Existence of Finitely Dominated CW-Complexes with $G_1(X) = \pi_1(X)$ and non-Vanishing Finiteness Obstruction

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Abstract: We show for a finite abelian group G and any element in the image of the Swan homomorphism $\mathrm{sw} : \mathbb{Z}/|G|^* \longrightarrow \widetilde{K}_0(\mathbb{Z}G)$ that it can be realized as the finiteness obstruction of a finitely dominated connected CW-complex X with fundamental group $\pi_1(X) = G$ such that $\pi_1(X)$ is equal to the subgroup $G_1(X)$ defined by Gottlieb. This is motivated by the observation that any H-space X satisfies $\pi_1(X) = G_1(X)$ and still the problem is open whether any finitely dominated H-space is up to homotopy finite.

The purpose of this note is to prove

Theorem 1 *Let G be a finite abelian group and η be any element in the image of the Swan homomorphism* $\mathrm{sw} : \mathbb{Z}/|G|^* \longrightarrow \widetilde{K}_0(\mathbb{Z}G)$*. Then there is a finitely dominated connected CW-complex X with the following properties:*

1. $G = \pi_1(X)$*;*

2. Gottlieb's subgroup $G_1(X) \subset \pi_1(X)$ *is equal to* $\pi_1(X)$*;*

3. Wall's finiteness obstruction $\widetilde{o}(X)$ *is* η*.*

Recall that a space X satisfies $\pi_1(X) = G_1(X)$ if and only if $\pi_1(X)$ is abelian and for each $w \in \pi_1(X)$ the associated deck transformation on the universal covering $l_w : \widetilde{X} \longrightarrow \widetilde{X}$, which is a $\pi_1(X)$-equivariant map, is $\pi_1(X)$-homotopic to the identity [2]. The *finiteness obstruction* of a finitely dominated CW-complex was introduced by Wall [10]. A survey about the finiteness obstruction and nilpotent and simple spaces is given in [7]. The *Swan homomorphism* $\mathrm{sw} : \mathbb{Z}/|G|^* \longrightarrow \widetilde{K}_0(\mathbb{Z}G)$ sends $\overline{r} \in \mathbb{Z}/|G|^*$ to the class $[\mathbb{Z}/r]$ of the $\mathbb{Z}G$-module given by the cyclic group $\mathbb{Z}/r$ with the trivial G-action for any representative $r \in \mathbb{Z}$ of $\overline{r}$. This $\mathbb{Z}G$-module has a finite projective $\mathbb{Z}G$-resolution P_* and for any such P_* the class $[\mathbb{Z}/r]$ is given by $\sum_{p \geq 0} (-1)^p \cdot [P_n]$ in $\widetilde{K}_0(\mathbb{Z}G)$ [8]. Computations of the image of the Swan homomorphism can be found for instance in [9].

The motivation for the study of the possible finiteness obstructions of finitely dominated CW-complexes X with $G_1(X) = \pi_1(X)$ comes from the to the author's knowledge still unsettled question whether a finitely dominated H-space is always up to homotopy finite. The point is that any H-space X satisfies $G_1(X) = \pi_1(X)$. Mislin has shown that a finitely dominated H-space is finite up to homotopy if its fundamental group is of square free order [5, Theorem II on page 375]. We mention that a nilpotent space is finitely dominated if and only if $H_i(X;\mathbb{Z})$ is finitely generated for all i and zero for sufficiently large i and that each H-space is nilpotent [4, Theorem A]. A space with $G_1(X) = \pi_1(X)$ is nilpotent, but the converse is not true.

If $\pi_1(X)$ is infinite, any finitely dominated CW-complex X with $G_1(X) = \pi_1(X)$ is homotopy equivalent to a space of the form $Z \times S^1$ and hence its finiteness obstruction vanishes by the product formula [3, Prop. 4.3 on page 153]. So it suffices to consider finitely dominated CW-complexes X with $G_1(X) = \pi_1(X)$ such that $\pi_1(X)$ is finite.

Theorem 1 has been proven in [6, Theorem 2.4, page 203] if one substitutes the second condition in Theorem 1 by requiring that X is simple. So Mislin gives an example for X where for each element $g \in G = \pi_1(X)$ the covering translation $l_g : \widetilde{X} \longrightarrow \widetilde{X}$ is homotopic to the identity where we require that it is G-equivariantly homotopic to the identity.

Now we prove Theorem 1. The construction of the space X is a variation of the one in [6, Theorem 2.2, page 201]. Fix an integer $n \geq 3$. We claim that there is a connected finite CW-complex A such that $G = \pi_1(A) = G_1(A)$ and $\widetilde{A}$ is $n+2$-connected. Since G is a product of cyclic groups, it suffices to treat the case where G is cyclic. Let G operate on $\mathbb{C}^{n+1}$ by multiplication with a primitive $|G|$-th root of unity. It induces a free G-action on the unit sphere S^{2n+1}. Take $A = S^{2n+1}/G$. For the given η in the image of the Swan homomorphism we can choose an odd natural number r such that $\mathrm{sw}(\overline{r}) = \eta$. Precisely as in [6, Theorem 2.2, page 201] we can attach n and $n+1$-cells to A to obtain a connected finitely dominated CW-complex X with η as finiteness obstruction such that $H_k(\widetilde{X}, \widetilde{A})$ is zero for $k \neq n$ and, for $k = n$, is $\mathbb{Z}G$-isomorphic to $\mathbb{Z}/r$ with the trivial G-action. It remains to prove for $g \in G$ that $l_g : \widetilde{X} \longrightarrow \widetilde{X}$ is G-homotopic to the identity. Notice that there is already a G-homotopy $h : \widetilde{A} \times I \longrightarrow \widetilde{A}$ between $l_g : \widetilde{A} \longrightarrow \widetilde{A}$ and the identity. Hence we have to extend $(l_g \coprod \mathrm{id}) \cup h : \widetilde{X} \times \partial I \cup \widetilde{A} \longrightarrow \widetilde{X}$ to a G-map $H : \widetilde{X} \times I \longrightarrow \widetilde{X}$.

