The Isomorphism Conjectures in general (Lecture IV)

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://131.220.77.52/lueck/

Bonn, August 2013

Flashback

- We introduced the Farrell-Jones Conjecture and the Baum-Connes Conjecture for torsion free groups:

$$\begin{array}{rcl} H_n(BG;\mathbf{K}_R) & \xrightarrow{\cong} & K_n(RG); \\ H_n(BG;\mathbf{L}_R^{\langle -\infty\rangle}) & \xrightarrow{\cong} & L_n^{\langle -\infty\rangle}(RG); \\ K_n(BG) & \xrightarrow{\cong} & K_n(C_r^*(G)). \end{array}$$

- We discussed applications of these conjectures such as to the Kaplansky Conjecture and the Borel Conjecture.
- Cliffhanger

Question (Arbitrary groups and rings)

Are there versions of the Farrell-Jones Conjecture for arbitrary groups and rings and of the Baum-Connes Conjecture for arbitrary groups?

Outline

- We introduce classifying spaces for families.
- We introduce equivariant homology theories.
- We state the Farrell-Jones Conjecture and the Baum-Connes Conjecture in general.
- We discuss further applications, such as the Novikov Conjecture.

Classifying spaces for families

Definition (Family of subgroups)

A family $\mathcal{F}$ of subgroups of G is a set of (closed) subgroups of G that is closed under conjugation and taking subgroups.

- Examples for $\mathcal{F}$ are:

$$\begin{array}{lcl} \mathcal{T}\mathrm{r} & = & \{\text{trivial subgroup}\}; \\ \mathcal{F}\mathrm{in} & = & \{\text{finite subgroups}\}; \\ \mathcal{VC}\mathrm{yc} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{A}\mathrm{ll} & = & \{\text{all subgroups}\}. \end{array}$$

Definition (Classifying G-CW-complex for a family of subgroups)

Let $\mathcal{F}$ be a family of subgroups of G. A model for the classifying G-CW-complex for the family $\mathcal{F}$ is a G-CW-complex $E_{\mathcal{F}}(G)$ with the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to E_{\mathcal{F}}(G)$.

- We abbreviate $\underline{E}G := E_{\mathcal{F}\text{in}}(G)$ and call it the universal G-CW-complex for proper G-actions.
- We abbreviate $EG := E_{\mathcal{T}\text{r}}(G)$ and $\underline{\underline{E}}G := E_{\mathcal{VC}\text{yc}}(G)$.

Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- *There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;*
- *Two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;*
- *A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all of its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is contractible.*

- A model for $E_{\mathcal{A}ll}(G)$ is G/G;
- $EG \to BG := G\backslash EG$ is the universal principal G-bundle for G-CW-complexes.
- Let $\mathcal{F} \subseteq \mathcal{G}$ be an inclusion of families of subgroups of G. Then there exists up to G-homotopy precisely one G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$.

Exercise

Let $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$ be the infinite dihedral group. Show that $\mathbb{R}$ with the obvious D_∞-action is a model for $\underline{E}D_\infty$.

Special models for $\underline{E}G$

- We want to illustrate that the space $\underline{E}G$ often has very nice geometric models and appears naturally in many interesting situations.

- The spaces $\underline{E}G$ are very interesting in their own right.

Theorem (Simplicial Model)

The geometric realization of the simplicial set whose k-simplices consist of $(k+1)$-tuples $(g_0, g_1, \ldots, g_k)$ of elements g_i in G is a model for $\underline{E}G$.

Theorem (Discrete subgroups of almost connected Lie groups)

Let L be a Lie group with finitely many path components and let $G \subseteq L$ be a discrete subgroup. Then L contains a maximal compact subgroup K which is unique up to conjugation, and L/K with the obvious left G-action is a model for $\underline{E}G$.

Theorem (Actions on CAT(0)-spaces)

Let X be a proper G-CW-complex. Suppose that X has the structure of a complete simply connected $\mathrm{CAT}(0)$*-space on which G acts by isometries.*
Then X is a model for $\underline{E}G$.

The result above contains as special case:

- isometric G-actions on simply connected complete Riemannian manifolds with non-positive sectional curvature;
- G-actions on trees.

Theorem (Rips complex)

Let G be a hyperbolic group. Then the barycentric subdivision of the Rips complex $P_d(G,S)'$ is a finite G-CW-model for $\underline{E}G$, for large enough d.

