A VANISHING THEOREM FOR TAUTOLOGICAL CLASSES OF ASPHERICAL MANIFOLDS

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Abstract. Tautological classes, or generalised Miller–Morita–Mumford classes, are basic characteristic classes of smooth fibre bundles, and have recently been used to describe the rational cohomology of classifying spaces of diffeomorphism groups for several types of manifolds. We show that rationally tautological classes depend only on the underlying topological block bundle, and use this to prove the vanishing of tautological classes for many bundles with fibre an aspherical manifold.

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1. Introduction

Spaces of automorphisms of manifolds have long been an active topic of research in topology, and various techniques have emerged for their study. In the case of high-dimensional manifolds, there are two competing approaches: On the one hand, one tries to understand the difference between the space of diffeomorphisms and the space of homotopy self-equivalences by introducing yet another space, the space of block diffeomorphisms, whose difference to homotopy equivalences is measured by surgery theory and whose difference to diffeomorphisms is measured, at least in a range depending only on the dimension of the manifold, in terms of Waldhausen's A-theory, see [WW89] for a modern approach. An example of this approach being successfully employed is [FH78], where Farrell and Hsiang investigate the rational homotopy type of various spaces of automorphisms, and in particular determine the rational homotopy groups of the space of homeomorphisms of aspherical manifolds in a range. This has a recent integral refinement in [ELP+16].

On the other hand, with the work of Madsen, Tillmann and Weiss on Mumford's conjecture, a new line of investigation emerged. This approach is often referred to as *scanning* and it tries to describe the cohomology of the classifying space of diffeomorphisms in terms of a certain Thom spectrum - an object accessible to the computational methods of algebraic topology. This method is particularly well suited to studying specific cohomology classes, the generalised Miller–Morita–Mumford classes. Since they are central to the present article let us briefly recall their definition.

Date: May 18, 2017.
2010 *Mathematics Subject Classification.* 55R20, 55R40, 55R60, 57P10.
Key words and phrases. aspherical closed manifolds, tautological classes.

Given a smooth, oriented fibre bundle $p\colon E \to B$ with typical fibre a compact, closed, oriented manifold M of dimension d, a coefficient ring R, and a characteristic class $c \in H^k(\mathrm{BSO}(d); R)$, the associated *Miller–Morita–Mumford class*, or *tautological class*, is the cohomology class

$$\kappa_c(p) = p_!(c(T_v p)) \in H^{k-d}(B; R)$$

obtained by applying the Gysin homomorphism $p_!$ associated to p to the class $c(T_v p) \in H^k(E; R)$ given by evaluating the characteristic class c on the vertical tangent bundle $T_v p$ of the map p. In particular, the tautological classes are defined on the universal smooth oriented fibre bundle with fibre M, whose base is $\mathrm{BDiff}^+(M)$, the classifying space for the topological group of orientation-preserving diffeomorphisms of M. This yields universal classes

$$\kappa_c(M) \in H^{k-d}(\mathrm{BDiff}^+(M); R).$$

These were originally considered when M is an oriented surface and were the subject of Mumford's conjecture describing the rational cohomology of the stable moduli space of Riemann surfaces and resolved in the work of Madsen, Tillmann, and Weiss [MW07, MT01]. In higher dimensions, tautological classes have been of recent interest due to the work of Galatius and Randal-Williams, culminating in [GRW16], which describes the rational cohomology of $\mathrm{BDiff}^+(M)$ in terms of tautological classes for certain simply connected manifolds M of dimension $2n \geq 6$, in a range bounded by roughly half the genus of M; the genus of M refers to the number of $S^n \times S^n$ connect-summands of M. In fact, already their work in [GRW14a] and [GRW14b] implies that any oriented $2n$-manifold of genus at least 11 has non-trivial tautological classes!

The goal of the present paper is to discuss tautological classes for aspherical manifolds. Such manifolds have vanishing genus (in the sense just described), so the the results mentioned above reveal nothing in this case. We will show that for a large class of aspherical manifolds M, including for instance non-positively curved manifolds and biquotients of Lie groups, almost all tautological classes vanish; more precisely our result is that for such M

$$0 = \kappa_c(p) \in H^{k-d}(B; \mathbb{Q}),$$

whenever $p\colon E \to B$ is an oriented, smooth M-fibre bundle with B simply connected and $c \in H^k(\mathrm{BSO}(d); \mathbb{Q})$ for $k \neq d$.

Our main theorem will be stated in terms of the following two conjectures.

Block Borel conjecture. *For a closed aspherical manifold M the canonical map $\widetilde{\mathrm{Top}}(M) \to \mathcal{G}(M)$ is a weak homotopy equivalence.*

Here $\widetilde{\mathrm{Top}}(M)$ denotes the realisation of the semi-simplicial set of *block homeomorphisms* of M, and $\mathcal{G}(M)$ denotes the space of self homotopy equivalences of M. For the purposes of this introduction the most important feature of this conjecture is that for manifolds of dimension at least 5 it is implied by the Farrell–Jones conjectures, and thus is known for large swathes of aspherical manifolds by the work of Bartels, Reich, Lück, and many others [BLR08a, BL10, KLR16].

Another input into our work is Burghelea's conjecture [Bur85], the part of which relevant for us reads as follows.

Central part of Burghelea's conjecture. *For a closed aspherical manifold M and a central element $g \in \pi_1(M)$ the rational cohomological dimension with trivial coefficients of $\pi_1(M)/\langle g\rangle$ is finite.*

This conjecture is not as well studied as the Farrell–Jones conjecture, but is still known to hold for a large class of groups. With these preliminaries out of the way we can state our main result.

Main theorem. *If an oriented, smooth, closed, aspherical manifold M of dimension d satisfies the central part of Burghelea's conjecture and the block Borel conjecture, then for all smooth M-fibre bundles $p\colon E \to B$ with trivial fibre transport, we have*

$$0 = \kappa_c(p) \in H^{k-d}(B; \mathbb{Q})$$

for all $c \in H^k(\mathrm{BSO}(d); \mathbb{Q})$ with $k \neq d$.

We can also state the conclusion of this theorem in terms of the universal fibre bundle of the type just described. To do so, let $\mathrm{Diff}_0(M) \leq \mathrm{Diff}(M)$ denote the subgroup of those diffeomorphisms isotopic to the identity. The classifying space $\mathrm{BDiff}_0(M)$ carries the universal smooth M-fibre bundle with trivial fibre transport, and applying the theorem to this bundle shows that

$$0 = \kappa_c(M) \in H^{k-d}(\mathrm{BDiff}_0(M); \mathbb{Q}),$$

whenever M and c satisfy the hypotheses above. In fact, this conclusion will hold on the classifying space of the slightly larger group consisting of diffeomorphisms *homotopic* to the identity.

This theorem in particular recovers several recent vanishing theorems of Bustamante, Farrell, and Jiang [BFJ16], but applies to a much wider class of manifolds. At the end of the paper we shall describe conditions on the fundamental group of an aspherical manifold which are known to imply that M satisfies both relevant conjectures.

This theorem is not the strongest or most general result that we prove, but is the most easily stated and has the least technical hypotheses. We shall prove similar vanishing results under conditions weaker than the block Borel conjecture, these will also hold for topological block bundles, in certain situations will extend to cover the case $k = d$, and we also have results for more general coefficients. To give some idea of these statements it will be helpful to first go through the main ingredients of the proof, but the strongest formulations will only be given in the body of the text.

1.1. **Characteristic classes for topological block bundles.** The first step in our proof is to show that tautological classes can be defined not just for smooth fibre bundles but for topological block bundles. This extends earlier work of Ebert and Randal-Williams [ERW14, Ran16], where among other things they show that rational tautological classes can be defined both for topological fibre bundles and for smooth block bundles.

To this end we will consider the universal oriented M-block bundle $\pi\colon \widetilde{E}^+(M) \to \mathrm{B}\widetilde{\mathrm{Top}}^+(M)$ and show that it has an oriented stable vertical tangent bundle $T_v^s(\pi)\colon \widetilde{E}^+(M) \to \mathrm{BSTop}$.We also construct a fibrewise Euler class $e^{fw}(\pi) \in H^d(\widetilde{E}^+(M); \mathbb{Z})$. In fact, we construct this class for any oriented fibration whose fibre is a Poincaré duality space of formal dimension d. By pulling cohomology classes back along the map

$$(T_v^s(\pi), e^{fw}(\pi))\colon \widetilde{E}^+(M) \longrightarrow \mathrm{BSTop} \times K(\mathbb{Z}, d)$$

and applying the Gysin homomorphism, we can associate

$$\kappa_c(M) = \pi_!((T_v^s(\pi), e^{fw}(\pi))^*(c)) \in H^{k-d}(\mathrm{B}\widetilde{\mathrm{Top}}^+(M); R)$$

to a cohomology class $c \in H^k(\mathrm{BSTop} \times K(\mathbb{Z}, d); R)$.

These define characteristic classes of oriented block bundles, and together with the stable vertical tangent bundle and fibrewise Euler class can be pulled back from the universal oriented M-block bundle to any other. On a block bundle $p\colon E \to B$ which arises from a smooth fibre bundle, $T_v^s(p)$ is the stabilisation of the vertical tangent bundle, $e^{fw}(p)$ is the Euler class of the vertical tangent bundle, and the Gysin homomorphism is the usual one, so these tautological classes reduce to those of the same name defined earlier. We will show that they also agree with the constructions of [ERW14] and [Ran16]. This comparison, in particular, shows that the classes defined in [ERW14] lie in the image of the Gysin homomorphism, a point not addressed in [ERW14] but essential for our work.

Recall now that $H^*(\mathrm{BSO}(d); \mathbb{Q})$ is generated by Pontryagin and Euler classes, and by work of Novikov, Kirby and Siebenmann the rational Pontryagin classes are pulled back from BSTop. Therefore, to establish a vanishing result for rational tautological classes it suffices to consider topological block bundles. That is, writing $\widetilde{\mathrm{Top}}_0(M) \leq \widetilde{\mathrm{Top}}(M)$ for the component of the identity, it is enough to show that

$$0 = \kappa_c(M) \in H^{k-d}(\mathrm{B}\widetilde{\mathrm{Top}}_0(M); \mathbb{Q})$$

for all $c \in H^k(\mathrm{BSTop} \times K(\mathbb{Z}, d); \mathbb{Q})$ such that $k \neq d$. Assuming the two conjectures stated earlier, we will show the vanishing of these classes, as we now explain.

1.2. **Vanishing results.** To this end let us fix a closed, connected, oriented, aspherical topological manifold M which satisfies the block Borel conjecture, i.e. such that the map

$$\widetilde{\mathrm{Top}}(M) \longrightarrow \mathcal{G}(M)$$

is a weak equivalence. This means that topological M-block bundles which are fibre homotopy equivalent are in fact equivalent as block bundles. As discussed in the last section the stable vertical tangent bundle of a manifold bundle only depends on the underlying topological block bundle, so it is fibre homotopy invariant among M-bundles. This conclusion was obtained in [BFJ16] by a slightly different route. Together with our construction of the fibrewise Euler class, it implies that rational tautological classes for M-fibre bundles are invariant under fibre homotopy equivalences and therefore vanish on fibre homotopically trivial bundles.

To obtain a criterion for fibre homotopy triviality note that for any connected, aspherical complex X a straightforward computation shows

$$\pi_k(G(X)) = \begin{cases} \mathrm{Out}(\pi_1(X)) & k = 0 \\ C(\pi_1(X)) & k = 1 \\ 0 & k \geq 2 \end{cases}$$

where Out denotes the outer automorphism group and C the centre of a given group. To obtain our results we will usually restrict to bundles with trivial fibre transport, so knowing that

$$\widetilde{\mathrm{Top}}_0(M) \longrightarrow \mathcal{G}_0(M)$$

is an equivalence will often suffice. We dub this weaker version of the block Borel conjecture the *identity block Borel conjecture*. In distinction with the block Borel conjecture, it is implied by the Farrell–Jones conjectures also when the aspherical manifold in question is of dimension 4. Now if $C(\pi_1(M)) = 0$, then $\mathrm{B}\mathcal{G}_0(M)$ is contractible; we refer to such manifolds as *centreless* and a block bundle with centreless, aspherical fibre is thus fibre homotopically trivial. We therefore find:

Theorem. *If M is a closed, oriented, aspherical, centreless manifold which satisfies the identity block Borel conjecture, then*

$$0 = \kappa_c(M) \in H^{k-d}(\mathrm{B}\widetilde{\mathrm{Top}}_0(M); R)$$

for all $c \in H^k(\mathrm{BSTop} \times K(\mathbb{Z}, d); R)$ such that $k \neq d$.

The consequences of this theorem for smooth manifold bundles, while not explicitly stated there, were essentially already obtained in [BFJ16]. And while the methods are similar as well our approach offers a novel perspective: The tautological classes of bundles with centreless, aspherical fibre and trivial fibre transport vanish because the universal space in which they are defined is contractible by the block Borel conjecture. The implications for the *integral* tautological classes of smooth fibre bundles are somewhat delicate, as $H^*(\mathrm{BSTop}; \mathbb{Z}) \to H^*(\mathrm{BSO}; \mathbb{Z})$ is not surjective. Instead of their vanishing, one only obtains (somewhat inexplicit) universal bounds on their order.

The condition that c not have degree d cannot be removed, as already observed in [BFJ16]: Because every bordism class can be represented by a negatively curved manifold, see [Ont14], for $c \in H^d(\mathrm{BSO}(d); \mathbb{Q})$ the classes $\kappa_c(M) = \langle c(TM), [M] \rangle$ do not generally vanish on aspherical manifolds. However, any negatively curved manifold is centreless: We will now see that stronger results may be obtained for an aspherical manifold whose fundamental group has non-trivial centre and in addition to being a Farrell–Jones group satisfies the central part of Burghelea's conjecture.

We begin by observing that by the identity block Borel conjecture the underlying fibration of the universal M-block bundle with trivial fibre transport is given by

$$\pi \colon \mathrm{B}(\Gamma/C(\Gamma)) \longrightarrow \mathrm{B}^2 C(\Gamma)$$

where we have abbreviated $\Gamma := \pi_1(M)$ and the map π classifies the central extension

$$1 \longrightarrow C(\Gamma) \longrightarrow \Gamma \longrightarrow \Gamma/C(\Gamma) \longrightarrow 1.$$

This observation relates the Gysin map for the universal block bundle over $\mathrm{B}\widetilde{\mathrm{Top}}_0(M)$ with the central part of Burghelea's conjecture, which we shall use to show the following.

Theorem. *If Γ is a rational Poincaré duality group of dimension d with non-trivial centre, which satisfies the central part of Burghelea's conjecture, then the Gysin map*

$$\pi_!\colon H^*(\mathrm{B}(\Gamma/C(\Gamma));\mathbb{Q}) \longrightarrow H^{*-d}(\mathrm{B}^2C(\Gamma);\mathbb{Q})$$

vanishes. If $C(\Gamma)$ is finitely generated, then the same statement holds integrally.

It seems to be an open problem whether the centre of the fundamental group of an aspherical manifold is finitely generated, though this is known for several classes of groups.

Corollary. *Let M be a closed, connected, oriented, aspherical manifold with non-trivial centre that satisfies the identity block Borel conjecture and the central part of Burghelea's conjecture. Then for any topological M-block bundle with trivial fibre transport*

$$0 = \kappa_c(M) \in H^{k-d}(\mathrm{B}\widetilde{\mathrm{Top}}_0(M);\mathbb{Q})$$

for all $c \in H^k(\mathrm{BSTop} \times K(\mathbb{Z}, d);\mathbb{Q})$. If $C(\pi_1(M))$ is finitely generated then the same statement holds integrally.

For this results our new approach via block bundles seems essential, as the universal fibration with fibre M (to which our algebraic vanishing result applies) need not have a fibre bundle representative. Since this corollary concerns *all* tautological classes, not just those of non-zero degree, it has content even for the bundle $M \to *$. In this case it implies that the Euler characteristic $\chi(M)$ and all Pontryagin numbers of M vanish. The vanishing of the Euler characteristic in the situation of the corollary was obtained by Gottlieb in [Got65] by more elementary means, without assuming either conjecture. We believe that the vanishing of Pontryagin numbers is new; it means that M represents an element of order at most 2 in the oriented cobordism ring. This should be contrasted with Ontaneda's result mentioned above.

The principal examples to which the corollary applies unconditionally are manifolds built as iterated bundles with fibres either non-positively curved manifolds or biquotients of Lie groups (that is manifolds of the form $\Gamma\backslash G/K$, where Γ is a cocompact lattice and K is a maximal compact subgroup). During the proof of the above theorem we will unearth slightly weaker finiteness conditions than Burghelea's that still allow the proof of vanishing of the Gysin map to go through. Chief among the examples we can cover this way is $S^1 \times M$, whenever $\pi_1(M)$ is a Farrell–Jones group.

This discussion leads us to formulate the following

Conjecture. *Let M be a closed, connected, oriented, aspherical manifold. If $C(\pi_1(M)) \neq 0$ then*

$$0 = \kappa_c \in H^*(\mathrm{B}\widetilde{\mathrm{Top}}_0(M); R)$$

for all $c \in H^*(\mathrm{BSTop} \times K(\mathbb{Z}, d); R)$.

Since $H^*(\mathrm{BSTop};\mathbb{Z}/2) \to H^*(\mathrm{BSO};\mathbb{Z}/2)$ is surjective, this conjecture in particular implies that an aspherical manifold with non-trivial centre is nullbordant.

Organisation of the paper. We begin Section 2 by recalling basics about block bundles and then construct the universal stable vertical tangent bundle in the latter half, the fibrewise Euler class for a fibration with Poincaré fibre in Section 3, and tautological classes for block bundles in Section 4. We also compare our definitions to previous ones. In Section 5 we review the homotopy type of the space of block homeomorphisms and its relation to the Farrell–Jones conjectures. Along the way we obtain the main theorem in the centreless case. To discuss aspherical manifolds whose centre is non-trivial, we introduce a plethora of finiteness conditions in Section 6, among them Burghelea's conjecture and untangle their relations, in particular, proving our main vanishing results. Finally, in Section 7 we discuss several classes of manifolds which satisfy both conjectures and indeed prove the vanishing of tautological classes for a few cases not covered by the existing literature on the Burghelea conjecture via intermediate finiteness assumptions introduced in Section 6. We end with counterexamples to some tempting strengthenings of our conjecture above and open questions encountered on the way.

Acknowledgements. We are happy to thank Diarmuid Crowley, Søren Galatius, and Wolfgang Steimle for several helpful discussions along the way. Furthermore, we want to heartily thank Alexander Engel and Michał Marcinkowski for making us aware of Burghelea's conjecture in the first place, which replaces a stronger assumption in an earlier version of this paper.

We would also like to thank the Mathematisches Forschungsinstitut Oberwolfach, where the seeds of the present project were sown over a lovely barbecue. WL and ML were supported by the ERC-grant 662400 'KL2MG-interactions'. ORW was supported by EPSRC grant EP/M027783/1 'Stable and unstable cohomology of moduli spaces'. ML and FH enjoyed support of the CRC 1085 'Higher invariants' at the University of Regensburg.

