

# *K*- and *L*-theory of group rings

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**Abstract.** This article will explore the *K*- and *L*-theory of group rings and their applications to algebra, geometry and topology. The Farrell-Jones Conjecture characterizes *K*- and *L*-theory groups. It has many implications, including the Borel and Novikov Conjectures for topological rigidity. Its current status, and many of its consequences are surveyed.

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## 0. Introduction

The algebraic *K*- and *L*-theory of group rings —  $K_n(RG)$  and  $L_n(RG)$  for a ring  $R$  and a group  $G$  — are highly significant, but are very hard to compute when  $G$  is infinite. The main ingredient for their analysis is the Farrell-Jones Conjecture. It identifies them with certain equivariant homology theories evaluated on the classifying space for the family of virtually cyclic subgroups of  $G$ . Roughly speaking, the Farrell-Jones Conjecture predicts that one can compute the values of these *K*- and *L*-groups for  $RG$  if one understands all of the values for  $RH$ , where  $H$  runs through the virtually cyclic subgroups of  $G$ .

Why is the Farrell-Jones Conjecture so important? One reason is that it plays an important role in the classification and geometry of manifolds. A second reason is that it implies a variety of well-known conjectures, such as the ones due to Bass, Borel, Kaplansky and Novikov. (These conjectures are explained in Section 1.) There are many groups for which these conjectures were previously unknown but are now consequences of the proof that they satisfy the Farrell-Jones Conjecture. A third reason is that most of the explicit computations of *K*- and *L*-theory of group rings for infinite groups are based on the Farrell-Jones Conjecture, since it identifies them with equivariant homology groups which are more accessible via standard tools from algebraic topology and geometry (see Section 5).

The rather complicated general formulation of the Farrell-Jones Conjecture is given in Section 3. The much easier, but already very interesting, special case of a torsionfree group is discussed in Section 2. In this situation the *K*- and *L*-groups are identified with certain homology theories applied to the classifying space  $BG$ .

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The recent proofs of the Farrell-Jones Conjecture for hyperbolic groups and CAT(0)-groups are deep and technically very involved. Nonetheless, we give a glimpse of the key ideas in Section 6. In each of these proofs there is decisive input coming from the geometry of the groups that is reminiscent of non-positive curvature. In order to exploit these geometric properties one needs to employ controlled topology and construct flow spaces that mimic the geodesic flow on a Riemannian manifold.

The class of groups for which the Farrell-Jones Conjecture is known is further extended by the fact that it has certain inheritance properties. For instance, subgroups of direct products of finitely many hyperbolic groups and directed colimits of hyperbolic groups belong to this class. Hence, there are many examples of exotic groups, such as groups with expanders, that satisfy the Farrell-Jones Conjecture because they are constructed as such colimits. There are of course groups for which the Farrell-Jones Conjecture has not been proved, like solvable groups, but there is no example or property of a group known that threatens to produce a counterexample. Nevertheless, there may well be counterexamples and the challenge is to develop new tools to find and construct them.

The status of the Farrell-Jones Conjecture is given in Section 4, and open problems are discussed in Section 7.

## 1. Some well-known conjectures

In this section we briefly recall some well-known conjectures. They address topics from different areas, including topology, algebra and geometric group theory. They have one — at first sight not at all obvious — common feature. Namely, their solution is related to questions about the  $K$ - and  $L$ -theory of group rings.

**1.1. Borel Conjecture.** A closed manifold  $M$  is said to be *topologically rigid* if every homotopy equivalence from a closed manifold to  $M$  is homotopic to a homeomorphism. In particular, if  $M$  is topologically rigid, then every manifold homotopy equivalent to  $M$  is homeomorphic to  $M$ . For example, the spheres  $S^n$  are topologically rigid, as predicted by the *Poincaré Conjecture*. A connected manifold is called *aspherical* if its homotopy groups in degree  $\geq 2$  are trivial. A sphere  $S^n$  for  $n \geq 2$  has trivial fundamental group, but its higher homotopy groups are very complicated. Aspherical manifolds, on the other hand, have complicated fundamental groups and trivial higher homotopy groups. Examples of closed aspherical manifolds are closed Riemannian manifolds with non-positive sectional curvature, and double quotients  $G \backslash L / K$  for a connected Lie group  $L$  with  $K \subseteq L$  a maximal compact subgroup and  $G \subseteq L$  a torsionfree cocompact discrete subgroup. More information about aspherical manifolds can be found, for instance, in [59].

**Conjecture 1.1** (Borel Conjecture). *Closed aspherical manifolds are topologically rigid.*

In particular the Borel Conjecture predicts that two closed aspherical manifolds are homeomorphic if and only if their fundamental groups are isomorphic.

Hence the Borel Conjecture may be viewed as the topological version of *Mostow rigidity*. One version of Mostow rigidity says that two hyperbolic closed manifolds of dimension  $\geq 3$  are isometrically diffeomorphic if and only if their fundamental groups are isomorphic.

It is not true that any homotopy equivalence of aspherical closed smooth manifolds is homotopic to a diffeomorphism. The  $n$ -dimensional torus for  $n \geq 5$  yields a counterexample (see [88, 15A]). Counterexamples with sectional curvature pinched arbitrarily close to  $-1$  are given in [29, Theorem 1.1].

For more information about topologically rigid manifolds which are not necessarily aspherical, the reader is referred to [48].

**1.2. Fundamental groups of closed manifolds.** The Borel Conjecture is a uniqueness result. There is also an existence part. The problem is to determine when a given group  $G$  is the fundamental group of a closed aspherical manifold. Let us collect some obvious conditions that a group  $G$  must satisfy so that  $G = \pi_1(M)$  for a closed aspherical manifold  $M$ . It must be finitely presented, since the fundamental group of any closed manifold is finitely presented. Since the cellular  $\mathbb{Z}G$ -chain complex of the universal covering of  $M$  yields a finite free  $\mathbb{Z}G$ -resolution of the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$ , the group  $G$  must be of type FP, i.e., the trivial  $\mathbb{Z}G$ -module  $\mathbb{Z}$  possesses a finite projective  $\mathbb{Z}G$ -resolution. Since  $\widetilde{M}$  is a model for the classifying  $G$ -space  $EG$ , Poincaré duality implies  $H^i(G; \mathbb{Z}G) \cong H_{\dim(M)-i}(\widetilde{M}; \mathbb{Z})$ , where  $H^i(G; \mathbb{Z}G)$  is the cohomology of  $G$  with coefficients in the  $\mathbb{Z}G$ -module  $\mathbb{Z}G$  and  $H_i(\widetilde{M}; \mathbb{Z})$  is the homology of  $\widetilde{M}$  with integer coefficients. Since  $\widetilde{M}$  is contractible,  $H^i(G; \mathbb{Z}G) = 0$  for  $i \neq \dim(M)$  and  $H^{\dim(M)}(G; \mathbb{Z}G) \cong \mathbb{Z}$ . Thus, a group  $G$  is called a *Poincaré duality group of dimension  $n$*  if  $G$  is finitely presented, is of type FP,  $H^i(G; \mathbb{Z}G) = 0$  for  $i \neq n$ , and  $H^n(G; \mathbb{Z}G) \cong \mathbb{Z}$ .

**Conjecture 1.2** (Poincaré duality groups). *A group  $G$  is the fundamental group of a closed aspherical manifold of dimension  $n$  if and only if  $G$  is a Poincaré duality group of dimension  $n$ .*

For more information about Poincaré duality groups, see [25, 42, 87].

**1.3. Novikov Conjecture.** Let  $G$  be a group and  $u: M \rightarrow BG$  be a map from a closed oriented smooth manifold  $M$  to  $BG$ . Let  $\mathcal{L}(M) \in \prod_{k \geq 0} H^k(M; \mathbb{Q})$  be the  $L$ -class of  $M$ , which is a certain polynomial in the Pontrjagin classes. Therefore it depends, a priori, on the tangent bundle and hence on the differentiable structure of  $M$ . For  $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ , define the *higher signature of  $M$  associated to  $x$  and  $u$*  to be the rational number

$$\text{sign}_x(M, u) := \langle \mathcal{L}(M) \cup u^*x, [M] \rangle.$$

We say that  $\text{sign}_x$  for  $x \in \prod_{n \geq 0} H^n(BG; \mathbb{Q})$  is *homotopy invariant* if, for two closed oriented smooth manifolds  $M$  and  $N$  with reference maps  $u: M \rightarrow BG$  and  $v: N \rightarrow BG$ , we have

$$\text{sign}_x(M, u) = \text{sign}_x(N, v)$$

whenever there is an orientation preserving homotopy equivalence  $f: M \rightarrow N$  such that  $v \circ f$  and  $u$  are homotopic.

