Hyperbolic groups with spheres as boundary

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Preview of the main result

Conjecture (Gromov (1994))

Let G be a hyperbolic group whose boundary is a sphere S^{n-1}. Then there is a closed aspherical manifold M with $\pi_1(M) \cong G$.

Theorem (Bartels-Lück-Weinberger (2011))

The Conjecture is true for $n \geq 6$.

We also deal with the questions:

- When is a Poincaré duality group the fundamental group of an aspherical closed manifold?
- When is an aspherical closed manifold a product?

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Hyperbolic spaces and hyperbolic groups

Definition (Hyperbolic space)

A δ-hyperbolic space X is a geodesic space whose geodesic triangles are all δ-thin.

A geodesic space is called hyperbolic if it is δ-hyperbolic for some $\delta > 0$.

- A geodesic space with bounded diameter is hyperbolic.
- A tree is 0-hyperbolic.
- A simply connected complete Riemannian manifold M with $\sec(M) \leq \kappa$ for some $\kappa < 0$ is hyperbolic.
- $\mathbb{R}^n$ is hyperbolic if and only if $n \leq 1$.

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Definition (Boundary of a hyperbolic space)

Let X be a hyperbolic space. Define its boundary ∂X to be the set of equivalence classes of geodesic rays. Put

$$\overline{X} := X \amalg \partial X.$$

- Two geodesic rays $c_1, c_2 \colon [0, \infty) \to X$ are called equivalent if there exists $C > 0$ satisfying $d_X\big(c_1(t), c_2(t)\big) \leq C$ for $t \in [0, \infty)$.

Lemma

There is a topology on $\overline{X}$ with the properties:

- *$\overline{X}$ is compact and metrizable;*
- *The subspace topology $X \subseteq \overline{X}$ is the given one;*
- *X is open and dense in $\overline{X}$.*

- Let M be a simply connected complete Riemannian manifold M with $\sec(M) \leq \kappa$ for some $\kappa < 0$. Then M is hyperbolic and $\partial M = S^{\dim(M)-1}$.

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Definition (Quasi-isometry)

A map $f\colon X \to Y$ of metric spaces is called a quasi-isometry if there exist real numbers $\lambda, C > 0$ satisfying:

- The inequality

$$\lambda^{-1} \cdot d_X(x_1, x_2) - C \leq d_Y\big(f(x_1), f(x_2)\big) \leq \lambda \cdot d_X(x_1, x_2) + C$$

  holds for all $x_1, x_2 \in X$;
- For every y in Y there exists $x \in X$ with $d_Y\big(f(x), y\big) < C$.

Lemma (Švarc-Milnor Lemma)

Let X be a geodesic space. Suppose that G acts properly, cocompactly and isometrically on X. Choose a base point $x \in X$. Then the map

$$f\colon G \to X, \quad g \mapsto gx$$

is a quasiisometry.

Lemma (Quasi-isometry invariance of the Cayley graph)

The quasi-isometry type of the Cayley graph of a finitely generated group is independent of the choice of a finite set of generators.

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The property "hyperbolic" is a quasi-isometry invariant of geodesic spaces.

Lemma (Quasi-isometry invariance of the boundary)

A quasi-isometry $f\colon X_1 \to X_2$ of hyperbolic spaces induces a homeomorphism

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Basic properties of hyperbolic groups

- A group G is hyperbolic if and only if it acts properly, cocompactly and isometrically on a hyperbolic space. In this case $\partial G = \partial X$.
- Let M be a closed Riemannian manifold with $\sec(M) < 0$. Then $\pi_1(M)$ is hyperbolic with $S^{\dim(M)-1}$ as boundary.
- If G is virtually torsionfree and hyperbolic, then $\mathrm{vcd}(G) = \dim(\partial G) + 1$.
- If the boundary of a hyperbolic groups contains an open subset homeomorphic to $\mathbb{R}^n$, then the boundary is homeomorphic to S^n.
- A subgroup of a hyperbolic group is either virtually cyclic or contains $\mathbb{Z} * \mathbb{Z}$ as subgroup. In particular $\mathbb{Z}^2$ is not a subgroup of a hyperbolic group.

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- A free product of two hyperbolic groups is again hyperbolic.
- A direct product of two finitely generated groups is hyperbolic if and only if one of the two groups is finite and the other is hyperbolic.
- The Rips complex of a hyperbolic group G is a cocompact model for its classifying space $\underline{E}G$ for proper actions. This implies that there is a model of finite type for BG and hence that G is finitely presented and that there are only finitely many conjugacy classes of finite subgroups.
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Gromov's Conjecture in low dimensions

Theorem (Casson-Jungreis (1994), Freden (1995), Gabai (1991))

A hyperbolic group has S^1 as boundary if and only if it is a Fuchsian group.

Conjecture (Cannon's Conjecture)

A hyperbolic group G has S^2 as boundary if and only if it acts properly, cocompactly and isometrically on $\mathbb{H}^3$.

Theorem (Bestvina-Mess (1991))

Let G be an infinite hyperbolic group which is the fundamental group of a closed irreducible 3-manifold M. Then M is hyperbolic and G satisfies Cannon's Conjecture.

