The Farrell-Jones Conjecture (Lecture IV)

Wolfgang Lück
Bonn
Germany
email wolfgang.lueck@him.uni-bonn.de
http://131.220.77.52/lueck/

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Outline

- We introduce equivariant homology theories.
- We give the general formulation of the Farrell-Jones Conjecture.
- We investigate how small one can make the relevant family of subgroups and prove a Transitivity Principle.
- We give a status report about the Farrell-Jones Conjecture.
- We report on the search for counterexamples.

Equivariant homology theories

Definition (G-homology theory)

A G-homology theory $\mathcal{H}_*$ is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n(X, A) \colon \mathcal{H}_n(X, A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Exercise (Transformations of G-homology theories)

Let $\mathcal{H}_*^G$ and $\mathcal{K}_*^G$ be two G-homology theories and let $T^G \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ be a natural transformation between them. Let $\mathcal{C}$ be a set of subgroups of G closed under conjugation such that for every $H \in \mathcal{C}$ and $n \in \mathbb{Z}$ the map $T_n^G(G/H) \colon \mathcal{H}_n^G(G/H) \to \mathcal{K}_n^G(G/H)$ is bijective.

Show that then for any G-CW-complex X with isotropy groups in $\mathcal{C}$ and any $n \in \mathbb{Z}$ the map

$$T_n^G(X) \colon \mathcal{H}_n^G(X) \to \mathcal{K}_n^G(X)$$

is bijective.

Theorem (Bredon homology)

Consider any covariant functor

$$M\colon \mathrm{Or}G \to \Lambda\text{-Modules}.$$

Then there is up to natural equivalence of G-homology theories precisely one G-homology theory $H_*^G(-,M)$, called Bredon homology, with the property that the covariant functor

$$H_n^G\colon \mathrm{Or}G \to \Lambda\text{-Modules}, \quad G/H \mapsto H_n^G(G/H)$$

is trivial for $n \neq 0$ and naturally equivalent to M for $n = 0$.

- Let M be the constant functor with value the Λ-module A. Then we get for every G-CW-complex X

$$H_n^G(X;M) \cong_\Lambda H_n(X/G;A)$$

Definition (Equivariant homology theory)

An equivariant homology theory $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$.

These are linked together with the following so called induction structure: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A), there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\operatorname{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \quad\to\quad \mathcal{H}_n^G(\operatorname{ind}_\alpha(X,A))$$

satisfying

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then $\operatorname{ind}_\alpha$ is a bijection;
- Compatibility with the boundary homomorphisms;
- Functoriality in α;
- Compatibility with conjugation.

Example (Equivariant homology theories)

- Given a non-equivariant homology theory $\mathcal{K}_*$, put

$$\begin{aligned} \mathcal{H}_*^G(X) &:= \mathcal{K}_*(X/G); \\ \mathcal{H}_*^G(X) &:= \mathcal{K}_*(EG \times_G X) \quad \text{(Borel homology)}. \end{aligned}$$

- Equivariant bordism $\Omega_*^?(X)$ (for proper G-spaces);
- Equivariant topological K-theory $K_*^?(X)$ (for proper G-spaces). It is two-periodic, i.e., $K_n^G(X) \cong K_{n+2}^G(X)$, and satisfies for a finite subgroup $H \subseteq G$

$$K_n^G(G/H) = \begin{cases} R_{\mathbb{C}}(H) & n \text{ even}; \\ \{0\} & n \text{ odd}. \end{cases}$$

Theorem (Lück-Reich)

Given a functor $\mathbf{E}\colon \mathrm{Groupoids} \to \mathrm{Spectra}$ *sending equivalences to weak equivalences, there exists an equivariant homology theory* $\mathcal{H}^?_*(-;\mathbf{E})$ *satisfying*

$$\mathcal{H}^H_n(pt) \cong \mathcal{H}^G_n(G/H) \cong \pi_n(\mathbf{E}(H)).$$

Theorem (Equivariant homology theories associated to K and L-theory, Davis-Lück)

