

Universal torsion, L^2 -invariants, polytopes and the Thurston norm

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Review of classical L^2 -invariants

- Given a group G , its **group von Neumann algebra** is defined to be

$$\mathcal{N}(G) = \mathcal{B}(L^2(G), L^2(G))^G = \overline{\mathbb{C}G}^{\text{weak}}.$$

- It comes with a natural **trace**

$$\text{tr}_{\mathcal{N}(G)}: \mathcal{N}(G) \rightarrow \mathbb{C}, \quad f \mapsto \langle f(e), e \rangle_{L^2(G)}.$$

- For $x = \sum_{g \in G} \lambda_g \cdot g \in \mathbb{C}G$ we have $\text{tr}_{\mathcal{N}(G)}(x) = \lambda_e$.
- If G is finite, $\mathcal{N}(G) = \mathbb{C}G$.
- If $G = \mathbb{Z}$, we get identifications

$$\begin{aligned} \mathcal{N}(\mathbb{Z}) &= L^\infty(S^1); \\ \text{tr}_{\mathcal{N}(\mathbb{Z})}(f) &= \int_{S^1} f(z) d\mu. \end{aligned}$$

- The trace yields via the Hattori-Stallings rank the dimension function with values in $[0, \infty)$ for finitely generated projective $\mathcal{N}(G)$ -modules due to **Murray-von Neumann**.
- This has been extended to the dimension function $\dim_{\mathcal{N}(G)}$ with values in $[0, \infty]$ for arbitrary $\mathcal{N}(G)$ -modules, which has still nice properties such as additivity and cofinality, by **Lück**.

Definition (L^2 -Betti number)

Let Y be a G -space. Define its n th L^2 -Betti number

$$b_n^{(2)}(Y; \mathcal{N}(G)) := \dim_{\mathcal{N}(G)}(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*^{\text{sing}}(Y))) \in [0, \infty].$$

- Sometimes we omit $\mathcal{N}(G)$ from the notation when G is obvious.

- This definition is the end of a long chain of generalizations of the original notion due to *Atiyah* which was motivated by index theory. He defined for a G -covering $\bar{M} \rightarrow M$ of a closed Riemannian manifold

$$b_n^{(2)}(\bar{M}) := \lim_{t \rightarrow \infty} \int_{\mathcal{F}} \text{tr}(e^{-t \cdot \bar{\Delta}_n}(\bar{X}, \bar{X})) \, d\text{vol}_{\bar{M}}.$$

- If G is finite, we have

$$b_n^{(2)}(\bar{X}) = \frac{1}{|G|} \cdot b_n(\bar{X}).$$

- If $G = \mathbb{Z}$, we have

$$b_n^{(2)}(\bar{X}) = \dim_{\mathbb{C}[\mathbb{Z}]^{(0)}}(\mathbb{C}[\mathbb{Z}]^{(0)} \otimes_{\mathbb{C}[\mathbb{Z}]} H_n(\bar{X}; \mathbb{C})) \in \mathbb{Z}.$$

- In the sequel **3-manifold** means a prime connected compact orientable 3-manifold with infinite fundamental group whose boundary is empty or a union of tori and which is not $S^1 \times D^2$ or $S^1 \times S^2$.

Theorem (Lott-Lück)

For every 3-manifold M all L^2 -Betti numbers $b_n^{(2)}(\tilde{M})$ vanish.

- We are interested in case where all L^2 -Betti numbers vanish, since then a very powerful secondary invariant comes into play, the so called **L^2 -torsion**.

