Universal torsion, L^2-invariants, polytopes and the Thurston norm

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Review of classical L^2-invariants

- Let $G \to \overline{X} \to X$ be a G-covering of a connected finite CW-complex X.
- The cellular chain complex of $\overline{X}$ is a finitely generated free $\mathbb{Z}G$-chain complex:

$$\dots \xrightarrow{c_{n-1}} \bigoplus_{I_n} \mathbb{Z}G \xrightarrow{c_n} \bigoplus_{i_{n-1}} \mathbb{Z}G \xrightarrow{c_{n-1}} \dots$$

- The associated L^2-chain complex

$$C_*^{(2)}(\overline{X}) := L^2(G) \otimes_{\mathbb{Z}G} C_*(\overline{X})$$

has Hilbert spaces with isometric linear G-action as chain modules and bounded G-equivariant operators as differentials

$$\dots \xrightarrow{c_{n-1}^{(2)}} \bigoplus_{I_n} L^2(G) \xrightarrow{c_n^{(2)}} \bigoplus_{i_{n-1}} L^2(G) \xrightarrow{c_{n-1}^{(2)}} \dots$$

Definition (L^2-homology and L^2-Betti numbers)

Define the n-th L^2-homology to be the Hilbert space

$$H_n^{(2)}(\overline{X}) := \ker(c_n^{(2)})/\overline{\operatorname{im}(c_{n+1}^{(2)})}.$$

Define the n-th L^2-Betti number

$$b_n^{(2)}(\overline{X}) := \dim_{\mathcal{N}(G)}\left(H_n^{(2)}(\overline{X})\right) \quad \in \mathbb{R}^{\geq 0}.$$

- The original notion is due to *Atiyah* an was motivated by index theory. He defined for a G-covering $\overline{M} \to M$ of a closed Riemannian manifold

$$b_n^{(2)}(\overline{M}) := \lim_{t \to \infty} \int_{\mathcal{F}} tr\left(e^{-t \cdot \overline{\Delta}_n}(\overline{x}, \overline{x})\right) d\mathrm{vol}_{\overline{M}}.$$

- If G is finite, we have

$$b_n^{(2)}(\overline{X}) = \frac{1}{|G|} \cdot b_n(\overline{X}).$$

- If $G = \mathbb{Z}$, we have

$$b_n^{(2)}(\overline{X}) = \dim_{\mathbb{C}[\mathbb{Z}]_{(0)}}\left(\mathbb{C}[\mathbb{Z}]_{(0)} \otimes_{\mathbb{C}[\mathbb{Z}]} H_n(\overline{X}; \mathbb{C})\right) \quad \in \mathbb{Z}.$$

- In the sequel 3-manifold means a prime connected compact orientable 3-manifold with infinite fundamental group whose boundary is empty or a union of tori and which is not $S^1 \times D^2$ or $S^1 \times S^2$.

Theorem (Lott-Lück)

For every 3-manifold M all L^2-Betti numbers $b_n^{(2)}(\widetilde{M})$ vanish.

- We are interested in the case where all L^2-Betti numbers vanish, since then a very powerful secondary invariant comes into play, the so called L^2-torsion.

- L^2-torsion can be defined analytical in terms of the spectrum of the Laplace operator, generalizing the notion of analytic Ray-Singer torsion. It can also be defined in terms of the cellular $\mathbb{Z}G$-chain complex, generalizing of the Reidemeister torsion.

- The definition of L^2-torsion is based on the notion of the Fuglede-Kadison determinant which is a generalization of the classical determinant to the infinite-dimensional setting. It is defined for an element $f \in \mathcal{N}(G)$ to be the non-negative real number
$$\det^{(2)}(f) = \exp\left(\frac{1}{2} \cdot \int \ln(\lambda)\, d\nu_{f^*f}\right) \in \mathbb{R}^{>0}$$
where ν_{f^*f} is the spectral measure of the positive operator f^*f.

- If G is finite, then $\det^{(2)}(f) = |\det(f)|^{1/|G|}$.