**Conjecture 1.3** (Novikov Conjecture). *Let  $G$  be a group. Then  $\text{sign}_x$  is homotopy invariant for all  $x \in \prod_{k \geq 0} H^k(BG; \mathbb{Q})$ .*

The Hirzebruch signature formula says that for  $x = 1$  the signature  $\text{sign}_1(M, c)$  coincides with the ordinary signature  $\text{sign}(M)$  of  $M$  if  $\dim(M) = 4n$ , and is zero if  $\dim(M)$  is not divisible by four. Obviously  $\text{sign}(M)$  depends only on the oriented homotopy type of  $M$  and hence the Novikov Conjecture 1.3 is true for  $x = 1$ .

A consequence of the Novikov Conjecture 1.3 is that for a homotopy equivalence  $f: M \rightarrow N$  of orientable closed manifolds, we get  $f_*\mathcal{L}(M) = \mathcal{L}(N)$  provided  $M$  and  $N$  are aspherical. This is surprising since it is not true in general. Often the  $L$ -classes are used to distinguish the homeomorphism or diffeomorphism types of homotopy equivalent closed manifolds. However, if one believes in the Borel Conjecture 1.1, then the map  $f$  above is homotopic to a homeomorphism and a celebrated result of Novikov [69] on the topological invariance of rational Pontrjagin classes says that  $f_*\mathcal{L}(M) = \mathcal{L}(N)$  holds for any homeomorphism of closed manifolds.

For more information about the Novikov Conjecture, see, for instance, [37, 47].

**1.4. Kaplansky Conjecture.** Let  $F$  be a field of characteristic zero. Consider a group  $G$ . Let  $g \in G$  be an element of finite order  $|g|$ . Set  $N_g = \sum_{i=1}^{|g|} g^i$ . Then  $N_g \cdot N_g = |g| \cdot N_g$ . Hence  $x = N_g/|g|$  is an idempotent, i.e.,  $x^2 = x$ . There are no other constructions known to produce idempotents different from 0 in  $FG$ . If  $G$  is torsionfree, this construction yields only the obvious idempotent 1. This motivates:

**Conjecture 1.4** (Kaplansky Conjecture). *Let  $F$  be a field of characteristic zero and let  $G$  be a torsionfree group. Then the group ring  $FG$  contains no idempotents except 0 and 1.*

**1.5. Hyperbolic groups with spheres as boundary.** Let  $G$  be a hyperbolic group. One can assign to  $G$  its boundary  $\partial G$ . For information about the boundaries of hyperbolic groups, the reader is referred to [16, 43, 60]. Let  $M$  be an  $n$ -dimensional closed connected Riemannian manifold with negative sectional curvature. Then its fundamental group  $\pi_1(M)$  is a hyperbolic group. The exponential map at a point  $x \in M$  yields a diffeomorphism  $\exp: T_x\mathbb{R}^n \rightarrow M$ , which sends 0 to  $x$ , and a linear ray emanating from 0 in  $T_x\mathbb{R}^n \cong \mathbb{R}^n$  is mapped to a geodesic ray in  $M$  emanating from  $x$ . Hence, it is not surprising that the boundary of  $\pi_1(M)$  is  $S^{\dim(M)-1}$ . This motivates (see Gromov [38, page 192]):

**Conjecture 1.5** (Hyperbolic groups with spheres as boundary). *Let  $G$  be a hyperbolic group whose boundary  $\partial G$  is homeomorphic to  $S^{n-1}$ . Then  $G$  is the fundamental group of an aspherical closed manifold of dimension  $n$ .*

This conjecture has been proved for  $n \geq 6$  by Bartels-Lück-Weinberger [9] using the proof of the Farrell-Jones Conjecture for hyperbolic groups (see [4]) and the topology of homology ANR-manifolds (see, for example, [17, 76]).

**1.6. Vanishing of the reduced projective class group.** Let  $R$  be an (associative) ring (with unit). Define its *projective class group*  $K_0(R)$  to be the abelian group whose generators are isomorphism classes  $[P]$  of finitely generated projective  $R$ -modules  $P$ , and whose relations are  $[P_0] + [P_2] = [P_1]$  for any exact sequence  $0 \rightarrow P_0 \rightarrow P_1 \rightarrow P_2 \rightarrow 0$  of finitely generated projective  $R$ -modules. Define the *reduced projective class group*  $\tilde{K}_0(R)$  to be the quotient of  $K_0(R)$  by the abelian subgroup  $\{[R^m] - [R^n] \mid n, m \in \mathbb{Z}, m, n \geq 0\}$ , which is the same as the abelian subgroup generated by the class  $[R]$ .

Let  $P$  be a finitely generated projective  $R$ -module. Then its class  $[P] \in \tilde{K}_0(R)$  is trivial if and only if  $P$  is *stably free*, i.e.,  $P \oplus R^r \cong R^s$  for appropriate integers  $r, s \geq 0$ . So the reduced projective class group  $\tilde{K}_0(R)$  measures the deviation of a finitely generated projective  $R$ -module from being stably free. Notice that stably free does not, in general, imply free.

A ring  $R$  is called *regular* if it is Noetherian and every  $R$ -module has a finite-dimensional projective resolution. Any principal ideal domain, such as  $\mathbb{Z}$  or a field, is regular.

**Conjecture 1.6** (Vanishing of the reduced projective class group). *Let  $R$  be a regular ring and let  $G$  be a torsionfree group. Then the change of rings homomorphism*

$$K_0(R) \rightarrow K_0(RG)$$

*is an isomorphism.*

*In particular  $\tilde{K}_0(RG)$  vanishes for every principal ideal domain  $R$  and every torsionfree group  $G$ .*

The vanishing of  $\tilde{K}_0(RG)$  contains valuable information about the finitely generated projective  $RG$ -modules over  $RG$ . In the case  $R = \mathbb{Z}$ , it also has the following important geometric interpretation.

Let  $X$  be a connected  $CW$ -complex. It is called *finite* if it consists of finitely many cells, or, equivalently, if  $X$  is compact. It is called *finitely dominated* if there is a finite  $CW$ -complex  $Y$ , together with maps  $i: X \rightarrow Y$  and  $r: Y \rightarrow X$ , such that  $r \circ i$  is homotopic to the identity on  $X$ . The fundamental group of a finitely dominated  $CW$ -complex is always finitely presented. While studying existence problems for spaces with prescribed properties (like group actions, for example), it is occasionally relatively easy to construct a finitely dominated  $CW$ -complex within a given homotopy type, whereas it is not at all clear whether one can also find a homotopy equivalent *finite*  $CW$ -complex. *Wall's finiteness obstruction*, a certain obstruction element  $\tilde{o}(X) \in \tilde{K}_0(\mathbb{Z}\pi_1(X))$ , decides this question.

The vanishing of  $\tilde{K}_0(\mathbb{Z}G)$ , as predicted in Conjecture 1.6 for torsionfree groups, has the following interpretation: For a finitely presented group  $G$ , the vanishing of  $\tilde{K}_0(\mathbb{Z}G)$  is equivalent to the statement that any connected finitely dominated  $CW$ -complex  $X$  with  $G \cong \pi_1(X)$  is homotopy equivalent to a finite  $CW$ -complex.

For more information about the finiteness obstruction, see [35, 49, 67, 86].

**1.7. Vanishing of the Whitehead group.** The *first algebraic  $K$ -group*  $K_1(R)$  of a ring  $R$  is defined to be the abelian group whose generators  $[f]$  are

conjugacy classes of automorphisms  $f: P \rightarrow P$  of finitely generated projective  $R$ -modules  $P$  and has the following relations. For each exact sequence  $0 \rightarrow (P_0, f_0) \rightarrow (P_1, f_1) \rightarrow (P_2, f_2) \rightarrow 0$  of automorphisms of finitely generated projective  $R$ -modules, there is the relation  $[f_0] - [f_1] + [f_2] = 0$ ; and for every two automorphisms  $f, g: P \rightarrow P$  of the same finitely generated projective  $R$ -module, there is the relation  $[f \circ g] = [f] + [g]$ . Equivalently,  $K_1(R)$  is the abelianization of the general linear group  $GL(R) = \operatorname{colim}_{n \rightarrow \infty} GL_n(R)$ .

An invertible matrix  $A$  over  $R$  represents the trivial element in  $K_1(R)$  if it can be transformed by elementary row and column operations and by stabilization,  $A \rightarrow A \oplus 1$  or the inverse, to the empty matrix.

Let  $G$  be a group, and let  $\{\pm g \mid g \in G\}$  be the subgroup of  $K_1(\mathbb{Z}G)$  given by the classes of  $(1, 1)$ -matrices of the shape  $(\pm g)$  for  $g \in G$ . The *Whitehead group*  $\operatorname{Wh}(G)$  of  $G$  is the quotient  $K_1(\mathbb{Z}G)/\{\pm g \mid g \in G\}$ .