- L^2 -torsion can be defined analytical in terms of the spectrum of the Laplace operator, generalizing the notion of **analytic Ray-Singer torsion**. It can also be defined in terms of the cellular $\mathbb{Z}G$ -chain complex, generalizing of the **Reidemeister torsion**.
- The definition of L^2 -torsion is based on the notion of the **Fuglede-Kadison determinant** which is a generalization of the classical determinant to the infinite-dimensional setting. It is defined for an element $f \in \mathcal{N}(G)$ to be the non-negative real number

$$\det^{(2)}(f) = \exp\left(\frac{1}{2} \cdot \int \ln(\lambda) d\nu_{f^*f}\right) \in \mathbb{R}^{>0}$$

where ν_{f^*f} is the spectral measure of the positive operator f^*f .

- If G is finite, then $\det^{(2)}(f) = |\det(f)|^{1/|G|}$.

Theorem (Lück-Schick)

Let M be a 3-manifold. Let M_1, M_2, \dots, M_m be the hyperbolic pieces in its Jaco-Shalen decomposition.

Then

$$\rho^{(2)}(\tilde{M}) := -\frac{1}{6\pi} \cdot \sum_{i=1}^m \text{vol}(M_i).$$

Definition ($K_1^w(\mathbb{Z}G)$)

Let $K_1^w(\mathbb{Z}G)$ be the abelian group given by:

- generators

If $f: \mathbb{Z}G^m \rightarrow \mathbb{Z}G^m$ is a $\mathbb{Z}G$ -map such that the induced $\mathcal{N}(G)$ -map $\mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^m$ map is a weak isomorphism, i.e., the dimensions of its kernel and cokernel are trivial, then it determines a generator $[f]$ in $K_1^w(\mathbb{Z}G)$.

- relations

$$\left[\begin{pmatrix} f_1 & * \\ 0 & f_2 \end{pmatrix} \right] = [f_1] + [f_2];$$
$$[g \circ f] = [f] + [g].$$

Define $\text{Wh}^w(G) := K_1^w(\mathbb{Z}G)/\{\pm g \mid g \in G\}$.

Definition (Universal L^2 -torsion)

Let $G \rightarrow \bar{X} \rightarrow X$ be a G -covering of a finite CW-complex. Suppose that \bar{X} is L^2 -acyclic, i.e., $b_n^{(2)}(\bar{X})$ vanishes for all $n \in \mathbb{Z}$.

Then its **universal L^2 -torsion** is defined as an element

$$\rho_u^{(2)}(\bar{X}) = \rho_u^{(2)}(\bar{X}; \mathcal{N}(G)) \in K_1^w(\mathbb{Z}G).$$

- We do not present the actual definition but it is similar to the definition of Whitehead torsion in terms of chain contractions, one has now to use weak chain contractions.
- It has indeed a universal property on the level of weakly acyclic finite based free $\mathbb{Z}G$ -chain complexes.
- By stating its main properties and its relation to classical invariant we want to convince the audience that this is a very interesting and powerful invariant which is worthwhile to be explored further.

- **Homotopy invariance**

Consider a pullback of G -coverings of finite CW -complexes with f a homotopy equivalence

$$\begin{array}{ccc} \bar{X} & \xrightarrow{\bar{f}} & \bar{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

Suppose that \bar{Y} is L^2 -acyclic.

Then \bar{X} is L^2 -acyclic and the obvious homomorphism

$$\mathrm{Wh}(G) \rightarrow \mathrm{Wh}^w(G)$$

sends the **Whitehead torsion** $\tau(\bar{f}: \bar{X} \rightarrow \bar{Y})$ to the difference of universal L^2 -torsions $\rho_u^{(2)}(\bar{Y}) - \rho_u^{(2)}(\bar{X})$.

- **Sum formula**

Let $G \rightarrow \bar{X} \rightarrow X$ be a G -covering of a finite CW-complex X . Let $X_i \subseteq X$ for $i = 0, 1, 2$ CW-subcomplexes with $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Let $\bar{X}_i \rightarrow X_i$ be given by restriction. Suppose that \bar{X}_i is L^2 -acyclic for $i = 0, 1, 2$.