Theorem (Teichmüller space)

Let $\Gamma_{g,r}^s$ be the mapping class group of an orientable compact surface of genus g with s punctures and r boundary components. Suppose $2g+s+r>2$.
Then the associated Teichmüller space is a model for $\underline{E}\Gamma_{g,r}^s$.

Theorem (Outer space)

The outer space due to Culler-Vogtmann is a model for $\underline{E}\,\mathrm{Out}(F_n)$.

Exercise

Find nice models for $\underline{E}SL_2(\mathbb{Z})$.

Equivariant homology theories

Definition (G-homology theory)

A G-homology theory $\mathcal{H}_*$ is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups together with natural transformations

$$\partial_n(X, A) \colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Equivariant homology theories

Definition (G-homology theory)

A G-homology theory $\mathcal{H}_*^G$ is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded abelian groups together with natural transformations

$$\partial_n^G(X,A)\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Definition (Equivariant homology theory)

An equivariant homology theory $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so called induction structure: given a group homomorphism $\alpha \colon H \to G$ and a H-CW-pair (X, A), there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha \colon \mathcal{H}_n^H(X, A) \quad \to \quad \mathcal{H}_n^G(\operatorname{ind}_\alpha(X, A))$$

satisfying:

- Bijectivity;
  If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha$ is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

Theorem (Equivariant homology theories and spectra over groupoids)

Given a functor $\mathbf{E}\colon \mathrm{Groupoids} \to \mathrm{Spectra}$ *sending equivalences to weak equivalences, there exists an equivariant homology theory* $\mathcal{H}_*^?(-;\mathbf{E})$ *satisfying*

$$\mathcal{H}_n^H(pt) \cong \mathcal{H}_n^G(G/H) \cong \pi_n(\mathbf{E}(H)).$$

Exercise

Is there an equivariant homology theory $\mathcal{H}_*^?$ *such that* $\mathcal{H}_n^G(G/H)$ *is* $\mathcal{K}_n(BH)$ *for a given non-equivariant homology theory* $\mathcal{K}$*?*

Theorem (Equivariant homology theories associated to K and L-theory)

Let R be a ring (with involution). There exist covariant functors

$$\begin{aligned}
\mathbf{K}_R \colon \text{Groupoids} &\to \text{Spectra};\\
\mathbf{L}_R^{\langle\infty\rangle} \colon \text{Groupoids} &\to \text{Spectra};\\
\mathbf{K}^{\text{top}} \colon \text{Groupoids}^{\text{inj}} &\to \text{Spectra},
\end{aligned}$$

with the following properties:

- *They respect equivalences;*
- *For every group G and all $n \in \mathbb{Z}$ we have*

$$\begin{aligned}
\pi_n(\mathbf{K}_R(G)) &\cong K_n(RG);\\
\pi_n(\mathbf{L}_R^{\langle-\infty\rangle}(G)) &\cong L_n^{\langle-\infty\rangle}(RG);\\
\pi_n(\mathbf{K}^{\text{top}}(G)) &\cong K_n(C_r^*(G)).
\end{aligned}$$

Example (Equivariant homology theories associated to K and L-theory)

We get equivariant homology theories

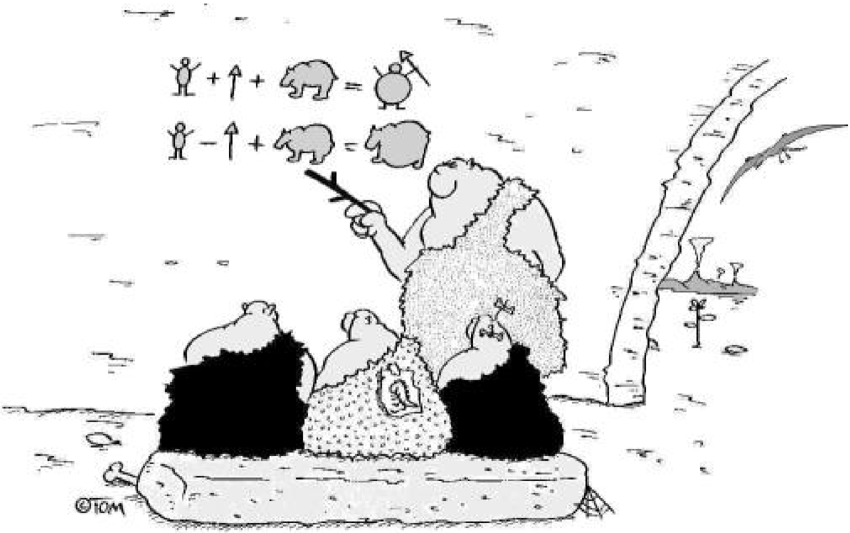
$$H_*^?(-;\mathbf{K}_R);$$
$$H_*^?(-;\mathbf{L}_R^{\langle -\infty\rangle});$$
$$H_*^?(-;\mathbf{K}^{\mathrm{top}}),$$