**Conjecture 1.7** (Vanishing of the Whitehead group). *The Whitehead group of a torsionfree group vanishes.*

This conjecture has the following geometric interpretation.

An  $n$ -dimensional *cobordism*  $(W; M_0, M_1)$  consists of a compact oriented  $n$ -dimensional smooth manifold  $W$  together with a disjoint decomposition  $\partial W = M_0 \amalg M_1$  of the boundary  $\partial W$  of  $W$ . It is called an *h-cobordism* if the inclusions  $M_i \rightarrow W$  for  $i = 0, 1$  are homotopy equivalences. An  $h$ -cobordism  $(W; M_0, M_1)$  is trivial if it is diffeomorphic relative  $M_0$  to the trivial  $h$ -cobordism  $(M_0 \times [0, 1], M_0 \times \{0\}, M_0 \times \{1\})$ . One can assign to an  $h$ -cobordism its *Whitehead torsion*  $\tau(W, M_0)$  in  $\operatorname{Wh}(\pi_1(M_0))$ .

**Theorem 1.8** ( $s$ -Cobordism Theorem). *Let  $M_0$  be a closed connected oriented smooth manifold of dimension  $n \geq 5$  with fundamental group  $\pi = \pi_1(M_0)$ . Then:*

- (i) *An  $h$ -cobordism  $(W; M_0, M_1)$  is trivial if and only if its Whitehead torsion  $\tau(W, M_0) \in \operatorname{Wh}(\pi)$  vanishes;*
- (ii) *For any  $x \in \operatorname{Wh}(\pi)$  there is an  $h$ -cobordism  $(W; M_0, M_1)$  with  $\tau(W, M_0) = x \in \operatorname{Wh}(\pi)$ .*

The  $s$ -Cobordism Theorem 1.8 is due to Barden, Mazur, Stallings. Its topological version was proved by Kirby and Siebenmann [45, Essay II]. More information about the  $s$ -Cobordism Theorem can be found, for instance, in [44], [52, Chapter 1], [66]. The Poincaré Conjecture of dimension  $\geq 5$  is a consequence of the  $s$ -Cobordism Theorem 1.8. The  $s$ -Cobordism Theorem 1.8 is an important ingredient in the surgery theory due to Browder, Novikov, Sullivan and Wall, which is the main tool for the classification of manifolds.

The  $s$ -Cobordism Theorem tells us that the vanishing of the Whitehead group, as predicted in Conjecture 1.7, has the following geometric interpretation: For a finitely presented group  $G$  the vanishing of the Whitehead group  $\operatorname{Wh}(G)$  is equivalent to the statement that every  $h$ -cobordism  $W$  of dimension  $\geq 6$  with fundamental group  $\pi_1(W) \cong G$  is trivial.

**1.8. The Bass Conjecture.** For a finite group  $G$  there is a well-known fact that the homomorphism from the complexification of the complex representation ring of  $G$  to the  $\mathbb{C}$ -algebra of complex-valued class functions on  $G$ , given by taking the character of a finite-dimensional complex representation, is an isomorphism. The Bass Conjecture aims at a generalization of this fact to arbitrary groups.

Let  $\text{con}(G)$  be the set of conjugacy classes  $(g)$  of elements  $g \in G$ . Denote by  $\text{con}(G)_f$  the subset of  $\text{con}(G)$  consisting of those conjugacy classes  $(g)$  for which each representative  $g$  has finite order. Let  $\text{class}_0(G)$  and  $\text{class}_0(G)_f$  respectively be the  $\mathbb{C}$ -vector spaces with the set  $\text{con}(G)$  and  $\text{con}(G)_f$  respectively as basis. This is the same as the  $\mathbb{C}$ -vector space of  $\mathbb{C}$ -valued functions on  $\text{con}(G)$  and  $\text{con}(G)_f$  with finite support. Define the *universal  $\mathbb{C}$ -trace* as

$$\text{tr}_{\mathbb{C}G}^u: \mathbb{C}G \rightarrow \text{class}_0(G), \quad \sum_{g \in G} \lambda_g \cdot g \mapsto \sum_{g \in G} \lambda_g \cdot (g).$$

It extends to a function  $\text{tr}_{\mathbb{C}G}^u: M_n(\mathbb{C}G) \rightarrow \text{class}_0(G)$  on  $(n, n)$ -matrices over  $\mathbb{C}G$  by taking the sum of the traces of the diagonal entries. Let  $P$  be a finitely generated projective  $\mathbb{C}G$ -module. Choose a matrix  $A \in M_n(\mathbb{C}G)$  such that  $A^2 = A$  and the image of the  $\mathbb{C}G$ -map  $r_A: \mathbb{C}G^n \rightarrow \mathbb{C}G^n$  given by right multiplication with  $A$  is  $\mathbb{C}G$ -isomorphic to  $P$ . Define the *Hattori-Stallings rank* of  $P$  as

$$\text{HS}_{\mathbb{C}G}(P) := \text{tr}_{\mathbb{C}G}^u(A) \in \text{class}_0(G).$$

The Hattori-Stallings rank depends only on the isomorphism class of the  $\mathbb{C}G$ -module  $P$  and induces a homomorphism  $\text{HS}_{\mathbb{C}G}: K_0(\mathbb{C}G) \rightarrow \text{class}_0(G)$ .

**Conjecture 1.9** ((Strong) Bass Conjecture for  $K_0(\mathbb{C}G)$ ). *The Hattori-Stalling rank yields an isomorphism*

$$\text{HS}_{\mathbb{C}G}: K_0(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow \text{class}_0(G)_f.$$

More information and further references about the Bass Conjecture can be found in [8, 0.5], [13],[54, Subsection 9.5.2], and [63, 3.1.3].

## 2. The Farrell-Jones Conjecture for torsionfree groups

**2.1. The  $K$ -theoretic Farrell-Jones Conjecture for torsionfree groups and regular coefficient rings.** We have already explained  $K_0(R)$  and  $K_1(R)$  for a ring  $R$ . There exist algebraic  $K$ -groups  $K_n(R)$ , for every  $n \in \mathbb{Z}$ , defined as the homotopy groups of the associated  $K$ -theory spectrum  $\mathbf{K}(R)$ . For the definition of higher algebraic  $K$ -theory groups and the (connective)  $K$ -theory spectrum see, for instance, [20, 74, 82, 85]. For information about negative  $K$ -groups, we refer the reader to [12, 32, 72, 73, 80, 82].

How can one come to a conjecture about the structure of the groups  $K_n(RG)$ ? Let us consider the special situation, where the coefficient ring  $R$  is regular. Then one gets isomorphisms

$$\begin{aligned} K_n(R[\mathbb{Z}]) &\cong K_n(R) \oplus K_{n-1}(R); \\ K_n(R[G * H]) \oplus K_n(R) &\cong K_n(RG) \oplus K_n(RH). \end{aligned}$$

Now notice that for any generalized homology theory  $\mathcal{H}$ , we obtain isomorphisms

$$\begin{aligned} \mathcal{H}_n(B\mathbb{Z}) &\cong \mathcal{H}_n(\{\bullet\}) \oplus \mathcal{H}_{n-1}(\{\bullet\}); \\ \mathcal{H}_n(B(G * H)) \oplus \mathcal{H}_n(\{\bullet\}) &\cong \mathcal{H}_n(BG) \oplus \mathcal{H}(BH). \end{aligned}$$

This and other analogies suggest that  $K_n(RG)$  may coincide with  $\mathcal{H}_n(BG)$  for an appropriate generalized homology theory. If this is the case, we must have  $\mathcal{H}_n(\{\bullet\}) = K_n(R)$ . Hence, a natural guess for  $\mathcal{H}_n$  is  $H_n(-; \mathbf{K}(R))$ , the homology theory associated to the algebraic  $K$ -theory spectrum  $\mathbf{K}(R)$  of  $R$ . These considerations lead to:

**Conjecture 2.1** (*K-theoretic Farrell-Jones Conjecture for torsionfree groups and regular coefficient rings*). *Let  $R$  be a regular ring and let  $G$  be a torsionfree group. Then there is an isomorphism*

$$H_n(BG; \mathbf{K}(R)) \xrightarrow{\cong} K_n(RG).$$

**Remark 2.2** (The Farrell-Jones Conjecture and the vanishing of middle  $K$ -groups). If  $R$  is a regular ring, then  $K_q(R) = 0$  for  $q \leq -1$ . Hence the Atiyah-Hirzebruch spectral sequence converging to  $H_n(BG; \mathbf{K}(R))$  is a first quadrant spectral sequence. Its  $E^2$ -term is  $H_p(BG; K_q(R))$ . The edge homomorphism at  $(0, 0)$  obviously yields an isomorphism  $H_0(BG; K_0(R)) \xrightarrow{\cong} H_0(BG; \mathbf{K}(R))$ . The Farrell-Jones Conjecture 2.1 predicts, because of  $H_0(BG; K_0(R)) \cong K_0(R)$ , that there is an isomorphism  $K_0(R) \xrightarrow{\cong} K_0(RG)$ . We have not specified the isomorphism appearing in the Farrell-Jones Conjecture 2.1 above. However, we remark that it is easy to check that this isomorphism  $K_0(R) \xrightarrow{\cong} K_0(RG)$  must be the change of rings map associated to the inclusion  $R \rightarrow RG$ . Thus, we see that the Farrell-Jones Conjecture 2.1 implies Conjecture 1.6.

