L^2-Betti numbers

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Outline

- We introduce L^2-Betti numbers.
- We present their basic properties and tools for their computation.
- We compute the L^2-Betti numbers of all 3-manifolds.
- We discuss the Atiyah Conjecture and the Singer Conjecture.

Basic motivation

- Given an invariant for finite *CW*-complexes, one can get much more sophisticated versions by passing to the universal covering and defining an analogue taking the action of the fundamental group into account.
- Examples:

Classical notion	generalized version
Homology with coefficients in $\mathbb{Z}$	Homology with coefficients in representations
Euler characteristic $\in \mathbb{Z}$	Walls finiteness obstruction in $K_0(\mathbb{Z}\pi)$
Lefschetz numbers $\in \mathbb{Z}$	Generalized Lefschetz invariants in $\mathbb{Z}\pi_\phi$
Signature $\in \mathbb{Z}$	Surgery invariants in $L_*(\mathbb{Z}G)$
—	torsion invariants

- We want to apply this principle to (classical) Betti numbers

$$b_n(X) := \dim_{\mathbb{C}}(H_n(X;\mathbb{C})).$$

- Here are two naive attempts which fail:
  - $\dim_{\mathbb{C}}(H_n(\widetilde{X};\mathbb{C}))$
  - $\dim_{\mathbb{C}\pi}(H_n(\widetilde{X};\mathbb{C}))$,
    where $\dim_{\mathbb{C}\pi}(M)$ for a $\mathbb{C}[\pi]$-module could be chosen for instance as $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}G} M)$.
- The problem is that $\mathbb{C}\pi$ is in general not Noetherian and $\dim_{\mathbb{C}\pi}(M)$ is in general not additive under exact sequences.
- We will use the following successful approach which is essentially due to Atiyah [1].

Group von Neumann algebras

- Throughout these lectures let G be a discrete group.
- Given a ring R and a group G, denote by RG or $R[G]$ the group ring.
- Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients r_g are non-zero.
- Addition is given by adding the coefficients.
- Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of $\cdot$).
- In general RG is a very complicated ring.

- Denote by $L^2(G)$ the Hilbert space of (formal) sums $\sum_{g \in G} \lambda_g \cdot g$ such that $\lambda_g \in \mathbb{C}$ and $\sum_{g \in G} |\lambda_g|^2 < \infty$.

Definition

Define the group von Neumann algebra

$$\mathcal{N}(G) := \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}$$

to be the algebra of bounded G-equivariant operators $L^2(G) \to L^2(G)$. The von Neumann trace is defined by

$$\mathrm{tr}_{\mathcal{N}(G)} \colon \mathcal{N}(G) \to \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

Example (Finite G)

If G is finite, then $\mathbb{C}G = L^2(G) = \mathcal{N}(G)$. The trace $\mathrm{tr}_{\mathcal{N}(G)}$ assigns to $\sum_{g \in G} \lambda_g \cdot g$ the coefficient λ_e.

Example ($G = \mathbb{Z}^n$)

Let G be $\mathbb{Z}^n$. Let $L^2(T^n)$ be the Hilbert space of L^2-integrable functions $T^n \to \mathbb{C}$. Fourier transform yields an isometric $\mathbb{Z}^n$-equivariant isomorphism

$$L^2(\mathbb{Z}^n) \xrightarrow{\cong} L^2(T^n).$$

Let $L^\infty(T^n)$ be the Banach space of essentially bounded measurable functions $f\colon T^n \to \mathbb{C}$. We obtain an isomorphism

$$L^\infty(T^n) \xrightarrow{\cong} \mathcal{N}(\mathbb{Z}^n), \quad f \mapsto M_f$$

where $M_f\colon L^2(T^n) \to L^2(T^n)$ is the bounded $\mathbb{Z}^n$-operator $g \mapsto g \cdot f$.

Under this identification the trace becomes

$$\operatorname{tr}_{\mathcal{N}(\mathbb{Z}^n)}\colon L^\infty(T^n) \to \mathbb{C}, \quad f \mapsto \int_{T^n} f d\mu.$$

von Neumann dimension

Definition (Finitely generated Hilbert module)

A finitely generated Hilbert $\mathcal{N}(G)$-module V is a Hilbert space V together with a linear isometric G-action such that there exists an isometric linear G-embedding of V into $L^2(G)^n$ for some $n \geq 0$.
A map of finitely generated Hilbert $\mathcal{N}(G)$-modules $f\colon V \to W$ is a bounded G-equivariant operator.