Then \bar{X} is L^2 -acyclic and we get

$$\rho_u^{(2)}(\bar{X}; \mathcal{N}(G)) = \rho_u^{(2)}(\bar{X}_1; \mathcal{N}(G)) + \rho_u^{(2)}(\bar{X}_2; \mathcal{N}(G)) - \rho_u^{(2)}(\bar{X}_0; \mathcal{N}(G)).$$

- There are **product formulas** or more general formulas for **fibrations** and for **proper S^1 -actions** which we will not state.
- Whenever an invariant comes from the universal torsion by applying a group homomorphism, these formulas automatically extend to the other invariant.

Example (Trivial G)

- Suppose that G is trivial. Then we get isomorphisms

$$\begin{aligned} K_1^w(\{1\}) &\xrightarrow{\cong} K_1(\mathbb{Q}) \xrightarrow{\det} \mathbb{Q}^\times \\ \text{Wh}^w(\{1\}) &\xrightarrow{\cong} \{r \in \mathbb{Q} \mid r > 0\}. \end{aligned}$$

- A finite based free \mathbb{Z} -chain complex C_* is L^2 -acyclic if and only if each of its homology groups is finite. Under the isomorphism above the universal torsion of C_* is sent to $\prod_{n \geq 0} |H_n(C_*)|^{(-1)^n}$.
- For finite CW -complexes we essentially have the same information as the **Milnor torsion**.

- If G is finite, we rediscover essentially the classical **Reidemeister torsion**.

The fundamental square and the Atiyah Conjecture

- The **fundamental square** is given by the following inclusions of rings

$$\begin{array}{ccc} \mathbb{Z}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G) \end{array}$$

- $\mathcal{U}(G)$ is the **algebra of affiliated operators**. Algebraically it is just the **Ore localization** of $\mathcal{N}(G)$ with respect to the multiplicatively closed subset of non-zero divisors.
- $\mathcal{D}(G)$ is the **division closure** of $\mathbb{Z}G$ in $\mathcal{U}(G)$, i.e., the smallest subring of $\mathcal{U}(G)$ containing $\mathbb{Z}G$ such that every element in $\mathcal{D}(G)$, which is a unit in $\mathcal{U}(G)$, is already a unit in $\mathcal{D}(G)$ itself.

- If G is finite, it is given by

$$\begin{array}{ccc}
 \mathbb{Z}G & \longrightarrow & \mathbb{C}G \\
 \downarrow & & \downarrow \text{id} \\
 \mathbb{Q}G & \longrightarrow & \mathbb{C}G
 \end{array}$$

- If $G = \mathbb{Z}$, it is given by

$$\begin{array}{ccc}
 \mathbb{Z}[\mathbb{Z}] & \longrightarrow & L^\infty(S^1) \\
 \downarrow & & \downarrow \\
 \mathbb{Q}[\mathbb{Z}]^{(0)} & \longrightarrow & L(S^1)
 \end{array}$$

- If G is elementary amenable torsionfree, then $\mathcal{D}(G)$ can be identified with the Ore localization of $\mathbb{Z}G$ with respect to the multiplicatively closed subset of non-zero elements.
- In general the Ore localization does not exist and in these cases $\mathcal{D}(G)$ is the right replacement.

Conjecture (Atiyah Conjecture for torsionfree groups)

Let G be a torsionfree group. It satisfies the **Atiyah Conjecture** if $\mathcal{D}(G)$ is a skew-field.

- A torsionfree group G satisfies the Atiyah Conjecture if and only if for any matrix $A \in M_{m,n}(\mathbb{Z}G)$ the von Neumann dimension

$$\dim_{\mathcal{N}(G)}(\ker(r_A: \mathcal{N}(G)^m \rightarrow \mathcal{N}(G)^n))$$

is an integer. In this case this dimension agrees with

$$\dim_{\mathcal{D}(G)}(r_A: \mathcal{D}(G)^m \rightarrow \mathcal{D}(G)^n).$$

- The Atiyah Conjecture implies the **Kaplansky Conjecture** saying that for any torsionfree group and field of characteristic zero F the group ring FG has no non-trivial zero-divisors.