The Atiyah-Hirzebruch spectral sequence yields an exact sequence  $0 \rightarrow K_1(R) \rightarrow H_1(BG; \mathbf{K}(R)) \rightarrow H_1(G, K_0(R)) \rightarrow 0$ . In the special case  $R = \mathbb{Z}$ , this reduces to an exact sequence  $0 \rightarrow \{\pm 1\} \rightarrow H_1(BG; \mathbf{K}(R)) \rightarrow G/[G, G] \rightarrow 0$ . This implies that the assembly map sends  $H_1(BG; \mathbf{K}(R))$  bijectively onto the subgroup  $\{\pm g \mid g \in G\}$  of  $K_1(\mathbb{Z}G)$ . Hence, the Farrell-Jones Conjecture 2.1 implies Conjecture 1.7.

**Remark 2.3** (The Farrell-Jones Conjecture and the Kaplansky Conjecture). The Farrell-Jones Conjecture 2.1 also implies the Kaplansky Conjecture 1.4 (see [8, Theorem 0.12]).

**Remark 2.4** (The conditions torsionfree and regular are needed in Conjecture 2.1). The version of the Farrell-Jones Conjecture 2.1 cannot be true without the assumptions that  $R$  is regular and  $G$  is torsionfree. The Bass-Heller-Swan decomposition yields an isomorphism  $K_n(R[\mathbb{Z}]) \cong K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R)$ ,

whereas  $H_n(B\mathbb{Z}; \mathbf{K}(R)) \cong K_n(R) \oplus K_{n-1}(R)$ . If  $R$  is regular, then  $NK_n(R)$  is trivial, but there are rings  $R$  with non-trivial  $NK_n(R)$ .

Suppose that  $R = \mathbb{C}$  and  $G$  is finite. Then  $H_0(BG; \mathbf{K}_{\mathbb{C}}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ , whereas  $K_0(\mathbb{C}G)$  is the complex representation ring of  $G$ , which is isomorphic to  $\mathbb{Z}$  if and only if  $G$  is trivial.

## 2.2. The $L$ -theoretic Farrell-Jones Conjecture for torsionfree groups.

There is also an  $L$ -theoretic version of Conjecture 2.1:

**Conjecture 2.5** ( $L$ -theoretic Farrell-Jones Conjecture for torsionfree groups). *Let  $R$  be a ring with involution and let  $G$  be a torsionfree group. Then there is an isomorphism*

$$H_n(BG; \mathbf{L}(R)^{\langle -\infty \rangle}) \xrightarrow{\cong} L_n^{\langle -\infty \rangle}(RG).$$

Here  $\mathbf{L}(R)^{\langle -\infty \rangle}$  is the periodic quadratic  $L$ -theory spectrum of the ring with involution  $R$  with decoration  $\langle -\infty \rangle$ , and  $L_n^{\langle -\infty \rangle}(R)$  is the  $n$ -th quadratic  $L$ -group with decoration  $\langle -\infty \rangle$ , which can be identified with the  $n$ -th homotopy group of  $\mathbf{L}_{RG}^{\langle -\infty \rangle}$ . For more information about the various types of  $L$ -groups and decorations and  $L$ -theory spectra we refer the reader to [18, 19, 75, 78, 79, 80, 81, 88]. Roughly speaking,  $L$ -theory deals with quadratic forms. For even  $n$ ,  $L_n(R)$  is related to the Witt group of quadratic forms and for odd  $n$ ,  $L_n(R)$  is related to automorphisms of quadratic forms. Moreover, the  $L$ -groups are four-periodic, i.e.,  $L_n(R) \cong L_{n+4}(R)$ .

**Theorem 2.6** (The Farrell-Jones Conjecture implies the Borel Conjecture in dimensions  $\geq 5$ ). *Suppose that a torsionfree group  $G$  satisfies Conjecture 2.1 and Conjecture 2.5 for  $R = \mathbb{Z}$ . Then the Borel Conjecture 1.1 holds for any closed aspherical manifold of dimension  $\geq 5$  whose fundamental group is isomorphic to  $G$ .*

*Sketch of proof.* The topological structure set  $\mathcal{S}^{\text{top}}(M)$  of a closed manifold  $M$  is defined to be the set of equivalence classes of homotopy equivalences  $f: M' \rightarrow M$ , with a topological closed manifold as its source and  $M$  as its target, for which  $f_0: M_0 \rightarrow M$  and  $f_1: M_1 \rightarrow M$  are equivalent if there is a homeomorphism  $g: M_0 \rightarrow M_1$  such that  $f_1 \circ g$  and  $f_0$  are homotopic. The Borel Conjecture 1.1 can be reformulated in the language of surgery theory to the statement that  $\mathcal{S}^{\text{top}}(M)$  consists of a single point if  $M$  is an aspherical closed topological manifold.

The *surgery sequence* of a closed topological manifold  $M$  of dimension  $n \geq 5$  is the exact sequence

$$\begin{aligned} \dots \rightarrow \mathcal{N}_{n+1}(M \times [0, 1], M \times \{0, 1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M)) \xrightarrow{\partial} \mathcal{S}^{\text{top}}(M) \\ \xrightarrow{\eta} \mathcal{N}_n(M) \xrightarrow{\sigma} L_n^s(\mathbb{Z}\pi_1(M)), \end{aligned} \quad (2.7)$$

which extends infinitely to the left. It is the fundamental tool for the classification of topological manifolds. (There is also a smooth version of it.) The map  $\sigma$  appearing in the sequence sends a normal map of degree one to its surgery obstruction. This map can be identified with the version of the  $L$ -theory assembly map, where one works with the 1-connected cover  $\mathbf{L}^s(\mathbb{Z})\langle 1 \rangle$  of  $\mathbf{L}^s(\mathbb{Z})$ . The map

$H_k(M; \mathbf{L}^s(\mathbb{Z})\langle 1 \rangle) \rightarrow H_k(M; \mathbf{L}^s(\mathbb{Z}))$  is injective for  $k = n$  and an isomorphism for  $k > n$ . Because of the  $K$ -theoretic assumptions (and the so-called Rothenberg sequence), we can replace the  $s$ -decoration with the  $\langle -\infty \rangle$ -decoration. Therefore the Farrell-Jones Conjecture 2.5 implies that the map  $\sigma: \mathcal{N}_n(M) \rightarrow L_n^s(\mathbb{Z}\pi_1(M))$  is injective and the map  $\mathcal{N}_{n+1}(M \times [0, 1], M \times \{0, 1\}) \xrightarrow{\sigma} L_{n+1}^s(\mathbb{Z}\pi_1(M))$  is bijective. Thus, by the surgery sequence,  $\mathcal{S}^{\text{top}}(M)$  is a point and hence the Borel Conjecture 1.1 holds for  $M$ . More details can be found in [36, pages 17, 18, 28], [79, Chapter 18].  $\square$

For more information about surgery theory, see [18, 19, 46, 52, 81, 88].

### 3. The general formulation of the Farrell-Jones Conjecture

**3.1. Classifying spaces for families.** Let  $G$  be a group. A *family*  $\mathcal{F}$  of subgroups of  $G$  is a set of subgroups which is closed under conjugation with elements of  $G$  and under taking subgroups. A  $G$ -CW-complex, all of whose isotropy groups belong to  $\mathcal{F}$  and whose  $H$ -fixed point sets are contractible for all  $H \in \mathcal{F}$ , is called a *classifying space for the family*  $\mathcal{F}$  and will be denoted  $E_{\mathcal{F}}(G)$ . Such a space is unique up to  $G$ -homotopy, because it is characterized by the property that for any  $G$ -CW-complex  $X$ , all whose isotropy groups belong to  $\mathcal{F}$ , there is precisely one  $G$ -map from  $X$  to  $E_{\mathcal{F}}(G)$  up to  $G$ -homotopy. These spaces were introduced by tom Dieck [84]. A functorial “bar-type” construction is given in [22, section 7].