Definition (von Neumann dimension)

Let V be a finitely generated Hilbert $\mathcal{N}(G)$-module. Choose a G-equivariant projection $p\colon L^2(G)^n \to L^2(G)^n$ with $\operatorname{im}(p) \cong_{\mathcal{N}(G)} V$.
Define the von Neumann dimension of V by

$$\dim_{\mathcal{N}(G)}(V) := \operatorname{tr}_{\mathcal{N}(G)}(p) := \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(p_{i,i}) \quad \in \mathbb{R}^{\geq 0}.$$

Example (Finite G)

For finite G a finitely generated Hilbert $\mathcal{N}(G)$-module V is the same as a unitary finite dimensional G-representation and

$$\dim_{\mathcal{N}(G)}(V) = \frac{1}{|G|} \cdot \dim_{\mathbb{C}}(V).$$

Example ($G = \mathbb{Z}^n$)

Let G be $\mathbb{Z}^n$. Let $X \subset T^n$ be any measurable set with characteristic function $\chi_X \in L^\infty(T^n)$. Let $M_{\chi_X} \colon L^2(T^n) \to L^2(T^n)$ be the $\mathbb{Z}^n$-equivariant unitary projection given by multiplication with χ_X. Its image V is a Hilbert $\mathcal{N}(\mathbb{Z}^n)$-module with

$$\dim_{\mathcal{N}(\mathbb{Z}^n)}(V) = \mathrm{vol}(X).$$

In particular each $r \in \mathbb{R}^{\geq 0}$ occurs as $r = \dim_{\mathcal{N}(\mathbb{Z}^n)}(V)$.

Definition (Weakly exact)

A sequence of Hilbert $\mathcal{N}(G)$-modules $U \xrightarrow{i} V \xrightarrow{p} W$ is weakly exact at V if the kernel $\ker(p)$ of p and the closure $\overline{\operatorname{im}(i)}$ of the image $\operatorname{im}(i)$ of i agree.

A map of Hilbert $\mathcal{N}(G)$-modules $f \colon V \to W$ is a weak isomorphism if it is injective and has dense image.

Example

The morphism of $\mathcal{N}(\mathbb{Z})$-Hilbert modules

$$M_{z-1} \colon L^2(\mathbb{Z}) = L^2(S^1) \to L^2(\mathbb{Z}) = L^2(S^1), \quad u(z) \mapsto (z-1) \cdot u(z)$$

is a weak isomorphism, but not an isomorphism.

Theorem (Main properties of the von Neumann dimension)

1. *Faithfulness*

   We have for a finitely generated Hilbert $\mathcal{N}(G)$-module V

   $$V = 0 \Longleftrightarrow \dim_{\mathcal{N}(G)}(V) = 0;$$

2. *Additivity*

   If $0 \to U \to V \to W \to 0$ is a weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules, then

   $$\dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V);$$

3. *Cofinality*

   Let $\{V_i \mid i \in I\}$ be a directed system of Hilbert $\mathcal{N}(G)$- submodules of V, directed by inclusion. Then

   $$\dim_{\mathcal{N}(G)}\left(\overline{\bigcup_{i \in I} V_i}\right) = \sup\{\dim_{\mathcal{N}(G)}(V_i) \mid i \in I\}.$$

L^2-homology and L^2-Betti numbers

Definition (L^2-homology and L^2-Betti numbers)

Let X be a connected CW-complex of finite type. Let $\widetilde{X}$ be its universal covering and $\pi = \pi_1(M)$. Denote by $C_*(\widetilde{X})$ its cellular $\mathbb{Z}\pi$-chain complex.

Define its cellular L^2-chain complex to be the Hilbert $\mathcal{N}(\pi)$-chain complex

$$C_*^{(2)}(\widetilde{X}) := L^2(\pi) \otimes_{\mathbb{Z}\pi} C_*(\widetilde{X}) = \overline{C_*(\widetilde{X})}.$$

Define its n-th L^2-homology to be the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\widetilde{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Define its n-th L^2-Betti number

$$b_n^{(2)}(\widetilde{X}) := \dim_{\mathcal{N}(\pi)} \left(H_n^{(2)}(\widetilde{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

Theorem (Main properties of L^2-Betti numbers)

Let X and Y be connected CW-complexes of finite type.

- *Homotopy invariance*

  If X and Y are homotopy equivalent, then

$$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{Y});$$

- *Euler-Poincaré formula*

  We have

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X});$$

- *Poincaré duality*

  Let M be a closed manifold of dimension d. Then

$$b_n^{(2)}(\widetilde{M}) = b_{d-n}^{(2)}(\widetilde{M});$$

Theorem (Continued)

- *Künneth formula*

$$b_n^{(2)}(\widetilde{X \times Y}) = \sum_{p+q=n} b_p^{(2)}(\widetilde{X}) \cdot b_q^{(2)}(\widetilde{Y});$$

- *Zero-th L^2-Betti number*

  We have

$$b_0^{(2)}(\widetilde{X}) = \frac{1}{|\pi|};$$

- *Finite coverings*

  If $X \to Y$ is a finite covering with d sheets, then

$$b_n^{(2)}(\widetilde{X}) = d \cdot b_n^{(2)}(\widetilde{Y}).$$

Example (Finite π)

If π is finite then

$$b_n^{(2)}(\widetilde{X}) = \frac{b_n(\widetilde{X})}{|\pi|}.$$

Example (S^1)

Consider the $\mathbb{Z}$-CW-complex $\widetilde{S^1}$. We get for $C_*^{(2)}(\widetilde{S^1})$

$$\ldots \to 0 \to L^2(\mathbb{Z}) \xrightarrow{M_{z-1}} L^2(\mathbb{Z}) \to 0 \to \ldots$$

and hence $H_n^{(2)}(\widetilde{S^1}) = 0$ and $b_n^{(2)}(\widetilde{S^1}) = 0$ for all ≥ 0.

