Groups, Geometry and Actions: Classifying spaces for families

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Introduction

- These slides cover parts of the course Groups, Geometry and Actions of the summer term 2010, but also contain some additional material which will not be presented in the lectures.
- In the actual talks more background information, more examples and more details are given on the blackboard.
- This will be an on demand course, i.e., the audience can choose what topic will be presented and also determine how much time shall be spent on it
- The first topic will be classifying spaces for families.

- Possible further topics are:
  1. A basic short introduction to homological algebra and group (co-)homology
  2. Free actions of finite groups on homotopy CW-spheres
  3. Introduction to Isomorphism Conjectures
  4. Introduction to geometric group theory
  5. Groups and L^2-invariants
- We will announce what topic is covered for which time period so that people may choose to attend a topic or not.
- I will put the slides on my homepage.
- There will be a Tutorial run by Roman Sauer.
- Next we have to decide on the forthcoming topics.

G-CW-complexes

Definition (G-CW-complex)

A *G-CW-complex* X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \ldots \subseteq X_n \subseteq \ldots \subseteq \bigcup_{n \geq 0} X_n = X$$

such that X carries the colimit topology with respect to this filtration, and X_n is obtained from X_{n-1} for each $n \geq 0$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{array}{ccc} \coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in I_n} q_i^n} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in I_n} Q_i^n} & X_n \end{array}$$

- A G-CW-complex X is the same as a CW-complex with a G-action such that for any open cell e with $g \cdot e \cap e \neq \emptyset$ we have $gx = x$ for all $x \in e$.

Example (1- and 2-dimensional sphere with various actions)

Example (Simplicial actions)

Let X be a simplicial complex. Suppose that G acts simplicially on X. Then G acts simplicially also on the barycentric subdivision X', and the G-space X' inherits the structure of a G-CW-complex.

Example (Smooth actions)

If G acts properly and smoothly on a smooth manifold M, then M inherits the structure of G-CW-complex.

Classifying spaces for families

Definition (Family of subgroups)

A *family $\mathcal{F}$ of subgroups* of G is a set of subgroups of G which is closed under conjugation and taking subgroups.

Examples for $\mathcal{F}$ are:

$$\begin{array}{lcl} \mathcal{TR} & = & \{\text{trivial subgroup}\}; \\ \mathcal{F}\text{in} & = & \{\text{finite subgroups}\}; \\ \mathcal{VC}\text{yc} & = & \{\text{virtually cyclic subgroups}\}; \\ \mathcal{ALL} & = & \{\text{all subgroups}\}. \end{array}$$

Definition (Classifying G-CW-complex for a family of subgroups, tom Dieck(1974))

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying G-CW-complex for the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ which has the following properties:

- All isotropy groups of $E_{\mathcal{F}}(G)$ belong to $\mathcal{F}$;
- For any G-CW-complex Y, whose isotropy groups belong to $\mathcal{F}$, there is up to G-homotopy precisely one G-map $Y \to X$.

We abbreviate $\underline{E}G := E_{\mathcal{F}\text{in}}(G)$ and call it the *universal G-CW-complex for proper G-actions*.

We abbreviate $\underline{\underline{E}}G := E_{\mathcal{VC}\text{yc}}(G)$.

We also write $EG = E_{\mathcal{TR}}(G)$.

Theorem (Homotopy characterization of $E_{\mathcal{F}}(G)$)

Let $\mathcal{F}$ be a family of subgroups.

- *There exists a model for $E_{\mathcal{F}}(G)$ for any family $\mathcal{F}$;*
- *Two models for $E_{\mathcal{F}}(G)$ are G-homotopy equivalent;*
- *A G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if all its isotropy groups belong to $\mathcal{F}$ and for each $H \in \mathcal{F}$ the H-fixed point set X^H is contractible.*

- Sketch of the proof

Trivial family

- We have $EG = \underline{E}G$ if and only if G is torsionfree.
- $G \to EG \to BG$ is the universal G-principal bundle.
- $BG := G \backslash EG$ is sometimes called the classifying space of G and is a model for the Eilenberg-MacLane space of type (G,1).
  It is unique up to homotopy.
- A closed oriented surface F_g of genus g is a model for $B\pi_1(F_g)$ if and only if $g \geq 1$.
- A closed orientable 3-manifold M is a model for $B\pi_1(M)$ if and only if its fundamental group is torsionfree, prime and different from $\mathbb{Z}$.
- A connected CW-complex is called aspherical if and only if $\pi_n(X) = 0$ for $n \geq 2$, or, equivalently, X is a model for $B\pi_1(X)$.