Definition (L^2-torsion)

Suppose that $\overline{X}$ is L^2-acyclic, i.e., all L^2-Betti numbers $b_n^{(2)}(\overline{X})$ vanish. Let $\Delta_n^{(2)} \colon C_n^{(2)}(\overline{X}) \to C_n^{(2)}(\overline{X})$ be the n-Laplace operator given by $c_{n+1}^{(2)} \circ \left(c_n^{(2)}\right)^* + \left(c_{n-1}^{(2)}\right)^* \circ c_n^{(2)}$.
Define the L^2-torsion

$$\rho^{(2)}(\overline{X}) := \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln\left(\det{}^{(2)}(\Delta_n^{(2)})\right) \in \mathbb{R}.$$

Theorem (Lück-Schick)

Let M be a 3-manifold. Let $M_1, M_2, \ldots, M_m$ be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then

$$\rho^{(2)}(\widetilde{M}) := -\frac{1}{3\pi} \cdot \sum_{i=1}^{m} \mathrm{vol}(M_i).$$

Universal L^2-torsion

Definition ($K_1^w(\mathbb{Z}G)$)

Let $K_1^w(\mathbb{Z}G)$ be the abelian group given by:

- generators

  If $f\colon \mathbb{Z}G^m \to \mathbb{Z}G^m$ is a $\mathbb{Z}G$-map such that the induced bounded G-equivariant $L^2(G)^m \to L^2(G)^m$ map is a weak isomorphism, i.e., the dimensions of its kernel and cokernel are trivial, then it determines a generator $[f]$ in $K_1^w(\mathbb{Z}G)$.
- relations

$$\begin{aligned} \left[\begin{pmatrix} f_1 & * \\ 0 & f_2 \end{pmatrix}\right] &= [f_1] + [f_2]; \\ [g \circ f] &= [f] + [g]. \end{aligned}$$

Define $\mathrm{Wh}^w(G) := K_1^w(\mathbb{Z}G)/\{\pm g \mid g \in G\}$.

Definition (Universal L^2-torsion)

Let $G \to \overline{X} \to X$ be a G-covering of a finite CW-complex. Suppose that $\overline{X}$ is L^2-acyclic, i.e., $b_n^{(2)}(\overline{X})$ vanishes for all $n \in \mathbb{Z}$.

Then its universal L^2-torsion is defined as an element

$$\rho_u^{(2)}(\overline{X}) \in K_1^w(\mathbb{Z}G).$$

- The universal L^2-torsion is defined by the same expression as the L^2-torsion, but now using the fact that the combinatorial Laplace operator can be thought of as an element in $K_1^w(\mathbb{Z}[G])$, namely by

$$\rho_u^{(2)}(\overline{X}) := \sum_{n \geq 0} (-1)^n \cdot n \cdot [\Delta_n^c] \quad \in K_1^w(\mathbb{Z}G).$$

for $\Delta_n^c := c_{n+1} \circ c_n^* + c_{n-1}^* \circ c_n$.

- The universal L^2-torsion is a simple homotopy invariant.
- It satisfies useful sum formulas and product formulas. There are also formulas for appropriate fibrations and S^1-actions.
- If G is finite, we rediscover essentially the classical Reidemeister torsion.
- Many other invariants come from the universal L^2-torsion by applying a homomorphism $K_1^w(\mathbb{Z}G) \to A$ of abelian groups.

  For instance, the Fuglede-Kadison determinant defines a homomorphism

  $$\det^{(2)} \colon \operatorname{Wh}^w(\mathbb{Z}G) \to \mathbb{R}$$

  which maps the universal L^2-torsion $\rho_u^{(2)}(\overline{X})$ to the (classical) L^2-torsion $\rho^{(2)}(\overline{X})$.

The fundamental square and the Atiyah Conjecture

- The fundamental square is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the algebra of affiliated operators. Algebraically it is just the Ore localization of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the division closure of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

- If G is finite, its is given by

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathbb{C}G \\ \downarrow & & \downarrow \mathrm{id} \\ \mathbb{Q}G & \longrightarrow & \mathbb{C}G \end{array}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\ \downarrow & & \downarrow \\ \mathbb{Q}[\mathbb{Z}]_{(0)} & \longrightarrow & L(S^1) \end{array}$$

- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.
- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the Atiyah Conjecture if $\mathcal{D}(G)$ is a skew-field.

- Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group G satisfies the Atiyah Conjecture if and only if for any G-covering $\overline{M} \to M$ of a closed Riemannian manifold of dimension d we have $b_n^{(2)}(\overline{M}) \in \mathbb{Z}$ for every $n \geq 0$.