Example ($\pi = \mathbb{Z}^d$)

Let X be a connected CW-complex of finite type with fundamental group $\mathbb{Z}^d$. Let $\mathbb{C}[\mathbb{Z}^d]^{(0)}$ be the quotient field of the commutative integral domain $\mathbb{C}[\mathbb{Z}^d]$. Then

$$b_n^{(2)}(\widetilde{X}) = \dim_{\mathbb{C}[\mathbb{Z}^d]^{(0)}} \left(\mathbb{C}[\mathbb{Z}^d]^{(0)} \otimes_{\mathbb{Z}[\mathbb{Z}^d]} H_n(\widetilde{X})\right)$$

Obviously this implies

$$b_n^{(2)}(\widetilde{X}) \in \mathbb{Z}.$$

- For a discrete group G we can consider more generally any free finite G-CW-complex $\overline{X}$ which is the same as a G-covering $\overline{X} \to X$ over a finite CW-complex X. (Actually proper finite G-CW-complex suffices.)

- The universal covering $p\colon \widetilde{X} \to X$ over a connected finite CW-complex is a special case for $G = \pi_1(X)$.

- Then one can apply the same construction to the finite free $\mathbb{Z}G$-chain complex $C_*(\overline{X})$. Thus we obtain the finitely generated Hilbert $\mathcal{N}(G)$-module

$$H_n^{(2)}(\overline{X};\mathcal{N}(G)) := H_n^{(2)}(L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})),$$

and define

$$b_n^{(2)}(\overline{X};\mathcal{N}(G)) := \dim_{\mathcal{N}(G)}\big(H_n^{(2)}(\overline{X};\mathcal{N}(G))\big) \in \mathbb{R}^{\geq 0}.$$

- Let $i\colon H \to G$ be an injective group homomorphism and C_* be a finite free $\mathbb{Z}H$-chain complex.
- Then $i_* C_* := \mathbb{Z}G \otimes_{\mathbb{Z}H} C_*$ is a finite free $\mathbb{Z}G$-chain complex.
- We have the following formula

$$\dim_{\mathcal{N}(G)}\big(H_n^{(2)}(L^2(G) \otimes_{\mathbb{Z}G} i_* C_*)\big) = \dim_{\mathcal{N}(H)}\big(H_n^{(2)}(L^2(H) \otimes_{\mathbb{Z}H} C_*)\big).$$

Lemma

If $\overline{X}$ is a finite free H-CW-complex, then we get

$$b_n^{(2)}(i_*\overline{X}; \mathcal{N}(G)) = b_n^{(2)}(\overline{X}; \mathcal{N}(H)).$$

- The corresponding statement is wrong if we drop the condition that i is injective.

- An example comes from $p\colon \mathbb{Z} \to \{1\}$ and $\widetilde{X} = \widetilde{S^1}$ since then $p_*\widetilde{S^1} = S^1$ and we have for $n = 0, 1$

$$b_n^{(2)}(\widetilde{S^1}; \mathcal{N}(\mathbb{Z})) = b_n^{(2)}(\widetilde{S^1}) = 0,$$

and

$$b_n^{(2)}(p_*\widetilde{S^1}; \mathcal{N}(\{1\})) = b_n(S^1) = 1.$$

The L^2-Mayer Vietoris sequence

Lemma

Let $0 \to C_*^{(2)} \xrightarrow{i_*^{(2)}} D_*^{(2)} \xrightarrow{p_*^{(2)}} E_*^{(2)} \to 0$ *be a weakly exact sequence of finite Hilbert* $\mathcal{N}(G)$*-chain complexes.*

Then there is a long weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$*-modules*

$$\cdots \xrightarrow{\delta_{n+1}^{(2)}} H_n^{(2)}(C_*^{(2)}) \xrightarrow{H_n^{(2)}(i_*^{(2)})} H_n^{(2)}(D_*^{(2)}) \xrightarrow{H_n^{(2)}(p_*^{(2)})} H_n^{(2)}(E_*^{(2)})$$
$$\xrightarrow{\delta_n^{(2)}} H_{n-1}^{(2)}(C_*^{(2)}) \xrightarrow{H_{n-1}^{(2)}(i_*^{(2)})} H_{n-1}^{(2)}(D_*^{(2)})$$
$$\xrightarrow{H_{n-1}^{(2)}(p_*^{(2)})} H_{n-1}^{(2)}(E_*^{(2)}) \xrightarrow{\delta_{n-1}^{(2)}} \cdots .$$

Lemma

Let

$$\begin{array}{ccc} \overline{X_0} & \longrightarrow & \overline{X_1} \\ \downarrow & & \downarrow \\ \overline{X_2} & \longrightarrow & \overline{X} \end{array}$$

be a cellular G-pushout of finite free G-CW-complexes, i.e., a G-pushout, where the upper arrow is an inclusion of a pair of free finite G-CW-complexes and the left vertical arrow is cellular.