- A torsionfree G satisfies the Atiyah Conjecture if and only if each of its finitely generated subgroups does.
- Fix a natural number $d \geq 5$. Then a finitely generated torsionfree group G satisfies the Atiyah Conjecture if and only if for any G -covering $\overline{M} \rightarrow M$ of a closed Riemannian manifold of dimension d we have $b_n^{(2)}(\overline{M}; \mathcal{N}(G)) \in \mathbb{Z}$ for every $n \geq 0$.
- There is also a version of the Atiyah Conjecture for groups with a bound on the order of its finite subgroups.
- However, there exist closed Riemannian manifolds whose universal coverings have an L^2 -Betti number which is irrational, see [Austin, Grabowski](#).

Theorem (Linnell, Schick)

- 1 *Let \mathcal{C} be the smallest class of groups which contains all free groups, is closed under extensions with elementary amenable groups as quotients and directed unions. Then every torsionfree group G which belongs to \mathcal{C} satisfies the Atiyah Conjecture.*
- 2 *If G is residually torsionfree elementary amenable, then it satisfies the Atiyah Conjecture.*

- This Theorem and results by Waldhausen show for the fundamental group π of a 3-manifold that it satisfies the Atiyah Conjecture and that $\text{Wh}(\pi)$ vanishes.

Identifying $K_1^w(\mathbb{Z}G)$ and $K_1(\mathcal{D}(G))$

Theorem (Linnell-Lück)

If G belongs to \mathcal{C} , then the natural map

$$K_1^w(\mathbb{Z}G) \xrightarrow{\cong} K_1(\mathcal{D}(G))$$

is an isomorphism.

- Its proof is based on identifying $\mathcal{D}(G)$ as an appropriate Cohn localization of $\mathbb{Z}G$ and the investigating localization sequences in algebraic K -theory.
- There is a **Dieudonné determinant** which induces an isomorphism

$$\det_D: K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times / [\mathcal{D}(G)^\times, \mathcal{D}(G)^\times].$$

- In particular we get for $G = \mathbb{Z}$

$$K_1^w(\mathbb{Z}[\mathbb{Z}]) \cong \mathbb{Q}[\mathbb{Z}]^{(0)} \setminus \{0\}.$$

- It turns out that then the universal torsion is the same as the **Alexander polynomial** of an infinite cyclic covering, as it occurs for instance in knot theory.

Twisting L^2 -invariants

- Consider a CW-complex X with $\pi = \pi_1(M)$. Fix an element $\phi \in H^1(X; \mathbb{Z}) = \text{hom}(\pi; \mathbb{Z})$.
- For $t \in (0, \infty)$, let $\phi^* \mathbb{C}_t$ be the 1-dimensional π -representation given by

$$w \cdot \lambda := t^{\phi(w)} \cdot \lambda \quad \text{for } w \in \pi, \lambda \in \mathbb{C}.$$

- One can **twist** the L^2 -chain complex of X with this representation, or, equivalently, apply the following ring homomorphism to the cellular $\mathbb{Z}G$ -chain complex before passing to the Hilbert space completion

$$\mathbb{C}G \rightarrow \mathbb{C}G, \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda \cdot t^{\phi(g)} \cdot g.$$

- Notice that for irrational t the relevant chain complexes do not have coefficients in $\mathbb{Q}G$ anymore and the **Determinant Conjecture** does not apply. Moreover, the Fuglede-Kadison determinant is in general not continuous.

- Thus we obtain the ϕ -twisted L^2 -torsion function

$$\rho(\tilde{X}; \phi): (0, \infty) \rightarrow \mathbb{R}$$

sending t to the \mathbb{C}_t -twisted L^2 -torsion.