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ANR-homology manifolds

Definition (Absolute neighborhood retract (ANR))

A topological space X is called absolute neighborhood retract (ANR) if it is normal and for every normal space Z, which contains X as a closed subset, there exists an open neighborhood U of X in Z together with a retraction of U onto X.

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Definition (Homology ANR-manifold)

A homology ANR-manifold X is an ANR satisfying:

- X has a countable basis for its topology;
- The topological dimension of X is finite;
- X is locally compact;
- for every $x \in X$ we have for the singular homology

$$H_i(X, X - \{x\}; \mathbb{Z}) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$$

If X is additionally compact, it is called a closed ANR-homology manifold.

There is also the notion of a compact ANR-homology manifold with boundary.

- Every closed topological manifold is a closed ANR-homology manifold.
- Let M be homology sphere with non-trivial fundamental group. Then its suspension ΣM is a closed ANR-homology manifold but not a topological manifold.

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Definition (Disjoint Disk Property (DDP))

A homology ANR-manifold M has the disjoint disk property (DDP), if for any $\epsilon > 0$ and maps $f, g \colon D^2 \to M$, there are maps $f', g' \colon D^2 \to M$ so that f' is ϵ-close to f, g' is ϵ-close to g and $f'(D^2) \cap g'(D^2) = \emptyset$

- A topological manifold of dimension ≥ 5 is a closed ANR-homology manifold, which has the DDP by transversality.

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Poincaré duality groups

Definition (Poincaré duality group)

A Poincaré duality group G of dimension n is a finitely presented group satisfying:

- G is of type FP;
- $H^i(G;\mathbb{Z}G) \cong \begin{cases} 0 & i \neq n; \\ \mathbb{Z} & i = n. \end{cases}$

Lemma

Let X be a closed aspherical ANR-homology manifold of dimension n. Then its fundamental group is a Poincaré duality group of dimension n.

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Theorem (Poincaré duality groups and ANR-homology manifolds Bartels-Lück-Weinberger (2011))

Let G be a torsionfree group. Suppose that its satisfies the K- and L-theoretic Farrell-Jones Conjecture. Consider $n \geq 6$.

Then the following statements are equivalent:

1. *G is a Poincaré duality group of dimension n;*
2. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*
3. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$ which has the DDP.*

If the first statements holds, then the homology ANR-manifold M appearing above is unique up to s-cobordism of ANR-homology manifolds.

The proof of the result above relies on

- Surgery theory as developed by Browder, Novikov, Sullivan, Wall for smooth manifolds and its extension to topological manifolds using the work of Kirby-Siebenmann.
- The algebraic surgery theory of Ranicki.
- The surgery theory for ANR-manifolds due to Bryant-Ferry-Mio-Weinberger and basic ideas of Quinn.
- The Farrell-Jones Conjecture.

The Farrell-Jones Conjecture

Conjecture (*K*-theoretic Farrell-Jones Conjecture for torsionfree groups)

The K-theoretic Farrell-Jones Conjecture with coefficients in the regular ring R for the torsionfree group G predicts that the assembly map

$$H_n(BG; \mathbf{K}_R) \to K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

- There is also a version for *L*-theory.
- The most general version called Full Farrell-Jones Conjecture makes sense for all groups and all possible coefficient rings and twistings and extensions with finite groups as quotient.

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- The most general version called Full Farrell-Jones Conjecture makes sense for all groups and all possible coefficient rings and twistings and extensions with finite groups as quotient.

Theorem (Bartels, Echterhoff, Farrell, Lück, Reich, Roushon, Rüping, Wegner, Wu)

Let $\mathcal{FJ}$ be the class of groups for which the Full Farrell-Jones Conjecture holds. Then $\mathcal{FJ}$ contains the following groups:

- *Hyperbolic groups belong to $\mathcal{FJ}$;*
- CAT(0)*-groups belong to $\mathcal{FJ}$;*
- *Virtually poly-cyclic groups belong to $\mathcal{FJ}$;*
- *Solvable groups belong to $\mathcal{FJ}$;*
- *Cocompact lattices in almost connected Lie groups belong to $\mathcal{FJ}$;*
- *All 3-manifold groups belong to $\mathcal{FJ}$;*
- *If R is a ring whose underlying abelian group is finitely generated free, then $SL_n(R)$ and $GL_n(R)$ belong to $\mathcal{FJ}$ for all $n \geq 2$;*
- *All arithmetic groups belong to $\mathcal{FJ}$.*
- *All Baumslag-Solitar groups belong to $\mathcal{FJ}$.*

Theorem (continued)

Moreover, $\mathcal{FJ}$ has the following inheritance properties:

- *If G_1 and G_2 belong to $\mathcal{FJ}$, then $G_1 \times G_2$ and $G_1 * G_2$ belong to $\mathcal{FJ}$;*
- *If H is a subgroup of G and $G \in \mathcal{FJ}$, then $H \in \mathcal{FJ}$;*
- *If $H \subseteq G$ is a subgroup of G with $[G : H] < \infty$ and $H \in \mathcal{FJ}$, then $G \in \mathcal{FJ}$;*
- *Let $\{G_i \mid i \in I\}$ be a directed system of groups (with not necessarily injective structure maps) such that $G_i \in \mathcal{FJ}$ for $i \in I$. Then $\operatorname{colim}_{i \in I} G_i$ belongs to $\mathcal{FJ}$;*

Theorem (Bestvina-Mess (1991))

A hyperbolic G is a Poincaré duality group of dimension n if and only if its boundary and S^{n-1} have the same Čech cohomology.

