The flow space associated to a CAT(0)-space (Lecture IV)

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Outline

- We introduce CAT(0)-spaces and CAT(0)-groups and state their main properties.
- We construct the flow space $\operatorname{FS}(X)$ associated to a CAT(0)-space and collect its main properties.
- We discuss the main flow estimate.

CAT(0)-spaces

Definition (CAT(0)-space)

A CAT(0)-space or Hadamard space is a geodesic complete metric space (X, d_X) such that any geodesic triangle Δ in X satisfies the CAT(0)-inequality

$$d_X(x, y) \le d_{\mathbb{R}^2}(\overline{x}, \overline{y})$$

for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y}$ in the comparison triangle $\overline{\Delta} \subseteq \mathbb{R}^2$.

- A metric space X is called geodesic if for any two points $x, y \in X$ there exists a geodesic segment joining x and y, i.e., an isometric embedding $c\colon [0, d_X(x, y)] \to X$ with $c(0) = x$ and $c(d_X(x, y)) = y$.

- A geodesic triangle Δ in X consists of three points p, q, r and a choice of geodesic segments $[p, q]$, $[p, r]$ and $[q, r]$.
- A comparison triangle $\overline{\Delta}$ for geodesic triangle Δ is a geodesic triangle $\overline{\Delta} \subseteq \mathbb{R}^2$ given by three points $\overline{p}, \overline{q}$ and $\overline{r}$ such that $d_X(p, q) = d_{\mathbb{R}^2}(\overline{p}, \overline{q})$, $d_X(p, r) = d_{\mathbb{R}^2}(\overline{p}, \overline{r})$, and $d_X(q, r) = d_{\mathbb{R}^2}(\overline{q}, \overline{r})$.
- If x belongs to the segment $[p, q]$, then its comparison point $\overline{x}$ is the point on the geodesic $[\overline{p}, \overline{q}]$ uniquely determined by $d_X(p, x) = d_{\mathbb{R}^2}(\overline{p}, \overline{x})$ and $d_X(x, q) = d_{\mathbb{R}^2}(\overline{x}, \overline{q})$.
- A simply connected complete Riemannian manifold with non-positive sectional curvature is a CAT(0)-space.

- There is a unique geodesic segment joining each pairs of points and this geodesic segment various continuously with its endpoints.
- If X is a CAT(0)-space, then X and every open ball and every closed ball in X are contractible.

Definition (Generalized geodesic)

Let (X, d_X) be a metric space. A continuous map $c\colon \mathbb{R} \to X$ is called a generalized geodesic if there are $c_-, c_+ \in \overline{\mathbb{R}} := \mathbb{R} \coprod \{-\infty, \infty\}$ satisfying

$$c_- \leq c_+, \quad c_- \neq \infty, \quad c_+ \neq -\infty,$$

such that c is locally constant on the complement of the interval $I_c := (c_-, c_+)$ and restricts to an isometry on I_c.

Definition (Boundary of a metric space)

Let X be a metric space. Two geodesic rays $c, c' \colon [0, \infty) \to X$ are called asymptotic if there exists a constant K with $d_X(c(t), c'(t)) \leq K$ for all $t \in [0, \infty)$. The boundary ∂X of X is the set of asymptotic equivalence classes of rays. Denote by $\overline{X} = X \amalg \partial X$ the disjoint union of X and ∂X.

Lemma

Let X be a $\mathrm{CAT}(0)$*-space and $c \colon [0, \infty) \to X$ be a geodesic ray. Then for every $x' \in X$ there is a unique geodesic ray $c' \colon [0, \infty) \to X$ with $c'(0) = x'$ such that c and c' are asymptotic.*

- In contrast to hyperbolic spaces it is in general not true for a $\mathrm{CAT}(0)$- space that for two distinct elements $y, z \in \partial X$ there exists a geodesic $c \colon \mathbb{R} \to X$ joining y and z.