Let R be a ring (with involution). There exist covariant functors

$$\begin{aligned} \mathbf{K}_R \colon \mathrm{Groupoids} &\to \mathrm{Spectra}; \\ \mathbf{L}_R^{\langle\infty\rangle} \colon \mathrm{Groupoids} &\to \mathrm{Spectra}, \end{aligned}$$

with the following properties:

- *They send equivalences of groupoids to weak equivalences of spectra;*
- *For every group G and all $n \in \mathbb{Z}$ we have*

$$\begin{aligned} \pi_n(\mathbf{K}_R(G)) &\cong K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle -\infty\rangle}(G)) &\cong L_n^{\langle -\infty\rangle}(RG). \end{aligned}$$

Example (Equivariant homology theories associated to K and L-theory)

We get equivariant homology theories

$$H_*^?(-;\mathbf{K}_R);$$
$$H_*^?(-;\mathbf{L}_R^{\langle -\infty\rangle}),$$

satisfying for $H \subseteq G$

$$\begin{array}{lclcl} H_n^G(G/H;\mathbf{K}_R) & \cong & H_n^H(\mathrm{pt};\mathbf{K}_R) & \cong & K_n(RH); \\ H_n^G(G/H;\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & H_n^H(\mathrm{pt};\mathbf{L}_R^{\langle -\infty\rangle}) & \cong & L_n^{\langle -\infty\rangle}(RH). \end{array}$$

The general formulation of the Farrell-Jones Conjecture

Conjecture (K-theoretic Farrell-Jones Conjecture)

The K-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map, which is the map induced by the projection $E_{\mathcal{VCYC}}(G) \to pt$,

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- The basic idea is to understand the K-theory of RG in terms of its values on RV for all virtually cyclic subgroups V and just reduce the computation for general G to the virtually cyclic subgroups $V \subseteq G$.

- In general the right hand side is the hard part and the left side is the more accessible part since for equivariant homology theories there are methods for its computations available, for instance spectral sequences and equivariant Chern characters.

- Often the assembly maps have a more structural geometric or analytic description, which are more sophisticated and harder to construct, but link the Farrell-Jones Conjecture to interesting problems in geometry, topology, algebra or operator theory and are relevant for proofs.

Conjecture (*L*-theoretic Farrell-Jones Conjecture)

The L-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{L}_R^{\langle -\infty \rangle}) \to H_n^G(pt, \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG)$$

is bijective for all $n \in \mathbb{Z}$.

Conjecture (Baum-Connes Conjecture)

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(E_{\mathcal{FIN}}(G)) \to K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

- The assembly maps can also be interpreted in terms of homotopy colimits, where the functor of interest evaluated at G is assembled from its values on subgroups belonging to the relevant family.
- For instance, for K-theory (and analogously for L-theory) we get an interpretation of the assembly map as the canonical map

$$\operatorname{hocolim}_{V \in \mathcal{VCYC}} \mathbf{K}(RV) \to \mathbf{K}(RG).$$

- There are other theories for which one can formulate Isomorphism Conjectures in an analogous way, e.g., pseudoisotopy, Waldhausen's A-theory, topological Hochschild homology, topological cyclic homology.

Exercise (Trivial case)

Show that the Farrell-Jones Conjecture is true for G if G is virtually cyclic.

- Obviously the Farrell-Jones Conjecture is more valuable if we can make the family $\mathcal{VCYC}$ smaller. In general this is not possible but in some special cases this can be done using the Transitivity Principal which we explain next.

Changing the family

Theorem (Transitivity Principle)

Let $\mathcal{F} \subseteq \mathcal{G}$ be two families of subgroups of G. Let $\mathcal{H}_*^?$ be an equivariant homology theory. Assume that for every element $H \in \mathcal{G}$ and $n \in \mathbb{Z}$ the assembly map

$$\mathcal{H}_n^H(E_{\mathcal{F}|_H}(H)) \to \mathcal{H}_n^H(pt)$$

is bijective, where $\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}$.