Further elementary examples

- We have $E_{\mathcal{F}}(G) = \text{pt}$ if and only if $\mathcal{F} = \mathcal{ALL}$.
- We have $\underline{E}G = \text{pt}$ if and only if G is finite.
- A model for $\underline{E}D_\infty$ is the real line with the obvious $D_\infty = \mathbb{Z} \rtimes \mathbb{Z}/2 = \mathbb{Z}/2 * \mathbb{Z}/2$-action.
  Every model for ED_∞ is infinite dimensional, e.g., the universal covering of $\mathbb{RP}^\infty \vee \mathbb{RP}^\infty$.
- The spaces $\underline{E}G$ are interesting in their own right and have often very nice geometric models which are rather small.
- On the other hand any CW-complex is homotopy equivalent to $G\backslash\underline{E}G$ for some group G (see Leary-Nucinkis (2001)).

The family of finite subgroups

- We want to illustrate that the space $\underline{E}G = \underline{E}G$ often has very nice geometric models and appear naturally in many interesting situations.
- Let $C_0(G)$ be the Banach space of complex valued functions of G vanishing at infinity with the supremum-norm. The group G acts isometrically on $C_0(G)$ by $(g \cdot f)(x) := f(g^{-1}x)$ for $f \in C_0(G)$ and $g, x \in G$.
  Let $PC_0(G)$ be the subspace of $C_0(G)$ consisting of functions f such that f is not identically zero and has non-negative real numbers as values.

Theorem (Operator theoretic model, Abels (1978))

The G-space $PC_0(G)$ is a model for $\underline{E}G$.

Theorem (Another operator theoretic model)

A model for $\underline{E}G$ is the space

$$X_G = \left\{ f \colon G \to [0,1] \;\middle|\; f \text{ has finite support}, \sum_{g \in G} f(g) = 1 \right\}$$

with the topology coming from the supremum norm.

Theorem (Simplicial Model)

Let $P_\infty(G)$ be the geometric realization of the simplicial set whose k-simplices consist of $(k+1)$-tupels $(g_0, g_1, \ldots, g_k)$ of elements g_i in G. This is a model for $\underline{E}G$.

- The spaces X_G and $P_\infty(G)$ have the same underlying sets but in general they have different topologies.
- The identity map induces a G-map $P_\infty(G) \to X_G$ which is a G-homotopy equivalence, but in general not a G-homeomorphism.

- The Rips complex $P_d(G,S)$ of a group G with a symmetric finite set S of generators for a natural number d is the geometric realization of the simplicial set whose set of k-simplices consists of $(k+1)$-tuples $(g_0, g_1, \ldots g_k)$ of pairwise distinct elements $g_i \in G$ satisfying $d_S(g_i, g_j) \leq d$ for all $i, j \in \{0, 1, \ldots, k\}$.
- The obvious G-action by simplicial automorphisms on $P_d(G,S)$ induces a G-action by simplicial automorphisms on the barycentric subdivision $P_d(G,S)'$.

Theorem (Rips complex, Meintrup-Schick (2002))

Let G be a discrete group with a finite symmetric set of generators. Suppose that (G,S) is δ-hyperbolic for the real number $\delta \geq 0$. Let d be a natural number with $d \geq 16\delta + 8$.
Then the barycentric subdivision of the Rips complex $P_d(G,S)'$ is a finite G-CW-model for $\underline{E}G$.

- Let $\Gamma_{g,r}^s$ be the mapping class group of an orientable compact surface F of genus g with s punctures and r boundary components.
  We will always assume that $2g + s + r > 2$, or, equivalently, that the Euler characteristic of the punctured surface F is negative.
- It is well-known that the associated Teichmüller space $\mathcal{T}_{g,r}^s$ is a contractible space on which $\Gamma_{g,r}^s$ acts properly.

Theorem (Teichmüller space)

The $\Gamma_{g,r}^s$-space $\mathcal{T}_{g,r}^s$ is a model for $\underline{E}\Gamma_{g,r}^s$.

- Let F_n be the free group of rank n.
- Denote by $\mathrm{Out}(F_n)$ the group of outer automorphisms of F_n, i.e., the quotient of the group of all automorphisms of F_n by the normal subgroup of inner automorphisms.
- Culler-Vogtmann (1996) have constructed a space X_n called outer space on which $\mathrm{Out}(F_n)$ acts with finite isotropy groups. It is analogous to the Teichmüller space of a surface with the action of the mapping class group of the surface.
- The space X_n contains a spine K_n which is an $\mathrm{Out}(F_n)$-equivariant deformation retraction.
- This space K_n is a simplicial complex of dimension $(2n-3)$ on which the $\mathrm{Out}(F_n)$-action is by simplicial automorphisms and cocompact.