- A generalized geodesic ray is a generalized geodesic c that is either a constant generalized geodesic or a non-constant generalized geodesic with $c_- = 0$.
- Fix a base point $x_0 \in X$ in the CAT(0)-space X. For every $x \in \overline{X}$, there is a unique generalized geodesic ray c_x such that $c(0) = x_0$ and $c(\infty) = x$. Define for $r > 0$ the canonical projection

$$\rho_r = \rho_{r,x_0} : \overline{X} \to \overline{B}_r(x_0)$$

by $\rho_r(x) := c_x(r)$.

Definition (Cone topology on $\overline{X}$.)

Let X be a CAT(0)-space. The sets $(\rho_r)^{-1}(V)$ with $r > 0$, V an open subset of $\overline{B}_r(x_0)$ are a basis for the cone topology on $\overline{X}$.

- The cone topology is independent of the choice of base point.
- A map f whose target is $\overline{X}$ is continuous if and only if $\rho_r \circ f$ is continuous for all r.
- $\overline{X}$ is a compact metrizable space.
- $\partial X \subseteq \overline{X}$ is closed and $X \subseteq \overline{X}$ is dense.
- The inclusion $X \to \overline{X}$ is a homeomorphism onto its image which is an open subset.
- If M is a simply connected complete n-dimensional Riemannian manifold with non-positive sectional curvature, then ∂M is S^{n-1}.
- There are closed topological manifolds M constructed by Davis-Januszkiewicz(1991) such that the universal covering $\widetilde{M}$ admits a $\pi_1(M)$-invariant CAT(0)-metric and $\partial\widetilde{M}$ is not homeomorphic to a sphere and $\widetilde{M}$ is not homeomorphic to $\mathbb{R}^n$.

CAT(0)-groups

Definition (CAT(0)-group)

A (discrete) group G is called a CAT(0)-group if it acts properly cocompactly and isometrically on a CAT(0)-space of finite topological dimension.

A CAT(0)-group G satisfies:

- There exists a finite model $\underline{E}G$.
- There is a model for BG of finite type;
- G is finitely presented;
- There are only finitely many conjugacy classes of finite subgroups;
- Every solvable subgroup is virtually $\mathbb{Z}^n$;
- The direct product of two CAT(0)-groups is again a CAT(0)-group;

- Limit groups in the sense of Sela are CAT(0)-groups;
- Coxeter groups are CAT(0)-groups;
- The word-problem and the conjugation-problem are solvable.

Question

Is every hyperbolic group a CAT(0)*-group?*

The flow space of a metric space

- Throughout this section let (X, d_X) be a metric space.

Definition (Flow space)

- Let $\mathrm{FS} = \mathrm{FS}(X)$ be the set of all generalized geodesics in X;
- We define a metric on $\mathrm{FS}(X)$ by

$$d_{\mathrm{FS}(X)}(c, d) := \int_{\mathbb{R}} \frac{d_X(c(t), d(t))}{2e^{|t|}} \, dt.$$

- Define a flow

$$\Phi \colon \mathrm{FS}(X) \times \mathbb{R} \to \mathrm{FS}(X)$$

by $\Phi_\tau(c)(t) = c(t + \tau)$ for $\tau \in \mathbb{R}$, $c \in \mathrm{FS}(X)$ and $t \in \mathbb{R}$.

Lemma

The map Φ *is a continuous flow and we have for* $c, d \in \mathrm{FS}(X)$ *and* $\tau, \sigma \in \mathbb{R}$

$$d_{\mathrm{FS}(X)}\big(\Phi_\tau(c), \Phi_\sigma(d)\big) \quad \leq \quad e^{|\tau|} \cdot d_{\mathrm{FS}(X)}(c, d) + |\sigma - \tau|.$$

Proof:

- We estimate for $c \in \mathrm{FS}(X)$ and $\tau \in \mathbb{R}$:

$$\begin{aligned} d_{\mathrm{FS}(X)}\big(c, \Phi_\tau(c)\big) &= \int_{\mathbb{R}} \frac{d_X\big(c(t), c(t+\tau)\big)}{2e^{|t|}}\, dt \\ &\leq \int_{\mathbb{R}} \frac{|\tau|}{2e^{|t|}}\, dt \\ &= |\tau| \cdot \int_{\mathbb{R}} \frac{1}{2e^{|t|}}\, dt \\ &= |\tau|. \end{aligned}$$

We estimate for $c, d \in \mathrm{FS}(X)$ and $\tau \in \mathbb{R}$

$$\begin{aligned}
d_{\mathrm{FS}(X)}\big(\Phi_\tau(c), \Phi_\tau(d)\big) &= \int_{\mathbb{R}} \frac{d_X\big(c(t+\tau), d(t+\tau)\big)}{2e^{|t|}}\, dt \\
&= \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t-\tau|}}\, dt \\
&\leq \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t|-|\tau|}}\, dt \\
&= e^{|\tau|} \cdot \int_{\mathbb{R}} \frac{d_X\big(c(t), d(t)\big)}{2e^{|t|}}\, dt \\
&= e^{|\tau|} \cdot d_{\mathrm{FS}(X)}(c, d).
\end{aligned}$$

The two inequalities above together with the triangle inequality imply for $c, d \in \mathrm{FS}(X)$ and $\tau, \sigma \in \mathbb{R}$

$$\begin{aligned} & d_{\mathrm{FS}(X)}\big(\Phi_\tau(c), \Phi_\sigma(d)\big) \\ = \quad & d_{\mathrm{FS}(X)}\big(\Phi_\tau(c), \Phi_{\sigma-\tau} \circ \Phi_\tau(d)\big) \\ \leq \quad & d_{\mathrm{FS}(X)}\big(\Phi_\tau(c), \Phi_\tau(d)\big) + d_{\mathrm{FS}(X)}\big(\Phi_\tau(d), \Phi_{\sigma-\tau} \circ \Phi_\tau(d)\big) \\ \leq \quad & e^{|\tau|} \cdot d_{\mathrm{FS}(X)}(c, d) + |\sigma - \tau|. \end{aligned}$$

□

Lemma

Let $c, d \colon \mathbb{R} \to X$ be generalized geodesics. Consider $t_0 \in \mathbb{R}$.

- $d_X\big(c(t_0), d(t_0)\big) \;\leq\; e^{|t_0|} \cdot d_{\mathrm{FS}}(c,d) + 2;$
- *If* $d_{\mathrm{FS}}(c,d) \leq 2e^{-|t_0|-1}$, *then*

$$d_X\big(c(t_0), d(t_0)\big) \;\leq\; \sqrt{4e^{|t_0|+1}} \cdot \sqrt{d_{\mathrm{FS}}(c,d)}.$$

  In particular, $c \mapsto c(t_0)$ *defines a uniform continuous map* $\mathrm{FS}(X) \to X$.

Proof of the first assertion

- We abbreviate $D := d_X(c(t_0), d(t_0))$.
- We get

$$d_X(c(t), d(t)) \geq D - d_X\big(c(t_0), c(t)\big) - d_X\big(d(t_0), d(t)\big) \geq D - 2 \cdot |t - t_0|.$$

- This implies

$$
\begin{aligned}
d_{\mathrm{FS}(X)}(c,d) &= \int_{-\infty}^{+\infty} \frac{d_X(c(t),d(t))}{2e^{|t|}}\,dt \\
&\geq \int_{-D/2+t_0}^{D/2+t_0} \frac{D-2\cdot|t-t_0|}{2e^{|t|}}\,dt \\
&= \int_{-D/2}^{D/2} \frac{D-2\cdot|t|}{2e^{|t+t_0|}}\,dt \\
&\geq \int_{-D/2}^{D/2} \frac{D-2\cdot|t|}{2e^{|t|+|t_0|}}\,dt \\
&= e^{-|t_0|}\cdot \int_{-D/2}^{D/2} \frac{D-2\cdot|t|}{2e^{|t|}}\,dt \\
&= e^{-|t_0|}\cdot \left(2\cdot e^{-D/2}+D-2\right) \\
&\geq e^{-|t_0|}\cdot (D-2). \quad \square
\end{aligned}
$$