2. A stable vertical tangent bundle for block bundles

In this section we shall remind the reader of the definition of a block bundle with fibre a manifold M, describe the classifying space for such block bundles and the universal block bundle, and construct the stable vertical normal bundle on its total space. For our applications we require this theory for topological manifolds and topological block bundles, but it can be developed in any category $\mathrm{Cat} \in \{\mathrm{Diff}, \mathrm{PL}, \mathrm{Top}\}$ and we shall do so in this generality.

Many of the necessary ideas already appeared in work of Ebert and Randal-Williams [ERW14], where models for the universal smooth block bundle were described, and it was shown that any smooth block bundle over a finite simplicial complex had a stable vertical tangent bundle. The argument given there was particular to vector bundles (gluing together explicit maps to Grassmannians defined on different blocks). Here we shall improve the result to hold for Cat block bundles and give a stable vertical Cat tangent bundle for the universal block bundle (whose base is *not* a finite simplicial complex).

The credulous reader not interested in the rather technical construction of the universal vertical tangent bundle may skip the entire section, except maybe the reminder on block bundles in Section 2.3 if warranted, since the techniques employed are entirely different from those of the remainder of the article. In particular, they will not miss out on anything else relevant.

2.1. **Notation and conventions.** For convenience we use the following notion. A *p-block space* is a space X with a reference map $\pi\colon X \to \Delta^p$ to the p-simplex. A morphism between p-block spaces (X, π) and (X', π') is a continuous map $f\colon X \to X'$ which *weakly* commutes with the reference map in the following sense: for each face $\tau \subset \Delta^p$, the map f sends $\pi^{-1}(\tau)$ into $\pi'^{-1}(\tau)$. If X and X' are Cat manifolds and f is a Cat isomorphism, we say it is a *p-block* Cat *isomorphism.*

If (X, π) is a p-block space then for each $i = 0, 1, 2, \dots, p$ we obtain a $(p-1)$-block space $d_i(X, \pi)$ by restriction to the ith face of Δ_p. More precisely, if $\Delta_i^{p-1} \subset \Delta^p$ denotes the face spanned by all vertices but the ith, then $d_i(X, \pi) = (\pi^{-1}(\Delta_i^{p-1}), \pi|_{\pi^{-1}(\Delta_i^{p-1})})$. We call this the *restriction* of X to the ith face of Δ^p.

We shall always implicitly consider spaces of the form $\Delta^p \times T$ to be p-block spaces with reference map given by projection to the first factor.

2.2. **Block diffeomorphisms.** For $i = 0, 1, \dots, p$ and $0 < \epsilon \leq 1$ let us write

$$\Delta_i^p(\epsilon) := \{(t_0, t_1, \dots, t_p) \in \Delta^p \mid 0 \leq t_i < \epsilon\}.$$

For any $0 < \epsilon \leq 1$ define a homeomorphism

$$\begin{aligned} h_i(\epsilon)\colon \Delta_i^p(\epsilon) &\longrightarrow \Delta_i^{p-1} \times [0, \epsilon) \\ (t_0, t_1, \dots, t_p) &\longmapsto (\tfrac{t_0}{1-t_i}, \tfrac{t_1}{1-t_i}, \dots, \tfrac{t_{i-1}}{1-t_i}, \tfrac{t_{i+1}}{1-t_i}, \dots, \tfrac{t_p}{1-t_i}; t_i). \end{aligned}$$

and a retraction $\pi_i(\epsilon) = \pi_1 \circ h_i(\epsilon)\colon \Delta_i^p(\epsilon) \to \Delta_i^{p-1}$.

2.2.1. **Definition.** A *collared p-block* Cat *isomorphism of* $\Delta^p \times M$ is a Cat isomorphism

$$f\colon \Delta^p \times M \longrightarrow \Delta^p \times M$$

which is also a p-block map, such that for each $i = 0, 1, \dots, p$ there is an $\epsilon > 0$ such that f preserves the set $\Delta_i^p(\epsilon) \times M$ and $h_i(\epsilon) \circ f|_{\Delta_i^p(\epsilon) \times M} \circ h_i(\epsilon)^{-1} = d_i(f) \times \mathrm{Id}_{[0,\epsilon)}$.

It is an elementary but tedious exercise to see that if f is a collared p-block Cat isomorphism of $\Delta^p \times M$ then $d_i(f)$ is a collared $(p-1)$-block Cat isomorphism of $\Delta^{p-1} \times M$. Thus there is a semi-simplicial group $\widetilde{\mathrm{Cat}}(M)_\bullet$ with p-simplices the set of collared p-block Cat isomorphisms of $\Delta^p \times M$, and face maps given by restriction. Because of the collaring condition it is easy to see that $\widetilde{\mathrm{Cat}}(M)_\bullet$ is Kan (see [BLR08a, Appendix A, Lemma 3.2] for an explicit construction of degeneracy maps for $\mathrm{Cat} \in \{\mathrm{Top}, \mathrm{Diff}\}$; a simplicial group is always Kan, giving one possible proof, but one can also give a direct proof). The classifying space $B\widetilde{\mathrm{Cat}}(M)$ is defined to be the geometric realisation of the bi-semi-simplicial set $N_\bullet \widetilde{\mathrm{Cat}}(M)_\bullet$ obtained by taking the levelwise nerve of the semi-simplicial group $\widetilde{\mathrm{Cat}}(M)_\bullet$. The definition in [ERW14] omitted the collaring condition, and it is unclear to us whether their version really is Kan as claimed at the end of [ERW14, Proposition 2.8]. Clearly the homotopy type of $B\widetilde{\mathrm{Cat}}(M)$ is not affected by this change.

2.3. **Block bundles and their moduli spaces.** Let K be a simplicial complex, and $\pi\colon E \to |K|$ be a continuous map. We recall the notion of a Cat block bundle structure on this map, with fibre a Cat manifold M. A *block chart* for E over $\sigma \subset |K|$ is a homeomorphism

$$h_\sigma\colon p^{-1}(\sigma) \longrightarrow \sigma \times M$$

such that for each face $\tau \leq \sigma$ the map $h_\sigma|_{p^{-1}(\tau)}$ sends $p^{-1}(\tau)$ homeomorphically to $\tau \times M$. A *block atlas* $\mathcal{A}$ for E is a set of block charts for E, at least one for each simplex of $|K|$, so that if $h_{\sigma_i}\colon p^{-1}(\sigma_i) \to \sigma_i \times M$, $i = 0, 1$, are two block charts then the composition

$$h_{\sigma_1} \circ h_{\sigma_0}^{-1}\colon (\sigma_0 \cap \sigma_1) \times M \longrightarrow (\sigma_0 \cap \sigma_1) \times M$$

is a p-block Cat isomorphism in the sense of Definition 2.2.1. A *block bundle structure* on $p\colon E \to |K|$ is a maximal block atlas.

It can be shown directly that concordance classes of block bundles over $|K|$ are classified by homotopy classes of maps $f\colon |K| \to B\widetilde{\mathrm{Cat}}(M)$, but for both the proof and geometric constructions, the following model for the classifying space is more convenient. It depends on $\mathrm{Cat} \in \{\mathrm{Diff}, \mathrm{Top}, \mathrm{PL}\}$, but we omit this from the notation.

2.3.1. **Definition.** Let $\mathcal{M}(M)_p^{\epsilon,n}$ denote the set of locally flat Cat submanifolds $W \subset \Delta^p \times \mathbb{R}^n$ (considered as p-block spaces via projection to the Δ^p factor) such that for each $i = 0, 1, \ldots, p$ we have

- (i) W is Cat transverse to $\Delta_i^{p-1} \times \mathbb{R}^n \subset \Delta^p \times \mathbb{R}^n$,
- (ii) $W \cap (\Delta_i^p(\epsilon) \times \mathbb{R}^n) = (\pi_i(\epsilon) \times \mathbb{R}^n)^{-1}(W \cap (\Delta_i^{p-1} \times \mathbb{R}^n))$, and
- (iii) there is a p-block Cat isomorphism $f\colon \Delta^p \times M \to W \subset \Delta^p \times \mathbb{R}^n$ which is collared in the sense that for each $i = 0, 1, \ldots, p$ the map f agrees with the map

$$\begin{aligned}(\Delta_i^{p-1} * \{e_i\}) \times M &\longrightarrow (\Delta_i^{p-1} * \{e_i\}) \times \mathbb{R}^n \\ ((1-t_i) \cdot w + t_i \cdot e_i, x) &\longmapsto ((1-t_i) \cdot w' + t_i \cdot e_i, x')\end{aligned}$$

  on $\Delta_i^p(\epsilon) \times M$, where $(w', x') = f|_{\Delta_i^{p-1} \times M}(w, x)$ and $e_i \in \mathbb{R}^n$ denotes the i-th unit vector.

Define face maps $d_i\colon \mathcal{M}(M)_p^{\epsilon,n} \to \mathcal{M}(M)_{p-1}^{\epsilon,n}$ by restricting W to the ith face of Δ^p, to give a semi-simplicial set $\mathcal{M}(M)_\bullet^n$. Put $\mathcal{M}(M)_\bullet^n = \bigcup_{\epsilon>0} \mathcal{M}(M)_\bullet^{\epsilon,n}$ and finally let $\mathcal{M}(M)_\bullet = \operatorname{colim}_{n\to\infty} \mathcal{M}(M)_\bullet^n$, under the evident comparison maps, and $\mathcal{M}(M) = |\mathcal{M}(M)_\bullet|$.

The semi-simplicial set $\mathcal{M}(M)_\bullet$ is Kan: given a $E \subset \Lambda_i^p \times \mathbb{R}^n$ defining a block bundle over a horn Λ_i^p to be extended to Δ^p, condition (ii) above gives an extension to an open neighbourhood of Λ_i^p, and a full extension may be obtained from this by choosing an isotopy from the identity map of Δ^p to a suitable embedding into this open neighbourhood.

To compare $\mathcal{M}(M)$ with $B\widetilde{\mathrm{Cat}}(M)$, we follow [ERW14, Proposition 2.3] and consider the bi-semi-simplicial set $X_{\bullet,\bullet}$ with (p,q)-simplices given by a $W \in B\widetilde{\mathrm{Cat}}(M)_q$ and a sequence

$$W \xleftarrow{f_0} \Delta^q \times M \xleftarrow{f_1} \Delta^q \times M \xleftarrow{f_2} \cdots \xleftarrow{f_p} \Delta^q \times M$$

of p-block Cat isomorphisms, where $f_1, \dots, f_p$ are collared in the usual sense, and f_0 is collared in the sense of the definition above. The face maps in the q direction are by restriction to faces, and those on the p direction are by composing the f_i or forgetting f_p. The augmentation map $X_{\bullet,q} \to \mathcal{M}(M)_q$, which just records W, has fibre over W isomorphic to $E_\bullet G$, where G is the group of the collared q-block Cat isomorphisms of $\Delta^q \times M$; thus $|X_{\bullet,q}| \xrightarrow{\simeq} \mathcal{M}(M)_q$. There is a map $X_{p,\bullet} \to N_p\widetilde{\mathrm{Cat}}(M)_\bullet$, which just records $(f_1, \dots, f_p)$. This is a Kan fibration of semi-simplicial sets, and as in the proof of [ERW14, Proposition 2.3] its fibre after geometric realisation can be described as the space of block embeddings of M into $\mathbb{R}^\infty$, which is contractible. In total this yields a preferred homotopy equivalence $\mathcal{M}(M) \simeq \mathrm{B}\widetilde{\mathrm{Cat}}(M)$.

Let us now describe the universal M-block bundle $\pi\colon \mathcal{E}(M) \to \mathcal{M}(M)$. Strictly speaking this will not be a block bundle as described in the beginning of this section, since $\mathcal{M}(M)$ is not a finite simplicial complex. We will, however, blur this distinction in the notation, as the pull back of π along a simplicial map from a finite simplicial complex is indeed a block bundle as in the proof of [ERW14, Proposition 2.7].

Let $\mathcal{E}(M)_p \subset \mathcal{M}(M)_p \times \Delta^p \times \mathbb{R}^\infty$ be the subspace of those triples $(W; t_0, \dots, t_p; x)$ for which $(t_0, \dots, t_p; x) \in W$, and let $\pi_p\colon \mathcal{E}(M)_p \to \mathcal{M}(M)_p \times \Delta^p$ denote the projection map. These assemble to a continuous map

$$\pi\colon \mathcal{E}(M) \longrightarrow \mathcal{M}(M)$$

where

$$\mathcal{E}(M) = \left(\coprod_{p \geq 0} \mathcal{E}(M)_p \right) / \sim$$

with $\sim$ the equivalence related generated by

$$(W; t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots t_p; x) \sim (d_i(W); t_0, \dots, t_{i-1}, t_{i+1}, \dots t_p; x)$$

and

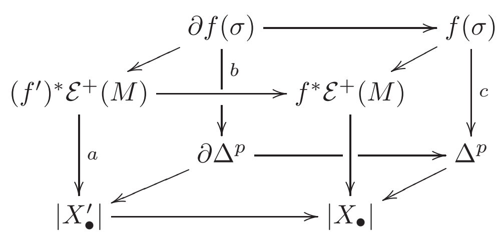
$$\mathcal{M}(M) = \left(\coprod_{p \geq 0} \Delta^p \times \mathcal{M}(M)_p \right) / \sim$$

the usual geometric realisation. The preimage of the simplex $\{W\} \times \Delta^p \subset \mathcal{M}(M)$ is $\{W\} \times W$, which is p-block Cat isomorphic to $\Delta^p \times M$.

We will now show that the map $\pi\colon \mathcal{E}(M) \to \mathcal{M}(M)$ is a weak quasi-fibration, in the sense that the comparison map $\pi^{-1}(v) \to \mathrm{hofib}_v(\pi)$ is a weak homotopy equivalence for any vertex $v \in \mathcal{M}(M)_0$, thereby directly identifying the underlying fibration of the universal block bundle. For future use, we formulate this in a slightly more general manner.

2.3.2. **Proposition.** *If $X_\bullet$ is a semi-simplicial set and $f\colon X_\bullet \to \mathcal{M}(M)_\bullet$ a semi-simplicial map, then the map $f^*\pi\colon f^*\mathcal{E}(M) \to |X_\bullet|$ is a weak quasi-fibration.*

Proof. Let us first suppose that $X_\bullet$ is a finite semi-simplicial set. We proceed by double induction on the dimension of $X_\bullet$ and the number of top-dimensional simplices. Firstly, if $|X_\bullet|$ is 0-dimensional then the claim clearly holds. Otherwise, let $\sigma \in X_p$ be a top-dimensional simplex and $X'_\bullet$ be the semi-simplicial set obtained by removing σ, and write $f' = f|_{X'_\bullet}$. Then $f(\sigma) \in \mathcal{M}(M)_p$ is a submanifold of $\Delta^p \times \mathbb{R}^\infty$ which is p-block isomorphic to $\Delta^p \times M$. Let us write $\partial f(\sigma) = f(\sigma) \cap (\partial \Delta^p \times \mathbb{R}^\infty)$. There is a cube



in which the top and bottom faces are homotopy push-outs. As $f(\sigma)$ is p-block isomorphic to $M \times \Delta^p$, the map c is a weak quasi-fibration; as $X'_\bullet$ has fewer top-dimensional simplices than $X_\bullet$

we may suppose by induction that a is a weak quasi-fibration; as $\partial\Delta^p$ is of lower dimension than $X_\bullet$ we may suppose by induction that b is a weak quasi-fibration. The left and back faces are cartesian, so as a, b, and c are weak quasi-fibrations it follows that they are homotopy cartesian. By Mather's First Cube Theorem [Mat76, Theorem 18] it follows that the front and right faces are also homotopy cartesian: as c (or a) is a weak quasi-fibration, it follows that $f^*\pi$ is too.

Now, if $X_\bullet$ is an arbitrary semi-simplicial set, let $v \in X_0$ and let $\mathcal{F}$ denote the directed set of finite sub-semi-simplicial sets $F_\bullet \subset X_\bullet$ which contain v. If we let $f^*\pi|_{|F_\bullet|} \colon f^*\mathcal{E}^+(M)|_{|F_\bullet|} \to |F_\bullet|$ denote the pullback of $f^*\pi$ along the inclusion $|F_\bullet| \to |X_\bullet|$, then as each compact subset of $|X_\bullet|$ lies in the geometric realisation of a finite sub-semi-simplicial set, the map

$$\operatorname*{hocolim}_{F_\bullet \in \mathcal{F}} \mathrm{hofib}_v(f^*\pi|_{|F_\bullet|}) \longrightarrow \mathrm{hofib}_v(f^*\pi)$$

is a weak homotopy equivalence. As each $f^*\pi|_{|F_\bullet|}$ is a weak quasi-fibration the left-hand side may be replaced with the homotopy colimit of the constant diagram $(f^*\pi)^{-1}(v)$, which shows that $(f^*\pi)^{-1}(v) \to \mathrm{hofib}_v(f^*\pi)$ is a weak homotopy equivalence. □

2.4. **The stable vertical normal bundle.** Our goal is to construct a stable Cat bundle on the total space $\mathcal{E}(M)$ of the universal block bundle $\pi\colon \mathcal{E}(M) \to \mathcal{M}(M)$. We shall focus on the unoriented case for simplicity, but there are no significant changes necessary to treat the oriented case. Our construction will be quite natural once we pull back the universal block bundle to a slightly different, but homotopy equivalent, base. In comparison to the previous section, we shall construct a model for $\mathcal{M}(M)$ which also encodes choices of Cat normal bundles. This will allow us to essentially follow the argument [ERW14, Proposition 3.2] using this model of the universal block bundle.

2.4.1. **Definition.** If $W \in \mathcal{M}(M)_p^{n,\epsilon}$, an *$\epsilon$-prepared normal* Cat *bundle* for W consists of an open neighbourhood $W \subset U \subset \Delta^p \times \mathbb{R}^n$, a retraction $r\colon U \to W$, and a Cat $\mathbb{R}^{n-d}$-bundle atlas $\mathcal{A}$ for r. In addition we require that r is a morphism of p-block spaces, and that for each $i = 0, 1, \ldots, p$

(i) $U \cap (\Delta_i^p(\epsilon) \times \mathbb{R}^n) = (\pi_i(\epsilon) \times \mathbb{R}^n)^{-1}(U \cap (\Delta_i^{p-1} \times \mathbb{R}^n))$,
(ii) the map r restricted to $U \cap (\Delta_i^p(\epsilon) \times \mathbb{R}^n)$ commutes with the ith barycentric coordinate t_i (which makes the left hand vertical map in the following diagram well defined), and

$$\begin{array}{ccc} U \cap (\Delta_i^p(\epsilon) \times \mathbb{R}^n) & \xrightarrow{\pi_i(\epsilon)\times\mathbb{R}^n} & U \cap (\Delta_i^{p-1} \times \mathbb{R}^n) \\ \Big\downarrow r|_{U\cap(\Delta_i^p(\epsilon)\times\mathbb{R}^n)} & & \Big\downarrow r|_{U\cap(\Delta_i^p\times\mathbb{R}^n)} \\ W \cap (\Delta_i^p(\epsilon) \times \mathbb{R}^n) & \xrightarrow{\pi_i(\epsilon)\times\mathbb{R}^n} & W \cap (\Delta_i^{p-1} \times \mathbb{R}^n) \end{array}$$

is a pullback of Cat $\mathbb{R}^{n-d}$-bundles (with the Cat bundle structure on both sides given by restriction of $\mathcal{A}$).