Theorem (Linnell, Schick)

1. *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions.*
   Then every torsionfree group G which belongs to $\mathcal{C}$ satisfies the Atiyah Conjecture.
2. *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

- This theorem and results by Waldhausen show for the fundamental group π of a 3-manifold (with the exception of some graph manifolds) that it satisfies the Atiyah Conjecture and that $\mathrm{Wh}(\pi)$ vanishes.

Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück)

If G belongs to $\mathcal{C}$, then the natural map

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G))$$

is an isomorphism.

- Its proof is based on identifying $\mathcal{D}(G)$ as an appropriate Cohn localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic K-theory.
- There is a Dieudonné determinant which induces an isomorphism

$$\det_D \colon K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times/[\mathcal{D}(G)^\times, \mathcal{D}(G)^\times].$$

- In particular we get for $G = \mathbb{Z}$

$$K_1^w(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Q}[\mathbb{Z}]_{(0)} \setminus \{0\}.$$

- It turns out that then the universal torsion is the same as the Alexander polynomial of an infinite cyclic covering, as it occurs for instance in knot theory.

Twisting L^2-invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X;\mathbb{Z}) = \hom(\pi;\mathbb{Z})$.
- For $t \in (0,\infty)$, let $\phi^*\mathbb{C}_t$ be the 1-dimensional π-representation given by

$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$

- One can twist the L^2-chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$-chain complex before passing to the Hilbert space completion

$$\mathbb{C}G \to \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$

- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the Determinant Conjecture does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.

- Thus we obtain the ϕ-twisted L^2-torsion function

$$\rho(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$$

  sending t to the $\mathbb{C}_t$-twisted L^2-torsion.

  Its value at $t = 1$ is just the L^2-torsion.

- On the analytic side this corresponds for closed Riemannian manifold M to twisting with the flat line bundle $\widetilde{M} \times_\pi \mathbb{C}_t \to M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π-action on $\mathbb{C}_t$ is not isometric.

Theorem (Lück)

Suppose that $\widetilde{X}$ is L^2-acyclic.

1. *The L^2 torsion function $\rho^{(2)} := \rho^{(2)}(\widetilde{X};\phi)\colon (0,\infty) \to \mathbb{R}$ is well-defined.*
2. *The limits $\lim_{t\to\infty} \frac{\rho^{(2)}(t)}{\ln(t)}$ and $\lim_{t\to 0} \frac{\rho^{(2)}(t)}{\ln(t)}$ exist and we can define the degree of ϕ*

   $$\deg(X;\phi) \in \mathbb{R}$$

   to be their difference.
3. *There is a ϕ-twisted Fuglede-Kadison determinant*

   $$\det^{(2)}_{\mathrm{tw},\phi}\colon K_1^w(\mathbb{Z}G) \to \operatorname{map}((0,\infty),\mathbb{R})$$

   which sends $\rho_u^{(2)}(\widetilde{X})$ to $\rho^{(2)}(\widetilde{X};\phi)$.

Definition (Thurston norm)

Let M be a 3-manifold and $\phi \in H^1(M;\mathbb{Z})$ be a class. Define its Thurston norm

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_-(F) = \sum_{C \in \pi_0(M)} \max\{-\chi(C), 0\}.$$

- Thurston showed that this definition extends to the real vector space $H^1(M;\mathbb{R})$ and defines a seminorm on it.
- If $F \to M \xrightarrow{p} S^1$ is a fiber bundle and $\phi = \pi_1(p)$, then

$$x_M(\phi) = \chi(F).$$

Theorem (Friedl-Lück)

Let M be a 3-manifold. Then for every $\phi \in H^1(M;\mathbb{Z})$ we get the equality

$$\deg(M;\phi) = x_M(\phi).$$

Polytopes

- Consider a finitely generated abelian free abelian group A. Let $A_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} A$ be the real vector space containing A as a spanning lattice;
- A polytope $P \subseteq A_{\mathbb{R}}$ is a convex bounded subset which is the convex hull of a finite subset S;
- It is called integral, if S is contained in A;
- The Minkowski sum of two polytopes P and Q is defined by

$$P + Q = \{p + q \mid p \in P, q \in Q\};$$

- It is cancellative, i.e., it satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$;