Lemma

The maps

$$\begin{array}{rcll} FS(X) - \mathrm{FS}(X)^{\mathbb{R}} & \to & \overline{\mathbb{R}}, & c \mapsto c_-; \\ FS(X) - \mathrm{FS}(X)^{\mathbb{R}} & \to & \overline{\mathbb{R}}, & c \mapsto c_+, \end{array}$$

are continuous.

Lemma

Let $(c_n)_{n \in \mathbb{N}}$ *be a sequence in* $\mathrm{FS}(X)$. *Then it converges uniformly on compact subsets to* $c \in \mathrm{FS}(X)$ *if and only if it converges to c with respect to* $d_{\mathrm{FS}(X)}$.

Lemma

The flow space $\mathrm{FS}(X)$ *is sequentially closed in the space of all maps* $\mathbb{R} \to X$ *with respect to the topology of uniform convergence on compact subsets.*

Definition (Proper metric space)

A metric space is called *proper* if every closed ball is compact.

Lemma

If (X, d_X) *is a proper metric space, then* $(\mathrm{FS}(X), d_{FS(X)})$ *is a proper metric space.*

Proof:

- Let $R > 0$ and $c \in \mathrm{FS}(X)$.
- It suffices to show that the closed ball $\overline{B}_R(c)$ in $\mathrm{FS}(X)$ is sequentially compact.
- Let $(c_n)_n \in \mathbb{N}$ be a sequence in $\overline{B}_R(c)$. There is $R' > 0$ such that $c_n(0) \in \overline{B}_{R'}(c(0))$. By assumption $\overline{B}_{R'}(c(0))$ is compact.
- Now we can apply the Arzelà-Ascoli Theorem.
- Thus after passing to a subsequence there is $d \colon \mathbb{R} \to X$ such that $c_n \to d$ uniformly on compact subsets.

Lemma

Let (X, d_X) be a proper metric space and $t_0 \in \mathbb{R}$. Then the evaluation map $\mathrm{FS}(X) \to X$ defined by $c \mapsto c(t_0)$ is uniformly continuous and proper.

Proof:

- We have already shown that the map is uniformly continuous
- To show that is is also proper, it suffices to show that preimages of closed balls have finite diameter.
- If $d_X(c(t_0), d(t_0)) \leq r$, then $d_X(c(t), d(t)) \leq r + 2|t - t_0|$. Thus

$$d_{\mathrm{FS}}(c, d) \leq \int_{\mathbb{R}} \frac{r + 2|t - t_0|}{2e^{|t|}} dt,$$

provided $d_X\big(c(t_0), d(t_0)\big) \leq r$. □

Lemma

Let G act isometrically, properly and cocompactly on the proper metric space (X, d_X). Then action of G on $(\mathrm{FS}(X), d_{\mathrm{FS}})$ is also isometric, proper and cocompact.

Proof:

- The action of G on $\mathrm{FS}(X)$ is isometric.
- The map $\mathrm{FS}(X) \to X$ defined by $c \mapsto c(0)$ is G-equivariant, continuous and proper.
- The existence of such a map implies that the G-action on $\mathrm{FS}(X)$ is also proper and cocompact. □

Lemma

The subspace $\mathrm{FS}(X)^{\mathbb{R}}$ is closed in $\mathrm{FS}(X)$.

