

Introduction to the Farrell-Jones Conjecture

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$K_0(R)$ and the Idempotent Conjecture

- Given a ring R and a group G , denote by RG or $R[G]$ the **group ring**.
- Elements are formal sums $\sum_{g \in G} r_g \cdot g$, where $r_g \in R$ and only finitely many of the coefficients r_g are non-zero.
- Addition is given by adding the coefficients.
- Multiplication is given by the expression $g \cdot h := g \cdot h$ for $g, h \in G$ (with two different meanings of \cdot).
- In general RG is a very complicated ring.
- An RG -module is the same as **G -representation** with coefficients in R , i.e., an R -module with G -action by R -linear maps.
- If $\bar{X} \rightarrow X$ is a G -covering of a CW -complex X , then the cellular chain complex of \bar{X} is a free $\mathbb{Z}G$ -chain complex.

- If g has finite order $|g|$ and F is a field of characteristic zero, then we get an idempotent in FG by

$$x = \frac{1}{|g|} \cdot \sum_{i=0}^{|g|-1} g^i.$$

- Are there other idempotents?

Conjecture (Idempotent Conjecture)

The *Kaplansky Conjecture* says that for a torsionfree group G and a field F of characteristic zero the elements 0 and 1 are the only idempotents in FG .

Definition (Projective class group $K_0(R)$)

Define the **projective class group** of a ring R

$$K_0(R)$$

to be the following abelian group:

- Generators are isomorphism classes $[P]$ of finitely generated projective R -modules P ;
- The relations are $[P_0] + [P_2] = [P_1]$ for every exact sequence $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$ of finitely generated projective R -modules.
- The assignment $P \mapsto [P] \in K_0(R)$ is the **universal additive invariant** or **dimension function** for finitely generated projective R -modules.

Definition (Reduced Projective class group $\tilde{K}_0(R)$)

The **reduced projective class group**

$$\tilde{K}_0(R) = \text{cok}(K_0(\mathbb{Z}) \rightarrow K_0(R))$$

is the quotient of $K_0(R)$ by the subgroup generated by the classes of finitely generated free R -modules.

- Let P be a finitely generated projective R -module. It is **stably free**, i.e., $P \oplus R^m \cong R^n$ for appropriate $m, n \in \mathbb{Z}$, if and only if $[P] = 0$ in $\tilde{K}_0(R)$.

Conjecture

If G is torsionfree, then $\tilde{K}_0(\mathbb{Z}G)$ and $\tilde{K}_0(FG)$ for a field F of characteristic zero vanish.

- The last conjecture implies the Idempotent Conjecture.

Definition (K_1 -group $K_1(R)$)

Define the K_1 -group of a ring R

$$K_1(R)$$

to be the abelian group whose generators are conjugacy classes $[f]$ of automorphisms $f: P \rightarrow P$ of finitely generated projective R -modules with the following relations:

- Given an exact sequence $0 \rightarrow (P_0, f_0) \rightarrow (P_1, f_1) \rightarrow (P_2, f_2) \rightarrow 0$ of automorphisms of finitely generated projective R -modules, we get $[f_0] + [f_2] = [f_1]$;
- $[g \circ f] = [f] + [g]$.

- We have the inclusion $GL_n(R) \rightarrow GL_{n+1}(R)$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.
- Put $GL(R) := \bigcup_{n \geq 1} GL_n(R)$.
- The obvious maps $GL_n(R) \rightarrow K_1(R)$ induce an isomorphism

$$GL(R)/[GL(R), GL(R)] \xrightarrow{\cong} K_1(R).$$

- An invertible matrix $A \in GL(R)$ can be reduced by **elementary row and column operations** and **(de-)stabilization** to the trivial empty matrix if and only if $[A] = 0$ holds in the **reduced K_1 -group**

$$\tilde{K}_1(R) := K_1(R)/\{\pm 1\} = \text{cok}(K_1(\mathbb{Z}) \rightarrow K_1(R)).$$

- The assignment $A \mapsto [A] \in K_1(R)$ can be thought of as the **universal determinant for R** .