The space  $E_{\mathcal{TR}}(G)$ , for  $\mathcal{TR}$  the family consisting of the trivial subgroup only, is the same as the space  $EG$ , which is by definition the total space of the universal  $G$ -principal bundle  $G \rightarrow EG \rightarrow BG$ , or, equivalently, the universal covering of  $BG$ . A model for  $E_{\mathcal{ALL}}(G)$ , for the family  $\mathcal{ALL}$  of all subgroups, is given by the space  $G/G = \{\bullet\}$  consisting of one point.

The space  $E_{\mathcal{Fin}}(G)$ , for  $\mathcal{Fin}$  the family of finite subgroups, is also known as *the classifying space for proper  $G$ -actions*, and is denoted by  $\underline{EG}$  in the literature. Recall that a  $G$ -CW-complex  $X$  is proper if and only if all of its isotropy groups are finite (see for instance [50, Theorem 1.23 on page 18]).

There are often nice models for  $\underline{EG}$ . If  $G$  is word hyperbolic in the sense of Gromov, then the Rips-complex is a finite model [65]. If  $G$  is a discrete subgroup of a Lie group  $L$  with finitely many path components, then for any maximal compact subgroup  $K \subseteq L$ , the space  $L/K$  with its left  $G$ -action is a model for  $\underline{EG}$ . More information about  $\underline{EG}$  can be found in [14, 27, 51, 58, 62].

Let  $\mathcal{VCyc}$  be the family of virtually cyclic subgroups, i.e., subgroups which are either finite or contain  $\mathbb{Z}$  as subgroup of finite index. We often abbreviate  $\underline{\underline{EG}} = E_{\mathcal{VCyc}}(G)$ .

**3.2.  $G$ -homology theories.** Fix a group  $G$ . A  $G$ -homology theory  $\mathcal{H}_*^G$  is a collection of covariant functors  $\mathcal{H}_n^G$  from the category of  $G$ -CW-pairs to the

category of abelian groups indexed by  $n \in \mathbb{Z}$  together with natural transformations

$$\partial_n^G(X, A): \mathcal{H}_n^G(X, A) \rightarrow \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$$

for  $n \in \mathbb{Z}$ , such that four axioms hold; namely,  $G$ -homotopy invariance, long exact sequence of a pair, excision, and the disjoint union axiom. The obvious formulation of these axioms is left to the reader or can be found in [53]. Of course a  $G$ -homology theory for the trivial group  $G = \{1\}$  is a homology theory (satisfying the disjoint union axiom) in the classical non-equivariant sense.

**Remark 3.1** ( $G$ -homology theories and spectra over  $\text{Or}(G)$ ). The *orbit category*  $\text{Or}(G)$  has as objects the homogeneous spaces  $G/H$  and as morphisms  $G$ -maps. Given a covariant functor  $\mathbf{E}$  from  $\text{Or}(G)$  to the category of spectra, there exists a  $G$ -homology theory  $\mathcal{H}_*^G$  such that  $\mathcal{H}_n^G(G/H) = \pi_n(\mathbf{E}(G/H))$  holds for all  $n \in \mathbb{Z}$  and subgroups  $H \subseteq G$  (see [22], [63, Proposition 6.3 on page 737]). For trivial  $G$ , this boils down to the classical fact that a spectrum defines a homology theory.

**3.3. The Meta-Isomorphism Conjecture.** Now we can formulate the following Meta-Conjecture for a group  $G$ , a family of subgroups  $\mathcal{F}$ , and a  $G$ -homology theory  $\mathcal{H}_*^G$ .

**Conjecture 3.2** (Meta-Conjecture). *The so-called assembly map*

$$A_{\mathcal{F}}: \mathcal{H}_n^G(E_{\mathcal{F}}(G)) \rightarrow \mathcal{H}_n^G(\text{pt}),$$

which is the map induced by the projection  $E_{\mathcal{F}}(G) \rightarrow \text{pt}$ , is an isomorphism for  $n \in \mathbb{Z}$ .

Notice that the Meta-Conjecture 3.2 is always true if we choose  $\mathcal{F} = \mathcal{ALL}$ . So given  $G$  and  $\mathcal{H}_*^G$ , the point is to choose  $\mathcal{F}$  as small as possible.

**3.4. The Farrell-Jones Conjecture.** Let  $R$  be a ring. Then one can construct for every group  $G$ , using Remark 3.1,  $G$ -homology theories  $H_*^G(-; \mathbf{K}_R)$  and  $H_*^G(-; \mathbf{L}_R^{\langle -\infty \rangle})$  satisfying  $H_n^G(G/H; \mathbf{K}_R) \cong K_n(RH)$  and  $H_n^G(G/H; \mathbf{L}_R^{\langle -\infty \rangle}) \cong L_n^{\langle -\infty \rangle}(RH)$ . The Meta-Conjecture 3.2 for  $\mathcal{F} = \mathcal{VCyc}$  is the Farrell-Jones Conjecture:

**Conjecture 3.3** (Farrell-Jones Conjecture). *The maps induced by the projection  $\underline{EG} \rightarrow G/G$  are, for every  $n \in \mathbb{Z}$ , isomorphisms*

$$\begin{aligned} H_n^G(\underline{EG}; \mathbf{K}_R) &\rightarrow H_n^G(G/G; \mathbf{K}_R) = K_n(RG); \\ H_n^G(\underline{EG}; \mathbf{L}_R^{\langle -\infty \rangle}) &\rightarrow H_n^G(G/G; \mathbf{L}_R^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(RG). \end{aligned}$$

The version of the Farrell-Jones Conjecture 3.3 is equivalent to the original version due to Farrell-Jones [30, 1.6 on page 257]. The decoration  $\langle -\infty \rangle$  cannot be replaced by the decorations  $h$ ,  $s$  or  $p$  in general, since there are counterexamples for these decorations (see [34]).

**Remark 3.4** (Generalized Induction Theorem). One may interpret the Farrell-Jones Conjecture as a kind of generalized induction theorem. A prototype of an induction theorem is Artin's Theorem, which essentially says that the complex representation ring of a finite group can be computed in terms of the representation rings of the cyclic subgroups. In the Farrell-Jones setting one wants to compute  $K_n(RG)$  and  $L_n^{(-\infty)}(RG)$  in terms of the values of these functors on virtually cyclic subgroups, where one has to take into account all the relations coming from inclusions and conjugations, and the values in degree  $n$  depend on all the values in degree  $k \leq n$  on virtually cyclic subgroups.

**Remark 3.5** (The choice of the family  $\mathcal{VCyc}$ ). One can show that, in general,  $\mathcal{VCyc}$  is the smallest family of subgroups for which one can hope that the Farrell-Jones Conjecture is true for all  $G$  and  $R$ . The family  $\mathcal{Fin}$  is definitely too small. Under certain conditions one can use smaller families, for instance,  $\mathcal{Fin}$  is sufficient if  $R$  is regular and contains  $\mathbb{Q}$ , and  $\mathcal{TR}$  is sufficient if  $R$  is regular and  $G$  is torsionfree. This explains that Conjecture 3.3 reduces to Conjecture 2.1 and Conjecture 2.5. More information about reducing the family of subgroups can be found in [3], [23], [24], [57, Lemma 4.2], [63, 2.2], [77].

Remarks 3.4 and 3.5 can be illustrated by the following consequence of the Farrell-Jones Conjecture 3.3: Given a field  $F$  of characteristic zero and a group  $G$ , the obvious map

$$\bigoplus_{H \subseteq G, |H| < \infty} K_0(FH) \rightarrow K_0(FG)$$

coming from the various inclusions  $H \subseteq G$  is surjective, and actually induces an isomorphism

$$\operatorname{colim}_{H \subseteq G, |H| < \infty} K_0(FH) \xrightarrow{\cong} K_0(FG).$$

**Remark 3.6** (The  $K$ -theoretic Farrell-Jones Conjecture and the Bass Conjecture). The  $K$ -theoretic Farrell-Jones Conjecture 3.3 implies the Bass Conjecture 3.3 (see [8, Theorem 0.9]).

**Remark 3.7** (Coefficients in additive categories). It is sometimes important to consider twisted group rings, where we take a  $G$ -action on  $R$  into account, or more generally, crossed product rings  $R * G$ . In the  $L$ -theory case we also want to allow orientation characters. All of these generalizations can be uniformly handled if one allows coefficients in an additive category. These more general versions of the Farrell-Jones Conjectures are explained for  $K$ -theory in [10] and for  $L$ -theory in [5]. These generalizations also encompass the so-called fibered versions. One of their main features is that they have much better inheritance properties, (e.g., passing to subgroups, direct and free products, directed colimits) than the untwisted version 3.3.

For proofs the coefficients are often dummy variables. In the right setup it does not matter whether one uses coefficients in a ring  $R$  or in an additive category.