- Let X be a metric space. For $c \in \mathrm{FS}(X)$ and $T \in [0, \infty]$, define $c|_{[-T,T]} \in \mathrm{FS}(X)$ by

$$c|_{[-T,T]}(t) := \begin{cases} c(-T) & \text{if } t \leq -T; \\ c(t) & \text{if } -T \leq t \leq T; \\ c(T) & \text{if } t \geq T. \end{cases}$$

- We denote by

$$\mathrm{FS}(X)_f := \left\{ c \in \mathrm{FS}(X) - \mathrm{FS}(X)^{\mathbb{R}} \;\middle|\; c_- > -\infty, c_+ < \infty \right\} \cup \mathrm{FS}(X)^{\mathbb{R}}$$

the subspace of finite geodesics.

Lemma

The map

$$H \colon \mathrm{FS}(X) \times [0,1] \to \mathrm{FS}(X)$$

defined by $H_\tau(c) := c|_{[\ln(\tau), -\ln(\tau)]}$ *is continuous and satisfies* $H_0 = \mathrm{id}_{\mathrm{FS}(X)}$ *and* $H_\tau(c) \in \mathrm{FS}(X)_f$ *for* $\tau > 0$.

The flow space of a CAT(0)-space

Example (Flow space of a manifold of non-positive sectional curvature)

- Let M be a simply connected complete Riemannian manifold of non-positive sectional curvature.
- Recall that M is a CAT(0)-space.
- Put

$$\begin{aligned} P &:= \{(a_-, a_+) \in \overline{R} \times \overline{R} \mid a_- < \infty, a_+ > -\infty, a_- \leq a_+\}; \\ \Delta &= \{(a, a) \in \overline{R} \times \overline{R} \mid -\infty < a < \infty\}. \end{aligned}$$

- Define maps

$$\begin{aligned} f\colon STM \times P &\to \mathrm{FS}(M), \quad (v, a_-, a_+) \mapsto c(v)_{[a_-, a_+]}; \\ p\colon STM \times \Delta &\to M, \quad (v, a) \mapsto c_v(a), \end{aligned}$$

where $c(v)\colon \mathbb{R} \to M$ is the geodesic determined by v.

Example (continued)

- The map f is compatible with the obvious flows.
- Then we obtain pushout

$$\begin{array}{ccc} STM \times \Delta & \xrightarrow{i} & STM \times P \\ \downarrow p & & \downarrow f \\ M & \xrightarrow{j} & \mathrm{FS}(M) \end{array}$$

  where i is the inclusion and $j\colon M \to \mathrm{FS}(X)$ sends x to const_x.
- In particular f induces a homeomorphism

$$STM \times (P - \Delta) \xrightarrow{\cong} \mathrm{FS}(M) - \mathrm{FS}(M)^{\mathbb{R}}.$$

Definition (End points of a geodesic)

For $c \in \mathrm{FS}(X)$ we define $c(\infty) \in \overline{X}$ by

$$c(\infty) := \lim_{t \to \infty} c(t) = \begin{cases} c(c_+) & \text{if } c_+ < \infty; \\ [c|_{[0,\infty)}] & \text{if } c_+ = \infty. \end{cases}$$

Define $c(-\infty)$ analogously.

Lemma

The maps

$$\begin{array}{rcll} FS(X) - \mathrm{FS}(X)^{\mathbb{R}} & \to & \overline{X}, & c \mapsto c(-\infty); \\ FS(X) - \mathrm{FS}(X)^{\mathbb{R}} & \to & \overline{X}, & c \mapsto c(\infty), \end{array}$$

are continuous.