Theorem (Spine of outer space)

The barycentric subdivision K_n' is a finite $(2n-3)$-dimensional model of $\underline{E}\,\mathrm{Out}(F_n)$.

Theorem (Lie groups)

Let L be a connected Lie group L, let $K \subseteq L$ be a maximal compact subgroup and let $G \subseteq L$ a discrete subgroup.
Then L/K with the obvious G-action is a model for $\underline{E}G$.

Theorem (CAT(0)-spaces)

A CAT(0)-space with proper isometric G-actions is a model for $\underline{E}G$.

- Examples for CAT(0)-spaces are connected Riemannian manifolds with non-positive sectional curvature and trees.

Example ($SL_2(\mathbb{Z})$)

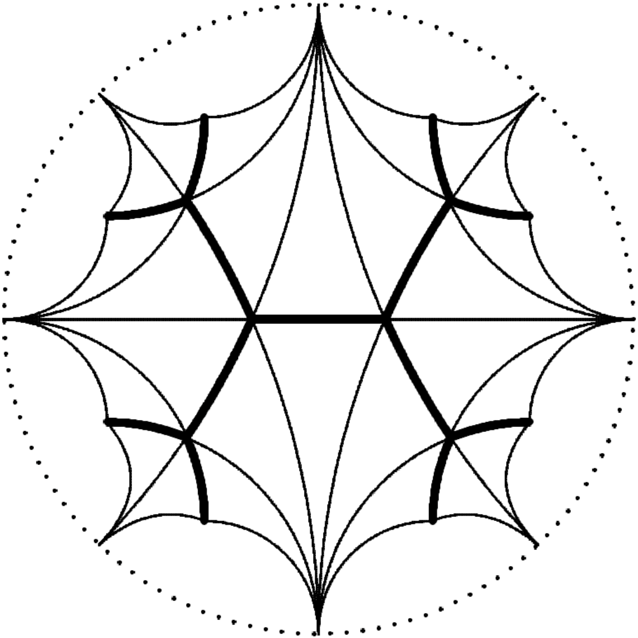
- In order to illustrate some of the general statements above we consider the special example $SL_2(\mathbb{Z})$.
- Let $\mathbb{H}^2$ be the 2-dimensional hyperbolic space. It is a simply-connected 2-dimensional Riemannian manifold, whose sectional curvature is constant -1. In particular it is a CAT(0)-space. The group $SL_2(\mathbb{Z})$ acts properly and isometrically by diffeomorphisms on the upper half-plane by Moebius transformations.
  Hence the $SL_2(\mathbb{Z})$-space $\mathbb{H}^2$ is a model for $\underline{E}SL_2(\mathbb{Z})$.

Example (continued)

- The group $SL_2(\mathbb{R})$ is a connected Lie group and $SO(2) \subseteq SL_2(\mathbb{R})$ is a maximal compact subgroup.
  Hence $SL_2(\mathbb{R})/SO(2)$ is a model for $\underline{E}SL_2(\mathbb{R})$
- The group $SL_2(\mathbb{R})$ acts by isometric diffeomorphisms on the upper half-plane by Moebius transformations. This action is proper and transitive and the isotropy group of $z = i$ is $SO(2)$.
  Hence the $SL_2(\mathbb{Z})$-manifolds $SL_2(\mathbb{R})/SO(2)$ and $\mathbb{H}^2$ are $SL_2(\mathbb{Z})$-diffeomorphic.

Example (continued)

- The group $SL_2(\mathbb{Z})$ is isomorphic to the amalgamated product $\mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6$. This implies that there is a tree on which $SL_2(\mathbb{Z})$ acts with finite stabilizers. The tree has alternately two and three edges emanating from each vertex. This is a 1-dimensional model for $\underline{E}SL_2(\mathbb{Z})$.
- The tree model and the other model given by $\mathbb{H}^2$ must be $SL_2(\mathbb{Z})$-homotopy equivalent. They can explicitly be related by the following construction.