**3.5. The Baum-Connes and the Bost Conjectures.** There also exists a  $G$ -homology theory  $H_*^G(-; \mathbf{K}_{C_r^*}^{\text{top}})$  with the property that  $H_n^G(G/H; \mathbf{K}_{C_r^*}^{\text{top}}) = K_n(C_r^*(H))$ , where  $K_n(C_r^*(H))$  is the topological  $K$ -theory of the reduced group  $C^*$ -algebra. For a proper  $G$ - $CW$ -complex  $X$ , the equivariant topological  $K$ -theory  $K_n^G(X)$  agrees with  $H_n^G(X; \mathbf{K}_{C_r^*}^{\text{top}})$ . The Meta-Conjecture 3.2 for  $\mathcal{F} = \mathcal{F}\text{in}$  is:

**Conjecture 3.8** (Baum-Connes-Conjecture). *The maps induced by the projection  $\underline{EG} \rightarrow G/G$*

$$K_n^G(\underline{EG}) = H_n^G(\underline{EG}; \mathbf{K}_{C_r^*}^{\text{top}}) \rightarrow H_n^G(G/G; \mathbf{K}_{C_r^*}^{\text{top}}) = K_n(C_r^*(G)).$$

are isomorphisms for every  $n \in \mathbb{Z}$ .

The original version of the Baum-Connes Conjecture is stated in [14, Conjecture 3.15 on page 254]. For more information about the Baum-Connes Conjecture, see, for instance, [40, 63, 68].

**Remark 3.9** (The relation between the conjectures of Novikov, Farrell-Jones and Baum-Connes). Both the  $L$ -theoretic Farrell-Jones Conjecture 3.3 and the Baum-Connes Conjecture 3.8 imply the Novikov Conjecture. See [47, Section 23], where the relation between the  $L$ -theoretic Farrell-Jones Conjecture 3.3 and the Baum-Connes Conjecture 3.8 is also explained.

## 4. The status of the Farrell-Jones Conjecture

**4.1. The work of Farrell-Jones and the status in 2004.** One of the highlights of the work of Farrell and Jones is their proof of the Borel Conjecture 1.1 for manifolds of dimension  $\geq 5$  which support a Riemannian metric of non-positive sectional curvature [31]. They were able to extend this result to cover compact complete affine flat manifolds of dimension  $\geq 5$  [33]. This was done by considering complete non-positively curved manifolds that are not necessarily compact. Further results by Farrell and Jones about their conjecture for  $K$ -theory and pseudo-isotopy can be found in [30]. For a detailed report about the status of the Baum-Connes Conjecture and Farrell-Jones Conjecture in 2004 we refer to [63, Chapter 5], where one can also find further references to relevant papers.

**4.2. Hyperbolic groups and CAT(0)-groups.** In recent years, the class of groups for which the Farrell-Jones Conjecture, and hence the other conjectures appearing in Section 1, are true has been extended considerably beyond fundamental groups of non-positively curved manifolds. In what follows, a *hyperbolic group* is to be understood in the sense of Gromov. A CAT(0)-group is a group that admits a proper isometric cocompact action on some CAT(0)-space of finite topological dimension.

**Theorem 4.1** (Hyperbolic groups). *The Farrell-Jones Conjecture with coefficients in additive categories (see Remark 3.7) holds for both  $K$ - and  $L$ -theory for every hyperbolic group.*

*Proof.* The  $K$ -theory part is proved in Bartels-Lück-Reich [7], the  $L$ -theory part in Bartels-Lück [4].  $\square$

**Theorem 4.2** (CAT(0)-groups).

- (i) *The  $L$ -theoretic Farrell-Jones Conjecture with coefficients in additive categories (see Remark 3.7) holds for every CAT(0)-group;*
- (ii) *The assembly map for the  $K$ -theoretic Farrell-Jones Conjecture with coefficients in additive categories (see Remark 3.7) is bijective in degrees  $n \leq 0$  and surjective in degree  $n = 1$  for every CAT(0)-group.*

*Proof.* This is proved in Bartels-Lück [4].  $\square$

For the proofs that the Farrell-Jones Conjecture implies the conjectures mentioned in Section 1, it suffices to know the statements appearing in Theorem 4.2. For instance Theorem 4.2 implies the Borel Conjecture for every closed aspherical manifold of dimension  $\geq 5$  whose fundamental group is a CAT(0)-group.

**4.3. Inheritance properties.** We have already mentioned that the version of the Farrell-Jones Conjecture with coefficients in additive categories (see Remark 3.7) does not only include twisted group rings and allow one to insert orientation homomorphisms, but it also has very valuable inheritance properties.

**Theorem 4.3** (Inheritance properties). *Let (A) be one of the following assertions for a group  $G$ :*

- *The  $K$ -theoretic Farrell-Jones Conjecture with coefficients in additive categories (see Remark 3.7) holds for  $G$ ;*
- *The  $K$ -theoretic Farrell-Jones Conjecture with coefficients in additive categories (see Remark 3.7) holds for  $G$  up to degree one, i.e., the assembly map is bijective in dimension  $n \leq 0$  and surjective for  $n = 1$ ;*
- *The  $L$ -theoretic Farrell-Jones Conjecture with coefficients in additive categories (see Remark 3.7) holds for  $G$ .*

*Then the following is true:*

- (i) *If  $G$  satisfies assertion (A), then also every subgroup  $H \subseteq G$  satisfies (A);*
- (ii) *If  $G_1$  and  $G_2$  satisfies assertion (A), then also the free product  $G_1 * G_2$  and the direct product  $G_1 \times G_2$  satisfy assertion (A);*
- (iii) *Let  $\pi: G \rightarrow Q$  be a group homomorphism. If  $Q$  satisfies (A) and for every virtually cyclic subgroup  $V \subseteq Q$ , its preimage  $\pi^{-1}(V)$  satisfies (A), then  $G$  satisfies assertion (A);*
- (iv) *Let  $\{G_i \mid i \in I\}$  be a directed system of groups (with not necessarily injective structure maps). If each  $G_i$  satisfies assertion (A), then the colimit  $\operatorname{colim}_{i \in I} G_i$  satisfies assertion (A).*

*Proof.* See [4, Lemma 2.3].  $\square$

**Examples.** Let  $\mathcal{FJ}$  be the class of groups satisfying both the  $K$ -theoretic and  $L$ -theoretic Farrell-Jones Conjecture with additive categories as coefficients (see Remark 3.7). Let  $\mathcal{FJ}_{\leq 1}$  be the class of groups which satisfy the  $L$ -theoretic Farrell-Jones Conjecture with additive categories as coefficients and the  $K$ -theoretic Farrell-Jones Conjecture with additive categories as coefficients up to degree one.

In view of the results above, these classes contain many groups which lie in the region *Hic Abundant Leones* in Martin Bridson’s universe of groups (see [15]). Theorem 4.1 and Theorem 4.3 (iv) imply that directed colimits of hyperbolic groups belong to  $\mathcal{FJ}$ . This class of groups contains a number of groups with unusual properties. Counterexamples to the Baum-Connes Conjecture with coefficients are groups with expanders [41]. The only known construction of such groups is as a directed colimit of hyperbolic groups (see [2]). Thus the Farrell-Jones Conjecture in  $K$ - and  $L$ -theory holds for the only presently known counterexamples to the Baum-Connes Conjecture with coefficients. (We remark that the formulation of the Farrell-Jones Conjecture we are considering allows for twisted group rings, so this includes the correct analog of the Baum-Connes Conjecture with coefficients.) The class of directed colimits of hyperbolic groups contains, for instance, a torsionfree non-cyclic group all whose proper subgroups are cyclic, constructed by Ol’shanskii [70]. Further examples are lacunary groups (see [71]).

Davis and Januszkiewicz used Gromov’s hyperbolization technique to construct exotic aspherical manifolds. They showed that for every  $n \geq 5$  there are closed aspherical  $n$ -dimensional manifolds such that their universal covering is a  $\text{CAT}(0)$ -space whose fundamental group at infinity is non-trivial [26, Theorem 5b.1]. In particular, these universal coverings are not homeomorphic to Euclidean space. Because these examples are non-positively curved polyhedron, their fundamental groups are  $\text{CAT}(0)$ -groups and belong to  $\mathcal{FJ}_{\leq 1}$ . There is a variation of this construction that uses the strict hyperbolization of Charney-Davis [21] and produces closed aspherical manifolds whose universal cover is not homeomorphic to Euclidean space and whose fundamental group is hyperbolic. All of these examples are topologically rigid.

*Limit groups* in the sense of Zela have been a focus of geometric group theory in recent years. Alibegović-Bestvina [1] have shown that limit groups are  $\text{CAT}(0)$ -groups.