- The two maps appearing above cannot be continuosly extended to $\mathrm{FS}(X)$ by the following observation.
- Let c be a generalized geodesic with $c_+ < \infty$ and $c_- = \infty$. Then

$$\begin{array}{rcl}
c(-\infty) & \neq & c(\infty); \\
d_{\mathrm{FS}}(c, \mathrm{const}_{c(\infty)}) & \leq & e^{c_+}/2; \\
\lim_{\tau \to \infty} \Phi_\tau(c) & = & \mathrm{const}_{c(\infty)}; \\
\left(\lim_{\tau \to \infty} \Phi_\tau(c)\right)(-\infty) & = & c(\infty); \\
\Phi_\tau(c)(-\infty) & = & c(-\infty) \quad \text{for all } \tau > 0; \\
\lim_{\tau \to \infty} \left(\Phi_\tau(c)(-\infty)\right) & = & c(-\infty).
\end{array}$$

Theorem (Embedding the flow space)

If X is proper as a metric space, then the map

$$E\colon \operatorname{FS}(X) - \operatorname{FS}(X)^{\mathbb{R}} \to \overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}$$

defined by $E(c) := (c_-, c(-\infty), c(0), c(\infty), c_+)$ is injective and continuous. It is a homeomorphism onto its image.

Lemma

If X is proper as a metric space and its covering dimension $\dim X$ is $\le N$, then $\dim \overline{X} \le N$.

Proof:

- Let $\mathcal{U} = \{U_i \mid i \in I\}$ be an open covering of $\overline{X}$.
- For every $x \in \overline{X}$ there are r_x, $W_x \subseteq \overline{B}_{r_x}(x_0)$ and $U_x \in \mathcal{U}$ such that $x \in \rho_{r_x}^{-1}(W_x) \subset U_x$.
- Since $\overline{X}$ is compact, a finite number of the sets $\rho_{r_x}^{-1}(W_x)$ cover $\overline{X}$.
- Note that $\rho_r = \rho_r|_{\overline{B}_{r'}(x_0)} \circ \rho_{r'}$ and hence $\rho_r^{-1}(W) = \rho_{r'}^{-1}(\rho_r|_{\overline{B}_{r'}(x_0)}(W))$ if $r' \ge r$.

Theorem (Dimension of the flow space)

Assume that X is proper and that $\dim X \leq N$. Then

$$\dim\bigl(FS(X) - FS(X)^{\mathbb{R}}\bigr) \leq 3N + 2.$$

Proof:

- Every compact subset K of $FS(X) - FS(X)^{\mathbb{R}}$ is homeomorphic to a compact subset of $\overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}$.
- Hence its topological dimension satisfies

$$\begin{aligned}\dim(K) &\leq \dim\bigl(\overline{R} \times \overline{X} \times X \times \overline{X} \times \overline{R}\bigr) \\ &= 2\dim(\overline{R}) + 2\dim(\overline{X}) + \dim(X) \leq 3N + 2.\end{aligned}$$

- One shows that $FS(X) - FS(X)^{\mathbb{R}}$ has a countable basis for its topology.
- Now $\dim\bigl(FS(X) - FS(X)^{\mathbb{R}}\bigr) \leq 3N + 2$ follows from standard result of dimension theory.

The homotopy action on $B_r(x_0)$

- The G-action on X induces an G-action on X.
- For technical reasons we will not take the space $\overline{X}$ as the space appearing in the axiomatic approach as we have done it for hyperbolic groups. We will take the closed ball $B_R(x_0)$ for some base point x_0 and some very large real number R.
- The prize to pay is that we do not obtain a G-action on $B_R(x_0)$ but at least the following homotopy G-action.

Definition (The homotopy G-action on $\overline{B}_R(x_0)$)

Define a homotopy G-action (φ^R, H^R) on $\overline{B}_R(x)$ as follows.

- For $g \in G$, we define the map

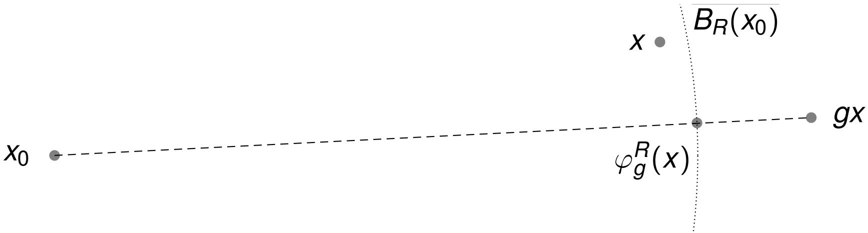
$$\varphi_g^R \colon \overline{B}_R(x_0) \to \overline{B}_R(x_0)$$

  by $\varphi_g^R(x) := \rho_{R,x_0}(gx)$.