Definition (Whitehead group)

The **Whitehead group** of a group G is defined to be

$$\text{Wh}(G) = K_1(\mathbb{Z}G) / \{\pm g \mid g \in G\}.$$

Lemma

We have $\text{Wh}(\{1\}) = \{0\}$.

Definition (h -cobordism)

An **h -cobordism** W is a compact manifold W whose boundary is the disjoint union $\partial_0 W \amalg \partial_1 W$ such that both inclusions $\partial_0 W \rightarrow W$ and $\partial_1 W \rightarrow W$ are homotopy equivalences.

An **h -cobordism over a closed manifold M** is an h -cobordism together with a diffeomorphism (or homeomorphism) $f: M \xrightarrow{\cong} \partial_0 W$.

Theorem (**s-Cobordism Theorem**, **Barden, Mazur, Stallings, Kirby-Siebenmann**)

Let M be a closed smooth or topological manifold of dimension ≥ 5 . Then the so called Whitehead torsion yields a bijection

$$\tau: \mathcal{H}(M) \xrightarrow{\cong} \text{Wh}(\pi_1(M))$$

where $\mathcal{H}(M)$ is the set of h -cobordisms over M modulo diffeomorphisms or homeomorphisms relativ M .

Conjecture (Poincaré Conjecture)

Let M be an n -dimensional topological manifold which is a homotopy sphere, i.e., homotopy equivalent to S^n .

Then M is homeomorphic to S^n .

Theorem (Freedman, Perelman, Smale)

The Poincaré Conjecture is true.

Proof.

We sketch the proof for $n \geq 6$. The proofs for $n = 3, 4$ are of different nature.

- Let M be a n -dimensional homotopy sphere.
- Let W be obtained from M by deleting the interior of two disjoint embedded disks D_0^n and D_1^n . Then W is a simply connected h -cobordism.
- Since $\text{Wh}(\{1\})$ is trivial, we can find a homeomorphism $f: W \xrightarrow{\cong} \partial D_0^n \times [0, 1]$ that is the identity on $\partial D_0^n = \partial D_0^n \times \{0\}$.
- By the **Alexander trick** we can extend the homeomorphism $f|_{\partial D_1^n}: \partial D_1^n \xrightarrow{\cong} \partial D_0^n$ to a homeomorphism $g: D_1^n \rightarrow D_0^n$.
- The three homeomorphisms $id_{D_0^n}$, f and g fit together to a homeomorphism $h: M \rightarrow D_0^n \cup_{\partial D_0^n \times \{0\}} \partial D_0^n \times [0, 1] \cup_{\partial D_0^n \times \{1\}} D_0^n$. The target is obviously homeomorphic to S^n .



- The argument above does not imply that for a smooth manifold M we obtain a diffeomorphism $g: M \rightarrow S^n$, since the Alexander trick does not work smoothly.
- Indeed, there exist so called **exotic spheres**, i.e., closed smooth manifolds which are homeomorphic but not diffeomorphic to S^n .

Conjecture (Vanishing of $\text{Wh}(G)$ for torsionfree G)

If G is torsionfree, then

$$\text{Wh}(G) = \{0\}.$$

Lemma

Let G be finitely presented and $d \geq 5$ be any natural number. Then the following statements are equivalent:

- The Whitehead group $\text{Wh}(G)$ vanishes;
- For one closed manifold M of dimension d with $G \cong \pi_1(M)$ every h -cobordism over M is trivial;
- For every closed manifold M of dimension d with $G \cong \pi_1(M)$ every h -cobordism over M is trivial.