We use the equivariant obstruction theory as developed in [1, II.3]. Because of the obstruction sequence [1, Theorem 3.10 on page 115] and [1, Theorem 3.17 on page 120] it suffices to show

1. The primary obstruction
$$\gamma((l_g \coprod \mathrm{id}) \cup h) \in H_G^{n+1}((\widetilde{X}, \widetilde{A}) \times (I, \partial I); \pi_n(\widetilde{X}))$$
vanishes;
2. $H_G^{k+1}((\widetilde{X}, \widetilde{A}) \times (I, \partial I); \pi_k(\widetilde{X}))$ is trivial for $k \geq n+1$.

In the sequel we will identify
$$H_G^{k+1}((\widetilde{X}, \widetilde{A}) \times (I, \partial I); \pi_k(\widetilde{X})) = H_G^k(\widetilde{X}, \widetilde{A}); \pi_k(\widetilde{X}))$$
by the suspension isomorphism. Recall that $H_G^n(\widetilde{X}, \widetilde{A}; \pi_n(\widetilde{X}))$ is the cohomology of the cochain complex $\hom_{\mathbb{Z}G}(C_*(\widetilde{X}, \widetilde{A}), \pi_n(\widetilde{X}))$. Since $\widetilde{A}$ is $n+2$-connected and X obtained from A by attaching cells of dimension greater or equal to n, we get an isomorphism

$$\begin{aligned} H_G^n(\widetilde{X}, \widetilde{A}; \pi_n(\widetilde{X})) &\xrightarrow{\cong} \hom_{\mathbb{Z}G}(H_n(\widetilde{X}, \widetilde{A}), \pi_n(\widetilde{X})) \\ &\xrightarrow{\cong} \hom_{\mathbb{Z}G}(H_n(\widetilde{X}, \widetilde{A}), \pi_n(\widetilde{X}, \widetilde{A})) \\ &\xrightarrow{\cong} \hom_{\mathbb{Z}G}(H_n(\widetilde{X}, \widetilde{A}), H_n(\widetilde{X}, \widetilde{A})). \end{aligned}$$

One easily checks that the primary obstruction $\gamma((l_g \coprod \mathrm{id}) \cup h)$ is sent under this isomorphism to $H_n(l_g) - \mathrm{id}$ (cf. [1, 3.18 and 3.19 on page 121]). Since G acts trivially on $H_n(\widetilde{X}, \widetilde{A})$ this difference and hence the primary obstruction vanish.

Since $\widetilde{X}$ is obtained from $\widetilde{A}$ by attaching cells in dimensions n and $n+1$, it remains to prove that $H_G^{n+1}(\widetilde{X}, \widetilde{A}; \pi_{n+1}(\widetilde{X}))$ vanishes. Since $H_k(\widetilde{X}, \widetilde{A})$ vanishes or is isomorphic to $\mathbb{Z}/r$ with odd r for all $k \geq 0$, it suffices to prove that $\pi_{n+1}(\widetilde{X})$ is a finite abelian 2-group. Denote by $\widetilde{X}_n$ the n-skeleton of the relative CW-complex $(\widetilde{X}, \widetilde{A})$. Consider the following part of the long exact sequence of a triple

$$\ldots \longrightarrow \pi_{n+1}(\widetilde{X}_n, \widetilde{A}) \longrightarrow \pi_{n+1}(\widetilde{X}, \widetilde{A}) \longrightarrow \pi_{n+1}(\widetilde{X}, \widetilde{X}_n)$$

$$\xrightarrow{\Delta} \pi_n(\widetilde{X}_n, \widetilde{A}) \longrightarrow \ldots$$

By the Hurewicz we can identify Δ with the $n+1$-differential in the cellular $\mathbb{Z}G$-chain complex of $(\widetilde{X}, \widetilde{A})$. Since $H_{n+1}(\widetilde{X}, \widetilde{A})$ vanishes, Δ is injective. Hence it suffices to show that $\pi_{n+1}(\widetilde{X}_n, \widetilde{A})$ is a finite abelian 2-group because $\pi_{n+1}(\widetilde{X}) \cong \pi_{n+1}(\widetilde{X}, \widetilde{A})$ is a quotient of $\pi_{n+1}(\widetilde{X}_n, \widetilde{A})$. Since $\widetilde{A}$ is $n+2$-connected, $\pi_{n+1}(\widetilde{X}_n, \widetilde{A})$ is isomorphic to $\pi_{n+1}(\widetilde{X}_n/\widetilde{A})$. As $\widetilde{X}_n/\widetilde{A}$ is a wedge of copies of S^n-s and $n \geq 3$ we conclude from the Freudenthal Suspension Theorem, that $\pi_{n+1}(\widetilde{X}/\widetilde{A})$ is isomorphic to a direct sum of copies of the stable homotopy group π_1^s which is $\mathbb{Z}/2$. This finishes the proof of Theorem 1.

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