Its value at $t = 1$ is just the L^2 -torsion.

- On the analytic side this corresponds for closed Riemannian manifold M to twisting with the flat line bundle $\tilde{M} \times_{\pi} \mathbb{C}_t \rightarrow M$. It is obvious that some work is necessary to show that this is a well-defined invariant since the π -action on \mathbb{C}_t is **not** isometric.

Theorem (Lück)

Suppose that \tilde{X} is L^2 -acyclic.

- 1 The L^2 torsion function $\rho^{(2)} := \rho^{(2)}(\tilde{X}; \phi): (0, \infty) \rightarrow \mathbb{R}$ is well-defined.
- 2 The limits $\lim_{t \rightarrow \infty} \frac{\rho^{(2)}(t)}{\ln(t)}$ and $\lim_{t \rightarrow 0} \frac{\rho^{(2)}(t)}{\ln(t)}$ exist and we can define the **degree of ϕ**

$$\deg(X; \phi) \in \mathbb{R}$$

to be their difference.

- 3 There is a **ϕ -twisted Fuglede-Kadison determinant**

$$\det_{\text{tw}, \phi}^{(2)}: K_1^w(\mathbb{Z}G) \rightarrow \text{map}((0, \infty), \mathbb{R})$$

which sends $\rho_u^{(2)}(\tilde{X})$ to $\rho^{(2)}(\tilde{X}; \phi)$.

Definition (Thurston norm)

Let M be a 3-manifold and $\phi \in H^1(M; \mathbb{Z})$ be a class. Define its **Thurston norm**

$$x_M(\phi) = \min\{\chi_-(F) \mid F \text{ embedded surface in } M \text{ dual to } \phi\}$$

where

$$\chi_i(F) = \sum_{C \in \pi_0(M)} \max\{-\chi(C), 0\}.$$

- **Thurston** showed that this definition extends to the real vector space $H^1(M; \mathbb{R})$ and defines a **seminorm** on it.
- If $F \rightarrow M \xrightarrow{p} S^1$ is a fiber bundle and $\phi = \pi_1(p)$, then

$$x_M(\phi) = \chi(F).$$

Theorem (Friedl-Lück)

Let M be a 3-manifold. Then for every $\phi \in H^1(M; \mathbb{Z})$ we get the equality

$$\deg(M; \phi) = x_M(\phi).$$

Polytope homomorphism

- Consider the projection

$$\text{pr}: G \rightarrow H_1(G)_f := H_1(G)/\text{tors}(H_1(G)).$$

Let K be its kernel.

- After a choice of a set-theoretic section of pr we get isomorphisms

$$\begin{aligned} \mathbb{Z}K * H_1(G)_f &\xrightarrow{\cong} \mathbb{Z}G; \\ S^{-1}(\mathcal{D}(K) * H_1(G)_f) &\xrightarrow{\cong} \mathcal{D}(G), \end{aligned}$$

where here and in the sequel S^{-1} denotes Ore localization with respect to the multiplicative closed set of non-trivial elements.

- Given $x = \sum_{h \in H_1(G)_f} u_h \cdot h \in \mathcal{D}(K) * H_1(G)_f$, define its **support**

$$\text{supp}(x) := \{h \in H_1(G)_f \mid u_h \neq 0\}.$$

- The convex hull of $\text{supp}(x)$ defines a **polytope**

$$P(x) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f = H^1(M; \mathbb{R}).$$

This is the non-commutative analogon of the **Newton polygon** associated to a polynomial in several variables.

- Recall the **Minkowski sum** of two polytopes P and Q

$$P + Q = \{p + q \mid p \in P, q \in Q\}$$

It satisfies $P_0 + Q = P_1 + Q \implies P_0 = P_1$.

Definition (Polytope group)

Let $\mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$ be the Grothendieck group of the abelian monoid of integral polytopes under the Minkowski sum modulo translations by elements in $H_1(G)_f$, where integral means that all extreme points lie on the lattice $H_1(G)_f$.