Then we obtain a long weakly exact sequence of finitely generated Hilbert $\mathcal{N}(G)$-modules

$$\begin{aligned} \cdots \to H_n^{(2)}(\overline{X_0};\mathcal{N}(G)) \to H_n^{(2)}(\overline{X_1};\mathcal{N}(G)) \oplus H_n^{(2)}(\overline{X_2};\mathcal{N}(G)) \\ \to H_n^{(2)}(\overline{X};\mathcal{N}(G)) \to H_{n-1}^{(2)}(\overline{X_0};\mathcal{N}(G)) \\ \to H_{n-1}^{(2)}(\overline{X_1};\mathcal{N}(G)) \oplus H_{n-1}^{(2)}(\overline{X_2};\mathcal{N}(G)) \to H_{n-1}^{(2)}(\overline{X};\mathcal{N}(G)) \to \cdots . \end{aligned}$$

Proof.

- From the cellular G-pushout we obtain an exact sequence of $\mathbb{Z}G$-chain complexes

$$0 \to C_*(\overline{X_0}) \to C_*(\overline{X_1}) \oplus C_*(\overline{X_2}) \to C_*(\overline{X}) \to 0.$$

- It induces an exact sequence of finite Hilbert $\mathcal{N}(G)$-chain complexes

$$0 \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X_0}) \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X_1}) \oplus L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X_2}) \\ \to L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X}) \to 0.$$

- Now apply the previous result.

□

Definition (L^2-acyclic)

A finite (not necessarily connected) CW-complex X is called L^2-acyclic, if $b_n^{(2)}(\widetilde{C}) = 0$ holds for every $C \in \pi_0(X)$ and $n \in \mathbb{Z}$.

- If X is a finite (not necessarily connected) CW-complex, we define

$$b_n^{(2)}(\widetilde{X}) := \sum_{C \in \pi_0(X)} b_n^{(2)}(\widetilde{C}) \in \mathbb{R}^{\geq 0}.$$

Definition (π_1-injective)

A map $X \to Y$ is called π_1-injective, if for every choice of base point in X the induced map on the fundamental groups is injective.

- Consider a cellular pushout of finite CW-complexes

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

such that each of the maps $X_i \to X$ is π_1-injective.

Lemma

We get under the assumptions above for any $n \in \mathbb{Z}$

- *If X_0 is L^2-acyclic, then*

$$b_n^{(2)}(\widetilde{X}) = b_n^{(2)}(\widetilde{X}_1) + b_n^{(2)}(\widetilde{X_2}).$$

- *If X_0, X_1 and X_2 are L^2-cyclic, then X is L^2-acyclic.*

Proof.

- Without loss of generality we can assume that X is connected.
- By pulling back the universal covering $\widetilde{X} \to X$ to X_i, we obtain a cellular $\pi = \pi_1(X)$-pushout

$$\begin{array}{ccc} \overline{X}_0 & \longrightarrow & \overline{X}_1 \\ \downarrow & & \downarrow \\ \overline{X}_2 & \longrightarrow & \widetilde{X} \end{array}$$

- Notice that $\overline{X}_i$ is in general not the universal covering of X_i.

□

Proof continued.

- Because of the associated long exact L^2-sequence and the weak exactness of the von Neumann dimension, it suffices to show for $n \in \mathbb{Z}$ and $i = 1, 2$

$$\begin{array}{rcl} H_n^{(2)}(\overline{X_0}; \mathcal{N}(\pi)) & = & 0; \\ b_n^{(2)}(\overline{X_i}; \mathcal{N}(\pi)) & = & b_n^{(2)}(\widetilde{X}_i). \end{array}$$

- This follows from π_1-injectivity, the lemma above about L^2-Betti numbers and induction, the assumption that X_0 is L^2-acyclic, and the faithfulness of the von Neumann dimension.

□

Some computations and results

Example (Finite self coverings)

We get for a connected CW-complex X of finite type, for which there is a selfcovering $X \to X$ with d-sheets for some integer $d \geq 2$,

$$b_n^{(2)}(\widetilde{X}) = 0 \quad \text{for } n \geq 0.$$

This implies for each connected CW-complex Y of finite type that $S^1 \times Y$ is L^2-acyclic.

Example (L^2-Betti number of surfaces)

- Let F_g be the orientable closed surface of genus $g \geq 1$.
- Then $|\pi_1(F_g)| = \infty$ and hence $b_0^{(2)}(\widetilde{F_g}) = 0$.
- By Poincaré duality $b_2^{(2)}(\widetilde{F_g}) = 0$.
- Since $\dim(F_g) = 2$, we get $b_n^{(2)}(\widetilde{F_g}) = 0$ for $n \geq 3$.
- The Euler-Poincaré formula shows

$$\begin{aligned} b_1^{(2)}(\widetilde{F_g}) &= -\chi(F_g) = 2g - 2; \\ b_n^{(2)}(\widetilde{F_0}) &= 0 \quad \text{for} \quad n \neq 1. \end{aligned}$$

Theorem (S^1-actions, Lück)

Let M be a connected compact manifold with S^1-action. Suppose that for one (and hence all) $x \in X$ the map $S^1 \to M, \quad z \mapsto zx$ is π_1-injective.