Example (continued)

- Divide the Poincaré disk into fundamental domains for the $SL_2(\mathbb{Z})$-action. Each fundamental domain is a geodesic triangle with one vertex at infinity, i.e., a vertex on the boundary sphere, and two vertices in the interior. Then the union of the edges, whose end points lie in the interior of the Poincaré disk, is a tree T with $SL_2(\mathbb{Z})$-action which is the tree model above. The tree is a $SL_2(\mathbb{Z})$-equivariant deformation retraction of the Poincaré disk. A retraction is given by moving a point p in the Poincaré disk along a geodesic starting at the vertex at infinity, which belongs to the triangle containing p, through p to the first intersection point of this geodesic with T.

The family of virtually cyclic subgroups

- In the case of the Farrell-Jones Conjecture we will have to deal with $\underline{\underline{E}}G = E_{\mathcal{VC}\mathrm{yc}}(G)$ instead of $\underline{E}G = E_{\mathcal{F}\mathrm{in}}(G)$.
- Unfortunately, $\underline{\underline{E}}G$ is much more complicated than $\underline{E}G$.

Example ($\underline{\underline{E}}\mathbb{Z}^n$)

- A model for $\underline{E}\mathbb{Z}^n$ is $\mathbb{R}^n$ with the free standard $\mathbb{Z}^n$-action.
- If we cross it with $\mathbb{R}$ with the trivial action, we obtain another model for $\underline{E}\mathbb{Z}^n$.
- Let $\{C_k \mid k \in \mathbb{Z}\}$ be the set of infinite cyclic subgroups of $\mathbb{Z}^n$. Then a model for $\underline{\underline{E}}\mathbb{Z}^n$ is obtained from $\mathbb{R}^n \times \mathbb{R}$ if we collapse for every $k \in \mathbb{Z}$ the n-dimensional real vector space $\mathbb{R}^n \times \{k\}$ to the $(n-1)$-dimensional real vector space $\mathbb{R}^n/V_C$, where V_C is the one-dimensional real vector space generated by the C-orbit through the origin.

Finiteness properties

- Finiteness properties of the spaces EG and $\underline{E}G$ have been intensively studied in the literature. We mention a few examples and results.
- If EG has a finite-dimensional model, the group G must be torsionfree.
- There are often finite models for $\underline{E}G$ for groups G with torsion.
- Often geometry provides small model for $\underline{E}G$ in cases, where the models for EG are huge.
- These small models can be useful for computations concerning BG itself.

Theorem (Discrete subgroups of Lie groups)

Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be a maximal compact subgroup K. Let $G \subseteq L$ be a discrete subgroup of L. Then L/K with the left G-action is a model for $\underline{E}G$.
Suppose additionally that G is virtually torsionfree, i.e., contains a torsionfree subgroup $\Delta \subseteq G$ of finite index.
Then we have for its virtual cohomological dimension

$$\operatorname{vcd}(G) \leq \dim(L/K).$$

Equality holds if and only if $G \backslash L$ is compact.

Theorem (A criterion for 1-dimensional models for BG, Stallings (1968), Swan (1969))

Let G be a discrete group. The following statements are equivalent:

- *There exists a 1-dimensional model for EG;*
- *There exists a 1-dimensional model for BG;*
- *The cohomological dimension of G is less or equal to one;*
- *G is a free group.*

Theorem (A criterion for 1-dimensional models for $\underline{E}G$, Dunwoody (1979))

Let G be a discrete group. Then there exists a 1-dimensional model for $\underline{E}G$ if and only if the cohomological dimension of G over the rationals $\mathbb{Q}$ is less or equal to one.

Theorem (Virtual cohomological dimension and $\dim(\underline{E}G)$, L. (2000))

Let G be a discrete group which is virtually torsionfree.

- *Then*

  $$\operatorname{vcd}(G) \le \dim(\underline{E}G)$$

  for any model for $\underline{E}G$.

- *Let $l \ge 0$ be an integer such that for any chain of finite subgroups $H_0 \subsetneq H_1 \subsetneq \ldots \subsetneq H_r$ we have $r \le l$.*
  Then there exists a model for $\underline{E}G$ of dimension $\max\{3, \operatorname{vcd}(G)\} + l$.

- The following problem has been stated by Brown (1979) and has created a lot of activities.

Problem

For which discrete groups G, which are virtually torsionfree, does there exist a G-CW-model for $\underline{E}G$ of dimension $\mathrm{vcd}(G)$?

- The results above do give some evidence for a positive answer.
- However, Leary-Nucinkis (2003) have constructed groups, where the answer is negative.