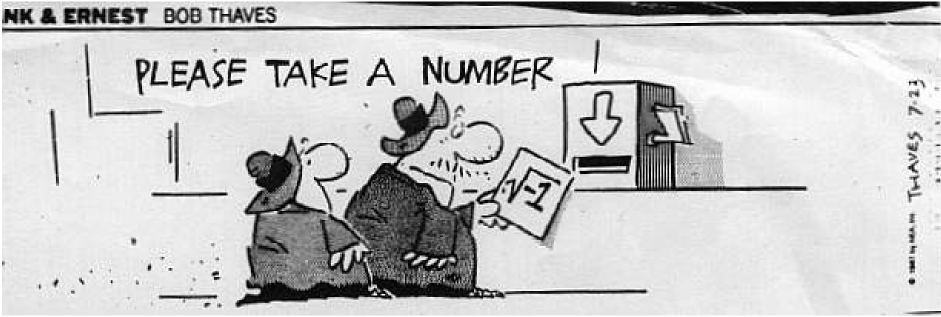
satisfying for $H \subseteq G$

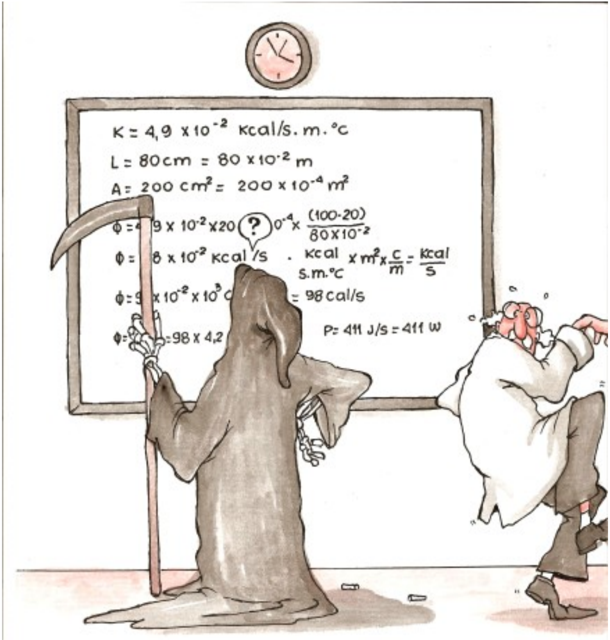
$$\begin{array}{lclcl} H_n^G(G/H;\mathbf{K}_R) & \cong & H_n^H(\mathrm{pt};\mathbf{K}_R) & \cong & K_n(RH); \\ H_n^G(G/H;\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & H_n^H(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & L_n^{\langle -\infty\rangle}(RH); \\ H_n^G(G/H;\mathbf{K}^{\mathrm{top}}) & \cong & H_n^H(\mathrm{pt};\mathbf{K}^{\mathrm{top}}) & \cong & K_n(C_r^*(H)). \end{array}$$

Mathematics

Mini-Break




NK & ERNEST BOB THAVES
PLEASE TAKE A NUMBER
√-1


K = 4,9 x 10-2 kcal/s.m.°c
L = 80cm = 80 x 10-2 m
A = 200 cm² = 200 x 10-4 m²
(100-20)
80x10-2
kcal
s.m.°C
x m² x C/m = kcal/s
= 98 cal/s
P = 411 J/s = 411 W

The general formulation of the Isomorphism Conjectures

Conjecture (K-theoretic Farrell-Jones-Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- The assembly map is the map induced by the projection $E_{\mathcal{VC}\mathrm{yc}}(G) \to \mathrm{pt}$.

Conjecture (*L*-theoretic Farrell-Jones-Conjecture)

The L-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for every $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture)

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{F}in}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(pt, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

is bijective for every $n \in \mathbb{Z}$.