Let  $G$  be a (not necessarily cocompact) lattice in  $SO(n, 1)$ , e.g., the fundamental group of a hyperbolic Riemannian manifold with finite volume. Then  $G$  acts properly cocompactly and isometrically on a  $\text{CAT}(0)$ -space by [16, Corollary 11.28 in Chapter II.11 on page 362], and hence belongs to  $\mathcal{FJ}_{\leq 1}$ .

## 5. Computational aspects

It is very hard to compute  $K_n(RG)$  or  $L_n^{(-\infty)}(RG)$  directly. It is easier to compute the source of the assembly map appearing in the Farrell-Jones Conjecture 3.3, since one can apply standard techniques for the computation of equivariant homology theories and there are often nice models for  $\underline{E}G$ . Rationally, equivariant Chern

characters, as developed in [53, 55, 56] give rather general answers. We illustrate this with the following result taken from [53, Example 8.11].

**Theorem 5.1.** *Let  $G$  be a group for which the Farrell-Jones Conjecture 3.3 holds for  $R = \mathbb{C}$ . Let  $T$  be the set of conjugacy classes  $(g)$  of elements  $g \in G$  of finite order. For an element  $g \in G$ , denote by  $C_G(g)$  the centralizer of  $g$ . Then we obtain isomorphisms*

$$\begin{aligned} \bigoplus_{p+q=n} \bigoplus_{(g) \in T} H_p(C_G(g); \mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) &\rightarrow \mathbb{C} \otimes_{\mathbb{Z}} K_n(\mathbb{C}G); \\ \bigoplus_{p+q=n} \bigoplus_{(g) \in T} H_p(C_G(g); \mathbb{C}) \otimes_{\mathbb{Z}} L_q^{(-\infty)}(\mathbb{C}) &\rightarrow \mathbb{C} \otimes_{\mathbb{Z}} L_n^{(-\infty)}(\mathbb{C}G), \end{aligned}$$

where we use the involutions coming from complex conjugation in the definition of  $L_q^{(-\infty)}(\mathbb{C})$  and  $L_n^{(-\infty)}(\mathbb{C}G)$ .

Integral computations can only be given in special cases. For example, the semi-direct product  $\mathbb{Z}^r \rtimes \mathbb{Z}/n$  cannot be handled in general. Not even its ordinary group homology is known, so it is not a surprise that the  $K$ - and  $L$ -theory of the associated group ring are unknown in general. Sometimes explicit answers can be found in the literature, see for instance [63, 8.3]. As an illustration we mention the following result which follows from Theorem 4.1 using [11, Theorem 1.3], and [64, Corollary 2.11 and Example 3.6].

**Theorem 5.2** (Torsionfree hyperbolic groups). *Let  $G$  be a torsionfree hyperbolic group. Let  $\mathcal{M}$  be a complete system of representatives of the conjugacy classes of maximal infinite cyclic subgroups of  $G$ .*

(i) *For every  $n \in \mathbb{Z}$ , there is an isomorphism*

$$H_n(BG; \mathbf{K}(R)) \oplus \bigoplus_{V \in \mathcal{M}} NK_n(R) \oplus NK_n(R) \xrightarrow{\cong} K_n(RG),$$

where  $NK_n(R)$  the Bass-Nil-groups of  $R$ ;

(ii) *For every  $n \in \mathbb{Z}$ , there is an isomorphism*

$$H_n(BG; \mathbf{L}^{(-\infty)}(R)) \xrightarrow{\cong} L_n^{(-\infty)}(RG).$$

Computations of  $L$ -groups of group rings are important in the classification of manifolds since they appear in the surgery sequence (2.7).

## 6. Methods of proof

Here is a brief sketch of the strategy of proof which has led to the results about hyperbolic groups and  $\text{CAT}(0)$ -groups mentioned above. It is influenced by ideas of Farrell and Jones. However, we have to deal with spaces that are not manifolds, and hence new ideas and techniques are required. A more detailed survey about methods of proof can be found in [4, Section 1], [6, Section 1], [7, Section 1], [61] and [63, Chapter 7].

**Assembly and forget control.** We have defined the assembly map appearing in the Farrell-Jones Conjecture as a map induced by the projection  $\underline{EG} \rightarrow G/G$ . A homotopy theoretic interpretation by homotopy colimits and a description in terms of the universal property that it is the best approximation from the left by a homology theory is presented in [22]. This interpretation is good for structural and computational aspects but is not helpful for actual proofs. For this purpose the interpretation of the assembly map as a *forget control map* is the right one. This fundamental idea is due to Quinn.

Roughly speaking, one attaches to a metric space certain categories, to these categories spectra and then takes their homotopy groups, where everything depends on a choice of certain control conditions which in some sense measure sizes of cycles. If one requires certain control conditions, one obtains the source of the assembly map. If one requires no control conditions, one obtains the target of the assembly map. The assembly map itself is forgetting the control condition.

One of the basic features of a homology theory is excision. It often comes from the fact that a representing cycle can be found with arbitrarily good control. An example is the technique of subdivision which allows to make the representing cycles for simplicial homology arbitrarily controlled. That is, the diameter of any simplex appearing with non-zero coefficient is very small. One may say that requiring control conditions amounts to implementing homological properties.

With this interpretation it is clear what the main task in the proof of surjectivity of the assembly map is: *achieve control*, i.e., manipulate cycles without changing their homology class so that they become sufficiently controlled. There is a general principle that a proof of surjectivity also gives injectivity, namely, proving injectivity means that one must construct a cycle whose boundary is a given cycle, i.e., one has to solve a surjectivity problem in a relative situation.

**Contracting maps and open coverings.** Contracting maps on suitable control spaces are very useful for gaining control. The idea is that the contraction improves the control of a cycle without changing its homology class if the contracting map is, roughly speaking, homotopic to the identity. Of course one has to choose the contracting maps and control spaces with care. If a  $G$ -space  $X$  has a fixed point, the projection to this fixed point is a contracting  $G$ -equivariant map, but it turns out that this is just enough to prove the trivial version of the Meta Conjecture, where the family  $\mathcal{F}$  is not  $\mathcal{VCyc}$  as desired, but rather consists of all subgroups.

Let  $\mathcal{F}$  be a family of subgroups and let  $X$  be a metric space with an isometric  $G$ -action. An  $\mathcal{F}$ -covering  $\mathcal{U}$  is an open covering  $\mathcal{U}$  such that  $gU \in \mathcal{U}$  holds for  $U \in \mathcal{U}, g \in G$ , for every  $U \in \mathcal{U}$  and  $g \in G$  we have  $gU \cap U \neq \emptyset \implies gU = U$ , and for every  $U \in \mathcal{U}$  the subgroup  $G_U = \{g \in G \mid gU = U\}$  belongs to  $\mathcal{F}$ . Associated to these data there is a map  $f_{\mathcal{U}}: X \rightarrow |\mathcal{U}|$  from  $X$  to the simplicial nerve of  $\mathcal{U}$ . The larger the Lebesgue number of  $\mathcal{U}$  is, the more contracting the map becomes with respect to the  $L^1$ -metric on  $|\mathcal{U}|$ , provided we are able to fix a uniform bound on its covering dimension (see [7, Proposition 5.3]).

Notice that the simplicial nerve carries a  $G$ -CW-complex structure and all its isotropy groups belong to  $\mathcal{F}$ . We see that  $\mathcal{F}$ -coverings can yield contracting maps,

as long as the covering dimension of the possible  $\mathcal{U}$  are uniformly bounded.

An axiomatic description of the properties such an equivariant covering has to fulfill can be found in [7, Section 1] and more generally in [4, Section 1]. The equivariant coverings satisfy conditions that are similar to those for finite asymptotic dimension, but with extra requirements about equivariance. A key technical paper for the construction of such equivariant coverings is [6], where the connection to asymptotic dimension is explained.

**Enlarging  $G$  and transfer.** Let us try to find  $\mathcal{F}$ -coverings for  $G$  considered as a metric space with the word metric. If we take  $\mathcal{U} = \{G\}$ , we obtain a  $G$ -invariant open covering with arbitrarily large Lebesgue number, but the open set  $G$  is an  $\mathcal{F}$ -set only if we take  $\mathcal{F}$  to be the family of all subgroups. If we take  $\mathcal{U} = \{\{g\} \mid g \in G\}$  and denote by  $\mathcal{TR}$  the family consisting only of the trivial subgroup, we obtain a  $\mathcal{TR}$ -covering of topological dimension zero, but the Lebesgue number is not very impressive, it's just 1. In order to increase the Lebesgue number, we could take large balls around each element. Since the covering has to be  $G$ -invariant, we could start with  $\mathcal{U} = \{B_R(g) \mid g \in G\}$ , where  $B_R(g)$  is the open ball of radius  $R$  around  $g$ . This is a  $G$ -invariant open covering with Lebesgue number  $R$ , but the sets  $B_R(g)$  are not  $\mathcal{F}$ -sets in general and the covering dimension grows with  $R$ .