Then M is L^2-acyclic.

Proof.

Each of the S^1-orbits S^1/H in M satisfies $S^1/H \cong S^1$. Now use induction over the number of cells $S^1/H_i \times D^n$ and a previous result using π_1-injectivity and the vanishing of the L^2-Betti numbers of spaces of the shape $S^1 \times X$. □

Theorem (S^1-actions on aspherical manifolds, Lück)

Let M be an aspherical closed manifold with non-trivial S^1-action. Then

1. *The action has no fixed points;*
2. *The map $S^1 \to M, \;\; z \mapsto zx$ is π_1-injective for $x \in M$;*
3. $b_n^{(2)}(\widetilde{M}) = 0$ *for* $n \geq 0$ *and* $\chi(M) = 0$.

Proof.

The hard part is to show that the second assertion holds, since M is aspherical. Then the first assertion is obvious and the third assertion follows from the previous theorem. □

Theorem (L^2-Hodge - de Rham Theorem, Dodziuk [2])

Let M be a closed Riemannian manifold. Put

$$\mathcal{H}^n_{(2)}(\widetilde{M}) = \{\widetilde{\omega} \in \Omega^n(\widetilde{M}) \mid \widetilde{\Delta}_n(\widetilde{\omega}) = 0, \; ||\widetilde{\omega}||_{L^2} < \infty\}$$

Then integration defines an isomorphism of finitely generated Hilbert $\mathcal{N}(\pi)$-modules

$$\mathcal{H}^n_{(2)}(\widetilde{M}) \xrightarrow{\cong} H^n_{(2)}(\widetilde{M}).$$

Corollary (L^2-Betti numbers and heat kernels)

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\; dvol\,.$$

where $e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{y})$ is the heat kernel on $\widetilde{M}$ and $\mathcal{F}$ is a fundamental domain for the π-action.

Theorem (hyperbolic manifolds, Dodziuk [3])

Let M be a hyperbolic closed Riemannian manifold of dimension d. Then:

$$b_n^{(2)}(\widetilde{M}) = \begin{cases} = 0 & , \text{if } 2n \neq d; \\ > 0 & , \text{if } 2n = d. \end{cases}$$

Proof.

A direct computation shows that $\mathcal{H}^p_{(2)}(\mathbb{H}^d)$ is not zero if and only if $2n = d$. Notice that M is hyperbolic if and only if $\widetilde{M}$ is isometrically diffeomorphic to the standard hyperbolic space $\mathbb{H}^d$. □

Corollary

Let M be a hyperbolic closed manifold of dimension d. Then

1. *If $d = 2m$ is even, then*

$$(-1)^m \cdot \chi(M) > 0;$$

2. *M carries no non-trivial S^1-action.*

Proof.

(1) We get from the Euler-Poincaré formula and the last result

$$(-1)^m \cdot \chi(M) = b_m^{(2)}(\widetilde{M}) > 0.$$

(2) We give the proof only for $d = 2m$ even. Then $b_m^{(2)}(\widetilde{M}) > 0$. Since $\widetilde{M} = \mathbb{H}^d$ is contractible, M is aspherical. Now apply a previous result about S^1-actions. □

Theorem (3-manifolds, Lott-Lück [7])

Let the 3*-manifold* M *be the connected sum* $M_1 \sharp \ldots \sharp M_r$ *of (compact connected orientable) prime* 3*-manifolds* M_j*. Assume that* $\pi_1(M)$ *is infinite. Then*

$$\begin{aligned}
b_1^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} - \chi(M) \\
&\quad + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|; \\
b_2^{(2)}(\widetilde{M}) &= (r-1) - \sum_{j=1}^{r} \frac{1}{|\pi_1(M_j)|} \\
&\quad + \left|\{C \in \pi_0(\partial M) \mid C \cong S^2\}\right|; \\
b_n^{(2)}(\widetilde{M}) &= 0 \quad \text{for } n \neq 1, 2.
\end{aligned}$$

Proof.

- We have already explained why a closed hyperbolic 3-manifold is L^2-acyclic.
- One of the hard parts of the proof is to show that this is also true for any hyperbolic 3-manifold with incompressible toral boundary.
- Recall that these have finite volume.
- One has to introduce appropriate boundary conditions and Sobolev theory to write down the relevant analytic L^2-deRham complexes and L^2-Laplace operators.
- A key ingredient is the decomposition of such a manifold into its core and a finite number of cusps.

□

Proof continued.