- The assembly maps can also be interpreted in terms of homotopy colimits, where the functor of interest evaluated at G is assembled from its values on subgroups belonging to the relevant family.

- For instance, K-theory, we get an interpretation of the assembly map as the canonical map

$$\operatorname{hocolim}_{V \in \mathcal{VC}\mathrm{yc}} \mathbf{K}(RV) \to \mathbf{K}(RG).$$

- There are other theories for which one can formulate Isomorphism Conjectures in an analogous way, e.g., pseudoisotopy, Waldhausen's A-theory, topological Hochschild homology, topological cyclic homology.

Further Conclusions of the Isomorphism Conjectures

Conjecture (Novikov Conjecture)

The Novikov Conjecture for G predicts for a closed oriented manifold M together with a map $f\colon M \to BG$ that for any $x \in H^*(BG)$ the higher signature

$$\operatorname{sign}_x(M, f) := \langle \mathcal{L}(M) \cup f^*x, [M] \rangle$$

is an oriented homotopy invariant of (M, f).

- For $x = 1$ this follows from Hirzebruch's signature formula

$$\text{sign}(M) := \langle \mathcal{L}(M), [M] \rangle.$$

- For a homotopy equivalence $f\colon M \to N$ of closed aspherical manifolds the Novikov Conjecture predicts $f^*\mathcal{L}(N) = \mathcal{L}(M)$.

- In this case it follows from the Borel Conjecture together with Novikov's Theorem about the topological invariance of rational Pontryagin classes.

Theorem (The Farrell-Jones, the Baum-Connes and the Novikov Conjecture)

Suppose that one of the following assembly maps

$$\begin{aligned} H_n^G(E_{\mathcal{VC}yc}(G), \mathbf{L}_R^{\langle -\infty \rangle}) &\to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG); \\ K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{F}in}(G), \mathbf{K}^{\mathrm{top}}) &\to H_n^G(pt, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G)), \end{aligned}$$

is rationally injective.
Then the Novikov Conjecture holds for the group G.

Theorem (Moody's Induction Conjecture)

Let F be a field of characteristic p. Suppose $G \in \mathcal{FJ}_K(R)$. Then:

- *If $p = 0$, the map given by induction from finite subgroups of G*

$$\operatorname*{colim}_{H \in \mathcal{F}in} K_0(FH) \to K_0(FG)$$

  is bijective;

- *If $p > 0$, then the map*

$$\operatorname*{colim}_{H \in \mathcal{F}in} K_0(FH)[1/p] \to K_0(FG)[1/p]$$

  is bijective.

Exercise

Compute $K_0(\mathbb{C}[\mathbb{Z}/3 \rtimes_\phi \mathbb{Z}])$ for $\phi = -\operatorname{id}\colon \mathbb{Z}/3 \to \mathbb{Z}/3$.

- The Farrell-Jones Conjecture for algebraic K-theory implies the Bass Conjecture.

- The Farrell-Jones Conjecture for algebraic K-theory is part of a program due to Linnell to prove the Atiyah Conjecture about the integrality of L^2-Betti numbers of closed Riemannian manifolds with torsion free fundamental groups.

- The Baum-Connes Conjecture implies the Stable Gromov-Lawson-Rosenberg Conjecture about the existence of Riemannian metrics with positive scalar curvature.

- The Farrell-Jones Conjecture for K and L-theory implies for a Poincaré duality group G of dimension ≥ 5 that it is the fundamental group of a closed ANR-homology manifold.

Theorem (Bartels-Lück-Weinberger)

Let G be a torsion free hyperbolic group and let n be an integer ≥ 6. The following statements are equivalent:

- *the boundary ∂G is homeomorphic to S^{n-1};*
- *there is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

- If the manifold above exists, it is unique up to homeomorphism by the Borel Conjecture.

- The Farrell-Jones Conjecture and Baum-Connes Conjecture are basic ingredients in concrete computations of K and L-groups.
- Such computations have interesting applications to problems in manifold theory and the classification of C^*-algebras.
- Depending on the theory under consideration one can sometimes choose a smaller family than $\mathcal{VC}$yc or $\mathcal{F}$in.

Cliffhanger

Question (Status)

For which groups are the Farrell-Jones Conjecture and the Baum-Connes Conjecture known to be true?
What are open interesting cases?

Question (Methods of proof)

What are the methods of proof?

To be continued

Stay tuned

Next talk: Today 11:50