A computation

- Let G be a discrete group. Let $\mathcal{MF}\text{in}$ be the subset of $\mathcal{F}\text{in}$ consisting of elements in $\mathcal{F}\text{in}$ which are maximal in $\mathcal{F}\text{in}$.
- Assume that G satisfies the following assertions:

  (M) Every non-trivial finite subgroup of G is contained in a unique maximal finite subgroup;

  (NM) $M \in \mathcal{MF}\text{in}, M \neq \{1\} \;\Rightarrow\; N_G M = M$.
- Here are some examples of groups G which satisfy conditions (M) and (NM):
  - Extensions $1 \to \mathbb{Z}^n \to G \to F \to 1$ for finite F such that the conjugation action of F on $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$;
  - Fuchsian groups;
  - One-relator groups G.

- For such a group there is a nice model for $\underline{E}G$ with as few non-free cells as possible.
- Let $\{(M_i) \mid i \in I\}$ be the set of conjugacy classes of maximal finite subgroups of $M_i \subseteq G$.
- By attaching free G-cells we get an inclusion of G-CW-complexes $j_1 \colon \coprod_{i \in I} G \times_{M_i} EM_i \to EG$.
- Define $\underline{E}G$ as the G-pushout

$$\begin{array}{ccc} \coprod_{i \in I} G \times_{M_i} EM_i & \xrightarrow{j_1} & EG \\ \downarrow u_1 & & \downarrow f_1 \\ \coprod_{i \in I} G/M_i & \xrightarrow{k_1} & \underline{E}G \end{array}$$

where u_1 is the obvious G-map obtained by collapsing each EM_i to a point.

- Next we explain why $\underline{E}G$ is a model for the classifying space for proper actions of G.
- Its isotropy groups are all finite. We have to show for $H \subseteq G$ finite that $\underline{E}G^H$ contractible.
- We begin with the case $H \neq \{1\}$. Because of conditions (M) and (NM) there is precisely one index $i_0 \in I$ such that H is subconjugated to M_{i_0} and is not subconjugated to M_i for $i \neq i_0$. We get

$$\left(\coprod_{i \in I} G/M_i\right)^H = (G/M_{i_0})^H = \text{pt}.$$

Hence $\underline{E}G^H = \text{pt}$.
- It remains to treat $H = \{1\}$. Since u_1 is a non-equivariant homotopy equivalence and j_1 is a cofibration, f_1 is a non-equivariant homotopy equivalence. Hence $\underline{E}G$ is contractible.

- Consider the pushout obtained from the G-pushout above by dividing the G-action

$$\begin{array}{ccc} \coprod_{i\in I} BM_i & \longrightarrow & BG \\ \downarrow & & \downarrow \\ \coprod_{i\in I} \mathrm{pt} & \longrightarrow & G\backslash \underline{E}G \end{array}$$

- The associated Mayer-Vietoris sequence yields

$$\ldots \to \widetilde{H}_{p+1}(G\backslash \underline{E}G) \to \bigoplus_{i\in I} \widetilde{H}_p(BM_i) \to \widetilde{H}_p(BG) \to \widetilde{H}_p(G\backslash \underline{E}G) \to \ldots$$

- In particular we obtain an isomorphism for $p \geq \dim(\underline{E}G) + 1$

$$\bigoplus_{i\in I} H_p(BM_i) \xrightarrow{\cong} H_p(BG).$$

Example (One-relator groups)

- Let $G = \langle s_1, s_2, \ldots s_g \mid r \rangle$ be a finitely generated one-relator-group.
- If G is torsionfree, the presentation complex associated to the presentation above is a 2-dimensional model for BG and we get

$$H_n(BG) = 0 \quad \text{for } n \geq 3.$$

- Now suppose that G is not torsionfree.

Example (continued)

- Let F be the free group with basis $\{q_1, q_2, \ldots, q_g\}$. Then r is an element in F. There exists an element $s \in F$ and an integer $m \geq 2$ such that $r = s^m$, the cyclic subgroup C generated by the class $\overline{s} \in Q$ represented by s has order m, any finite subgroup of G is subconjugated to C and for any $g \in G$ the implication $g^{-1}Cg \cap C \neq 1 \Rightarrow g \in C$ holds.
- Hence G satisfies (M) and (NM).
- There is an explicit two-dimensional model for $\underline{E}G$ with one 0-cell $G/C \times D^0$, g 1-cells $G \times D^1$ and one free 2-cell $G \times D^2$.
- We conclude for $n \geq 3$

$$H_n(BC) \cong H_n(BG).$$