- We have $P(x \cdot y) = P(x) + P(y)$ for $x, y \in (\mathcal{D}(K) * H_1(G)_f)$.
- Hence we can define a homomorphism of abelian groups

$$P': \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f) \right)^{\times} \rightarrow \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f),$$

by sending $x \cdot y^{-1}$ to $[P(x)] - [P(y)]$.

- The composite

$$\begin{aligned}
 K_1^w(\mathbb{Z}G) &\xrightarrow{\cong} K_1(\mathcal{D}(G)) \xrightarrow{\cong} \mathcal{D}(G)^\times \xrightarrow{\cong} \left(S^{-1}(\mathcal{D}(K) * H_1(G)_f) \right)^\times \\
 &\xrightarrow{P'} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)
 \end{aligned}$$

factories to the **polytope homomorphism**

$$P: \text{Wh}^w(G) \rightarrow \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f).$$

Definition (Thurston polytope)

Let M be a 3-manifold. Define the **Thurston polytope** to be subset of $H^1(M; \mathbb{R})$

$$T(M) := \{\phi \in H^1(M; \mathbb{R}) \mid x_M(\phi) \leq 1\}.$$

Theorem (Friedl-Lück)

Let M be a 3-manifold. Then the image of the universal L^2 -torsion $\rho_u^{(2)}(\tilde{M})$ under the polytope homomorphism

$$P: \text{Wh}^w(\pi_1(M)) \rightarrow \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f)$$

is represented by the dual of the Thurston polytope, which is an integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H_1(\pi_1(M))_f = H_1(M; \mathbb{R}) = H^1(M; \mathbb{R})^*$.

Summary

- We can assign to a finite CW -complex X its **universal L^2 -torsion**

$$\rho^{(2)}(\tilde{X}) \in \text{Wh}^w(\pi),$$

provided that \tilde{X} is L^2 -acyclic and π satisfies the Atiyah Conjecture.

- These assumptions are always satisfied for 3-manifolds.
- The Alexander polynomial is a special case.
- One can read off from the universal L^2 -torsion a **polytope** which for a 3-manifold is the dual of the **Thurston polytope**.
- One can twist the L^2 -torsion by a cycle $\phi \in H^1(M)$ and obtain a **L^2 -torsion function** from which one can read off the **Thurston norm**.

Definition (L^2 -Euler characteristic)

Let Y be a G -space. Suppose that

$$h^{(2)}(Y; \mathcal{N}(G)) := \sum_{n \geq 0} b_n^{(2)}(Y; \mathcal{N}(G)) < \infty.$$

Then we define its L^2 -Euler characteristic

$$\chi^{(2)}(Y; \mathcal{N}(G)) := \sum_{n \geq 0} (-1)^n \cdot b_n^{(2)}(Y; \mathcal{N}(G)) \in \mathbb{R}.$$

Definition (ϕ - L^2 -Euler characteristic)

Let X be a connected CW -complex. Suppose that \tilde{X} is L^2 -acyclic. Consider an epimorphism $\phi: \pi = \pi_1(M) \rightarrow \mathbb{Z}$. Let K be its kernel. Suppose that G is torsionfree and satisfies the Atiyah Conjecture.

Define the ϕ - L^2 -Euler characteristic

$$\chi^{(2)}(\tilde{X}; \phi) := \chi^{(2)}(\tilde{X}; \mathcal{N}(K)) \in \mathbb{R}.$$

- Notice that \tilde{X}/K is not a finite CW -complex. Hence it is not obvious but true that $h^{(2)}(\tilde{X}; \mathcal{N}(K)) < \infty$ and $\chi^{(2)}(\tilde{X}; \phi)$ is a well-defined real number.
- The ϕ - L^2 -Euler characteristic has a bunch of good properties, it satisfies for instance a **sum formula**, **product formula** and is **multiplicative** under finite coverings.