2.4.2. **Definition.** Let $\mathcal{M}'(M)_\bullet^{\epsilon,n}$ denote the semi-simplicial set with p-simplices given by tuples $(W, U, r, \mathcal{A})$ of a $W \in \mathcal{M}(M)_p^{\epsilon,n}$ and an ϵ-prepared normal bundle $(U, r, \mathcal{A})$. The ith face map is given by restricting all three pieces of data to $\Delta_i^{p-1} \times \mathbb{R}^n$. Again, let $\mathcal{M}'(M)_\bullet^n = \bigcup_{\epsilon>0} \mathcal{M}'(M)_\bullet^{\epsilon,n}$. There are maps $\mathcal{M}'(M)_\bullet^n \to \mathcal{M}'(M)_\bullet^{n+1}$ given by sending (W, U, r) to $(W, U \times \mathbb{R}, r \circ \mathrm{proj}_U)$ and we let $\mathcal{M}'(M)_\bullet = \operatorname*{colim}_{n\to\infty} \mathcal{M}'(M)_\bullet^n$, and $\mathcal{M}'(M) = |\mathcal{M}'(M)_\bullet|$.

2.4.3. **Lemma.** *The semi-simplicial map $\mathcal{M}'(M)_\bullet \to \mathcal{M}(M)_\bullet$, given by forgetting the bundle data, is a weak homotopy equivalence on geometric realisation.*

Proof. We shall show that the map has vanishing relative homotopy groups. Our main tool is the relative stable existence and uniqueness theorem for normal Cat microbundles, and the Cat microbundle representation theorem. We have explained that $\mathcal{M}(M)_\bullet$ is Kan, and the same argument shows that $\mathcal{M}'(M)_\bullet$ is too, so a relative homotopy class may be described by a submanifold $W \subset \Delta^p \times \mathbb{R}^n$ such that $W|_{\partial\Delta^p}$ is comes with a prepared normal Cat bundle given by $W|_{\partial\Delta^p} \subset U_\partial \subset \partial\Delta^p \times \mathbb{R}^n$, $r_\partial\colon U_\partial \to W|_{\partial\Delta^p}$, and $\mathcal{A}_\partial$. In order to show that this relative homotopy class is trivial, it will be sufficient to show that (after perhaps increasing n) the prepared normal bundle $(U_\partial, r_\partial, \mathcal{A}_\partial)$ is the restriction of a prepared normal bundle for W'.

Choose $\epsilon > 0$ so that the given data lie in $\mathcal{M}'(M)^{\epsilon,n}$ or $\mathcal{M}(M)^{\epsilon,n}$. The product structures given by Definition 2.3.1 (ii) and Definition 2.4.1 (ii) give an extension of $(U_\partial, r_\partial, \mathcal{A}_\partial)$ to a normal Cat bundle of $W|_{\partial W(\epsilon/2)}$, where $\partial W(\epsilon/2) = \cup_{i=0}^{p} W \cap (\Delta^p_i(\frac{1}{2}\epsilon) \times \mathbb{R}^n)$. Furthermore, the submanifold $W|_{\Delta^p \setminus \Delta^p(\epsilon)} \subset (\Delta^p \setminus \Delta^p(\epsilon)) \times \mathbb{R}^n$ has a normal Cat microbundle (after perhaps increasing n) [KS77, p. 204], and this may be represented by a Cat $\mathbb{R}^{n-d}$-bundle (by Kister–Mazur [Kis64] for Top, Kuiper–Lashof [KL66] for PL, and the tubular neighbourhood theorem for Diff). These yield Cat normal $\mathbb{R}^{n-d}$-bundles over the boundary of

$$W|_{\Delta^p(\epsilon)\setminus\Delta^p(\epsilon/2)} \cong W|_{\partial\Delta^p} \times [\epsilon/2, \epsilon] \subset \partial\Delta^p \times \mathbb{R}^n \times [\epsilon/2, \epsilon].$$

By stable uniqueness of Cat normal microbundles, and of representing Cat $\mathbb{R}^{n-d}$-bundles, there is an extension of the Cat normal $\mathbb{R}^{n-d}$-bundles over the boundary to the whole of $W|_{\Delta^p(\epsilon)\setminus\Delta^p(\epsilon/2)}$. Gluing these three Cat normal $\mathbb{R}^{n-d}$-bundles together shows that $(U_\partial, r_\partial, \mathcal{A}_\partial)$ is the restriction of a prepared normal bundle for W. ☐

Let us write $\mathcal{E}'(M)^n_p \subset \mathcal{M}'(M)^n_p \times \Delta^p \times \mathbb{R}^n$ for the subspace of those $(W, U, r, \mathcal{A}; t_0, \dots, t_p; x)$ such that $(t_0, \dots, t_p; x) \in W$, and $U^n_p \subset \mathcal{M}'(M)^n_p \times \Delta^p \times \mathbb{R}^n$ be the subspace of those tuples $(W, U, r, \mathcal{A}; t_0, \dots, t_p; x)$ such that $(t_0, \dots, t_p; x) \in U$. We define

$$\mathcal{E}'(M)^n := |\mathcal{E}(M)^n_\bullet| := \left(\coprod_{p\geq 0} \mathcal{E}(M)^n_p\right) / \sim \qquad\qquad |U^n_\bullet| := \left(\coprod_{p\geq 0} U^n_p\right) / \sim$$

where in both cases $\sim$ is generated by

$$(W, U, r, \mathcal{A}; t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots t_p; x) \sim (d_i(W, U, r, \mathcal{A}); t_0, \dots, t_{i-1}, t_{i+1}, \dots t_p; x).$$

There are maps $r^n_p \colon U^n_p \to \mathcal{E}'(M)^n_p$ given by

$$r^n_p(W, U, r, \mathcal{A}; t_0, \dots, t_p; x) = (W, U, r, \mathcal{A}; r(t_0, \dots, t_p; x))$$

which assemble to a map $r^n \colon |U^n_\bullet| \to |\mathcal{E}'(M)^n_\bullet|$.

2.4.4. **Lemma.** *The map $r^n \colon |U^n_\bullet| \to |\mathcal{E}'(M)^n_\bullet|$ has the structure of a* Cat *$\mathbb{R}^{n-d}$-bundle ν_{n-d}, and the restriction of ν_{n-d} to $|\mathcal{E}'(M)^{n-1}_\bullet| \subset |\mathcal{E}'(M)^n_\bullet|$ is canonically isomorphic to $\nu_{n-1-d} \times \mathbb{R}$.*

Proof. A p-simplex

$$\sigma = (W_\sigma, U_\sigma, r_\sigma, \mathcal{A}_\sigma) \in \mathcal{M}'(M)^n_p$$

determines a map $\sigma \colon \Delta^p \to |\mathcal{M}'(M)^n_\bullet|$, so that $\sigma^*|\mathcal{E}'(M)^n_\bullet| = W_\sigma$. The map $r^n \colon |U^n_\bullet| \to |\mathcal{E}'(M)^n_\bullet|$ pulled back to this is precisely $r_\sigma \colon U_\sigma \to W_\sigma$, which is a locally trivial Cat $\mathbb{R}^{n-d}$-bundle via the atlas $\mathcal{A}_\sigma$. Now let

$$|\mathcal{E}'(M)^n_\bullet|^{(k)} = \left(\coprod_{p=0}^{k} \mathcal{E}'(M)^n_p\right) / \sim$$

denote the k-skeleton, similarly $|U^n_\bullet|^{(k)}$, and suppose given a Cat atlas $\mathcal{A}^{(k)}$ for $r^{(k)} \colon |U^n_\bullet|^{(k)} \to |\mathcal{E}'(M)^n_\bullet|^{(k)}$ which over each simplex $(W, U, r, \mathcal{A})$ restricts to the atlas $\mathcal{A}$ for $r \colon U \to W$. For each $(k+1)$-simplex

$$\sigma = (W_\sigma, U_\sigma, r_\sigma, \mathcal{A}_\sigma) \in \mathcal{M}'(M)^n_{k+1}$$

there is an $\epsilon > 0$ such that for each $i = 0, 1, \dots, p$ we have

$$W_\sigma \cap (\Delta^p_i(\epsilon) \times \mathbb{R}^n) = \pi_i(\epsilon)^{-1}(W_\sigma \cap (\Delta^{p-1}_i \times \mathbb{R}^n))$$

and

$$U_\sigma \cap (\Delta^p_i(\epsilon) \times \mathbb{R}^n) = \pi_i(\epsilon)^{-1}(U_\sigma \cap (\Delta^{p-1}_i \times \mathbb{R}^n))$$

and on this set r commutes with the ith barycentric coordinate t_i and satisfies $\pi_i(\epsilon) \circ r_\sigma = r_\sigma \circ \pi_i(\epsilon)$. In particular, the inclusion $\partial W_\sigma \to \partial^\epsilon W_\sigma$, where

$$\partial W_\sigma = \bigcup_{i=0}^{p} W_\sigma \cap (\Delta^{p-1}_i \times \mathbb{R}^n) \qquad \text{and} \qquad \partial^\epsilon W_\sigma = \bigcup_{i=0}^{p} W_\sigma \cap (\Delta^p_i(\epsilon) \times \mathbb{R}^n),$$

has a retraction ρ_σ such that $U_\sigma|_{\partial^\epsilon W_\sigma} \cong \rho_\sigma^* U_\sigma|_{\partial W_\sigma}$ as Cat $\mathbb{R}^{n-d}$-bundles. Thus $\rho_\sigma^*(\mathcal{A}^{(k)})$ gives a Cat atlas over $\partial^\epsilon W_\sigma$ which is compatible with $\mathcal{A}_\sigma$. This shows that there is an atlas $\mathcal{A}^{(k+1)}$ for $r^{(k+1)} \colon |U^n_\bullet|^{(k+1)} \to |\mathcal{E}'(M)^n_\bullet|^{(k+1)}$ extending the atlas $\mathcal{A}^{(k)}$ for $r^{(k)}$.

Gluing together the sets $\partial^\epsilon W_\sigma$ for all $(k+1)$-simplices σ gives an open subset

$$V^{(k)} \subseteq |\mathcal{E}'(M)^n_\bullet|^{(k+1)}$$

containing $|\mathcal{E}(M)^n_\bullet|^{(k)}$. The retractions ρ_σ glue together to a retraction

$$\rho^{(k)} \colon V^{(k)} \longrightarrow |\mathcal{E}'(M)^n_\bullet|^{(k)}$$

such that

$$|U^N_\bullet|^{(k+1)}|_{V^{(k)}} \cong (\rho^{(k)})^* |U^N_\bullet|^{(k)}$$

as Cat $\mathbb{R}^{n-d}$-bundles. A point $x \in |\mathcal{E}'(M)^n_\bullet|^{(k)}$ has an open neighbourhood

$$V_x = V^{(k)} \cup (\rho^{(k+1)})^{-1}(V^{(k)}) \cup (\rho^{(k+1)} \circ \rho^{(k+2)})^{-1}(V^{(k)}) \cup \cdots \subset |\mathcal{E}'(M)^n_\bullet|$$

which retracts to $|\mathcal{E}'(M)^n_\bullet|^{(k)}$ via

$$\rho_x = \rho^{(k)} \cup (\rho^{(k)} \circ \rho^{(k+1)}) \cup (\rho^{(k)} \circ \rho^{(k+1)} \circ \rho^{(k+2)}) \cup \cdots,$$

and $|U^n_\bullet||_{V_x} \cong \rho_x^* |U^n_\bullet|^{(k)}$ as Cat $\mathbb{R}^{n-d}$-bundles. This proves the first part; the second part is immediate from the formula for the map $\mathcal{E}'(M)^{n-1}_\bullet \to \mathcal{E}'(M)^n_\bullet$. □

Note that $\mathcal{E}'(M)^n = |\mathcal{E}'(M)^n_\bullet|$ is paracompact by a similar argument to that which shows that a cell complex is paracompact, and hence the Cat $\mathbb{R}^{n-d}$-bundle ν_{n-d} is numerable, so is classified by a map $\nu_{n-d} : \mathcal{E}'(M)^n \to \mathrm{BCat}(n-d)$. We thus obtain a diagram

$$\begin{array}{ccccccccc}
\longrightarrow & \mathcal{E}'(M)^n & \longrightarrow & \mathcal{E}'(M)^{n+1} & \longrightarrow & \mathcal{E}'(M)^{n+2} & \longrightarrow \\
& \downarrow{\scriptstyle \nu_{n-d}} & & \downarrow{\scriptstyle \nu_{n+1-d}} & & \downarrow{\scriptstyle \nu_{n+2-d}} & \\
\longrightarrow & \mathrm{BCat}(n-d) & \longrightarrow & \mathrm{BCat}(n+1-d) & \longrightarrow & \mathrm{BCat}(n+2-d) & \longrightarrow
\end{array}$$

in which each square homotopy commutes up to a preferred homotopy class of homotopies, and so taking (homotopy) colimits we obtain a map $\nu_v \mathcal{E}'(M) \colon \mathcal{E}'(M) \to \mathrm{BCat}$. Now, the square

$$\begin{array}{ccc}
\mathcal{E}'(M) & \longrightarrow & \mathcal{E}(M) \\
\downarrow & & \downarrow \\
\mathcal{M}'(M) & \longrightarrow & \mathcal{M}(M)
\end{array}$$

is homotopy cartesian by Proposition 2.3.2 so the top map is a weak equivalence. Thus we may transfer the map $\nu_v \mathcal{E}'(M)$ to a map

$$\nu_v \mathcal{E}(M) \colon \mathcal{E}(M) \longrightarrow \mathrm{BCat}$$

classifying what we shall call the Cat *stable vertical normal bundle*. We call its stable inverse the Cat *stable vertical tangent bundle*, and denote it $T^s_v \mathcal{E}(M)$.

2.5. **Comparisons.** Let us finally compare this definition with both the usual vertical tangent bundle of a fibre bundle, and the stable bundle constructed in [ERW14].

The vertical tangent bundles of fibre bundles. The simplest way to make this comparison is to produce a model $\mathcal{B}(M)$ for $\mathrm{BCat}(M)$ akin to $\mathcal{M}(M)$ by realising the semi simplicial set with p-simplices the locally flat Cat submanifolds $W \subset \Delta^p \times \mathbb{R}^\infty$ so that the map to the first factor is a Cat M-bundle. Just as in the case of block bundles there is a version $\mathcal{B}'(M)$ of this construction where manifolds are equipped with choices of tubular neighbourhoods $(U, r, \mathcal{A})$ as before, where one additionally insists that the map $r \colon U \to W$ is fibrewise over Δ^p. This space $\mathcal{B}'(M)$ has a forgetful map to $\mathcal{M}'(M)$, and the pullback of $\mathcal{E}'(M) \to \mathcal{M}'(M)$ to $\mathcal{B}'(M)$ gives a universal M-fibre bundle $\mathcal{F}'(M) \to \mathcal{B}'(M)$, to which the stable vertical normal bundle $\nu_v \mathcal{E}'(M)$ can be pulled back. The vertical tangent bundle of $\mathcal{F}'(M) \to \mathcal{B}'(M)$ is a stable inverse to this, by construction.

2.5.1. *Remark.* This comparison proves that the stable vertical tangent bundle of a topological manifold bundle only depends on its underlying block bundle and thus our constructions recover [BFJ16, Theorem G]: Their *strong Borel conjecture* is well-known to imply our block Borel conjecture (we will explain this in the proof of Proposition 5.1.1) and implies that fibre homotopy equivalent M-(block-)bundles are equivalent as block bundles, so must have isomorphic stable vertical tangent bundles.

The stable vertical tangent bundle of [ERW14]*.* The authors of that paper considered a smooth block bundle $(p\colon E \to |K|, \mathcal{A})$ with base the geometric realisation of a finite simplicial complex K. In [ERW14, Proposition 3.2] they constructed a stable vertical tangent bundle by choosing embeddings $e\colon E \to |K| \times \mathbb{R}^n$ and $a\colon |K| \to \mathbb{R}^k$ satisfying certain properties, and hence constructing a continuous map $E \to Gr_{d+k}(\mathbb{R}^{n+k})$: the $(d+k)$-dimensional vector bundle classified by this map is called $t_{E,e,a}$, and is the stable vertical tangent bundle; the $(n-d)$-dimensional vector bundle classified by this map is called $n_{E,e,a}$, and is the stable vertical normal bundle.

If the classifying map for a smooth block bundle $(p\colon E \to |K|, \mathcal{A})$ is factored up to homotopy as $|K| \to |\mathcal{M}(M)^n_\bullet| \to |\mathcal{M}(M)_\bullet|$, then the block bundle is concordant to a $(p'\colon E' \to |K|, \mathcal{A}')$ which comes equipped with an embedding $e'\colon E' \to |K| \times \mathbb{R}^n$ a neighbourhood $E' \subset U' \subset |K| \times \mathbb{R}^n$, and a retraction $r'\colon U' \to E'$ which has the structure of a smooth $\mathbb{R}^{n-d}$-bundle. This yields a $(n-d)$-dimensional vector bundle on E', and this is isomorphic to $n_{E',e',a'}$ for any choice of $a'\colon |K| \to \mathbb{R}^k$. In particular, the associated $t_{E,e,a}$ is stably isomorphic to the stable vertical tangent bundle constructed here.

Stable vertical tangent bundles of block bundles over manifolds. Given a block bundle over a triangulated manifold, one may describe its stable vertical tangent bundle in terms of the tangent bundles of the base and total space, as follows.

2.5.2. **Lemma.** *Let $|K| \xrightarrow{\cong} B$ be a* PL *triangulation of a* Cat *manifold (compatible in the smooth or piecewise linear cases), and $(p\colon E \to |K|, \mathcal{A})$ be a* Cat *block bundle. Then E has the structure of a* Cat *manifold, and the stable vertical* Cat *tangent bundle is equivalent to $TE - p^*TB$.*

Proof. Let us first show that E inherits a Cat manifold structure. The stars $\mathrm{St}(v) \subset |K|$ of vertices $v \in K$ have interiors which form an open cover of $|K|$, so their preimages $p^{-1}(\mathrm{St}(v))$ have interiors which form an open cover of E and hence it is enough to give (compatible) Cat manifold structures to these. We have

$$p^{-1}(\mathrm{St}(v)) = \bigcup_{\sigma \ni v} W_\sigma$$

where W_σ is the block over σ. There are Cat isomorphisms $W_\sigma \cong \sigma \times M$. As mentioned earlier, the semi-simplicial group $\widetilde{\mathrm{Cat}}(M)_\bullet$ is Kan so that we may choose such Cat isomorphisms in increasing order of $\dim(\sigma)$, extending those which have already been chosen on faces of σ (we use here that all simplices of $\mathrm{St}(v)$ have a free face). This gives a block Cat isomorphism $p^{-1}(\mathrm{St}(v)) \cong \mathrm{St}(v) \times M$, and hence induces a Cat manifold structure on $p^{-1}(\mathrm{St}(v))$.