Motivation and Statement of the Farrell-Jones Conjecture for torsionfree groups

- There are K -groups $K_n(R)$ for every $n \in \mathbb{Z}$.
- Can one identify $K_n(RG)$ with more accessible terms?
- If G_0 and G_1 are torsionfree and R is regular, one gets isomorphisms

$$\begin{aligned}K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ \tilde{K}_n(R[G_0 * G_1]) &\cong \tilde{K}_n(RG_0) \oplus \tilde{K}_n(RG_1).\end{aligned}$$

- If \mathcal{H} is any (generalized) homology theory, then

$$\begin{aligned}\mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\text{pt}) \oplus \mathcal{H}_{n-1}(\text{pt}); \\ \tilde{\mathcal{H}}_n(B(G_0 * G_1)) &\cong \tilde{\mathcal{H}}_n(BG_0) \oplus \tilde{\mathcal{H}}_n(BG_1).\end{aligned}$$

- Question: Can we find \mathcal{H}_* with $\mathcal{H}_n(BG) \cong K_n(RG)$, provided that G is torsionfree and R is regular.
- Of course such \mathcal{H}_* has satisfy $\mathcal{H}_n(\text{pt}) = K_n(R)$.
- So the only reasonable candidate is $H_n(-; \mathbf{K}_R)$.

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups and regular rings)

The *K*-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the *assembly map*

$$H_n(BG; \mathbf{K}_R) \rightarrow K_n(RG)$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an *L*-theory version.

The Borel Conjecture and other applications

- The conjectures above about the vanishing of $\tilde{K}_0(\mathbb{Z}G)$ and $\text{Wh}(G)$ for torsionfree G do follow from the Farrell-Jones Conjecture above.
- The idea of the proof is to study the **Atiyah-Hirzebruch spectral sequence** converging to $H_n(BG; \mathbf{K}_R)$ whose E^2 -term is given by

$$E_{p,q}^2 = H_p(BG, K_q(R)),$$

using

$$K_n(\mathbb{Z}) = \begin{cases} \{0\} & n \leq -1; \\ \mathbb{Z} & n = 0; \\ \{\pm 1\} & n = 1. \end{cases}$$

Conjecture (Borel Conjecture)

The *Borel Conjecture for G* predicts that for two aspherical closed manifolds M and N with $\pi_1(M) \cong \pi_1(N) \cong G$ any homotopy equivalence $M \rightarrow N$ is homotopic to a homeomorphism.

- In particular the Borel Conjecture predicts that two aspherical closed manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

- The Borel Conjecture can be viewed as the topological version of **Mostow rigidity**.
A special case of Mostow rigidity says that any homotopy equivalence between closed hyperbolic manifolds of dimension ≥ 3 is homotopic to an isometric diffeomorphism.
- The Borel Conjecture is not true in the smooth category by results of **Farrell-Jones**.
- The Borel Conjecture follows in dimension ≥ 5 from the Farrell-Jones Conjecture.

There are many other applications of the Farrell-Jones Conjecture, for instance:

- Characterization of hyperbolic groups with spheres as boundary.
- Fibered maps between closed manifolds.
- Classification of certain classes of manifolds with infinite fundamental group.
- Rational computations of $\pi_n(\text{Diff}(M))$ in a range for an aspherical closed manifold M .
- **Novikov Conjecture.**
- **Bass Conjecture.**
- **Moody's Induction Conjecture.**

The general version the Farrell-Jones Conjecture

- One can formulate a version of the Farrell-Jones Conjecture which makes sense for all groups G and all rings R .

Conjecture (*K*-theoretic Farrell-Jones-Conjecture)

The *K*-theoretic Farrell-Jones Conjecture with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\text{VCyc}}(G), \mathbf{K}_R) \rightarrow H_n^G(\text{pt}, \mathbf{K}_R) = K_n(RG).$$

is bijective for every $n \in \mathbb{Z}$.

- There is also an *L-theory* version.
- One can also allow *twisted group rings* and *orientation characters*.
- In the sequel the *Full Farrell-Jones Conjecture* refers to the most general version for both *K-theory* and *L-theory*, namely, with coefficients in additive *G*-categories (with involution) and finite wreath products.
- All conjecture or results mentioned in this talk follow from the Full Farrell-Jones Conjecture.