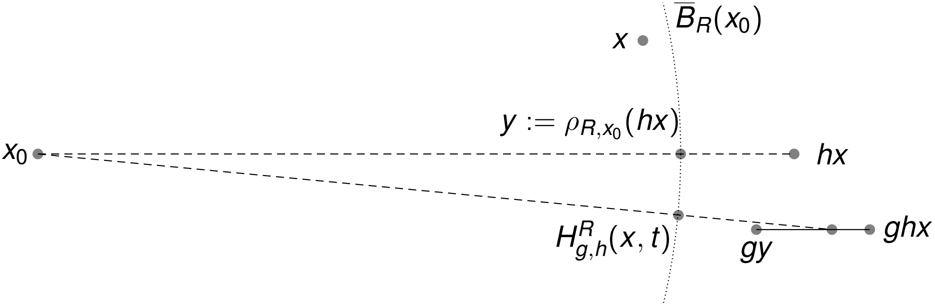
- For $g, h \in G$ we define the homotopy

$$H_{g,h}^R \colon \varphi_g^R \circ \varphi_h^R \simeq \varphi_{gh}^R$$

  by $H_{g,h}^R(x,t) := \rho_{R,x_0}\big(t \cdot (ghx) + (1-t) \cdot (g \cdot \rho_{R,x_0}(hx))\big)$.



$\overline{B_R(x_0)}$
x
gx
x_0
$\varphi_g^R(x)$


$\overline{B}_R(x_0)$
x
$y := \rho_{R,x_0}(hx)$
x_0
hx
$H^R_{g,h}(x,t)$
gy
ghx

- It turns out that the more obvious homotopy given by convex combination $(x, t) \mapsto t \cdot \varphi^R_{gh}(x) + (1-t) \cdot \varphi^R_g \circ \varphi^R_h(x)$ is not appropriate for our purposes.
- Notice that $H^R_{g,h}$ is indeed a homotopy from $\varphi^R_g \circ \varphi^R_h$ to φ_{gh} since

$$\begin{aligned} H^R_{g,h}(x,0) &= \rho_{R,x_0}\big(0 \cdot (ghx) + 1 \cdot (g \cdot \rho_{R,x_0}(hx))\big) \\ &= \rho_{R,x_0}\big(g \cdot \rho_{R,x_0}(hx)\big) \\ &= \varphi^R_g \circ \varphi^R_h(x), \end{aligned}$$

and

$$\begin{aligned} H^R_{g,h}(x,1) &= \rho_{R,x_0}\big(1 \cdot (ghx) + 0 \cdot (g \cdot \rho_{R,x_0}(hx))\big) \\ &= \rho_{R,x_0}(ghx) \\ &= \varphi^R_{gh}(x). \end{aligned}$$

The map ι

Definition (The map ι)

Define the map

$$\iota \colon G \times X \to \mathrm{FS}(X)$$

by sending $(g, x) \in G \times X$ to the generalized geodesic $c_{gx_0, gx}$ from gx_0 to gx.

The flow estimate

Theorem (The flow estimate)

Let $\beta, L > 0$. For all $\delta > 0$ there are $T, r > 0$ with the following property: For $x_1, x_2 \in X$ with $d_X(x_1, x_2) \leq \beta$, $x \in \overline{B}_{r+L}(x_1)$ there is $\tau \in [-\beta, \beta]$ such that

$$d_{FS}\big(\Phi_T(c_{x_1,\rho_{r,x_1}(x)}), \Phi_{T+\tau}(c_{x_2,\rho_{r,x_2}(x)})\big) \;\leq\; \delta.$$