- This can be used to write the L^2-Betti number as an integral over a fundamental domain $\mathcal{F}$ of finite volume, where the integrand is given by data depending on $\mathbb{H}^3$ only:

$$b_n^{(2)}(\widetilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}(e^{-t\widetilde{\Delta}_n}(\tilde{x},\tilde{x}))\, d\mathrm{vol}\,.$$

- Since $\mathbb{H}^3$ has a lot of symmetries, the integrand does not depend on $\tilde{x}$ and is a constant C_n depending only on $\mathbb{H}^3$.

- Hence we get

$$b_n^{(2)}(\widetilde{M}) = C_n \cdot \mathrm{vol}(M).$$

- From the closed case we deduce $C_n = 0$.

□

Proof continued.

- Next we show that any Seifert manifold with infinite fundamental group is L^2-acyclic.
- This follows from the fact that such a manifold is finitely covered by the total space of an S^1-bundle $S^1 \to E \to F$ over a surface with injective $\pi_1(S^1) \to \pi_1(E)$ using previous results.
- In the next step one shows that any irreducible 3-manifold M with incompressible or empty boundary and infinite fundamental group is L^2-acyclic.
- Recall that by the Thurston Geometrization Conjecture we can find a family of incompressible tori which decompose M into hyperbolic and Seifert pieces. The tori and all these pieces are L^2-acyclic.
- Now the claim follows from the L^2-Mayer Vietoris sequence.

□

Proof continued.

- In the next step one shows that any irreducible 3-manifold M with incompressible boundary and infinite fundamental group satisfies $b_1^{(2)}(\widetilde{M}) = -\chi(M)$ and $b_n^{(2)}(\widetilde{M}) = 0$ for $n \neq 1$.

- This follows by considering $N = M \cup_{\partial M} M$ using the L^2-Mayer-Vietoris sequence, the already proved fact that N is L^2-acyclic and the previous computation of the L^2-Betti numbers for surfaces.

- In the next step one shows that any irreducible 3-manifold M with infinite fundamental group satisfies $b_1^{(2)}(\widetilde{M}) = -\chi(M)$ and $b_n^{(2)}(\widetilde{M}) = 0$ for $n \neq 1$.

□

Proof continued.

- This is reduced by an iterated application of the Loop Theorem to the case where the boundary is incompressible. Namely, using the Loop Theorem one gets an embedded disk $D^2 \subseteq M$ along which one can decompose M as $M_1 \cup_{D^2} M_2$ or as $M_1 \cup_{S^0 \times D^2} D^1 \times D^2$ depending on whether D^2 is separating or not.
- Since the only prime 3-manifold that is not irreducible is $S^1 \times S^2$, and every manifold M with finite fundamental group satisfies the result by a direct inspection of the Betti numbers of its universal covering, the claim is proved for all prime 3-manifolds.
- Finally one uses the L^2-Mayer Vietoris sequence to prove the claim in general using the prime decomposition.

□

Corollary

Let M be a 3-manifold. Then M is L^2-acyclic if and only if one of the following cases occur:

- *M is an irreducible 3-manifold with infinite fundamental group whose boundary is empty or toral.*
- *M is $S^1 \times S^2$ or $\mathbb{RP}^3 \sharp \mathbb{RP}^3$.*

Corollary

Let M be a compact n-manifold such that $n \leq 3$ and its fundamental group is torsionfree.

Then all its L^2-Betti numbers are integers.

Theorem (mapping tori, Lück [9])

Let $f\colon X \to X$ be a cellular selfhomotopy equivalence of a connected CW-complex X of finite type. Let T_f be the mapping torus. Then

$$b_n^{(2)}(\widetilde{T_f}) = 0 \quad \textit{for } n \geq 0.$$

Proof.

- As $T_{f^d} \to T_f$ is up to homotopy a d-sheeted covering, we get

$$b_n^{(2)}(\widetilde{T_f}) = \frac{b_n^{(2)}(\widetilde{T_{f^d}})}{d}.$$

□

Proof continued.

- If $\beta_n(X)$ is the number of n-cells, then there is up to homotopy equivalence a CW-structure on T_{f^d} with $\beta_n(T_{f^d}) = \beta_n(X) + \beta_{n-1}(X)$. We have

$$b_n^{(2)}(\widetilde{T_{f^d}}) = \dim_{\mathcal{N}(G)}\left(H_n^{(2)}(C_n^{(2)}(\widetilde{T_{f^d}}))\right) \leq \dim_{\mathcal{N}(G)}\left(C_n^{(2)}(\widetilde{T_{f^d}})\right) = \beta_n(T_{f^d}).$$

- This implies for all $d \geq 1$

$$b_n^{(2)}(\widetilde{T}_f) \leq \frac{\beta_n(X) + \beta_{n-1}(X)}{d}.$$

- Taking the limit for $d \to \infty$ yields the claim.

□

- Let M be an irreducible manifold M with infinite fundamental group and empty or incompressible toral boundary which is not a closed graph manifold.
- Agol proved the Virtually Fibering Conjecture for such M.
- This implies by the result above that M is L^2-acyclic.