Then the relative assembly map induced by the up to G-homotopy unique G-map $E_{\mathcal{F}}(G) \to E_{\mathcal{G}}(G)$

$$\mathcal{H}_n^G(E_{\mathcal{F}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G))$$

is bijective for all $n \in \mathbb{Z}$.

Proof.

- The projection $E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G) \to E_{\mathcal{F}}(G)$ is a G-homotopy equivalence. Hence it suffices to show that the projection $\mathrm{pr}\colon E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G) \to E_{\mathcal{G}}(G)$ induces for every $n \in \mathbb{Z}$ a bijection

$$\mathcal{H}_n^G(\mathrm{pr})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times E_{\mathcal{G}}(G)) \to \mathcal{H}_n^G(E_{\mathcal{G}}(G)).$$

- We prove the more general statement that for any G-CW-complex X with isotropy groups in $\mathcal{G}$ the projection $\mathrm{pr}_X\colon E_{\mathcal{F}}(G) \times X \to X$ induces for every $n \in \mathbb{Z}$ an isomorphism

$$\mathcal{H}_n^G(\mathrm{pr}_X)\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times X) \to \mathcal{H}_n^G(X).$$

- Since this can be interpreted as a transformation of G-homology theories in X, it suffices to show that for every $H \in \mathcal{G}$ and every $n \in \mathbb{Z}$ we get an isomorphism

$$\mathcal{H}_n^G(\mathrm{pr}_{G/H})\colon \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) \to \mathcal{H}_n^G(G/H).$$

Proof (continued).

- The are G-homeomorphisms

$$\operatorname{ind}_H^G \operatorname{res}_G^H E_{\mathcal{F}}(G) = G \times_H E_{\mathcal{F}}(G) \xrightarrow{\cong} E_{\mathcal{F}}(G) \times G/H$$

  sending (g, x) to (gx, gH) and $\operatorname{ind}_H^G \mathrm{pt} = G \times_H \mathrm{pt} \xrightarrow{\cong} G/H$.

- The induction structure yields a commutative diagram for pr the projection with isomorphisms as vertical maps

$$\begin{array}{ccc} \mathcal{H}_n^H(\operatorname{res}_G^H E_{\mathcal{F}}(G)) & \xrightarrow{\mathcal{H}_n^H(\mathrm{pr})} & \mathcal{H}^H(\mathrm{pt}) \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{H}_n^G(E_{\mathcal{F}}(G) \times G/H) & \xrightarrow[\mathcal{H}_n^G(\mathrm{pr}_{G/H})]{} & \mathcal{H}_n^G(G/H). \end{array}$$

- Since $\operatorname{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}|_H}(H)$, the claim follows.

Exercise (Restriction and classifying space for families)

Let G be a group and $\mathcal{F}$ be a family of subgroups of G. Let $H \subseteq G$ be a subgroup and define $\mathcal{F}|_H = \{K \cap H \mid K \in \mathcal{F}\}$.

Show that $\mathcal{F}|_H$ is a family of subgroups of H and that $\operatorname{res}_G^H E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}|_H}(H)$.

Example (Passage from $\mathcal{FIN}$ to $\mathcal{VCYC}$ for the Baum-Connes Conjecture)

- The Baum-Connes Conjecture is known to be true for virtually cyclic groups.
- The Transitivity Principle implies that the relative assembly

$$K_n^G(E_{\mathcal{FIN}}(G)) \xrightarrow{\cong} K_n^G(E_{\mathcal{VCYC}}(G))$$

  is bijective for all $n \in \mathbb{Z}$.
- Hence it does not matter in the context of the Baum-Connes Conjecture whether we consider the family $\mathcal{FIN}$ or $\mathcal{VCYC}$.
- Bartels-Lück have shown that in the Baum-Connes Conjecture one can replace $\mathcal{FIN}$ by the family of finite cyclic subgroups.