By Lemma 2.4.3 we may suppose that $(p\colon E \to |K|, \mathcal{A})$ is classified by a map to some $|\mathcal{M}(M)^n_\bullet|$, so we have a neighbourhood $E \subset U \subset |K| \times \mathbb{R}^n$ and a retraction $r\colon U \to E$ equipped with the structure of a Cat $\mathbb{R}^{n-d}$-bundle. By the same argument as above, U has a Cat manifold structure making it an open submanifold of $|K| \times \mathbb{R}^n$. By the uniqueness theorem for stable normal (micro)bundles [KS77, p. 204], this must be isomorphic to the normal bundle of $E \subset |K| \times \mathbb{R}^n$, which is stably $TE - p^*TB$. □

3. An Euler class for fibrations with Poincaré fibre

In [Ran16, Section 2] Randal-Williams constructs a fibrewise Euler class for a fibration $p\colon E \to B$ in which B is a finite complex, the fibre F is an oriented Poincaré duality space of formal dimension d, and the fibration is oriented in the sense that the monodromy action of $\pi_1(B)$ is trivial. However, the line of argument used essentially that B is a finite cell complex, so cannot be used to obtain an Euler class for the fibration

$$F \longrightarrow \mathrm{B}\mathcal{G}^+_*(F) \longrightarrow \mathrm{B}\mathcal{G}^+(F),$$

which is easily checked to be the universal one. To define an Euler class also when the base B is not necessarily a finite complex – e.g. $\mathrm{B}\mathcal{G}^+(F)$ and $\mathrm{B}\widetilde{\mathrm{Top}}^+(M)$ – we shall give a different argument using parametrised stable homotopy theory.

3.1. **The fibrewise Euler class.** To motivate our construction, let us recall one definition of the Euler class of a Poincaré duality space F. If $D_F\colon H^*(F) \to H_{d-*}(F)$ is the Poincaré duality map for F, and $\Delta\colon F \to F \times F$ is the diagonal map, then the Euler class of F may be described as $e(F) = \Delta^* D_{F\times F}\Delta_* D_F(1)$. We will mimic this definition for a fibration $p\colon E \to B$ with Poincaré fibre F using parametrised (co)homology.

Consider the category Sp_B of spectra parametrised over B. Suppressing usual subscripts to declutter the notation as no other base will be considered in this section let $\mathrm{H}\mathbb{Z}$ and $\mathbb{S}$ denote the trivial parametrised Eilenberg–Mac Lane and sphere spectrum, respectively. Similarly, we have dropped the notation for a disjoint base point or section. We have the diagonal map $\Delta\colon E \to E\times_B E$ and we claim that Poincaré duality for F, and the orientability hypothesis, yield a *fibrewise Poincaré duality* equivalence

$$D_E^{fw}\colon \Sigma^d F_B(E, \mathrm{H}\mathbb{Z}) \xrightarrow{\simeq} E \wedge_B \mathrm{H}\mathbb{Z}$$

of $\mathrm{H}\mathbb{Z}$-modules, and similarly for $E \times_B E$. Granted this claim, the diagram

$$\begin{aligned}\mathbb{S}^d \xrightarrow{\Sigma^d 1} \Sigma^d F_B(E,\mathrm{H}\mathbb{Z}) \xrightarrow[\simeq]{D_E^{fw}} E\wedge_B \mathrm{H}\mathbb{Z} \xrightarrow{\Delta\wedge_B \mathrm{H}\mathbb{Z}} (E\times_B E)\wedge_B \mathrm{H}\mathbb{Z} \\ \xleftarrow[\simeq]{D^{fw}_{E\times_B E}} \Sigma^{2d} F_B(E\times_B E,\mathrm{H}\mathbb{Z}) \xrightarrow{F_B(\Delta,\mathrm{H}\mathbb{Z})} \Sigma^{2d}F_B(E,\mathrm{H}\mathbb{Z})\end{aligned}$$

represents a well-defined element

$$e^{fw}(p) := \Delta^*(D^{fw}_{E\times_B E})^{-1}\Delta_* D_E^{fw}(1) \in [\mathbb{S}^{-d}, F_B(E,\mathrm{H}\mathbb{Z})]_B \cong [\mathbb{S}^{-d}\wedge_B E, \mathrm{H}\mathbb{Z}]_B \cong H^d(E;\mathbb{Z}).$$

See [GGR17, Theorem 5.6] for a related discussion.

3.1.1. **Definition.** The class $e^{fw}(p) \in H^d(E;\mathbb{Z})$ so constructed is the *fibrewise Euler class* of the oriented fibration $p\colon E \to B$.

It remains to establish the equivalence D_E^{fw}. As in ordinary Poincaré duality, it will be given by cap product with a fundamental class. Consider the parametrised Atiyah–Hirzebruch spectral sequence

$$H^i(B;(E\wedge_B \mathrm{H}\mathbb{Z})^j) \longrightarrow (E\wedge_B\mathrm{H}\mathbb{Z})^{i+j}(B)$$

based on the parametrised spectrum $E\wedge_B \mathrm{H}\mathbb{Z}$, compare e.g. [MS06, Theorem 20.4.1] (with $X = B$ and $J = E\wedge_B \mathrm{H}\mathbb{Z}$ and $\mathscr{L}^*(B, E\wedge_B\mathrm{H}\mathbb{Z})$ abbreviated to $(E\wedge_B \mathrm{H}\mathbb{Z})^*$). The fundamental classes

$$[E_b] \in H_d(E_b;\mathbb{Z}) \cong H^0(b;(E_b\wedge \mathrm{H}\mathbb{Z})^{-d})$$

of the fibres E_b assemble to a class $[E]_B \in H^0(B;(E\wedge_B\mathrm{H}\mathbb{Z})^{-d})$ on the second page by the orientation hypothesis. The spectral sequence is concentrated in rows $-d,\dots,0$ and positive columns, so the remaining groups on the diagonal $i+j=-d$ are zero, and $[E]_B$ is a permanent cycle for degree reasons as well, so abusing notation we obtain a unique class

$$[E]_B \in (E\wedge_B\mathrm{H}\mathbb{Z})^{-d}(B) = [\mathbb{S}^d, E\wedge_B \mathrm{H}\mathbb{Z}]_B.$$

We may thus form the cap product with $[E]_B$, that is the map

$$\begin{aligned}\mathbb{S}^d\wedge_B F_B(E,\mathrm{H}\mathbb{Z}) \xrightarrow{[E]_B\wedge \mathrm{Id}} (E\wedge_B \mathrm{H}\mathbb{Z})\wedge_B F_B(E,\mathrm{H}\mathbb{Z}) \\ \xrightarrow{\Delta} E\wedge_B E\wedge_B \mathrm{H}\mathbb{Z}\wedge_B F_B(E,\mathrm{H}\mathbb{Z}) \xrightarrow{ev} E\wedge_B\mathrm{H}\mathbb{Z}\wedge_B\mathrm{H}\mathbb{Z} \longrightarrow E\wedge_B \mathrm{H}\mathbb{Z},\end{aligned}$$

which is the sought after map D_E^{fw}; that it is an equivalence may be checked on fibres, where it reduces to ordinary Poincaré duality.

3.1.2. *Remark.* Clearly the above construction works for any R-oriented fibration $p\colon E \to B$ as well, where R is a multiplicative cohomology theory: Whenever p is equipped with an R-orientation in the sense that the fibre F is an R-oriented Poincaré complex and the monodromy action of $\pi_1(B)$ on $R^d(F)$ is trivial, one obtains a class $e^{fw}(p) \in R^d(E)$ that restricts to the R-Euler class of F in each fibre by the same formula we used above. Also it is possible to drop the orientability assumption by working with twists (a.k.a. local coefficient systems for ordinary cohomology).

3.2. **Comparisons.** Again we compare our construction to both the classical case and the definition of [Ran16].

The Euler class of the vertical tangent bundle. Suppose that $p\colon E \to B$ is an oriented topological fibre bundle with fibre a d-dimensional manifold M, with B a CW-complex. The data $(\pi_1\colon E \times_B E \to E, \Delta\colon E \to E \times_B E)$ defines the vertical tangent topological microbundle T_pE over E. As B is a CW-complex, E is paracompact, so by [Hol67] it contains a Euclidean $\mathbb{R}^d$-bundle, i.e. there is an open neighbourhood $E \overset{s}{\hookrightarrow} U \subset E \times_B E$ with a projection $r\colon U \to E$ over B which is a Euclidean $\mathbb{R}^d$-bundle. Writing U_B^+ for the fibrewise 1-point compactification, there is a fibrewise collapse map

$$c\colon E \times_B E \longrightarrow U_B^+.$$

The composition

$$E \xrightarrow{\Delta} E \times_B E \xrightarrow{c} U_B^+ \xrightarrow{q} U_B^+/B = \mathrm{Th}(U)$$

pulls back the Thom class $u \in H^d(\mathrm{Th}(U); \mathbb{Z})$ to the Euler class $e(T_pE)$ of T_pE.

To compare this with the definition above, consider the map

$$d\colon U_B^+ \longrightarrow E \wedge_B U_B^+$$

of ex-spaces over B, induced by the diagonal map of U, which fits into a commutative diagram

$$\begin{array}{ccc}
(E \times_B E) \wedge_B \mathrm{H}\mathbb{Z} & \xrightarrow{\Delta_{E\times_B E}\wedge \mathrm{H}\mathbb{Z}} & (E \times_B E \times_B E \times_B E) \wedge_B \mathrm{H}\mathbb{Z} \\
\downarrow{\scriptstyle c\wedge \mathrm{H}\mathbb{Z}} & & \\
U_B^+ \wedge_B \mathrm{H}\mathbb{Z} & & \downarrow{\scriptstyle E\times E\times c\wedge \mathrm{H}\mathbb{Z}} \\
\downarrow{\scriptstyle d\wedge \mathrm{H}\mathbb{Z}} & & \\
E \wedge_B U_B^+ \wedge_B \mathrm{H}\mathbb{Z} & \xrightarrow{\Delta_E \wedge U_B^+ \wedge \mathrm{H}\mathbb{Z}} & (E \times_B E) \wedge_B U_B^+ \wedge_B \mathrm{H}\mathbb{Z} \\
\downarrow{\scriptstyle E\wedge q^* u} & & \downarrow{\scriptstyle E\wedge q^* u} \\
E \wedge_B \mathrm{H}\mathbb{Z} & \xrightarrow{\Delta_E \wedge \mathrm{H}\mathbb{Z}} & (E \times_B E) \wedge_B \mathrm{H}\mathbb{Z}.
\end{array}$$

Precomposing this with the map

$$\mathbb{S}^{2d} \xrightarrow{[E\times_B E]_B} (E \times_B E) \wedge_B \mathrm{H}\mathbb{Z}$$

by definition gives $[E \times_B E]_B \cap c^* q^* u$ along the top. Under the equivalence $E \to U$ we have

$$c_*[E \times_B E]_B \cap q^* u = [E]_B \in [\mathbb{S}^d, E \wedge_B \mathrm{H}\mathbb{Z}]_B;$$

by definition of $[E]_B$ this can be checked by restriction to a single fibre, where it reduces to Thom's description of Poincaré duals of submanifolds. Composition along the bottom is therefore $\Delta_*([E]_B)$. Hence $c^* q^* u = (D^{fw}_{E\times_B E})^{-1} \Delta_*([E]_B) \in H^d(E; \mathbb{Z})$, and so

$$e(T_pE) = s^* q^*(u) = \Delta^* c^* q^*(u) = \Delta^* (D^{fw}_{E\times_B E})^{-1} \Delta_* D^{fw}_E(1) = e^{fw}(p).$$

The Euler class of [Ran16]. The construction in [Ran16, Section 2] follows the proof of the 'Fibre Inclusion Theorem' of Casson–Gottlieb [CG77]: by embedding B into some $\mathbb{R}^n$, taking a regular neighbourhood, and doubling it, we may find an embedding $i\colon B \to B'$ into an oriented smooth n-manifold and a retraction $r\colon B' \to B$. Then $E' := r^*E \to B'$ is a fibration with oriented Poincaré base and fibre, so E' is also oriented Poincaré, by [Got79]. Let us write $D_{E'}\colon H^*(E') \to H_{n+d-*}(E')$ for the Poincaré duality isomorphism. Similarly, $E' \times_{B'} E'$ is Poincaré with duality isomorphism $D_{E' \times_B E'}$, and using the diagonal map $\Delta\colon E' \to E' \times_{B'} E'$ we can form

$$e(E') := \Delta^* D^{-1}_{E' \times_B E'} \Delta_* D_{E'}(1) \in H^d(E'; \mathbb{Z})$$

The Euler class $e(E) \in H^d(E; \mathbb{Z})$ is then defined by restriction along $E \to E'$.

The key step in comparing this definition with ours is the following, which makes use of the notion of Costenoble–Waner duality. Suppose $p\colon E \to B$ is a fibration with n-dimensional oriented manifold base, and write $r\colon B \to *$ for the constant map. Let $\nu_B \in \mathrm{Sp}_B$ be the Spivak normal fibration of B, suspended to have dimension 0. Then ν_B is the Costenoble–Waner dual of $\mathbb{S}^{-n}$ by [MS06, Theorem 18.6.1], and so by [MS06, Proposition 18.1.5] we have an equivalence of spectra

$$r_!(E \wedge_B \mathrm{H}\mathbb{Z} \wedge_B \nu_B) \simeq \Sigma^n r_*(E \wedge_B \mathrm{H}\mathbb{Z})$$

and $\mathrm{H}\mathbb{Z} \wedge_B \nu_B \simeq \mathrm{H}\mathbb{Z}$ from the orientation of B. The left-hand side is thus equivalent to $E \wedge \mathrm{H}\mathbb{Z}$. Under the assumption that the fibres of p are oriented Poincaré of dimension d we have the equivalence D_E^{fw}, and so

$$r_*(E \wedge_B \mathrm{H}\mathbb{Z}) \simeq \Sigma^d r_*(F_B(E, \mathrm{H}\mathbb{Z})) \simeq \Sigma^d F(E, \mathrm{H}\mathbb{Z}).$$

Via the Costenoble–Waner equivalence our equivalence D_E^{fw} becomes the ordinary Poincaré duality D_E for E.

4. Tautological characteristic classes of block bundles

In the rest of the paper we shall be interested in *oriented* block bundles. That is, we will assume that M is oriented, and consider block bundles $(p\colon E \to |K|, \mathcal{A})$ for which the transition maps are orientation preserving. These are classified by analogous spaces

$$\mathrm{B}\widetilde{\mathrm{Cat}}^+(M) \simeq \mathcal{M}^+(M),$$

where the p-simplices of $\mathcal{M}^+(M)_\bullet$ are *oriented* submanifolds $W \subset \Delta^p \times \mathbb{R}^n$ which are p-block Cat^+ isomorphic to $\Delta^p \times M$. Forgetting the orientation defines a map $f\colon \mathcal{M}^+(M) \to \mathcal{M}(M)$, which defines the universal oriented block bundle $\mathcal{E}^+(M) = f^*\mathcal{E}(M)$ with projection

$$\pi\colon \mathcal{E}^+(M) \longrightarrow \mathcal{M}^+(M)$$

for which we will now define tautological classes. Again the case of interest for us is that of topological block bundles, but our methods work just as well in the smooth and piecewise linear categories, so we work in that generality.

4.1. **The tautological classes.** By Proposition 2.3.2, the map π is a weak quasi-fibration, i.e. $\pi^{-1}(v) \to \mathrm{hofib}_v(\pi)$ is a weak homotopy equivalence for any vertex $v \in \mathcal{M}^+(M)_0$. As $\pi^{-1}(v) \cong M$, the Serre spectral sequence for the map π (replaced by a fibration) takes the form

$$H^p(\mathcal{M}^+(M); \mathcal{H}^q(M; R)) \longrightarrow H^{p+q}(\mathcal{E}^+(M); R).$$

Since the block bundle is oriented, the local system $\mathcal{H}^d(M; R)$ is trivialised for any ring R and so this spectral sequence defines a Gysin homomorphism

$$\pi_!\colon H^k(\mathcal{E}^+(M); R) \longrightarrow H^{k-d}(\mathcal{M}^+(M); R).$$

The stable vertical tangent bundle constructed in Section 2.4, together with the fibrewise Euler class constructed in Section 3, give a map

$$(T_v^s\mathcal{E}^+(M), e^{fw}(M))\colon \mathcal{E}^+(M) \longrightarrow \mathrm{BSCat} \times K(\mathbb{Z}, d)$$

for any d-dimensional Cat manifold M. Using the equivalence $H^*(\mathcal{M}^+(M); R) \cong H^*(\mathrm{B}\widetilde{\mathrm{Cat}}^+(M); R)$ discussed in Section 2.3 we obtain:

4.1.1. **Definition.** The *universal tautological characteristic classes*

$$\kappa_c(M) := \pi_!((T_v^s\mathcal{E}^+(M), e^{fw}(M))^*(c))$$

define a homomorphism

$$\kappa_-(M)\colon H^k(\mathrm{BSCat} \times K(\mathbb{Z}, d); R) \longrightarrow H^{k-d}(\mathrm{B}\widetilde{\mathrm{Cat}}^+(M); R).$$

4.2. **Comparisons.** These classes agree with the classes defined in [ERW14] and also restrict to the classical tautological classes for the universal smooth fibre bundle. We record this explicitly in the following propositions.

4.2.1. **Proposition.** *The square*

$$\begin{array}{ccc} H^*(\mathrm{BSTop} \times K(\mathbb{Z}, d); R) & \xrightarrow{\kappa_-(M)} & H^{*-d}(\mathrm{B}\widetilde{\mathrm{Top}}^+(M); R) \\ \downarrow & & \downarrow \\ H^*(\mathrm{BSO}(d); R) & \xrightarrow{\kappa_-(M)} & H^{*-d}(\mathrm{BDiff}^+(M); R) \end{array}$$

commutes.

The implications of this statement depend on the coefficient ring R, mostly due to the fact that relevant properties of the left vertical map depend on the choice of coefficients: The work of Kirby–Siebenmann implies that the map $\mathrm{BSO} \to \mathrm{BSTop}$ is a rational equivalence and thus the left vertical map in the diagram is a surjection when R is $\mathbb{Q}$. Therefore, all *rational* tautological classes in $H^*(\mathrm{BDiff}^+(M); \mathbb{Q})$ are in the image of the upper composition and we recover [BFJ16, Corollaries C.1 & G.1].

Integrally, it is not clear that every tautological class should lie in the image of the upper composition, as the work of Kirby–Siebenmann describes the fibre of the map $\mathrm{BSO} \to \mathrm{BSTop}$ in terms of groups of homotopy spheres. Since these are finite, some multiple of every class in $H^*(\mathrm{BSO}(d); \mathbb{Z})$ lies in the image of the left vertical map and in principle these multiples can be determined in terms of the orders of the groups of homotopy spheres; we shall refrain from carrying this out. But, in particular, our vanishing results will imply bounds on the order of κ_c independent of the fibre manifold.