The fundamental square and the Atiyah Conjecture

Conjecture (Atiyah Conjecture for torsionfree finitely presented groups)

Let G be a torsionfree finitely presented group. We say that G satisfies the Atiyah Conjecture if for any closed Riemannian manifold M with $\pi_1(M) \cong G$ we have for every $n \geq 0$

$$b_n^{(2)}(\widetilde{M}) \in \mathbb{Z}.$$

- All computations presented above support the Atiyah Conjecture.

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

- If G is finite, its is given by

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{C}G \\ \downarrow & & \downarrow \mathrm{id} \\ \mathbb{Q}G & \longrightarrow & \mathbb{C}G \end{array}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[\mathbb{Z}]^{(0)} & \longrightarrow & L(S^1) \end{array}$$

- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.

- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- A torsionfree group G satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension

$$\dim_{\mathcal{N}(G)}\big(\ker\big(r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n\big)\big)$$

  is an integer. In this case this dimension agrees with

$$\dim_{\mathcal{D}(G)}\big(\ker\big(r_A \colon \mathcal{D}(G)^m \to \mathcal{D}(G)^n\big)\big).$$

- The general version above is equivalent to the one stated before if G is finitely presented.

- The Atiyah Conjecture implies the Zero-divisor Conjecture due to Kaplansky saying that for any torsionfree group and field of characteristic zero F the group ring FG has no non-trivial zero-divisors.

- There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.

- However, there exist closed Riemannian manifolds whose universal coverings have an L^2-Betti number which is irrational, see Austin, Grabowski [4].

Theorem (Linnell [6], Schick [11])

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.*
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

Strategy to prove the Atiyah Conjecture

1. Show that $K_0(\mathbb{C}) \to K_0(\mathbb{C}G)$ is surjective
   (This is implied by the Farrell-Jones Conjecture)
2. Show that $K_0(\mathbb{C}G) \to K_0(\mathcal{D}(G))$ is surjective.
3. Show that $\mathcal{D}(G)$ is semisimple.

Approximation

- In general there are no relations between the Betti numbers $b_n(X)$ and the L^2-Betti numbers $b_n^{(2)}(\widetilde{X})$ for a connected CW-complex X of finite type except for the Euler Poincaré formula

$$\chi(X) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{X}) = \sum_{n \geq 0} (-1)^n \cdot b_n(X).$$

- Given an integer $l \geq 1$ and a sequence $r_1, r_2, \ldots, r_l$ of non-negative rational numbers, we can construct a group G such that BG is of finite type and

$$\begin{array}{rclr} b_n^{(2)}(BG) & = & r_n & \text{for } 1 \leq n \leq l; \\ b_n^{(2)}(BG) & = & 0 & \text{for } l+1 \leq n; \\ b_n(BG) & = & 0 & \text{for } n \geq 1. \end{array}$$

- For any sequence $s_1, s_2, \ldots$ of non-negative integers there is a CW-complex X of finite type such that for $n \geq 1$

$$\begin{array}{rcl} b_n(X) & = & s_n; \\ b_n^{(2)}(\widetilde{X}) & = & 0. \end{array}$$

Theorem (Approximation Theorem, Lück [8])

Let X be a connected CW-complex of finite type. Suppose that π is residually finite, i.e., there is a nested sequence

$$\pi = G_0 \supset G_1 \supset G_2 \supset \ldots$$

of normal subgroups of finite index with $\cap_{i \geq 1} G_i = \{1\}$. Let X_i be the finite $[\pi : G_i]$-sheeted covering of X associated to G_i.

Then for any such sequence $(G_i)_{i \geq 1}$

$$b_n^{(2)}(\widetilde{X}) = \lim_{i \to \infty} \frac{b_n(X_i)}{[G : G_i]}.$$

- Ordinary Betti numbers are not multiplicative under finite coverings, whereas the L^2-Betti numbers are. With the expression

$$\lim_{i\to\infty} \frac{b_n(X_i)}{[G : G_i]},$$

we try to force the Betti numbers to be multiplicative by a limit process.

- The theorem above says that L^2-Betti numbers are asymptotic Betti numbers. It was conjectured by Gromov.

Definition (Deficiency)

Let G be a finitely presented group. Define its deficiency

$$\operatorname{defi}(G) := \max\{g(P) - r(P)\}$$

where P runs over all presentations P of G and $g(P)$ is the number of generators and $r(P)$ is the number of relations of a presentation P.

Example

- The free group F_g has the obvious presentation $\langle s_1, s_2, \dots s_g \mid \emptyset \rangle$ and its deficiency is realized by this presentation, namely $\operatorname{defi}(F_g) = g$.
- If G is a finite group, $\operatorname{defi}(G) \leq 0$.
- The deficiency of a cyclic group $\mathbb{Z}/n$ is 0, the obvious presentation $\langle s \mid s^n \rangle$ realizes the deficiency.
- The deficiency of $\mathbb{Z}/n \times \mathbb{Z}/n$ is -1, the obvious presentation $\langle s, t \mid s^n, t^n, [s, t] \rangle$ realizes the deficiency.