Corollary

Let G be a torsionfree word-hyperbolic group. Let $n \geq 6$.
Then the following statements are equivalent:

1. *The boundary ∂G has the integral Čech cohomology of S^{n-1};*
2. *G is a Poincaré duality group of dimension n;*
3. *There exists a closed aspherical n-dimensional ANR-homology manifold M with $\pi_1(M) \cong G$;*
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Quinn's resolution obstruction

Theorem (Quinn (1987))

There is an invariant $\iota(M) \in 1 + 8\mathbb{Z}$ for homology ANR-manifolds with the following properties:

- *if $U \subset M$ is an open subset, then $\iota(U) = \iota(M)$;*
- *$i(M \times N) = i(M) \cdot i(N)$;*
- *Let M be a homology ANR-manifold of dimension ≥ 5. Then M is a topological manifold if and only if M has the DDP and $\iota(M) = 1$.*

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Question

Does the Quinn obstruction always vanishes for aspherical closed homology ANR-manifolds?

- If the answer is yes, we can replace "closed ANR-homology manifold" by "closed topological manifold" in the theorem above.
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Theorem (Quasi-isometry invariance of Quinn's resolution obstruction Bartels-Lück-Weinberger (2011))

Let G_1 and G_2 be torsionfree hyperbolic groups.

- *Let G_1 and G_2 be quasi-isometric. Then G_1 is a Poincaré duality group of dimension n if and only G_2 is;*
- *Let M_i be an aspherical closed ANR-homology manifold with $\pi_1(M_i) \cong G_i$ for $i = 1, 2$. If ∂G_1 and ∂G_2 are homeomorphic, then the Quinn obstructions of M_1 and M_2 agree;*
- *Let G_1 and G_2 be quasi-isometric. Then there exists an aspherical closed topological manifold M_1 with $\pi_1(M_1) = G_1$ if and only if there exists an aspherical closed topological manifold M_2 with $\pi_1(M_2) = G_2$.*

Hyperbolic groups with spheres as boundary

Theorem (Hyperbolic groups with spheres as boundary Bartels-Lück-Weinberger (2011))

Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. Then the following statements are equivalent:

1. *The boundary ∂G is homeomorphic to S^{n-1};*
2. *There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.*

If the first statement is true, the manifold appearing above is unique up to homeomorphism.

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Exotic Examples

By hyperbolization techniques due to Charney, Davis,Januskiewicz one can find the following examples:

Examples (Exotic universal coverings)

Given $n \geq 5$, there are aspherical closed topological manifolds M of dimension n with hyperbolic fundamental group $G = \pi_1(M)$ satisfying:

- The universal covering $\widetilde{M}$ is not homeomorphic to $\mathbb{R}^n$ and ∂G is not homeomorphic to S^{n-1}.
- M is smooth and $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ but ∂G is not S^{n-1}.

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Example (No smooth structures)

For every $k \geq 2$ there exists a torsionfree hyperbolic group G with $\partial G \cong S^{4k-1}$ such that there is no aspherical closed smooth manifold M with $\pi_1(M) \cong G$. In particular G is not the fundamental group of a closed smooth Riemannian manifold with $\sec(M) < 0$.

Example (No triangulation)

For any $n \geq 6$ there exists a hyperbolic group which is the fundamental group of an aspherical topological manifold but not the fundamental group of an aspherical triangulable topological manifold.

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Direct product decompositions of aspherical closed manifolds

Theorem (Product decomposition Lück (2010))

Let M be a closed aspherical manifold of dimension n with $n \neq 3, 4$ with fundamental group $G = \pi_1(M)$ together with a product decomposition

$$p_1 \times p_2 \colon G \xrightarrow{\cong} G_1 \times G_2.$$

Suppose that G satisfy the Farrell-Jones Conjecture and that the cohomological dimension of G_1 and G_2 is different from 3, 4 and 5.

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Theorem (continued)

Then

1. *There are topological closed aspherical manifolds M_1 and M_2 together with maps $f_i \colon M \to M_i$ for $i = 1, 2$ such that*

$$f = f_1 \times f_2 \colon M \to M_1 \times M_2$$

   is a homeomorphism and $\pi_1(f_i) = p_i$.

2. *The decomposition above is unique up to homeomorphism.*

Problems

- Can one give an example of a hyperbolic group (with torsion) whose boundary is a sphere, such that the group does not act properly discontinuously on some contractible manifold?
- Let $p\colon M \to N$ be a map of aspherical closed manifolds whose homotopy fiber is homotopy equivalent to a connected CW-complex of finite type.
  When is p homotopy equivalent to the projection of a locally trivial fiber bundle with a connected closed aspherical topological manifold as typical fiber?

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