- Let $f: X \rightarrow X$ be a selfhomotopy equivalence of a connected finite CW-complex. Let T_f be its mapping torus. The projection $T_f \rightarrow S^1$ induces an epimorphism $\phi: \pi_1(T_f) \rightarrow \mathbb{Z} = \pi_1(S^1)$.

Then \tilde{T}_f is L^2 -acyclic and we get

$$\chi^{(2)}(\tilde{T}_f; \phi) = \chi(X).$$

Theorem (Friedl-Lück)

Let M be a 3-manifold and $\phi: \pi_1(M) \rightarrow \mathbb{Z}$ be an epimorphism. Then

$$-\chi^{(2)}(\tilde{M}; \phi) = x_M(\phi).$$

Higher order Alexander polynomials

- Higher order Alexander polynomials were introduced for a covering $G \rightarrow \overline{M} \rightarrow M$ of a 3-manifold by Harvey and Cochran, provided that G occurs in the rational derived series of $\pi_1(M)$.
- At least the degree of these polynomials is a well-defined invariant of M and G .
- We can identify the degree with the ϕ - L^2 -Euler characteristic.
- Thus we can extend this notion of degree also to the universal covering of M and can prove the conjecture that the degree coincides with the Thurston norm.

Theorem (Lück)

Let $f: X \rightarrow X$ be a self homotopy equivalence of a finite connected CW-complex. Let T_f be its mapping torus.

Then all L^2 -Betti numbers $b_n^{(2)}(\tilde{T}_f)$ vanish.

Definition (Universal torsion for group automorphisms)

Let $f: G \rightarrow G$ be a group automorphism of the group G . Suppose that there is a finite model for BG and G satisfies the Atiyah Conjecture. Then we can define the **universal L^2 -torsion** of f by

$$\rho_u^{(2)}(f) := \rho^{(2)}(\tilde{T}_f; \mathcal{N}(G \rtimes_f \mathbb{Z})) \in \text{Wh}^w(G \rtimes_f \mathbb{Z})$$

- This seems to be a very powerful invariant which needs to be investigated further.

- It has nice properties, e.g., it depends only on the conjugacy class of f , satisfies a **sum formula** and a formula for **exact sequences**.
- If G is amenable, it vanishes.
- If G is the fundamental group of a compact surface F and f comes from an automorphism $a: F \rightarrow F$, then T_f is a 3-manifold and a lot of the material above applies.
- For instance, if a is irreducible, $\rho_U^{(2)}(f)$ detects whether a is **pseudo-Anosov** since we can read off the sum of the volumes of the hyperbolic pieces in the Jaco-Shalen decomposition of T_f .

- Suppose that $H_1(f) = \text{id}$. Then there is an obvious projection

$$\text{pr}: H_1(G \rtimes_f \mathbb{Z})_f = H_1(G)_f \times \mathbb{Z} \rightarrow H_1(G)_f.$$

Let

$$P(f) \in \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

be the image of $\rho_U^{(2)}(f)$ under the composite

$$\text{Wh}^w(G \rtimes \mathbb{Z}) \xrightarrow{P} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G \rtimes_f \mathbb{Z})) \xrightarrow{\mathcal{P}(\text{pr})} \mathcal{P}(\mathbb{R} \otimes_{\mathbb{Z}} H_1(G)_f)$$

- What are the main properties of this polytope? In which situations can it be explicitly computed? The case, where F is a finitely generated free group, is of particular interest.

Summary (continued)

- The Thurston norm can also be read off from an L^2 -Euler characteristic.
- The higher order Alexander polynomials due to Harvey and Cochran are special cases of the universal L^2 -torsion and we can prove the conjecture that their degree is the Thurston norm.
- The universal L^2 -torsion seems to give an interesting invariant for elements in $\text{Out}(F_n)$ and mapping class groups.