Example (deficiency and free products)

The deficiency is not additive under free products by the following example due to Hog-Lustig-Metzler. The group

$$(\mathbb{Z}/2 \times \mathbb{Z}/2) * (\mathbb{Z}/3 \times \mathbb{Z}/3)$$

has the obvious presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = t_0^2 = [s_0, t_0] = s_1^3 = t_1^3 = [s_1, t_1] = 1 \rangle$$

One may think that its deficiency is -2. However, it turns out that its deficiency is -1 realized by the following presentation

$$\langle s_0, t_0, s_1, t_1 \mid s_0^2 = 1, [s_0, t_0] = t_0^2, s_1^3 = 1, [s_1, t_1] = t_1^3, t_0^2 = t_1^3 \rangle.$$

Lemma

Let G be a finitely presented group. Then

$$\operatorname{defi}(G) \;\leq\; 1 - |G|^{-1} + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Proof.

We have to show for any presentation P that

$$g(P) - r(P) \;\leq\; 1 - b_0^{(2)}(G) + b_1^{(2)}(G) - b_2^{(2)}(G).$$

Let X be a CW-complex realizing P. Then

$$\chi(X) = 1 - g(P) + r(P) = b_0^{(2)}(\widetilde{X}) + b_1^{(2)}(\widetilde{X}) - b_2^{(2)}(\widetilde{X}).$$

Since the classifying map $X \to BG$ is 2-connected, we get

$$\begin{array}{rcll} b_n^{(2)}(\widetilde{X}) & = & b_n^{(2)}(G) & \quad \text{for } n = 0, 1; \\ b_2^{(2)}(\widetilde{X}) & \geq & b_2^{(2)}(G). & \end{array}$$

Theorem (Deficiency and extensions, Lück)

Let $1 \to H \xrightarrow{i} G \xrightarrow{q} K \to 1$ *be an exact sequence of infinite groups. Suppose that G is finitely presented and H is finitely generated. Then:*

1. $b_1^{(2)}(G) = 0$;
2. $\operatorname{defi}(G) \leq 1$;
3. *Let M be a closed oriented* 4*-manifold with G as fundamental group. Then*
$$|\operatorname{sign}(M)| \leq \chi(M).$$

Conjecture (Singer Conjecture)

If M is an aspherical closed manifold, then

$$b_n^{(2)}(\widetilde{M}) = 0 \quad \textit{if } 2n \neq \dim(M).$$

If M is a closed Riemannian manifold with negative sectional curvature, then

$$b_n^{(2)}(\widetilde{M}) \begin{cases} = 0 & \textit{if } 2n \neq \dim(M); \\ > 0 & \textit{if } 2n = \dim(M). \end{cases}$$

- The computations presented above do support the Singer Conjecture.
- Under certain negative pinching conditions the Singer Conjecture has been proved by Ballmann-Brüning, Donnelly-Xavier, Jost-Xin.
- The Singer Conjecture gives also evidence for the Atiyah Conjecture.

- Because of the Euler-Poincaré formula

$$\chi(M) = \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(\widetilde{M})$$

the Singer Conjecture implies the following conjecture provided that M has non-positive sectional curvature.

Conjecture (Hopf Conjecture)

If M is a closed Riemannian manifold of even dimension with sectional curvature $\operatorname{sec}(M)$, then

$$\begin{array}{rclcrcl}
(-1)^{\dim(M)/2} \cdot \chi(M) & > & 0 & \text{if} & \operatorname{sec}(M) & < & 0; \\
(-1)^{\dim(M)/2} \cdot \chi(M) & \geq & 0 & \text{if} & \operatorname{sec}(M) & \leq & 0; \\
\chi(M) & = & 0 & \text{if} & \operatorname{sec}(M) & = & 0; \\
\chi(M) & \geq & 0 & \text{if} & \operatorname{sec}(M) & \geq & 0; \\
\chi(M) & > & 0 & \text{if} & \operatorname{sec}(M) & > & 0.
\end{array}$$

Definition (Kähler hyperbolic manifold)

A Kähler hyperbolic manifold is a closed connected Kähler manifold M whose fundamental form ω is $\widetilde{d}$(bounded), i.e. its lift $\widetilde{\omega} \in \Omega^2(\widetilde{M})$ to the universal covering can be written as $d(\eta)$ holds for some bounded 1-form $\eta \in \Omega^1(\widetilde{M})$.

Theorem (Gromov [5])

Let M be a closed Kähler hyperbolic manifold of complex dimension c. Then

$$\begin{aligned} b_n^{(2)}(\widetilde{M}) &= 0 \quad \textit{if } n \neq c; \\ b_n^{(2)}(\widetilde{M}) &> 0; \\ (-1)^m \cdot \chi(M) &> 0; \end{aligned}$$

- Let M be a closed Kähler manifold. It is Kähler hyperbolic if it admits some Riemannian metric with negative sectional curvature, or, if, generally $\pi_1(M)$ is word-hyperbolic and $\pi_2(M)$ is trivial.
- A consequence of the theorem above is that any Kähler hyperbolic manifold is a projective algebraic variety.

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