4.2.2. **Proposition.** *Under the maps*

$$\mathrm{BTop}^+(M) \longrightarrow \mathrm{B}\widetilde{\mathrm{Top}}^+(M) \quad \textit{and} \quad \mathrm{B}\widetilde{\mathrm{Diff}}^+(M) \longrightarrow \mathrm{B}\widetilde{\mathrm{Top}}^+(M)$$

the tautological classes just defined restrict to those of Ebert and Randal-Williams.

Proof of Propositions 4.2.1 & 4.2.2. Proposition 4.2.1 follows immediately from Sections 2.5 and 3.2. For Proposition 4.2.2 let us restrict to the case of rational coefficients and simply remark that the other cases work similarly. We start with the easier case of block diffeomorphisms. As discussed in Section 2.5, the stable vertical tangent bundle we constructed coincides with the one of [ERW14]. Likewise, the discussion after Definition 3.1.1 shows that our fibrewise Euler class restricts to that of [Ran16]. By [Ran16, Lemma 2.2 (iv)] the claim follows. So let us now consider the case of topological bundles. By construction (see [ERW14, Proposition 4.2]) it suffices to treat the case of a manifold base, in which Lemma 2.5.2 implies that their vertical tangent bundle stabilises to our bundle and [Ran16, Lemma 2.2 (ii) & (iv)] shows that the two definitions of the Euler class agree (again these results are stated for smooth bundles in [Ran16], but their proofs make no use of that). Now the same reasoning as before applies. □

Finally, let us warn the reader that they should resist the temptation to think that the tautological classes in $H^*(\mathrm{B}\widetilde{\mathrm{Top}}^+(M); R)$ behave like their counterparts in $H^*(\mathrm{BTop}^+(M); R)$ or $H^*(\mathrm{BDiff}^+(M); R)$ as there is no reason for the homomorphism

$$\kappa_-(M)\colon H^*(\mathrm{BSTop} \times K(\mathbb{Z}, d); R) \longrightarrow H^{*-d}(\mathrm{B}\widetilde{\mathrm{Top}}^+(M); R)$$

to factor through $H^*(\mathrm{BSTop}(d); R)$. Indeed, in the smooth case, one of the key results of [Ran16, Proposition 3.1] implies

$$\kappa_{e^2}(W_g) \neq \kappa_{p_n}(W_g) \in H^{2n}(\mathrm{B}\widetilde{\mathrm{Diff}}^+(W_g); \mathbb{Q})$$

for $W_g = (S^n \times S^n)^{\# g}$ and large $g \in \mathbb{N}$ and in [ERW14, Theorem 3] the authors construct an 8-manifold M with

$$0 \neq \kappa_{p_5}(M) \in H^{12}(\mathrm{B}\widetilde{\mathrm{Diff}}^+(M); \mathbb{Q}).$$

While these results exclude the analogous factorisation in the case of smooth block bundles, note that they do not suffice to actually exclude it for topological block bundles. Indeed, by work of Weiss [Wei15], neither $e^2 = p_n \in H^{4n}(\mathrm{BSTop}(2n); \mathbb{Q})$ nor $0 = p_m \in H^{4n}(\mathrm{BSTop}(2n); \mathbb{Q})$ for $m \geq n$ hold in general. In fact, it seems to be our knowledge of $H^*(\mathrm{BSTop}(n); \mathbb{Q})$ that prevents us from disproving this factorisation.

In a similar direction, our methods do not lift *all* rational tautological classes of topological fibre bundles, as the ring $H^*(\mathrm{BSTop}(n); \mathbb{Q})$ is not generated by Euler and Pontryagin classes, which may be seen as follows. Using that $\frac{\mathrm{STop}(n)}{\mathrm{SO}(n)}$ is rationally $(n+1)$-connected (by [KS77, p. 246] and the fact that $\frac{\mathrm{STop}}{\mathrm{SO}}$ is rationally contractible), Morlet's identification

$$\mathrm{BDiff}_\partial(D^n) \simeq \Omega_0^n \left(\frac{\mathrm{STop}(n)}{\mathrm{SO}(n)} \right),$$

see [KS77, p. 241], and Farrell–Hsiang's calculation [FH78] of the rational homotopy groups of $\mathrm{BDiff}_\partial(D^n)$, it follows for n odd that

$$\pi_i \left(\frac{\mathrm{STop}(n)}{\mathrm{SO}(n)} \right) \otimes \mathbb{Q} = \begin{cases} 0 & 0 < i < n+4 \\ \mathbb{Q} & i = n+4 \end{cases}$$

and therefore $H^{n+4}\left(\frac{\mathrm{STop}(n)}{\mathrm{SO}(n)}; \mathbb{Q}\right) \cong \mathbb{Q}$. In the Serre spectral sequence for the fibre sequence

$$\frac{\mathrm{STop}(n)}{\mathrm{SO}(n)} \longrightarrow \mathrm{BSO}(n) \to \mathrm{BSTop}(n)$$

one immediately finds the transgression

$$H^{n+4}\left(\frac{\mathrm{STop}(n)}{\mathrm{SO}(n)}; \mathbb{Q}\right) \longrightarrow H^{n+5}(\mathrm{BSTop}(n); \mathbb{Q})$$

injective. By definition its image vanishes in $H^{n+5}(\mathrm{BSO}(n); \mathbb{Q})$ and therefore does not stabilise to $H^{n+5}(\mathrm{BSTop}; \mathbb{Q})$ as the map $H^{n+5}(\mathrm{BSTop}; \mathbb{Q}) \longrightarrow H^{n+5}(\mathrm{BSO}(n); \mathbb{Q})$ is an isomorphism with both sides given by the same products of Pontryagin classes.

5. Block homeomorphisms of aspherical manifolds

In the previous sections we have established that the tautological classes of smooth manifold bundles extend to topological block bundles. As this paper aims to understand the tautological classes for aspherical manifolds, we will now discuss the homotopy type of the space $\mathrm{B}\widetilde{\mathrm{Top}}(M)$ provided M is aspherical. This depends on what are called the full Farrell–Jones conjectures, see Section 5.2. We will call a group satisfying them a *Farrell–Jones group*.

5.1. **The block Borel conjecture.** Let M be an aspherical manifold and recall from the introduction that M is said to satisfy the *block Borel conjecture* if the canonical map

$$\iota \colon \mathrm{B}\widetilde{\mathrm{Top}}(M) \longrightarrow \mathrm{B}\mathcal{G}(M)$$

is a weak equivalence. As a slight modification, we will say that M satisfies the *identity block Borel conjecture* if the map

$$\iota_0 \colon \mathrm{B}\widetilde{\mathrm{Top}}_0(M) \longrightarrow \mathrm{B}\mathcal{G}_0(M)$$

is a weak equivalence.

5.1.1. **Proposition.** *Let M be an aspherical manifold whose fundamental group is a Farrell–Jones group. If the dimension of M is at least 5, then the block Borel conjecture holds for M. If the dimension of M is 4, then the map ι is a 1-coequivalence, in particular the identity block Borel conjecture holds for M.*

This proposition implies that the block Borel conjecture holds for a very large class of aspherical manifolds, see Theorem 5.2.1.

Proof. Let us first sketch the argument, following [BL10, Proposition 0.3], for $\dim(M) \geq 5$. We denote by $\mathcal{G}(M)/\widetilde{\mathrm{Top}}(M)$ the fibre of the comparison map ι. From surgery theory one has isomorphisms

$$\pi_k(\mathcal{G}(M)/\widetilde{\mathrm{Top}}(M)) \cong \mathcal{S}_\partial^{\mathrm{Top}}(M \times \Delta^k)$$

to the higher structure sets of M appearing in the surgery exact sequence

$$\cdots \longrightarrow L^q_{d+k+1}(\mathbb{Z}[\pi_1(M)]) \longrightarrow \mathcal{S}_\partial^{\mathrm{Top}}(M \times \Delta^k) \longrightarrow \mathcal{N}_\partial^{\mathrm{Top}}(M \times \Delta^k) \xrightarrow{\sigma} L^q_{d+k}(\mathbb{Z}[\pi_1(M)]) \longrightarrow \cdots$$

and an inclusion $\pi_0(\mathcal{G}(M)/\widetilde{\mathrm{Top}}(M)) \cong \mathcal{S}^{\mathrm{Top}}(M)$. By the work of Ranicki, σ can be identified with the assembly map

$$\mathrm{L}^q\mathbb{Z}\langle 1\rangle_{d+k}(M) \longrightarrow \mathrm{L}^q_{d+k}(\mathbb{Z}\pi_1(M))$$

in (free) quadratic L-theory, see [Ran92, Theorem 18.5] where $\mathbb{L}.$ denotes our $\mathrm{L}^q\mathbb{Z}\langle 1\rangle$. As explained in [BL10] the Farrell–Jones conjecture in K- and L-theory imply that for an aspherical manifold

$$\mathrm{L}^q\mathbb{Z}_*(M) \longrightarrow \mathrm{L}^q_*(\mathbb{Z}\pi_1(M))$$

is an isomorphism (the K-theoretic Farrell–Jones conjecture for $\pi_1(M)$ is used to change from universally decorated to free L-theory as explained in the proof of [BL10, Proposition 0.3 (i)]). Since the map

$$\mathrm{L}^q\mathbb{Z}\langle 1\rangle_*(M) \longrightarrow \mathrm{L}^q\mathbb{Z}_*(M)$$

is injective in degree d and an isomorphism in higher degrees, $\mathcal{S}_\partial^{\mathrm{Top}}(M \times \Delta^k)$ is trivial for all k. It follows that $\mathcal{G}(M)/\widetilde{\mathrm{Top}}(M)$ is contractible and hence that ι is a homotopy equivalence.

To prove the statement in dimension 4 apply the arguments above to $M \times \Delta^1$, to see that $\pi_k(\mathcal{G}(M)/\widetilde{\mathrm{Top}}(M))$ vanishes for $k \geq 1$. Therefore ι is a 1-coequivalence, i.e. it is injective on π_1 and bijective on π_i for $i \geq 2$. Passing to universal covers then proves that M satisfies the identity block Borel conjecture. ☐

5.1.2. *Remark.* For our purposes it will often suffice to know that the map $\mathrm{B}\widetilde{\mathrm{Top}}_0(M) \to \mathrm{B}\mathcal{G}_0(M)$ is an equivalence after inverting 2, or even rationally. To obtain this weaker statement one need not assume the full Farrell–Jones conjectures, a variant of the L-theoretic conjecture is enough, as we will explain in Proposition 5.2.4.

5.1.3. *Remark.* We do not know the validity of the block Borel conjecture for aspherical manifolds of dimension less than 4. The vanishing results for tautological classes, however, are known in dimensions at most 3 anyway (at least rationally in case of dimension 3). Indeed, in dimension 1 this is a straight forward calculation since S^1-bundles can always be given linear structures, in dimension 2, the orientable aspherical manifolds are exactly the surfaces Σ_g with $g \geq 1$. For $g \geq 2$ the space $\mathrm{BDiff}(\Sigma_g)$ has contractible components by a result of [EE69], so there is nothing to prove. For the torus T^2 one uses that the maps

$$\mathrm{BDiff}(T^2) \longrightarrow \mathrm{BTop}(T^2) \longrightarrow \mathrm{B}\mathcal{G}(T^2)$$

are homotopy equivalences and that the left translation map $\mathrm{B}T^2 \to \mathrm{B}\mathcal{G}(T^2)$ factors over $\mathrm{BDiff}(T^2)$ and is an equivalence onto the identity component. Then one uses the same calculation as in the case of principal S^1-bundles. For 3-dimensional manifolds there is a general vanishing result due to Ebert, see [Ebe13, Corollary 1.3]: He proves that rational, tautological classes vanish in the cohomology of $\mathrm{BDiff}^+(M)$ even for non-aspherical M. For more details, see the discussion in [BFJ16, p. 10].

5.2. **The Farrell–Jones conjectures.** Let us report now on the status of the Farrell–Jones conjectures to convince the reader that their assumption is not too restrictive. We state the following result for the class $\mathcal{FJ}$ of Farrell–Jones groups, that is those groups which satisfy the full Farrell-Jones conjectures; this terminology refers to the following version, the details of which are explained in [Lüc15, Section 11]: A group G is contained in $\mathcal{FJ}$ if for every finite group F and every additive $G \wr F$-category $\mathcal{A}$ (with involution in the L-theoretic case) both maps

$$\mathrm{K}\mathcal{A}_n^{G\wr F}(E_{\mathrm{vc}}G \wr F) \longrightarrow \mathrm{K}_n(\mathcal{A}; G \wr F) \quad \text{and} \quad \mathrm{L}\mathcal{A}_n^{G\wr F}(E_{\mathrm{vc}}G \wr F) \longrightarrow \mathrm{L}_n(\mathcal{A}; G \wr F)$$

are isomorphisms for all $n \in \mathbb{Z}$. Here, K denotes non-connective K-theory and L universally decorated, that is $\langle -\infty \rangle$, L-theory and E_{vc} denotes the classifying space for the family of virtually cyclic subgroups.

This version of the Farrell–Jones conjecture contains the more classical form saying that for any ring R and a discrete group G, the assembly maps

$$\mathrm{K}R_*^G(E_{\mathrm{vc}}G) \longrightarrow \mathrm{K}_*(RG) \quad \text{and} \quad \mathrm{L}R_*^G(E_{\mathrm{vc}}G) \longrightarrow \mathrm{L}_*(RG)$$

are isomorphisms. For a torsion-free group these two conjectures imply that

$$\mathrm{L}\mathbb{Z}_*(\mathrm{B}G) \longrightarrow \mathrm{L}_*(\mathbb{Z}G)$$

is an equivalence for the decoration needed in the proof of Proposition 5.1.1: Any torsion-free virtually cyclic group is in fact infinite cyclic (this follows easily from [FJ95, Lemma 2.5]) so $E_{\mathrm{vc}}G = E_{\mathrm{cyc}}G$. But the relative assembly map with universal decorations

$$\mathrm{L}\mathbb{Z}_*(\mathrm{B}G) \longrightarrow \mathrm{L}\mathbb{Z}_*^G(E_{\mathrm{cyc}}G)$$

is an isomorphism for every group, see [LR05, Proposition 2.10 (ii)]. Finally, the K-theoretic conjecture allows one change to the desired decorations.

5.2.1. **Theorem.** *The class $\mathcal{FJ}$ has the following properties*

- *(i) It contains hyperbolic groups and finite dimensional* Cat(0)*-groups;*
- *(ii) it contains virtually solvable groups;*
- *(iii) it contains (not necessarily cocompact) lattices in almost connected Lie groups;*
- *(iv) it contains S-arithmetic groups;*
- *(v) it is closed under passing to subgroups;*
- *(vi) it is closed under taking finite products, free products, and directed colimits;*
- *(vii) it is almost closed under extensions, more precisely, let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that for any cyclic subgroup $C \subseteq Q$ the group $p^{-1}(C)$ belongs to $\mathcal{FJ}$ and that the group Q belongs to $\mathcal{FJ}$. Then G belongs to $\mathcal{FJ}$;*
- *(viii) if H is a finite index subgroup of G, and H is in $\mathcal{FJ}$, then also G is in $\mathcal{FJ}$.*

For a more complete and detailed status of the Farrell–Jones conjectures we refer the reader to [RV17, BL10, BFL14, KLR16] and the references therein.

If one is willing to neglect 2-torsion, then the L-theoretic Farrell–Jones conjecture has further useful properties. For this we need to recall the following version known as the *fibered* Farrell–Jones conjecture, see [BLR08b, Section 2.1]. For a fixed group G its L-theoretic version after inverting 2 states that for any group homomorphism $\phi\colon H \to G$ the assembly map

$$\mathrm{L}R_*^H(E_{\phi^*(\mathrm{vc})}H)[\tfrac{1}{2}] \longrightarrow \mathrm{L}_*(RH)[\tfrac{1}{2}]$$

is an isomorphisms; here, for any family $\mathcal{F}$ of subgroups of G and a homomorphism $\phi\colon H \to G$, we denote by $\phi^*(\mathcal{F})$ the family of subgroups $K \subseteq H$, such that $\phi(H) \in \mathcal{F}$. As explained previously, after inverting 2, the L-theory spectra with different decorations become equivalent, see [LR05, Remark 1.22], therefore we do not need to consider the K-theoretic analogue. We will denote by $\mathcal{LFJ}_{\mathrm{vc}}^{\mathrm{fib}}[\frac{1}{2}]$ the class of groups that satisfy this conjecture. Since one can choose the homomorphism $\mathrm{id}\colon G \to G$, it contains the classical version of the assembly maps with 2 inverted.

The better closure properties of $\mathcal{LFJ}_{\mathrm{vc}}^{\mathrm{fib}}[\frac{1}{2}]$ as compared to $\mathcal{FJ}$ stem from the following lemma. For this let us denote by $\mathcal{LFJ}_{\mathrm{fin}}^{\mathrm{fib}}[\frac{1}{2}]$ the class of groups satisfying the variant of the fibered conjecture which takes into account the family of finite subgroups rather than the family of virtually cyclic subgroups.

5.2.2. **Lemma.** *We have that* $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}] = \mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{fin}}[\frac{1}{2}]$.

Proof. By [BLR08b, Theorem 2.4] this follows if we can show that every virtually cyclic group V is contained in $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{fin}}[\frac{1}{2}]$. From [FJ95, Lemma 2.5] it follows that V sits inside a short exact sequence

$$1 \longrightarrow F \longrightarrow V \longrightarrow S \longrightarrow 1$$

in which F is a finite group and S is either infinite cyclic or infinite dihedral, i.e. isomorphic to $D_\infty \cong \mathbb{Z}/2 * \mathbb{Z}/2$. By [BLR08b, Lemma 2.9] and the fact that finite groups are clearly contained in $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{fin}}[\frac{1}{2}]$, it hence suffices to prove that $\mathbb{Z}$ and D_∞ are contained in $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{fin}}[\frac{1}{2}]$.

To do so let $\phi\colon K \to S$ be a homomorphism with S either $\mathbb{Z}$ or D_∞. We need to show that K satisfies the L-theoretic Farrell–Jones conjecture after inverting 2 with respect to the family $\phi^*(\mathrm{fin})$. Factor ϕ as

$$K \xrightarrow{\psi} \mathrm{Im}(\phi) \xrightarrow{i} S$$

and observe that $\phi^*(\mathrm{fin}) = \psi^*(\mathrm{fin})$. For the image of ϕ – as for every subgroup of D_∞ – there are three possibilities: It is either finite, infinite cyclic or infinite dihedral. If the image of ϕ is finite, $\phi^*(\mathrm{fin})$ is the family of all subgroups and thus K clearly satisfies this isomorphism conjecture and otherwise we are reduced to considering a *surjection* $\phi\colon K \to S$. Notice that the space $E_{\mathrm{fin}}S$ acquires a K-action through the homomorphism ϕ and that $E_{\mathrm{fin}}S$ is a model for $E_{\phi^*(\mathrm{fin})}K$ when ϕ is surjective. Let K_0 denote the kernel of ϕ.