Status of the Full Farrell-Jones Conjecture

Theorem (Bartels, Farrell, Kammeyer, Lück, Reich, Rüping, Wegner)

Let \mathcal{FJ} be the class of groups for which the Full Farrell-Jones Conjecture holds. Then \mathcal{FJ} contains the following groups:

- Hyperbolic groups;
- CAT(0)-groups;
- Solvable groups,
- (Not necessarily uniform) lattices in almost connected Lie groups;
- Fundamental groups of (not necessarily compact) d -dimensional manifolds (possibly with boundary) for $d \leq 3$.
- Subgroups of $GL_n(\mathbb{Q})$ and of $GL_n(F[t])$ for a finite field F .
- All S -arithmetic groups.

Theorem (continued)

Moreover, \mathcal{FJ} has the following inheritance properties:

- If G_1 and G_2 belong to \mathcal{FJ} , then $G_1 \times G_2$ and $G_1 * G_2$ belong to \mathcal{FJ} ;
 - If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;
 - If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;
 - Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\text{colim}_{i \in I} G_i$ belongs to \mathcal{FJ} ;
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- Many more mathematicians have made important contributions to the Farrell-Jones Conjecture, e.g., **Bökstedt, Carlsson, Davis, Ferry, Hambleton, Hsiang, Jones, Linnell, Madsen, Pedersen, Quinn, Ranicki, Rognes, Rosenthal, Tessera, Varisco, Weinberger, Yu.**

The Farrell-Jones Conjecture is open for:

- mapping class groups;
- $\text{Out}(F_n)$;
- amenable groups;
- Thompson's groups;
- $G = F_n \rtimes \mathbb{Z}$.

- There are many **constructions of groups with exotic properties** which arise as colimits of hyperbolic groups.
- One example is the construction of **groups with expanders** due to **Gromov**, see **Arzhantseva-Delzant**. These yield **counterexamples** to the **Baum-Connes Conjecture with coefficients** due to **Higson-Lafforgue-Skandalis**.
- However, our results show that these groups do satisfy the Full Farrell-Jones Conjecture and hence also the other conjectures mentioned above.
- We have no good candidate for a group (or for a property of groups) for which the Farrell-Jones Conjecture may fail.

- **Davis-Januszkiewicz** have constructed exotic aspherical closed manifolds using **hyperbolization techniques**. For instance there are examples which do **not admit a triangulation** or whose **universal covering is not homeomorphic to Euclidean space**.
- However, in all cases the universal coverings are CAT(0)-spaces and the fundamental groups are CAT(0)-groups. Hence they satisfy the Full Farrell-Jones Conjecture and in particular the Borel Conjecture in dimension ≥ 5 .

- The assembly map can be thought of an **approximation** of the algebraic K - or L -theory **by a homology theory**.
- The basic feature between the left and right side of the assembly map is that on the left side one has **excision** which is not present on the right side.
- In general excision is available if one can make **representing cycles small**.
- A best illustration for this is the proof of excision for simplicial or singular homology based on **subdivision** whose effect is to make the support of cycles arbitrary small.

- Then the basic goal of the proof is obvious: Find a procedure to make the support of a representing cocycle as small as possible without changing its class.
- Suppose that $G = \pi_1(M)$ for a closed Riemannian manifold with negative sectional curvature.
- The idea is to use the **geodesic flow** on the universal covering to gain the necessary control.
- We will briefly explain this in the case, where the universal covering is the two-dimensional hyperbolic space \mathbb{H}^2 .

- Consider two points with coordinates (x_1, y_1) and (x_2, y_2) in the upper half plane model of two-dimensional hyperbolic space. We want to use the geodesic flow to make their distance smaller in a functorial fashion. This is achieved by letting these points flow towards the boundary at infinity along the geodesic given by the vertical line through these points, i.e., towards infinity in the y -direction.
- There is a fundamental problem: if $x_1 = x_2$, then the distance between these points is unchanged. Therefore we make the following prearrangement. Suppose that $y_1 < y_2$. Then we first let the point (x_1, y_1) flow so that it reaches a position where $y_1 = y_2$. Inspecting the hyperbolic metric, one sees that the distance between the two points (x_1, τ) and (x_2, τ) goes to zero if τ goes to infinity. This is the basic idea to gain control in the negatively curved case.