Example (Passage from $\mathcal{FIN}$ to $\mathcal{VCYC}$ for the Farrell-Jones Conjecture)

The Bass-Heller Swan decomposition

$$K_{n-1}(R) \oplus K_n(R) \oplus \mathrm{NK}_n(R) \oplus \mathrm{NK}_n(R)) \xrightarrow{\cong} K_n(R[t,t^{-1}]) \cong K_n(R[\mathbb{Z}])$$

and the isomorphism

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z};\mathbf{K}_R) = H_n^{\mathbb{Z}}(E\mathbb{Z};\mathbf{K}_R) = H_n^{\{1\}}(S^1,\mathbf{K}_R) = K_{n-1}(R) \oplus K_n(R)$$

show that

$$H_n^{\mathbb{Z}}(\underline{E}\mathbb{Z};\mathbf{K}_R) \to H_n^{\mathbb{Z}}(\mathrm{pt};\mathbf{K}_R) = K_n(R\mathbb{Z})$$

is bijective if and only if $\mathrm{NK}_n(R) = 0$.

- Hence in the Farrell-Jones setting one has to pass to $\mathcal{VCYC}$ and cannot use the easier to handle family $\mathcal{FIN}$.

Exercise (Torsionfree groups and regular rings)

Show that the K-theoretic Farrell-Jones Conjecture implies the version for torsionfree groups and regular rings.

Theorem (L-theory and torsionfree groups)

If the torsionfree group G satisfies the L-theoretic Farrell-Jones Conjecture, then it satisfies the L-theoretic Farrell-Jones Conjecture for torsionfree groups.

Proof.

- The Shaneson splitting shows the assembly map

$$H_n^{\mathbb{Z}}(E\mathbb{Z}; \mathbf{L}_R^{\langle -\infty \rangle}) \to L_n^{\langle -\infty \rangle}(R\mathbb{Z})$$

  is bijective for $n \in \mathbb{Z}$.
- Since every infinite torsionfree virtually cyclic group is isomorphic to $\mathbb{Z}$, we conclude from the Transitivity Principle that for any torsionfree group G and $n \in \mathbb{Z}$ the map

$$H_n^{\mathbb{Z}}(EG; \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} H_n^{\mathbb{Z}}(E_{\mathcal{VCYC}}(G); \mathbf{L}_R^{\langle -\infty \rangle})$$

  is bijective.
- The induction structure yields an isomorphism

$$H_n^G(EG; \mathbf{L}_R^{\langle -\infty \rangle}) \xrightarrow{\cong} H_n(BG; \mathbf{L}_R^{\langle -\infty \rangle}).$$

□

Proof (continued).

- Hence for a torsionfree group G the assembly map

$$H_n(BG;\mathbf{L}_R^{\langle -\infty\rangle}) \to L_n^{\langle -\infty\rangle}(\mathbb{Z}G)$$

is bijective, if the L-theoretic Farrell-Jones Conjecture holds for G.

□

Exercise (Computation of certain L-groups)

Compute $L_n(\mathbb{Z}G)$ for $n \in \mathbb{Z}$ for the group

$$G = \langle a_i, b_i, i = 1, 2, \cdots, k \mid \prod_{i=1}^{k}[a_i, b_i]\rangle.$$

- An infinite virtually cyclic group G is called of type I if it admits an epimorphism onto $\mathbb{Z}$ and of type II otherwise.
- A virtually cyclic group is of type II if and only if it admits an epimorphism onto D_∞.
- Let $\mathcal{VCYC}_I$ or $\mathcal{VCYC}_{II}$ respectively be the family of subgroups which are either finite or which are virtually cyclic of type I or II respectively.

Exercise (Virtually cyclic subgroups)

Show that an infinite virtually cyclic group G is of type I if and only if its abelianization is infinite.

