimorphism $F_0 \rightarrow P$.

Let $g: AG \rightarrow P/M$ be any AG -map. The map $\mathcal{N}(G) \otimes_{AG} (f \circ g)$ factorizes through $\mathcal{N}(G) \otimes_{AG} P/M$. Hence its image has von Neumann dimension zero because $\dim_{\mathcal{N}(G)}$ is additive [8, Theorem 6.7] and $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} P/M) = 0$ holds by assumption. Since the von Neumann algebra $\mathcal{N}(G)$ is semi-hereditary (see [8, Theorem 6.5 and Theorem 6.7]), the image of $\mathcal{N}(G) \otimes_{AG} (f \circ g)$ is a finitely generated projective $\mathcal{N}(G)$ -module, whose von Neumann dimension is zero, and hence is the zero-module. Therefore $\mathcal{N}(G) \otimes_{AG} (f \circ g)$ is the zero map. Since $AG \rightarrow \mathcal{N}(G)$ is injective, $f \circ g$ is trivial. This implies that f is trivial since g is any AG -map. \square

Lemma 5. *Let A be a ring with $\mathbb{Z} \subset A \subset \mathbb{C}$. Suppose that there is a positive integer d such that the order of any finite subgroup of G divides d and that the Atiyah Conjecture holds for A and G . Let N be a AG -module. Suppose that $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} N) < \infty$. Then there is a finitely generated AG -submodule $M \subset N$ with $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} N/M) = 0$.*

Proof: Since N is the colimit of the directed system of its finitely generated AG -modules $\{M_i \mid i \in I\}$ and tensor products commute with colimits, we get $\operatorname{colim}_{i \in I} \mathcal{N}(G) \otimes_{AG} N/M_i = 0$. Additivity (see [8, Theorem 6.7]) implies $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} N/M_i) < \infty$ for all $i \in I$ since $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} N) < \infty$ holds by assumption. We conclude from Additivity and Cofinality (see [8, Theorem 6.7]) and the fact that the functor colimit over a directed system of modules is exact

$$\inf\{\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} N/M_i) \mid i \in I\} = 0.$$

The assumption about G implies using [8, Lemma 10.10 (4)]

$$d \cdot \dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} N/M_i) \in \mathbb{Z}.$$

Hence there must be an index $i \in I$ with $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{AG} N/M_i) = 0$. \square

Proof of Theorem 2: A finitely generated free group has obviously homological dimension ≤ 1 . Since homology is compatible with colimits over directed systems (in contrast to cohomology), we get for every group G , which is the directed union of the family of subgroups $\{G_i \mid i \in I\}$, and every $\mathbb{Z}G$ -module M

$$H_n(G; M) = \operatorname{colim}_{i \in I} H_n(G_i; \operatorname{res}_i M),$$

where $\operatorname{res}_i M$ is the restriction of M to a $\mathbb{Z}G_i$ -module. Hence any locally free group has homological dimension ≤ 1 .

Suppose that G has homological dimension ≤ 1 . Let $H \subset G$ be a finitely generated subgroup. Then the homological dimension of H is ≤ 1 . Since each countably presented flat module is of projective dimension ≤ 1 [1, Lemma 4.4], we conclude that the cohomological dimension of H is ≤ 2 . We can choose an exact sequence $0 \rightarrow P \rightarrow \mathbb{Z}H^s \rightarrow \mathbb{Z}H \rightarrow \mathbb{Z}$, where s is the number

of generators and P is projective. Since the homological dimension is ≤ 1 , the induced map $\mathcal{N}(H) \otimes_{\mathcal{N}(H)} P \rightarrow \mathcal{N}(H) \otimes_{\mathcal{N}(H)} \mathbb{Z}H^s$ is injective and hence $\dim_{\mathcal{N}(H)}(\mathcal{N}(H) \otimes_{\mathcal{N}(H)} P) \leq \dim_{\mathcal{N}(H)}(\mathcal{N}(H) \otimes_{\mathcal{N}(H)} \mathbb{Z}H^s) = s$. Suppose that G satisfies the Atiyah Conjecture. Since G cannot contain a non-trivial finite subgroup, H also satisfies the Atiyah Conjecture, and Lemma 4 and Lemma 5 imply that P is finitely generated. Hence H is of type FP . Since each finitely presented flat module is projective [1, Lemma 4.4], the cohomological dimension and the homological dimension agree for groups of type FP . Hence H has cohomological dimension 1. A result of Stallings [10] implies that H is free. \square

In [3] the notion of Hirsch length for an elementary amenable group was defined, generalizing that of the Hirsch length of a solvable group. This was used in the proof of [5, Corollary 2] to show that an elementary amenable group of finite cohomological dimension is virtually solvable with finite Hirsch number, see [4, Theorem 1.11] for further details. We can now state

Lemma 6. *If G is an elementary amenable group of homological dimension ≤ 2 , then G is metabelian.*

Proof: A group is metabelian if and only if each finitely generated subgroup is metabelian. Hence we can assume without loss of generality that G is finitely generated. Then by the above remarks and [1, Theorem 7.10(a)], G is virtually solvable of Hirsch length ≤ 2 .

If G has Hirsch length 1, then G is infinite cyclic, so we may assume that G has Hirsch length 2. Let N denote the Fitting subgroup of G (so N is generated by the nilpotent normal subgroups of G and is a locally nilpotent normal subgroup).

Suppose that N has finite index in G . Then N is finitely generated and is therefore free abelian of rank 2. Also G/N acts faithfully by conjugation on N (a torsion-free group with a central subgroup of finite index must be abelian). If $g \in G \setminus N$, then $g^r \in N \setminus 1$ for some positive integer r and thus g fixes a nonidentity element of N . We deduce that $|G/N| \leq 2$ and it follows that G is metabelian.

On the other hand if N has infinite index in G , then it has Hirsch length 1. Hence every finitely generated subgroup of N is trivial or isomorphic to \mathbb{Z} . This implies that N is abelian and any automorphism of finite order $f: N \rightarrow N$ has the property that $f(x) \in \{x, -x\}$ holds for $x \in N$. Since the group G/N acts faithfully by conjugation on N and is virtually cyclic, we conclude that G/N is isomorphic to \mathbb{Z} or $\mathbb{Z} \times \mathbb{Z}/2$. Hence G is metabelian. \square

Proof of Theorem 3: Since G has cohomological dimension 2, it certainly has homological dimension at most 2 and so by Lemma 6 is metabelian. A result of Gildenhuys [2, Theorem 5], states that a solvable group G of cohomological dimension 2 has a presentation of the form $\langle x, y; y^{-1}xy = x^n \rangle$ for some $n \in \mathbb{Z}$ if G is finitely generated and is a non-cyclic subgroup of the additive group \mathbb{Q} if G is not finitely generated. Given a torsion free finitely generated one-relator group G , the finite two-dimensional CW -complex associated to a presentation

with finitely many generators and one non-trivial relation is a model for BG (see [7, Chapter III §§9 -11]). This finishes the proof of Theorem 3. \square

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