We will now proceed by studying the two cases separately. Let us first assume that $S = \mathbb{Z}$. As $\mathbb{Z}$ has no finite subgroups a model for the space $E_{\phi^*(\mathrm{fin})}K$ is given by $E\mathbb{Z}$ and as a K-CW complex is given by the pushout

$$\begin{array}{ccc} K/K_0 \times S^0 & \longrightarrow & K/K_0 \\ \downarrow & & \downarrow \\ K/K_0 \times D^1 & \longrightarrow & E_{\phi^*(\mathrm{fin})}K \end{array}$$

Consider then the diagram

$$\begin{array}{ccccccccc} \cdots \to & \mathrm{L}R^K_*(K/K_0) & \to & \mathrm{L}R^K_*(K/K_0) & \to & \mathrm{L}R^K_*(E_{\phi^*(\mathrm{fin})}K) & \to & \mathrm{L}R^K_{*-1}(K/K_0) & \to \cdots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \cdots \to & \mathrm{L}_*(RK_0)[\frac{1}{2}] & \to & \mathrm{L}_*(RK_0)[\frac{1}{2}] & \to & \mathrm{L}_*(RK)[\frac{1}{2}] & \to & \mathrm{L}_{*-1}(RK_0)[\frac{1}{2}] & \to \cdots \end{array}$$

where the upper horizontal sequence is the exact sequence induced by applying $\mathrm{L}R^K_*(-)$ to the above pushout, the lower horizontal sequence is the exact sequence of [Ran73, page 413] and the vertical arrows are given by the assembly map. By definition of the equivariant homology theory $\mathrm{L}R^K_*(-)$, the assembly maps involving only K/K_0 become isomorphisms after inverting 2 in the domain. We deduce that the map

$$\mathrm{L}R^K_*(E_{\phi^*(\mathrm{fin})}K)[\tfrac{1}{2}] \longrightarrow \mathrm{L}_*(RK)[\tfrac{1}{2}]$$

is an isomorphism from the 5-lemma.

To address the case $S = D_\infty$ note that a model for $E_{\mathrm{fin}}(D_\infty)$ is given by the pushout of D_∞-CW complexes

$$\begin{array}{ccc} D_\infty \times S^0 & \longrightarrow & D_\infty/C_1 \amalg D_\infty/C_2 \\ \downarrow & & \downarrow \\ D_\infty \times D^1 & \longrightarrow & E_{\mathrm{fin}}(D_\infty) \end{array}$$

where $C_1 = \mathbb{Z}/2 * \{e\}$ and $C_2 = \{e\} * \mathbb{Z}/2$ are the canonical subgroups. As explained above, it follows that a model for $E_{\phi^*(\mathrm{fin})}K$ is given by the pushout of K-CW complexes

$$\begin{array}{ccc} K/K_0 \times S^0 & \longrightarrow & K/K_1 \amalg K/K_2 \\ \big\downarrow & & \big\downarrow \\ K/K_0 \times D^1 & \longrightarrow & E_{\phi^*(\mathrm{fin})}K \end{array}$$

where $K_0 = \ker(\phi)$ and $K_i = \phi^{-1}(C_i)$ for $i = 1, 2$. We consider the diagram

$$\begin{array}{ccccccc} \cdots \mathrm{L}R^K_*(K/K_0) & \longrightarrow & \mathrm{L}R^K(K/K_1) \oplus \mathrm{L}R^K_*(K/K_2) & \longrightarrow & \mathrm{L}R^K_*(E_{\phi^*(\mathrm{fin})}K) & \longrightarrow & \mathrm{L}R^K_{*-1}(K/K_0) \cdots \\ \big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\ \cdots \mathrm{L}_*(RK_0)[\tfrac{1}{2}] & \longrightarrow & \mathrm{L}_*(RK_1)[\tfrac{1}{2}] \oplus \mathrm{L}_*(RK_2)[\tfrac{1}{2}] & \longrightarrow & \mathrm{L}_*(RK)[\tfrac{1}{2}] & \longrightarrow & \mathrm{L}_{*-1}(RK_0)[\tfrac{1}{2}] \cdots \end{array}$$

where again the upper horizontal sequence is the exact sequence induced by applying $\mathrm{L}R^K_*(-)$ to the above pushout, whereas the lower horizontal sequence is the exact sequence of [Cap74, Corollary 6]. The vertical maps are again the assembly maps and thus isomorphisms at the terms involving only homogeneous spaces in the source. The 5-lemma again finishes the proof. □

5.2.3. **Proposition.** *The class $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$ of groups G satisfying the fibered L-theoretic Farrell–Jones conjecture after inverting 2 has the following properties:*

(i) Farrell–Jones groups lie in $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$,
(ii) elementary amenable groups are contained in $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$,
(iii) it is closed under passage to subgroups, directed colimits, and amalgamated products, and
(iv) it is almost closed under extensions, more precisely, let $1 \to K \to G \to Q \to 1$ be an extension of groups. Suppose that for any finite subgroup $C \subseteq Q$ the group $p^{-1}(C)$ belongs to $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$ and that the group Q belongs to $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$. Then G belongs to $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$.

Note that, in particular, $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$ is closed under extensions of torsion-free groups, an observation we will make use of in the discussion following Proposition 7.1.5 and Proposition 7.1.9.

Proof. Statement (i) is proven in [BR07, Corollary 4.3]. In [BLR08b, Lemma 2.12] it is shown that elementary amenable groups are contained in $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{fin}}[\frac{1}{2}]$, even without inverting 2. By Lemma 5.2.2 we deduce (ii). Part (iii) is [BLR08b, Lemma 2.5 & Theorem 2.7] for subgroups and directed colimits. The closure property under amalgamated product follows from [Cap74, Corollary 6] using the fact that for a surjective group homomorphism $\phi\colon K \to G$ a decomposition $G = G_1 *_{G_0} G_2$ induces a decomposition $K = \phi^{-1}(G_1) *_{\phi^{-1}(G_0)} \phi^{-1}(G_2)$, similar to the argument in Lemma 5.2.2. It hence remains to prove part (iv). This follows immediately from [BLR08b, Lemma 1.9] using Lemma 5.2.2. □

5.2.4. **Proposition.** *Let M be an aspherical manifold of dimension at least 4. If $\pi_1(M)$ is contained in $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$, then the identity block Borel conjecture holds after inverting 2 for M.*

Proof. The difference between the decorations in L-theory disappears after inverting 2 (by [LR05, Remark 1.22]). From the identification of the surgery obstruction with the assembly map, as expounded in the proof of Proposition 5.1.1, one therefore obtains that the groups $\mathcal{S}^{\mathrm{Top}}_\partial(M \times \Delta^k)[\frac{1}{2}]$ vanish. For $k > 0$ this implies that $\pi_k(\mathcal{G}(M)/\widetilde{\mathrm{Top}}(M))[\frac{1}{2}] = 0$ (the fundamental group is abelian by [Ran92, Theorem 18.5]); note that no statement about the components can be deduced even if M is of dimension greater that 4. We conclude that $\iota\colon \mathrm{B}\widetilde{\mathrm{Top}}(M) \to \mathrm{B}\mathcal{G}(M)$ induces an isomorphism on higher homotopy groups after inverting 2; passing to universal covers then yields the claim. □

5.3. **Block homeomorphisms of aspherical manifolds.** The block Borel conjecture implies a strong computational result, namely a full understanding of the homotopy type of the space $\mathrm{B}\widetilde{\mathrm{Top}}(M)$, as we will see in Corollary 5.3.2. This result, together with the fact that the tautological classes are defined in $H^*(\mathrm{B}\widetilde{\mathrm{Top}}^+(M))$, as discussed in Definition 4.1.1, is key to our approach to understanding the tautological classes for aspherical manifolds.

To proceed, we record the following well-known proposition, compare [Got65, section III].

5.3.1. **Proposition.** *Let Γ be a group. Then there is a canonical fibre sequence*

$$\mathrm{B}^2 C(\Gamma) \longrightarrow \mathrm{B}\mathcal{G}(\mathrm{B}\Gamma) \longrightarrow \mathrm{BOut}(\Gamma)$$

where $C(\Gamma)$ denotes the centre of Γ and $\mathrm{Out}(\Gamma)$ denotes the group of outer automorphisms of Γ.

From this point onwards, we will let Γ be the fundamental group of an aspherical manifold M. We can draw the following corollary.

5.3.2. **Corollary.** *Let M be an aspherical manifold satisfying the block Borel conjecture. Then there is a fibre sequence*

$$\mathrm{B}^2 C(\Gamma) \longrightarrow \mathrm{B}\widetilde{\mathrm{Top}}(M) \longrightarrow \mathrm{BOut}(\Gamma).$$

Recall that we call an aspherical manifold M *centreless* if $C(\Gamma) = 0$. We immediately obtain:

5.3.3. **Corollary.** *Let M be an closed, connected, oriented, centreless, aspherical, manifold of dimension d, that satisfies the identity block Borel conjecture. Then*

$$0 = \kappa_c(M) \in H^{k-d}(\mathrm{B}\widetilde{\mathrm{Top}}_0(M); R)$$

for all $c \in H^k(\mathrm{BSTop} \times K(\mathbb{Z}, d); R)$ with $k \neq d$.

One large class of examples of centreless aspherical manifolds is given by those admitting a metric of negative sectional curvature. Just as in [BFJ16] one can strengthen our results for such manifolds. To this end recall, that the fundamental group of a negatively curved manifolds is *hyperbolic*. We thus collect two relevant features of hyperbolic groups.

5.3.4. **Proposition.** *A torsion-free hyperbolic group different from $\mathbb{Z}$ has trivial centre.*

This proposition is well-known, but as we had difficulties finding it in the literature, we also record a short proof.

Proof. Suppose that the centre of such a group Γ is non-trivial, and let $x \in C(\Gamma)$ be a non-trivial element. By [BH99, Corollary 3.10] we have that $\langle x \rangle$ has finite index in its centraliser, but since x is central its centraliser is the whole of Γ, and so Γ is virtually infinite cyclic. As explained earlier, it follows directly from [FJ95, Lemma 2.5] that a torsion-free virtually cyclic group is in fact infinite cyclic. □

Combining this with Gromov's theorem [Gro87, Theorem 5.4.A] that an aspherical manifold of dimension at least 3 with hyperbolic fundamental group Γ has $\mathrm{Out}(\Gamma)$ finite, we obtain:

5.3.5. **Corollary.** *Let M be a closed, oriented, aspherical manifold of dimension $d \geq 4$ with hyperbolic fundamental group. Then*

$$H^*\big(\mathrm{B}\widetilde{\mathrm{Top}}^+(M); \mathbb{Q}\big) = \mathbb{Q}.$$

In particular,

$$0 = \kappa_c(M) \in H^{k-d}(\mathrm{B}\widetilde{\mathrm{Top}}^+(M); \mathbb{Q})$$

for all $c \in H^k(\mathrm{BSTop} \times K(\mathbb{Z}, d); \mathbb{Q})$ with $k \neq d$.

5.3.6. *Remark.* The assumption on the fundamental group of M can of course be relaxed: the conclusion of the corollary holds whenever M is a centreless manifold whose fundamental group is a Farrell–Jones group that has rationally acyclic, e.g. finite, outer automorphism group. In a similar vein, one can ask for $\mathrm{Out}(\pi_1(M))$ to have finite rational cohomological dimension. In this case one still obtains that the tautological classes of non-zero degree are nilpotent in $H^*\big(\mathrm{B}\widetilde{\mathrm{Top}}^+(M); \mathbb{Q}\big)$, a claim we will make again later.

5.3.7. *Remark.* The stronger statement explained in Remark 5.3.6 recovers [BFJ16, Theorem F] where such a vanishing is proven for smooth bundles with fibre a non-positively curved centreless manifold whose fundamental group has finite outer automorphism group: fundamental groups of non-positively curved manifolds are Cat(0) and thus Farrell–Jones groups. Notice that the contents of Remark 5.3.6 for smooth bundles are also implied by [BFJ16, Corollary G.1].

6. Vanishing criteria for tautological classes of aspherical manifolds

In this section we shall introduce Burghelea's conjecture and mostly restrict to rational coefficients throughout. We will prove our main theorem from the introduction and discuss its integral refinement at the end of the section. For a group Γ, we denote by $C(\Gamma)$ its centre, and for an element $g \in \Gamma$, we denote by $C_\Gamma(g)$ its centraliser in Γ. Furthermore, $\mathrm{cd}_\mathbb{Q}$ denotes the rational cohomological dimension and $\mathrm{cd}_\mathbb{Q}^{\mathrm{tr}}$ denotes the rational cohomological dimension with trivial coefficients. We will say that a group Γ of type F is called an *oriented rational Poincaré duality group* [Bro82, VIII, Section 10] of formal dimension d, if $H^k(\Gamma;\mathbb{Q}\Gamma)$ is concentrated in degree d, and there is isomorphic to $\mathbb{Q}$ with trivial Γ action. In this case, cap product with a generator of this group yields an isomorphism

$$H^k(\Gamma; M) \longrightarrow H_{d-k}(\Gamma; M)$$

for any $\mathbb{Q}\Gamma$-module M.

6.1. Relating tautological classes to Burghelea's conjecture.

The basic ingredient into our study of tautological classes for not necessarily centreless aspherical manifolds is the following lemma.

6.1.1. **Lemma.** *For an aspherical manifold M with fundamental group Γ, the universal M-fibration over $\mathrm{B}\mathcal{G}_0(M)$ is given by*

$$\mathrm{B}\Gamma \longrightarrow \mathrm{B}(\Gamma/C(\Gamma)) \xrightarrow{\pi} \mathrm{B}^2 C(\Gamma),$$

where π classifies the central extension

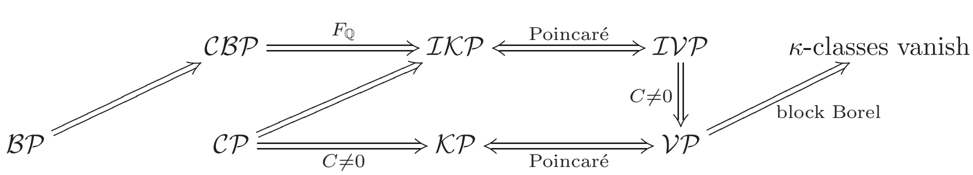
$$1 \longrightarrow C(\Gamma) \longrightarrow \Gamma \longrightarrow \Gamma/C(\Gamma) \longrightarrow 1.$$

If M satisfies the identity block Borel conjecture then the above also describes the underlying fibration of the universal M-block bundle over $\mathrm{B}\widetilde{\mathrm{Top}}_0(M)$. In order to show the vanishing of κ-classes it will therefore suffice to show that

$$\pi_!\colon H^k(\mathrm{B}(\Gamma/C(\Gamma)); R) \longrightarrow H^{k-d}(\mathrm{B}^2 C(\Gamma); R)$$

is the zero map, which is precisely what we will do in this section for $R = \mathbb{Q}$.

To this end we will introduce various finiteness conditions and already want to offer following diagram to sum up the various implications among them:



Here the fundamental group of an aspherical manifold lies in $\mathcal{CBP}$ if and only if it satisfies the central part of Burghelea's conjecture as stated in the introduction, thereby establishing our main theorem. The other terms are introduced throughout the section. To get started, let us axiomatise the conclusion we want to obtain.

6.1.2. **Definition.** Let $\mathcal{VP}$ (vanishing property) denote the class of oriented rational Poincaré duality groups of some dimension d for which the Gysin map

$$\pi_!\colon H^*(\mathrm{B}(\Gamma/C(\Gamma)); \mathbb{Q}) \longrightarrow H^{*-d}(\mathrm{B}^2 C(\Gamma); \mathbb{Q})$$

vanishes. Similarly, let $\mathcal{IVP}$ (individual vanishing property) consist of those oriented rational Poincaré duality groups for which

$$\rho_!\colon H^*(\mathrm{B}(\Gamma/\langle g\rangle);\mathbb{Q}) \longrightarrow H^{*-d}(\mathrm{B}^2\mathbb{Z};\mathbb{Q})$$

vanishes for each central $g \in \Gamma$ of infinite order individually; here $\rho\colon \mathrm{B}(\Gamma/\langle g\rangle) \longrightarrow \mathrm{B}^2\mathbb{Z}$ classifies the extension given by g.

Assuming the identity block Borel conjecture, $\pi_1(M)$ lying in $\mathcal{VP}$ implies the vanishing of *all* κ-classes in $H^*(\mathrm{B}\widetilde{\mathrm{Top}}_0(M);\mathbb{Q})$ for any oriented, aspherical manifold via Lemma 6.1.1. We mainly introduce the class $\mathcal{IVP}$ to connect our conjecture from the introduction to Burghelea's, see below. To start this off we have the following.

6.1.3. **Proposition.** *A group Γ in $\mathcal{IVP}$ lies in $\mathcal{VP}$ if and only if $C(\Gamma)\otimes\mathbb{Q} \neq 0$.*

Proof. If $C(\Gamma)\otimes\mathbb{Q} = 0$, the rational Gysin map is isomorphic to the Gysin map for the trivial fibration $\mathrm{B}\Gamma \to *$ which by Poincaré duality for Γ is non-zero in degree d and thus Γ does not lie in $\mathcal{VP}$.

To prove the converse observe that $H^*(\mathrm{B}^2C(\Gamma);\mathbb{Q})$ is the symmetric algebra on the finite dimensional graded vector space $\mathrm{Hom}(C(\Gamma),\mathbb{Q})[2]$: Its dimension equals the rational cohomological dimension of $C(\Gamma)$, which is bounded by that of its ambient group Γ. Now suppose that $x \in H^k(\mathrm{B}(\Gamma/C(\Gamma));\mathbb{Q})$ has $\pi_!(x) \neq 0 \in H^{k-d}(\mathrm{B}^2C(\Gamma);\mathbb{Q})$. Then we claim that there is an embedding $i\colon \mathbb{Z} \to C(\Gamma)$ such that $(\mathrm{B}^2 i)^*\pi_!(x) \neq 0 \in H^{k-d}(\mathrm{B}^2\mathbb{Z};\mathbb{Q})$. Assuming this claim for the moment, we consider the diagram

$$\begin{array}{ccc} \mathrm{B}(\Gamma/\mathbb{Z}) & \xrightarrow{\rho} & \mathrm{B}^2\mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{B}(\Gamma/i)} & & \downarrow{\scriptstyle \mathrm{B}^2 i} \\ \mathrm{B}(\Gamma/C(\Gamma)) & \xrightarrow{\pi} & \mathrm{B}^2C(\Gamma) \end{array}$$

which exhibits $\mathrm{B}(\Gamma/\mathbb{Z})$ as a homotopy pullback. Therefore the diagram

$$\begin{array}{ccc} H^*(\mathrm{B}(\Gamma/C(\Gamma));\mathbb{Q}) & \xrightarrow{\pi_!} & H^{*-d}(\mathrm{B}^2C(\Gamma);\mathbb{Q}) \\ \downarrow{\scriptstyle \mathrm{B}(\Gamma/i)^*} & & \downarrow{\scriptstyle (\mathrm{B}^2 i)^*} \\ H^*(\mathrm{B}(\Gamma/\mathbb{Z});\mathbb{Q}) & \xrightarrow{\rho_!} & H^{*-d}(\mathrm{B}^2\mathbb{Z};\mathbb{Q}) \end{array}$$

commutes, which implies $\rho_!(\mathrm{B}(\Gamma/i)^*(x)) = (\mathrm{B}^2 i)^*\pi_!(x) \neq 0$, a contradiction as $\rho_!$ is zero by assumption.