One of the main ideas is not to cover  $G$  itself, but to enlarge  $G$  to  $G \times \overline{X}$  for an appropriate compactification  $\overline{X}$  of a certain contractible metric space  $X$  that has an isometric proper cocompact  $G$ -action. This allows us to spread out the open sets and avoid having too many intersections. This strategy has also been successfully used in measurable group theory, where the role of the topological space  $\overline{X}$  is played by a probability space with measure preserving  $G$ -action (see Gromov [39, page 300]).

The elements under consideration lie in  $K$ - or  $L$ -theory spaces associated to the control space  $G$ . Using a transfer they can be lifted to  $G \times \overline{X}$ . (This step corresponds in the proofs of Farrell and Jones to the passage to the sphere tangent bundle.) We gain control there and then push the elements down to  $G$ . Since the space  $\overline{X}$  is contractible, its Euler characteristic is 1 and hence the composite of the push-down map with the transfer map is the identity on the  $K$ -theory level. On the  $L$ -theory level one needs something with signature 1. On the algebra level this corresponds to the assignment of a finitely generated projective  $\mathbb{Z}$ -module  $P$  to its *multiplicative hyperbolic form*  $H_{\otimes}(P)$ . It is given by replacing  $\oplus$  by  $\otimes$  in the standard definition of a hyperbolic form, i.e., the underlying  $\mathbb{Z}$ -module is  $P^* \otimes P$  and the symmetric form is given by the formula  $(\alpha, p) \otimes (\beta, q) \mapsto \alpha(q) \cdot \beta(p)$ . Notice that the signature of  $H_{\otimes}(\mathbb{Z})$  is 1 and taking the multiplicative hyperbolic form yields an isomorphism of rings  $K_0(\mathbb{Z}) \rightarrow L^0(\mathbb{Z})$ .

We can construct  $\mathcal{VCyc}$ -coverings that are contracting in the  $G$ -direction but will actually expand in the  $\overline{X}$ -direction. The latter defect can be compensated for because the transfer yields elements over  $G \times \overline{X}$  with arbitrarily good control in the  $\overline{X}$ -direction.

**Flows.** To find such coverings of  $G \times \overline{X}$ , it is crucial to construct, for hyperbolic and CAT(0)-spaces, flow spaces  $\text{FS}(X)$  which are the analog of the geodesic flow on a simply connected Riemannian manifold with negative or non-positive sectional curvature. One constructs appropriate coverings on  $\text{FS}(X)$ , often called *long and thin coverings*, and then pulls them back with a certain map  $G \times \overline{X} \rightarrow \text{FS}(X)$ . The flow is used to improve a given covering. The use flow spaces to gain control is one of the fundamental ideas of Farrell and Jones (see for instance [28]).

Let us look at a special example to illustrate the use of a flow. Consider two points with coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$  in the upper half plane model of two-dimensional hyperbolic space. We want to use the geodesic flow to make their distance smaller in a functorial fashion. This is achieved by letting these points flow towards the boundary at infinity along the geodesic given by the vertical line through these points, i.e., towards infinity in the  $y$ -direction. There is a fundamental problem: if  $x_1 = x_2$ , then the distance of these points is unchanged. Therefore we make the following prearrangement. Suppose that  $y_1 < y_2$ . Then we first let the point  $(x_1, y_1)$  flow so that it reaches a position where  $y_1 = y_2$ . Inspecting the hyperbolic metric, one sees that the distance between the two points  $(x_1, \tau)$  and  $(x_2, \tau)$  goes to zero if  $\tau$  goes to infinity. This is the basic idea to gain control in the negatively curved case.

Why is the non-positively curved case harder? Again, consider the upper half plane, but this time equip it with the flat Riemannian metric coming from Euclidean space. Then the same construction makes sense, but the distance between two points  $(x_1, \tau)$  and  $(x_2, \tau)$  is unchanged if we change  $\tau$ . The basic first idea is to choose a focal point far away, say  $f := ((x_1 + x_2)/2, \tau + 169356991)$ , and then let  $(x_1, \tau)$  and  $(x_2, \tau)$  flow along the rays emanating from them and passing through the focal point  $f$ . In the beginning the effect is indeed that the distance becomes smaller, but as soon as we have passed the focal point the distance grows again. Either one chooses the focal point very far away or uses the idea of moving the focal point towards infinity while the points flow. Roughly speaking, we are suggesting the idea of a *dog and sausage* principle. We have a dog, and attached to it is a long stick pointing in front of it with a delicious sausage on the end. The dog will try to reach the sausage, but the sausage is moving away according to the movement of the dog, so the dog will never reach the sausage. (The dog will become long and thin this way, but this is a different effect). The problem with this idea is obvious, we must describe this process in a functorial way and carefully check all the estimates to guarantee the desired effects.

## 7. Open Problems

**7.1. Virtually poly-cyclic groups, cocompact lattices and 3-manifold groups.** It is conceivable that our methods can be used to show that virtually poly-cyclic groups belong to  $\mathcal{FJ}$  or  $\mathcal{FJ}_{\leq 1}$ . This already implies the same conclusion for cocompact lattices in almost connected Lie groups following ideas of Farrell-Jones [30] and for fundamental groups of (not necessarily compact)

3-manifolds (possibly with boundary) following ideas of Roushon [83].

**7.2. Solvable groups.** Show that solvable groups belong to  $\mathcal{FJ}$  or  $\mathcal{FJ}_{\leq 1}$ . In view of the large class of groups belonging to  $\mathcal{FJ}$  or  $\mathcal{FJ}_{\leq 1}$ , it is very surprising that it is not known whether a semi-direct product  $A \rtimes_{\varphi} \mathbb{Z}$  for a (not necessarily finitely generated) abelian group  $A$  belongs to  $\mathcal{FJ}$  or  $\mathcal{FJ}_{\leq 1}$ . The problem is the possibly complicated dynamics of the automorphism  $\varphi$  of  $A$ .

Such groups are easy to handle in the Baum-Connes setting, where one can use the long exact Wang sequence for topological  $K$ -theory associated to a semi-direct product. Such a sequence does not exist for algebraic  $K$ -theory, and new contributions involving Nil-terms occur.

**7.3. Other open cases.** Show that mapping class groups,  $\text{Out}(F_n)$  and Thompson's groups belong to  $\mathcal{FJ}$  or  $\mathcal{FJ}_{\leq 1}$ . The point here is not that this has striking consequence in and of itself, but rather their proofs will probably give more insight in the Farrell-Jones Conjecture and will require some new input about the geometry of these groups which may be interesting in its own right.

A very interesting open case is  $SL_n(\mathbb{Z})$ . The main obstacle is that  $SL_n(\mathbb{Z})$  does not act cocompactly isometrically properly on a  $\text{CAT}(0)$ -space; the canonical action on  $SL_n(\mathbb{R})/SO(n)$  is proper and isometric and of finite covolume but not cocompact. The Baum-Connes Conjecture is also open for  $SL_n(\mathbb{Z})$ .

**7.4. Searching for counterexamples.** There is no group known for which the Farrell-Jones Conjecture is false. There has been some hope that groups with expanders may yield counterexamples, but this hope has been dampened since colimits of hyperbolic groups satisfy it. At the moment one does not know any property of a group which makes it likely to produce a counterexample. The same holds for the Borel Conjecture. Many of the known exotic examples of closed aspherical manifolds are known to satisfy the Borel Conjecture.

In order to find counterexamples one seems to need completely new ideas, maybe from random groups or logic.

**7.5. Pseudo-isotopy.** Extend our results to pseudo-isotopy spaces. There are already interesting results for these proved by Farrell-Jones [30].

**7.6. Transfer of methods.** The Baum-Connes Conjecture is unknown for all  $\text{CAT}(0)$ -groups. Can one use the techniques of the proof of the Farrell-Jones Conjecture for  $\text{CAT}(0)$ -groups to prove the Baum-Connes Conjecture for them? In particular it is not at all clear how the transfer methods in the Farrell-Jones setting carry over to the Baum-Connes case. In the other direction, the Dirac-Dual Dirac method, which is the main tool for proofs of the Baum-Connes Conjecture, lacks an analog on the Farrell-Jones side.

**7.7. Classification of (non-aspherical) manifolds.** The Farrell-Jones Conjecture is also very useful when one considers not necessarily aspherical manifolds. Namely, because of the surgery sequence (2.7), it gives an interpretation

of the structure set as a relative homology group. So it simplifies the classification of manifolds substantially and opens the door to explicit answers in favorable interesting cases. Here, a lot of work can and will have to be done.

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