Theorem (Lück, Quinn, Reich)

The following maps are bijective for all $n \in \mathbb{Z}$

$$\begin{array}{rcl} H_n^G(E_{\mathcal{VCY}_I}(G);\mathbf{K}_R) & \to & H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R); \\ H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) & \to & H_n^G(E_{\mathcal{VCY}_I}(G);\mathbf{L}_R^{\langle -\infty\rangle}). \end{array}$$

Theorem (Cappell, Grunewald, Waldhausen)

- *The following maps are bijective for all $n \in \mathbb{Z}$.*

$$\begin{array}{rcl} H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q} & \to & H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Q}; \\ H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle})\left[\frac{1}{2}\right] & \to & H_n^G(E_{\mathcal{VCY}}(G);\mathbf{L}_R^{\langle -\infty\rangle})\left[\frac{1}{2}\right]; \end{array}$$

- *If R is regular and $\mathbb{Q} \subseteq R$, then for all $n \in \mathbb{Z}$ we get a bijection*

$$H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \to H_n^G(E_{\mathcal{VCY}}(G);\mathbf{K}_R).$$

Exercise ($K_0(FG)$ and finite subgroups of G)

Let F be a field of characteristic zero. Suppose that G satisfies the K-theoretic Farrell-Jones Conjecture. Show that the obvious map

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \to K_0(FG)$$

is surjective.

Theorem (Bartels)

For every $n \in \mathbb{Z}$ the two maps

$$\begin{array}{rcl} H_n^G(E_{\mathcal{FIN}}(G); \mathbf{K}_R) & \to & H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{K}_R); \\ H_n^G(E_{\mathcal{FIN}}(G); \mathbf{L}_R^{\langle -\infty \rangle}) & \to & H_n^G(E_{\mathcal{VCYC}}(G); \mathbf{L}_R^{\langle -\infty \rangle}), \end{array}$$

are split injective.

- Hence we get (natural) isomorphisms

$$H_n^G(E_{\mathcal{VCYC}}(G);\mathbf{K}_R) \cong H_n^G(E_{\mathcal{FIN}}(G);\mathbf{K}_R) \oplus H_n^G(E_{\mathcal{VCYC}}(G),\underline{E}G;\mathbf{K}_R);$$

and

$$H_n^G(E_{\mathcal{VCYC}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \cong H_n^G(E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle}) \oplus H_n^G(E_{\mathcal{VCYC}}(G),\underline{E}G;\mathbf{L}_R^{\langle -\infty\rangle}).$$

- The analysis of the terms $H_n^G(E_{\mathcal{VCYC}}(G),E_{\mathcal{FIN}}(G);\mathbf{K}_R)$ and $H_n^G(E_{\mathcal{VCYC}}(G),E_{\mathcal{FIN}}(G);\mathbf{L}_R^{\langle -\infty\rangle})$ boils down to investigating Nil-terms and UNil-terms in the sense of Waldhausen and Cappell.

Status of the Farrell-Jones Conjecture

- There is also a version, which we will call the Full Farrell-Jones Conjecture, which works for all groups and rings and where one can even allow twisted group rings and non-trivial orientation homomorphisms in the L-theory case.

- For the experts, we mean the Farrell-Jones Conjecture for both K- and L-theory with coefficients in equivariant additive categories and with finite wreath products.

- The Full Farrell-Jones Conjecture implies the Farrell-Jones Conjectures stated above.

- One decisive advantage of the Full Farrell-Jones Conjecture is that it has much better inheritance properties than the Farrell-Jones Conjecture itself.