To prove the claim we consider a general non-zero element $y \neq 0 \in H^{2n}(\mathrm{B}^2C(\Gamma);\mathbb{Q}) = \mathrm{Sym}^n_{\mathbb{Q}}(\mathrm{Hom}(C(\Gamma),\mathbb{Q}))$. Such a y is a non-zero polynomial function on $C(\Gamma)\otimes\mathbb{Q}$ so since $C(\Gamma)\otimes\mathbb{Q}$ is non-zero, there must be some non-zero element $v \in C(\Gamma)\otimes\mathbb{Q}$ on which y does not vanish: as y is homogeneous it does not vanish on the entire line spanned by v except at the origin. Such a line contains the non-trivial image of an element $w \in C(\Gamma)$, and the homomorphism $i\colon \mathbb{Z} \to C(\Gamma)$ defined by w has the desired properties, since i^* precisely corresponds to restriction of functions to the line spanned by v. □

Let us now recall Burghelea's conjecture in full, see [Bur85]. We first state its conclusion in an axiomatic way, since the known cases go beyond Burghelea's original conjecture.

6.1.4. **Definition.** Let $\mathcal{BP}$ (Burghelea property) denote the class of groups Γ that satisfy the following: For any element $g \in \Gamma$ of infinite order we have that the limit of

$$\ldots \longrightarrow H_{*+4}(C_\Gamma(g)/\langle g\rangle;\mathbb{Q}) \xrightarrow{-\cap e} H_{*+2}(C_\Gamma(g)/\langle g\rangle;\mathbb{Q}) \xrightarrow{-\cap e} H_*(C_\Gamma(g)/\langle g\rangle;\mathbb{Q})$$

vanishes, where $e \in H^2(C_\Gamma(g)/\langle g\rangle;\mathbb{Q})$ is the Euler class of the central extension

$$1 \longrightarrow \mathbb{Z} \xrightarrow{g} C_\Gamma(g) \longrightarrow C_\Gamma(g)/\langle g\rangle \longrightarrow 1.$$

Let furthermore $\mathcal{CBP}$ (central Burghelea property) denote the class of groups where the same conclusion need only hold for central elements.

6.1.5. **Conjecture** (Burghelea). *Any group of type F is in $\mathcal{BP}$.*

Recall that a group is said to be of type F if there exists a model of its classifying space which is a finite complex, in particular the fundamental group of any aspherical manifold is of type F. We will review known results about Burghelea's conjecture in the final chapter. For now let it suffice to say, that it is known to be true for several classes of groups and that, while some groups are known to lie outside of $\mathcal{BP}$, none of them are of type F. In order to connect Burghelea's conjecture to ours we need yet another definition.

6.1.6. **Definition.** Let $\mathcal{KP}$ (kernel property) denote the class of groups Γ, such that $\mathrm{cd}_{\mathbb{Q}}(\Gamma) < \infty$ and the map

$$\pi^*\colon H^*(\mathrm{B}^2 C(\Gamma); \mathbb{Q}) \longrightarrow H^*(\mathrm{B}(\Gamma/C(\Gamma)); \mathbb{Q})$$

is not injective. Similarly, let $\mathcal{IKP}$ (individual kernel property) denote those Γ with $\mathrm{cd}_{\mathbb{Q}}(\Gamma) < \infty$, such that for each central $g \in \Gamma$ of infinite order the induced map

$$\rho^*\colon H^*(\mathrm{B}^2\mathbb{Z}; \mathbb{Q}) \longrightarrow H^*(\mathrm{B}(\Gamma/\langle g\rangle; \mathbb{Q})$$

is not injective.

6.1.7. *Remark.* The non-injectivity of the map in question for $\mathcal{IKP}$ is equivalent to the nilpotence of the Euler class of the extension

$$1 \longrightarrow \mathbb{Z} \xrightarrow{g} \Gamma \longrightarrow \Gamma/\langle g\rangle \longrightarrow 1,$$

which in turn is equivalent to the a priori stronger statement that $\mathrm{cd}^{\mathrm{tr}}_{\mathbb{Q}}(\Gamma/\langle g\rangle) < \infty$. This last equivalence follows from the Gysin sequence: It shows that multiplication with the Euler class is an isomorphism in degrees greater than the cohomological dimension of Γ. But as the Euler class is nilpotent, this can only happen if $\mathrm{cd}^{\mathrm{tr}}_{\mathbb{Q}}(\Gamma/\langle g\rangle) < \infty$. We conclude that Γ lies in $\mathcal{IKP}$ if and only if $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$ and $\mathrm{cd}^{\mathrm{tr}}_{\mathbb{Q}}(\Gamma/\langle g\rangle)$ are finite for each central $g \in \Gamma$ of infinite order.

6.1.8. **Proposition.** *Let $\Gamma \in \mathcal{CBP}$ be a group with $\mathrm{cd}_{\mathbb{Q}}(\Gamma) < \infty$, whose rational homology is of finite type. Then $\Gamma \in \mathcal{IKP}$.*

This proposition implies that for a type F group the central part of Burghelea's conjecture in the introduction is indeed equivalent to being in $\mathcal{CBP}$, thereby justifying the name.

Proof. Let $g \in \Gamma$ be central of infinite order. Since $\Gamma \in \mathcal{BP}$ we obtain that $\lim_{-\cap e} H_*(\Gamma/\langle g\rangle; \mathbb{Q}) = 0$. We compute

$$\begin{aligned} \big(\operatorname*{colim}_{-\cup e} H^*(\Gamma/\langle g\rangle; \mathbb{Q})\big)^* &\cong \lim_{-\cup e} \big(H^*(\Gamma/\langle g\rangle; \mathbb{Q})\big)^* \\ &\cong \lim_{-\cap e} H_*(\Gamma/\langle g\rangle; \mathbb{Q}) \\ &= 0 \end{aligned}$$

where the second isomorphism uses the finiteness assumption on $H_*(\Gamma; \mathbb{Q})$, which passes to $\Gamma/\langle g\rangle$ since the extension is central. Restricting to even degrees we find that

$$0 = \operatorname*{colim}_{-\cup e} H^{2*}(\Gamma/\langle g\rangle; \mathbb{Q}) \cong H^{2*}(\Gamma/\langle g\rangle; \mathbb{Q})\left[\tfrac{1}{e}\right]$$

which implies that $e \in H^2(\Gamma/\langle g\rangle; \mathbb{Q})$ is nilpotent. □

6.1.9. **Theorem.** *An oriented rational Poincaré duality group lies in $\mathcal{IVP}$ if and only if it lies in $\mathcal{IKP}$. The same statement holds for $\mathcal{VP}$ and $\mathcal{KP}$.*

6.1.10. **Corollary.** *Let M be an oriented aspherical manifold with fundamental group Γ. If Γ has non-trivial centre and satisfies both the Burghelea and identity block Borel conjecture then*

$$0 = \kappa_c(M) \in H^*(\mathrm{B}\widetilde{\mathrm{Top}}_0(M); \mathbb{Q})$$

for all $c \in H^*(\mathrm{BSTop} \times K(\mathbb{Z}, d); \mathbb{Q})$.

In fact the assumption may be weakened slightly. Instead of the identity block Borel conjecture it suffices to assume that the restriction $\mathrm{B}\iota_0 \colon \mathrm{B}\widetilde{\mathrm{Top}}_0(M) \longrightarrow \mathrm{B}\mathcal{G}_0(M)$ is a rational equivalence: The Serre spectral sequence then implies that the comparison map on total spaces of the universal block bundle and fibration, respectively, induces an isomorphism in rational cohomology. Therefore the κ-classes may in this case be lifted to $H^*(\mathrm{B}\mathcal{G}_0(M);\mathbb{Q})$ where they vanish by Theorem 6.1.9.

We saw in Proposition 5.2.4 and Proposition 5.2.3 that this weaker assumption is known for some more groups than the identity block Borel conjecture.

Proof of Theorem 6.1.9. Let C denote either $\langle g\rangle$ for g a central element of infinite order or the entire centre of Γ which is forced to be non-trivial by both $\Gamma \in \mathcal{KP}$ and $\Gamma \in \mathcal{VP}$.

The if direction admits the following simple proof. Let $0 \neq x \in H^*(\mathrm{B}^2C;\mathbb{Q})$ and $y \in H^*(\mathrm{B}(\Gamma/C);\mathbb{Q})$ such that $\pi^*(x) = 0$. Then we have

$$0 = \pi_!(\pi^*(x) \cup y) = x \cup \pi_!(y) \in H^*(\mathrm{B}^2C;\mathbb{Q}).$$

Since this ring is a domain it follows that $\pi_!(y) = 0$ and since y is arbitrary we have $\pi_! = 0$ as required.

The proof of the converse will need a bit of preparation and will in fact provide another proof of the implication just established. Consider the Serre spectral sequence for the fibration

$$\mathrm{B}\Gamma \longrightarrow \mathrm{B}(\Gamma/C) \xrightarrow{\pi} \mathrm{B}^2C$$

for C either $\langle g\rangle$ or the entire centre of Γ. On the one hand the Gysin homomorphism $\pi_!$ being trivial is equivalent by definition to the vanishing of the top row $SS_\infty^{*,d}$, where d is the degree of duality for Γ, so that $SS_\infty^{*,d}$ is a subgroup of $SS_2^{*,d}$. The projection having a kernel on the other hand is equivalent to the edge homomorphism $SS_2^{*,0} \to SS_\infty^{*,0}$ having a kernel. We will show that the two types of relevant differentials are dual to one another. We encourage the reader to draw the situation for $C = \langle g\rangle$.

Observe now that for arbitrary $y \in H^n(\mathrm{B}^2C;\mathbb{Q})$, $D \in H_m(\mathrm{B}^2C;\mathbb{Q})$ and $z \in SS_{i-1}^{m-n+i,i-1}$, such that $\mu_\Gamma \times y$ is an $(i-1)$-cycle, where $[\Gamma]$ and μ_Γ denote the homological and cohomological fundamental classes of Γ, respectively, we can calculate:

$$\begin{aligned}
\langle d_i(\mu_\Gamma \times y) \cap ([\Gamma] \times D), z\rangle_i &= \langle d_i((\mu_\Gamma \times y) \cap ([\Gamma] \times D)), z\rangle_i \\
&= \langle (\mu_\Gamma \times y) \cap ([\Gamma] \times D), d^i(z)\rangle_i \\
&= \langle (\mu_\Gamma \times y) \cap ([\Gamma] \times D), 1 \times x\rangle_2 \\
&= \langle 1 \times (y \cap D), 1 \times x\rangle_2 \\
&= \pm\langle y \cap D, x\rangle \\
&= \pm\langle D, y \cup x\rangle
\end{aligned}$$

where $d^i(z) = 1 \times x$ for some $x \in H^{m-n}(\mathrm{B}^2C;\mathbb{Q})$ and $\langle -,-\rangle_j$ denotes the Kronecker pairing of the homology and cohomology spectral sequences on page j; note that $[\Gamma] \times D$ is a permanent cycle for degree reasons, so all manipulations are indeed valid.

Suppose now for the one implication that $d^i(\mu_\Gamma \times y) = 0$ for some y and all i, i.e. y is in the image of the Gysin homomorphism. Then for all D and z we find $\langle D, y \cup x\rangle = 0$. Since D was arbitrary we find $y \cup x = 0$ for all z. Since $H^*(\mathrm{B}^2C;\mathbb{Q})$ is a domain, this means either $y = 0$ or $d^i(z) = 0$ for all z, which clearly gives the desired statement. Conversely, suppose that $d_i(z) = 0$ for all z as above. Then we find

$$d^i(\mu_\Gamma \times y) \cap ([\Gamma] \times D) = 0$$

for all i. Since D was arbitrary, Poincaré duality implies that already

$$d^i(\mu_\Gamma \times y) = 0$$

for all i, and therefore the Gysin homomorphism is even surjective. □

Let us discuss one more condition, which is in spirit similar to the strengthening of Burghelea's conjecture we gave right after Remark 6.1.7.

6.1.11. **Definition.** Let $\mathcal{CP}$ denote the class of groups Γ with $\mathrm{cd}_{\mathbb{Q}}(\Gamma) < \infty$ and $\mathrm{cd}^{\mathrm{tr}}_{\mathbb{Q}}(\Gamma/C(\Gamma)) < \infty$.

We do not know of an aspherical manifold whose fundamental group is not contained in $\mathcal{CP}$ and therefore pose as a question in Section 7.3 whether this is generally the case. One might in fact guess that $\mathrm{B}(\Gamma/C(\Gamma))$ is a Poincaré complex for the fundamental group Γ of an aspherical manifold. While this is true whenever $\Gamma/C(\Gamma)$ is of type F, in the next section we will give a counterexample to show that it need not have type F in general.

6.2. **Integral results.** Many of the above results are true integrally under an additional assumption: that $C(\Gamma)$ is finitely generated. Whether this holds for any Γ the fundamental group of an aspherical manifold seems to be an open problem, which we pose as a question in Section 7.3 as well. For a group Γ of type F_∞ the condition $\Gamma \in \mathcal{IKP}$ in fact implies that the integral map

$$\pi^*\colon H^*(\mathrm{B}^2\mathbb{Z};\mathbb{Z}) \xrightarrow{\rho(g)^*} H^*(\mathrm{B}(\Gamma/\langle g\rangle);\mathbb{Z})$$

has non-trivial kernel: For both sides the rational cohomology is the rationalisation of the integral cohomology, since the all homology groups are finitely generated: For the left hand side this is immediate and for the right hand side it can be obtained from the Serre spectral sequence. Then $\Gamma \in \mathcal{IKP}$ immediately implies the existence of an element $0 \neq x \in H^*(\mathrm{B}^2\mathbb{Z};\mathbb{Z})$ such that $\pi^*(x)$ is torsion in $H^*(\mathrm{B}(\Gamma/\langle g\rangle);\mathbb{Z})$, whence an appropriate multiple of x gives a non-zero element in the kernel of π^* since $H^*(\mathrm{B}^2\mathbb{Z};\mathbb{Z})$ is torsion-free.

Now the if direction of Theorem 6.1.9 remains valid integrally, more specifically, a Poincaré duality group Γ that lies in $\mathcal{IKP}$ also lies in the obvious integral version of $\mathcal{IVP}$. The argument really only used that $H^*(\mathrm{B}^2\mathbb{Z})$ is a domain, which holds for both $\mathbb{Z}$ and $\mathbb{Q}$ coefficients. So far no assumption on the centre of Γ entered. The proof that Poincaré duality groups in the integral version of $\mathcal{IVP}$ have vanishing integral Gysin map, however, uses that an element in $H^*(\mathrm{B}^2C(\Gamma))$, which vanishes under the restriction to $H^*(\mathrm{B}^2\mathbb{Z})$ for all embeddings $\mathbb{Z} \to C(\Gamma)$, has to be trivial. While true rationally for an arbitrary abelian group of finite rank (which the centre of a rational Poincaré duality group always is), the same is not clear integrally unless $C(\Gamma)$ is finitely generated. For example, outside degree zero, $H^*(\mathrm{B}^2\mathbb{Q};\mathbb{Z})$ is concentrated in odd degrees in each of which it is $\mathrm{Ext}(\mathbb{Q},\mathbb{Z})$. A similar issue arises in the comparison between the rational and integral versions of $\mathcal{KP}$, where finite generation again saves the day. We obtain:

6.2.1. **Theorem.** *If M is an oriented aspherical manifold, which satisfies the Burghelea conjecture and the identity block Borel conjecture and whose centre is non-trivial and finitely generated, then*

$$0 = \kappa_c(M) \in H^*(\mathrm{B}\widetilde{\mathrm{Top}}_0(M);\mathbb{Z})$$

for all $c \in H^*(\mathrm{BSTop} \times K(\mathbb{Z}, d);\mathbb{Z})$.

6.2.2. *Remark.* Again the integral vanishing of all tautological classes for smooth bundles is not implied by this result, as not every Pontryagin class lies in the image of the forgetful map $H^*(\mathrm{BSTop};\mathbb{Z}) \to H^*(\mathrm{BSO};\mathbb{Z})$. Just as in the discussion after Proposition 4.2.1 one does, however, obtain uniform bounds on their order.

6.3. **A useful lemma.** Finally we have the following convenient criterion for being in $\mathcal{KP}$:

6.3.1. **Lemma.** *Let Γ be a group with $\mathrm{cd}_{\mathbb{Q}}(\Gamma) < \infty$. If the map $C(\Gamma) \to \Gamma^{\mathrm{ab}} \otimes \mathbb{Q}$ is non-trivial, then*

$$\pi^*\colon H^2(\mathrm{B}^2C(\Gamma);\mathbb{Q}) \longrightarrow H^2(\mathrm{B}(\Gamma/C(\Gamma));\mathbb{Q})$$

is not injective, in particular $\Gamma \in \mathcal{KP}$.

Proof. We consider the Serre spectral sequence for the fibration

$$\mathrm{B}C(\Gamma) \longrightarrow \mathrm{B}\Gamma \xrightarrow{q} \mathrm{B}(\Gamma/C(\Gamma)).$$

Let r be the rank of $C(\Gamma)$ which is bounded by $\mathrm{cd}_{\mathbb{Q}}(\Gamma)$. Clearly the image of π^* is contained in $\ker(q^*)$, which therefore by assumption contains an r-dimensional subspace. On the other hand $\ker(q^*)$ is the image of the differential

$$d_2\colon H^1(\mathrm{B}C(\Gamma);\mathbb{Q}) \longrightarrow H^2(\mathrm{B}(\Gamma/C(\Gamma));\mathbb{Q}).$$

Since $\dim_{\mathbb{Q}}(H^1(\mathrm{B}C(\Gamma);\mathbb{Q})) = r$ it follows that this differential is an isomorphism. Therefore $H^1(\mathrm{B}\Gamma;\mathbb{Q}) \to H^1(\mathrm{B}(C(\Gamma));\mathbb{Q})$ is the zero map and the lemma follows by dualising. ☐

7. Examples and counterexamples

In this section we will discuss explicit examples of manifolds whose tautological classes vanish and in particular satisfy our conjecture. We also provide counterexamples to a few possible extensions. For the reader's convenience let us first recall our conjecture.

Conjecture. *Let M be a closed, connected, oriented, aspherical manifold. If $C(\pi_1(M)) \neq 0$ then*

$$0 = \kappa_c \in H^*(\mathrm{B}\widetilde{\mathrm{Top}}_0(M); R)$$

for all $c \in H^*(\mathrm{BSTop} \times K(\mathbb{Z}, d); R)$.

7.1. **Examples.** We want to start out with some rather abstract examples that satisfy our conjecture with $R = \mathbb{Q}$.