- The Newton polytope

$$N(p) \subseteq \mathbb{R}^n$$

of a polynomial

$$p(t_1, t_2, \dots, t_n) = \sum_{i_1,\dots,i_n} a_{i_1,i_2,\dots,i_n} \cdot t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}$$

in n variables $t_1, t_2, \dots, t_n$ is defined to be the convex hull of the elements $\{(i_1, i_2, \dots i_n) \in \mathbb{Z}^n \mid a_{i_1,i_2,\dots,i_n} \neq 0\}$;

- One has

$$N(p \cdot q) = N(p) + N(q).$$

Definition (Polytope group)

Let $\mathcal{P}(A)$ be the Grothendieck group of the abelian monoid of integral polytopes in $A_{\mathbb{R}}$.

- For $A = \mathbb{Z}^n$ we obtain a well-defined homomorphism of abelian groups

$$\left(\mathbb{Q}[\mathbb{Z}^n]_{(0)}\right)^{\times} \to \mathcal{P}(A), \quad \frac{p}{q} \mapsto [N(p)] - [N(q)].$$

Polytope homomorphism

- Consider the projection

$$\mathrm{pr}\colon G \to H_1(G)_f := H_1(G)/\operatorname{tors}(H_1(G)).$$

Let K be its kernel.

- After a choice of a set-theoretic section of pr we get isomorphisms

$$\begin{aligned} \mathbb{Z}K * H_1(G)_f &\xrightarrow{\cong} \mathbb{Z}G; \\ S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big) &\xrightarrow{\cong} \mathcal{D}(G), \end{aligned}$$

where here and in the sequel S^{-1} denotes Ore localization with respect to the multiplicative closed set of non-trivial elements.

- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its support

$$\operatorname{supp}(x) := \{h \in H_1(G)_f \mid h \in H_1(G)_f), u_h \neq 0\}.$$

- The convex hull of $\operatorname{supp}(x)$ defines a polytope

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f = H_1(M; \mathbb{R}).$$

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in (\mathcal{D}(K) * H_1(G)_f$.
- Hence we can define a homomorphism of abelian groups

$$P' \colon \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f)\right)^{\times} \to \mathcal{P}(H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

- The composite

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times \xrightarrow{\cong} \Big(S^{-1}\big(\mathcal{D}(K) * H_1(G)_f\big)\Big)^\times \xrightarrow{P'} \mathcal{P}(H_1(G)_f)$$

factories to the polytope homomorphism

$$P\colon \operatorname{Wh}^w(G) \to \mathcal{P}(H_1(G)_f).$$

Definition (Thurston polytope)

Let M be a 3-manifold. Define the Thurston polytope to be subset of $H^1(M;\mathbb{R})$

$$T(M) := \{\phi \in H^1(M;\mathbb{R}) \mid x_M(\phi) \leq 1\}.$$

Theorem (Friedl-Lück)

Let M be a 3-manifold. Then the image of the universal L^2-torsion $\rho_u^{(2)}(\widetilde{M})$ under the polytope homomorphism

$$P\colon \operatorname{Wh}^w(\pi_1(M)) \to \mathcal{P}(H_1(\pi_1(M))_f)$$

is represented by the dual of the Thurston polytope, which is an integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f = H_1(M;\mathbb{R}) = H^1(M;\mathbb{R})^*$.

Higher order Alexander polynomials

- Higher order Alexander polynomials were introduced for a covering $G \to \overline{M} \to M$ of a 3-manifold by Harvey and Cochran, provided that G occurs in the rational derived series of $\pi_1(M)$.

- At least the degree of these polynomials is a well-defined invariant of M and G.

- We can extend this notion of degree also to the universal covering of M and can prove the conjecture that the degree coincides with the Thurston norm.

Group automorphisms

Theorem (Lück)

Let $f\colon X \to X$ be a self homotopy equivalence of a finite connected CW-complex. Let T_f be its mapping torus.

Then all L^2-Betti numbers $b_n^{(2)}(\widetilde{T_f})$ vanish.