Theorem (Bartels, Bestvina, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- *Hyperbolic groups;*
- CAT(0)*-groups;*
- *Solvable groups,*
- *(Not necessarily uniform) lattices in almost connected Lie groups;*
- *Fundamental groups of (not necessarily compact) d-dimensional manifolds (possibly with boundary) for $d \leq 3$.*
- *Subgroups of $GL_n(\mathbb{Q})$ and of $GL_n(F[t])$ for a finite field F.*
- *All S-arithmetic groups.*
- *mapping class groups.*

Theorem (continued)

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*
- *Let $1 \to K \to G \xrightarrow{p} Q \to 1$ be an exact sequence. Suppose that Q and $p^{-1}(V)$ for every virtually cyclic subgroup $V \subseteq Q$ belong to $\mathcal{FJ}$. Then also G belongs to $\mathcal{FJ}$.*

Exercise (Groups in $\mathcal{FJ}$)

Show that the following groups G belong to $\mathcal{FJ}$

- *There is an extension* $1 \to K \to G \to Q \to 1$ *for virtually solvable H and hyperbolic Q.*
- $G = \langle x, y \mid yxy^{-1}xy = xyx^{-1}yx \rangle$.
- *G is locally finite.*
- *G is the fundamental group of the total space of a bundle whose fiber and base space are manifolds of dimension* ≤ 2.

Problem (Open cases)

It is unknown whether $\mathcal{FJ}$ contains the following classes of groups:

- *(Elementary) amenable groups;*
- *Residually finite groups;*
- *Fundamental groups of complete Riemannian manifolds with non-positive sectional curvature;*
- *Linear groups;*
- *One-relator-groups;*
- *Semidirect products $F_n \rtimes \mathbb{Z}$;*
- $\operatorname{Out}(F_n)$*;*
- *Thompson's groups.*

Problem (Open inheritance properties)

It is not known whether $\mathcal{FJ}$ is closed under:

- *amalgamated products.*
- *HNN-extensions.*
- *Infinite direct products.*

- Limit groups in the sense of Zela are CAT(0)-groups. (Alibegovic-Bestvina).
- There are many constructions of groups with exotic properties which arise as colimits of hyperbolic groups.
- One example is the construction of groups with expanders due to Gromov, see Arzhantseva-Delzant. These yield counterexamples to the Baum-Connes Conjecture with coefficients due to Higson-Lafforgue-Skandalis.
- However, our results show that these groups do satisfy the Full Farrell-Jones Conjecture.

- Many groups of the region 'Hic abundant leones' in the universe of groups in the sense of Bridson do satisfy the Full Farrell-Jones Conjecture.
- We do not know a (prominent) property of groups, e.g., Kazdhan's property (T), selfsimilarity, unsolvable word problem, unsolvable conjugacy problem, . . . , for which the intersection of the class of groups with this property and of $\mathcal{FJ}$ is empty.
- Hence we have no promising candidate or property of a group for which one may hope to disprove the Farrell-Jones Conjecture.
- Probably finding a counterexample may not be given by looking at a concrete group, but may come from very general methods and non explicite methods such as random groups or model theory.

- There are certain non-trivial results which are known to be true for all groups and which are consequences of the Farrell-Jones Conjecture.

Theorem (Lück-Roerdam)

Let $H \subseteq G$ be a normal finite subgroup. Then the canonical map

$$\mathrm{Wh}(H) \otimes_{\mathbb{Z}G} \mathbb{Z} \to \mathrm{Wh}(G)$$

is rationally injective.

Theorem (Yu)

If R is the ring of all Schatten class operators on an infinite dimensional and separable Hilbert space, then the assembly map appearing in the K-theoretic Farrell-Jones Conjecture is rationally injective.

- It is unlikely that it will be decided in the near future whether the Farrell-Jones Conjecture is true or not.
- There exists a universal finitely presented group U that contains all finitely presented groups as subgroups. There is such a group U with 14 generators and 42 relations.
- If I could choose a group for which I can prove the Farrell-Jones Conjecture, it would be U.

Exercise (Universal finitely presented group)

Show that the Full Farrell-Jones Conjecture holds for all groups if it holds for a universal finitely presented group U.

To be continued

Stay tuned