7.1.1. **Theorem.** *Let M be an oriented aspherical manifold with fundamental group Γ. If Γ belongs to one of the following classes of groups, then Γ belongs to $\mathcal{IKP}$ and our conjecture holds for M rationally.*

(i) Cocompact lattices in almost connected Lie groups,
(ii) hyperbolic and Cat(0)*-groups,*
(iii) solvable groups, linear groups over $\mathbb{Q}$,
(iv) groups of polynomial growth and arithmetic groups, and
(v) elementary amenable groups.

Proof. If the dimension of M is smaller than 4, then as explained Remark 5.1.3 the vanishing of rational tautological classes is known anyhow. We claim that all groups in the above list are Farrell–Jones groups (except elementary amenable ones but those are covered by Proposition 5.2.3), hence any such M satisfies the identity block Borel conjecture (at least with 2 inverted) provided the dimension of M is at least 4. The groups not covered by Theorem 5.2.1 are groups of polynomial growth which are virtually solvable and hence are Farrell–Jones groups, and linear groups over $\mathbb{Q}$ which are Farrell–Jones groups by [Rüp16]. Hence it suffices to verify that all above groups satisfy Burghelea's conjecture. This has been done in the following references: (i) is dealt with in [EM16, Theorem 4.27], for (ii) see [EM16, Corollary 4.8 and Corollary 4.9], (iii) is [Eck86, Theorem 2.4], (iv) is [Ji95, Theorem 4.3] and (v) is [EM16, Theorem 4.20]. ☐

Actually, in the case of Cat(0)-groups we do have an integral result.

7.1.2. **Proposition.** *If M is an orientable aspherical manifold of dimension at least 4 whose fundamental group is* Cat(0)*, then our conjecture holds for M with $R = \mathbb{Z}$.*

Proof. Cat(0)-groups are semihyperbolic, see [BH99, Corollary 4.8] and hence have finitely generated centre, see [BH99, Proposition 4.15; (3)]. Thus Theorem 6.2.1 applies. ☐

7.1.3. *Remark.* Even though we consider Burghelea's conjecture the bottleneck of our work, rather than the Farrell–Jones conjectures, there do exist groups for which Burghelea's conjecture is known and the Farrell–Jones conjectures are not, e.g. linear groups over arbitrary fields of characteristic 0 [Eck86, Theorem 2.4].

Concretely, we obtain the following consequences.

7.1.4. **Corollary.** *Our conjecture holds*

(i) rationally for oriented aspherical manifolds of the form $\Gamma\backslash G/K$, where G is a connected Lie group, K is a maximal compact subgroup and Γ is a cocompact lattice in G, and
(ii) integrally for oriented aspherical manifolds admitting a metric of non-positive sectional curvature, in particular tori.

The rational case for smooth torus bundles is also covered in [BFJ16, Corollary D.1] and the vanishing even holds in all of $H^*(\mathrm{BDiff}^+(T^n);\mathbb{Q})$.

Proof. (i) is a direct consequence of Theorem 7.1.1. (ii) is a special case of Proposition 7.1.2 and thus holds even integrally, because the fundamental group of such manifolds are Cat(0). We are not aware of results about the finite generation of the centre of cocompact lattices in Lie groups, hence (i) is only a rational result. □

As indicated all the groups appearing in Theorem 7.1.1 are contained in $\mathcal{IKP}$. In addition to these examples we have the following result.

7.1.5. **Proposition.** *The class $\mathcal{IKP}$ has the following properties.*

(i) It contains centreless groups of finite rational cohomological dimension, finitely generated abelian groups and finite groups,
(ii) it is closed under extensions, and
(iii) if a group Γ has a finite index subgroup K which lies in $\mathcal{IKP}$, then Γ lies in $\mathcal{IKP}$ as well.

This purely group theoretic statement has the following geometric interpretation: Together with Proposition 5.2.3 and Proposition 5.2.4 item (ii) verifies our conjecture rationally for total spaces of fibre bundles provided the fundamental groups of both base and fibre are in both $\mathcal{IKP}$ and $\mathcal{LFJ}^{\mathrm{fib}}_{\mathrm{vc}}[\frac{1}{2}]$. So, for instance, by our previously established results, our conjecture holds for an arbitrary torus bundle over a non-positively curved manifold, or vice versa, and iterates of those. Similarly, part (iii) enables passage from the total space of a finite cover to the base.

Proof of Proposition 7.1.5. For (i) simply observe that both classes of groups are contained in $\mathcal{CP}$.

For (ii), we consider a short exact sequence of groups

$$1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{q} Q \longrightarrow 1$$

and assume that K and Q are contained in the class $\mathcal{IKP}$. Let $g \in \Gamma$ be a central element of infinite order. We need to show that the map $\mathrm{B}(\Gamma/\langle g\rangle) \xrightarrow{\rho(g)} \mathrm{B}^2\mathbb{Z}$ classifying the central extension

$$1 \longrightarrow \langle g\rangle \longrightarrow \Gamma \longrightarrow \Gamma/\langle g\rangle \longrightarrow 1$$

has a non-trivial kernel in rational cohomology. We distinguish two cases, namely whether or not the central element $q(g) \in Q$ has infinite order. We begin with the case where $q(g)$ is of infinite order in Q. If this is the case we have a commutative diagram

$$\begin{array}{ccc} \mathrm{B}(\Gamma/\langle g\rangle) & \longrightarrow & \mathrm{B}^2\mathbb{Z} \\ \downarrow & & \| \\ \mathrm{B}(Q/\langle q(g)\rangle) & \longrightarrow & \mathrm{B}^2\mathbb{Z}. \end{array}$$

Since Q lies in the class $\mathcal{IKP}$ it follows that the lower horizontal map has a non-trivial kernel in rational cohomology. By the commutativity of the diagram the same follows for the upper horizontal map.

Now let us assume that $q(g)$ is of finite order, say n, and observe that this makes g^n a central element of K. Consider the short exact sequence of groups

$$1 \longrightarrow K/\langle g^n\rangle \longrightarrow \Gamma/\langle g^n\rangle \longrightarrow \Gamma/K \longrightarrow 1.$$

Since $K \in \mathcal{IKP}$ we obtain an element

$$0 \neq x \in \ker(\rho(g^n)^*\colon H^*(\mathrm{B}^2\mathbb{Z};\mathbb{Q}) \longrightarrow H^*(\mathrm{B}(K/\langle g^n\rangle);\mathbb{Q})).$$

Since $\Gamma/K \cong Q$ the group Γ/K has finite rational cohomological dimension, say d. Thus from the Serre spectral sequence we deduce that

$$0 \neq x^{d+1} \in \ker(\rho(g^n)^*\colon H^*(\mathrm{B}^2\mathbb{Z};\mathbb{Q}) \longrightarrow H^*(\mathrm{B}(\Gamma/\langle g^n\rangle);\mathbb{Q})).$$

To finish the proof of (ii) it hence suffices to verify the following

Claim. *If the map $\rho(g^n)$ has a non-trivial kernel on rational cohomology for some non-zero integer n, then so does the map $\rho(g)$.*

Proof of Claim. We consider the sequence of subgroups $\langle g^n \rangle \subseteq \langle g \rangle \subseteq \Gamma$ and conclude that the map

$$\mathrm{B}(\Gamma/\langle g^n \rangle) \longrightarrow \mathrm{B}(\Gamma/\langle g \rangle)$$

is a rational equivalence as its fibre $\mathrm{B}(\mathbb{Z}/n)$ is rationally contractible. From the commutative diagram

$$\begin{array}{ccc} \mathrm{B}(\Gamma/\langle g^n \rangle) & \longrightarrow & \mathrm{B}^2\mathbb{Z} \\ \downarrow & & \downarrow {\scriptstyle \cdot n} \\ \mathrm{B}(\Gamma/\langle g \rangle) & \longrightarrow & \mathrm{B}^2\mathbb{Z} \end{array}$$

and the fact that the right vertical map induces an isomorphism in rational cohomology we conclude the claim. □

To obtain (iii), let $K \subseteq \Gamma$ be a finite index subgroup with $K \in \mathcal{IKP}$ and $g \in \Gamma$ a central element of infinite order. Since K has finite index in Γ it follows that there exists an n such that g^n lies in K, and thus is a central element of infinite order in K. Therefore by assumption the composite

$$\rho(g^n)^*\colon H^*(\mathrm{B}^2\mathbb{Z};\mathbb{Q}) \longrightarrow H^*(\mathrm{B}(\Gamma/\langle g^n \rangle;\mathbb{Q}) \longrightarrow H^*(\mathrm{B}(K/\langle g^n \rangle;\mathbb{Q})$$

has non-trivial kernel. Now, the map $\mathrm{B}K \to \mathrm{B}\Gamma$ has finite, discrete homotopy fibres, so the second map in the composite is injective. The claim proven earlier now gives the desired conclusion. □

7.1.6. *Remark.* Let M be an aspherical manifold of dimension at least 5 satisfying the block Borel conjecture and Γ be its fundamental group. Suppose the conjecture is true rationally for M and that $\mathrm{Out}(\Gamma)$ has finite rational cohomological dimension. Then it follows easily from the Serre spectral sequence for the fibration

$$\mathrm{B}\widetilde{\mathrm{Top}}_0(M) \longrightarrow \mathrm{B}\widetilde{\mathrm{Top}}(M) \longrightarrow \mathrm{BOut}(\Gamma)$$

that the tautological classes in the rational cohomology of $\mathrm{B}\widetilde{\mathrm{Top}}(M)$ are nilpotent.

This is for instance the case if the fundamental group of M is nilpotent: Since nilpotent groups are solvable, we know that our conjecture is satisfied. Moreover, a finitely generated nilpotent group is polycyclic, hence by [BG06, Theorem 1.1] its outer automorphism group is arithmetic and thus has finite rational cohomological dimension, see [Bor75].

We provide two more examples of manifolds satisfying our conjecture using Lemma 6.3.1, i.e. $\mathcal{KP}$ instead of Burghelea and $\mathcal{IKP}$.

7.1.7. **Proposition.** *Let N be an oriented aspherical manifold and assume that $\pi_1(N)$ is a Farrell–Jones group. Let $e \in H^2(N;\mathbb{Z})$ be a torsion class, and let M be the total space of the principal S^1-bundle classified by e. Then our conjecture holds rationally for M.*

Proof. We first prove that M satisfies the block Borel conjecture. For this we observe that M is finitely covered by the trivial S^1-bundle over N, as e is a torsion class. Thus $\pi_1(M)$ contains $\pi_1(N) \times \mathbb{Z}$ as a finite index subgroup and hence is a Farrell–Jones group. To prove the proposition we consider the Serre spectral sequence for the fibration

$$S^1 \longrightarrow M \longrightarrow N.$$

By inspection, the map $H_1(S^1;\mathbb{Q}) \to H_1(M;\mathbb{Q})$ is non-zero, since e is rationally zero. We deduce the proposition from Lemma 6.3.1 using the the centrality of the inclusion $\pi_1(S^1) \to \pi_1(M)$. □

7.1.8. **Corollary.** *Let N be an closed, oriented, aspherical manifold such that $\pi_1(N)$ is a Farrell–Jones group. Then our conjecture holds rationally for $M = N \times S^1$.*

Let us close this section by considering mapping tori. Let $\varphi\colon M \to M$ be a orientation preserving homeomorphism of an oriented aspherical manifold M. Then φ determines an outer automorphism φ_* of $\pi_1(M)$. Picking a representing automorphism $\hat{\varphi}\colon \pi_1(M) \to \pi_1(M)$, there is an isomorphism between the fundamental group of the mapping torus M_φ, which is again oriented and aspherical, and $\pi_1(M) \rtimes_{\hat{\varphi}} \mathbb{Z}$. The centre of such a semi-direct product is readily computed to be

$$\{(g, n) \mid \hat{\varphi}^n = c_{g^{-1}}, \hat{\varphi}(g) = g\}$$

and so using Lemma 6.3.1 we obtain:

7.1.9. **Proposition.** *Let $\varphi\colon M \to M$ be an automorphism of an oriented, aspherical manifold M, such that $\pi_1(M)$ is a Farrell–Jones group. Assume that the induced automorphism on $\pi_1(M)$ admits a representative $\hat{\varphi}$ such that $\hat{\varphi}^n$ is conjugation by an element of $\pi_1(M)$ fixed by $\hat{\varphi}$ for some $n > 0$. Then our conjecture holds rationally for M_φ.*

To complete the proof note that the rational version of the identity block Borel conjecture holds for $\pi_1(M_\varphi) \cong \pi_1(M) \rtimes_{\hat{\varphi}} \mathbb{Z}$ by Proposition 5.2.3 and Proposition 5.2.4. The assumption is in particular satisfied for a finite order automorphism with a fixed point, e.g. the identity, thereby giving another proof that $S^1 \times M$ satisfies our conjecture rationally, whenever $\pi_1(M)$ is a Farrell–Jones group. We leave it as an exercise to the reader to check that the condition given on φ is actually independent of the representative $\hat{\varphi}$ chosen.

7.2. **Counterexamples.** Finally, we want to give examples that show the statement of our conjecture cannot generally be sharpened by all that much. To this end recall Madsen–Weiss' solution of Mumford's conjecture, stating that for Σ_g an oriented surface of genus g, the map

$$\kappa_-(\Sigma_g)\colon \mathrm{Sym}^*(H^{*>2}(\mathrm{BSO}(2);\mathbb{Q})[-2]) \longrightarrow H^*(\mathrm{BDiff}^+(\Sigma_g);\mathbb{Q})$$

is an isomorphism for $* \leq \frac{2g-2}{3}$.

Taking product bundles of high genus surface bundles therefore produces manifold bundles with aspherical fibre of arbitrarily high dimension for which tautological classes are rationally non-zero in arbitrarily high cohomological degree. This shows that the passage from $\mathrm{B}\widetilde{\mathrm{Top}}^+(M)$ to $\mathrm{B}\widetilde{\mathrm{Top}}_0(M)$ in the formulation of the conjecture cannot be avoided entirely.

Next, let us discuss the tautological classes associated to $p \in H^*(\mathrm{BSO};\mathbb{Q})$ with $|p| = \dim(M)$. These are classes

$$\kappa_p(M) \in H^0(\mathrm{B}\widetilde{\mathrm{Top}}(M);\mathbb{Q}) \cong \mathbb{Q}.$$

By restricting to a single point in $\mathrm{B}\widetilde{\mathrm{Top}}(M)$ we see that these classes coincide with the characteristic numbers of M, more specifically

$$\kappa_p(M) = \langle p(TM), [M]\rangle.$$

But by a result of [Ont14], any smooth oriented bordism class is represented by a manifold with negative sectional curvature. All manifolds constructed this way are centreless, a property generally predicted for non-nullbordant manifolds by our conjecture. This example shows that the restriction $k \neq d$ cannot be removed from our conjecture for centreless manifolds.

As stated at the end of Section 6.1 one might be tempted to hope that for any aspherical manifold M, the space $\mathrm{B}(\Gamma/C(\Gamma))$ is a Poincaré complex. Indeed, this is true when $\Gamma/C(\Gamma)$ is of type F by a 3-for-2 property for Poincaré spaces for fibrations of *finite* complexes. But it turns out that the group $\Gamma/C(\Gamma)$ is not even torsion-free in general: For example for M one of the manifolds constructed in [CWY13] as counterexamples to a conjecture about free S^1-actions on aspherical manifolds with non-trivial centre, $\Gamma/C(\Gamma)$ contains a non-trivial element of order 2, and thus cannot even admit a finite dimensional model of its classifying space. While we do not know whether the fundamental groups of these manifold are Farrell–Jones groups, we still have:

7.2.1. **Proposition.** *The aspherical manifolds constructed by Cappell–Weinberger–Yu just mentioned have fundamental groups in $\mathcal{IKP}$.*

Proof. There exist 2-fold covers of these manifolds which are of the form $S^1 \times V$, where V is an aspherical manifold with centreless fundamental group, see [CWY13], essentially by construction. We deduce that $\pi_1(S^1 \times V)$ lies in the class $\mathcal{IKP}$. Proposition 7.1.5 part (iii) yields the claim. □

7.3. **Several open questions.** We want to finish with some open problems that would be interesting to address. The first question seems to be a known open problem:

1. *Question.* Let M be a closed, connected, aspherical manifold. Is the centre of its fundamental group finitely generated?

Naturally, this is obvious for centreless, e.g. hyperbolic, groups, but it is also true for torsion-free nilpotent groups, which (unless trivial themselves) always have non-trivial centre, see [Neo15, Proposition 6.19]. Furthermore, as mentioned earlier, Cat(0)-groups have finitely generated centre.

2. *Question.* Are fundamental groups of aspherical manifolds always contained in $\mathcal{CP}$?

As mentioned earlier, this is true for several classes of groups, and would yield a general proof of the rational part of our conjecture that is independent of results on Burghelea's conjecture.

A positive answer the next question would provide a geometric reason for $\Gamma \in \mathcal{IKP}$.

3. *Question.* Let M be an oriented, aspherical manifold with fundamental group Γ and let $g \in \Gamma$ be a central element. Is there a finite cover $N \to M$ such that $g \in \pi_1(N)$ and such that g is realised by a principal S^1-action on N?

Note that the passage to a finite cover really is necessary. The examples from [CWY13] are aspherical manifolds M such that the quotient $\Gamma/C(\Gamma)$ contains an element of order 2.

4. *Question.* Do the tautological classes of a smooth, oriented aspherical manifold M vanish in the cohomology of $\mathrm{BDiff}_0(M)$ with arbitrary coefficients?

5. *Question.* Do the tautological classes vanish in the cohomology of $\widetilde{\mathrm{BTop}}^+(M)$ if M is oriented, aspherical, and either odd dimensional or its fundamental group has non-trivial centre?

We expect the answer to both parts of the question to be no, but did not find a counterexample.

7.3.1. *Remark.* To produce a counterexample one cannot take an even dimensional bundle, e.g. a bundle with fibre a product of surfaces as discussed above or even a point (which has nontrivial κ_1), and cross it with another bundle whose fibres are odd dimensional tori: as mentioned after Corollary 7.1.4 it is shown in [BFJ16, Theorem D], that tautological classes of torus bundles vanish even in $H^*(\mathrm{BDiff}^+(T^n);\mathbb{Q})$. In fact, for a smooth torus bundle over a smooth manifold, the authors even show that the Pontryagin classes of the vertical tangent bundle vanish in the cohomology of the total space. By contrast, since $\mathrm{cd}_{\mathbb{Q}}(\mathrm{Out}(\pi_1(T^n))) = \frac{n(n-1)}{2}$ our methods imply only the nilpotence of the tautological classes.

6. *Question.* Suppose M is an oriented aspherical manifold whose fundamental group has a non-trivial centre. Is M null bordant?

If M satisfies our conjecture rationally, then it follows that M is a torsion element in the bordism ring. Notice that even if M satisfies the integral version of our conjecture we cannot yet deduce that M is null bordant. Really, we are asking about Stiefel–Whitney numbers of aspherical manifolds and how their triviality relies on the non-triviality of the centre of its fundamental group.

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