Definition (Universal torsion for group automorphisms)

Let $f\colon G \to G$ be a group automorphism of the group G. Suppose that there is a finite model for BG, the Whitehead group $\mathrm{Wh}(G)$ vanishes, and G satisfies the Atiyah Conjecture. Then we can define the universal L^2-torsion of f by

$$\rho_u^{(2)}(f) := \rho^{(2)}(\widetilde{T_f}; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in \mathrm{Wh}^w(G \rtimes_f \mathbb{Z})$$

- This seems to be a very powerful invariant which needs to be investigated further.

- It has nice properties, e.g., it depends only on the conjugacy class of f, satisfies a sum formula and a formula for exact sequences.
- If G is amenable, it vanishes.
- If G is the fundamental group of a compact surface F and f comes from an automorphism $a\colon F \to F$, then T_f is a 3-manifold and a lot of the material above applies.
- For instance, if a is irreducible, $\rho_u^{(2)}(f)$ detects whether a is pseudo-Anosov since we can read off the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition of T_f.

- Suppose that $H_1(f) = \mathrm{id}$. Then there is an obvious projection

$$\mathrm{pr}\colon H_1(G \rtimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \to H_1(G)_f.$$

  Let

$$P(f) \in \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

  be the image of $\rho_u^{(2)}(f)$ under the composite

$$\mathrm{Wh}^w(G \rtimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G \rtimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}(\mathrm{pr})} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

- What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where F is a finitely generated free group, is of particular interest.

L^2-Euler characteristic

Definition (L^2-Euler characteristic)

Let Y be a G-space. Suppose that

$$h^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0} b_n^{(2)}(Y;\mathcal{N}(G)) < \infty.$$

Then we define its L^2-Euler characteristic

$$\chi^{(2)}(Y;\mathcal{N}(G)) := \sum_{n\geq 0}(-1)^n \cdot b_n^{(2)}(Y;\mathcal{N}(G)) \quad \in \mathbb{R}.$$

Definition (ϕ-L^2-Euler characteristic)

Let X be a connected CW-complex. Suppose that $\widetilde{X}$ is L^2-acyclic. Consider an epimorphism $\phi\colon \pi = \pi_1(M) \to \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ-L^2-Euler characteristic

$$\chi^{(2)}(\widetilde{X};\phi) := \chi^{(2)}(\widetilde{X};\mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that $\widetilde{X}/K$ is not a finite CW-complex. Hence it is not obvious but true that $h^{(2)}(\widetilde{X};\mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\widetilde{X};\phi)$ is a well-defined real number.

- The ϕ-L^2-Euler characteristic has a bunch of good properties, it satisfies for instance a sum formula, product formula and is multiplicative under finite coverings.

- Let $f\colon X \to X$ be a selfhomotopy equivalence of a connected finite CW-complex. Let T_f be its mapping torus. The projection $T_f \to S^1$ induces an epimorphism $\phi\colon \pi_1(T_f) \to \mathbb{Z} = \pi_1(S^1)$.

  Then $\widetilde{T_f}$ is L^2-acyclic and we get

$$\chi^{(2)}(\widetilde{T_f};\phi) = \chi(X).$$

Theorem (Friedl-Lück)

Let M be a 3-manifold and $\phi\colon \pi_1(M) \to \mathbb{Z}$ be an epimorphism. Then

$$-\chi^{(2)}(\widetilde{M};\phi) = x_M(\phi).$$

Summary

- We can assign to a finite CW-complex X its universal L^2-torsion

$$\rho^{(2)}(\widetilde{X}) \in \mathrm{Wh}^w(\pi),$$

provided that $\widetilde{X}$ is L^2-acyclic and π satisfies the Atiyah Conjecture.

- These assumptions are always satisfied for 3-manifolds.
- The Alexander polynomial is a special case.
- One can twist the L^2-torsion by a cycle $\phi \in H^1(M)$ and obtain a L^2-torsion function from which one can read of the Thurston norm.
- One can read of from the universal L^2-torsion a polytope which for a 3-manifold is the dual of the Thurston polytope.

Summary (continued)

- The Thurston norm can also be read of from an L^2-Euler characteristic.
- The higher order Alexander polynomials due to Harvey and Cochrane are special cases of the the universal L^2-torsion and we can prove the conjecture that their degree is the Thurston norm.
- The universal L^2-torsion seems to give an interesting invariant for elements in $\mathrm{Out}(F_n)$ and mapping class groups.