L^2-EULER CHARACTERISTICS AND THE THURSTON NORM

STEFAN FRIEDL AND WOLFGANG LÜCK

Abstract. We assign to a finite CW-complex and an element in its first cohomology group a twisted version of the L^2-Euler characteristic and study its main properties. In the case of an irreducible orientable 3-manifold with empty or toroidal boundary and infinite fundamental group we identify it with the Thurston norm. We will use the L^2-Euler characteristic to address the problem whether the existence of a map inducing an epimorphism on fundamental groups implies an inequality of the Thurston norms.

0. Introduction

0.1. **The (μ, ϕ)-L^2-Euler characteristic.** Let X be a finite connected CW-complex and let $\mu\colon \pi_1(X) \to G$ and $\phi\colon G \to \mathbb{Z}$ be group homomorphisms. We say that (μ, ϕ) is an *L^2-acyclic Atiyah pair* if the nth L^2-Betti number $b_n^{(2)}(\overline{X}; \mathcal{N}(G))$ of the G-covering $\overline{X} \to X$ associated to μ vanishes for all $n \geq 0$, and G is torsion-free and satisfies the Atiyah Conjecture. (We will discuss the Atiyah Conjecture in Section 3). Then one can define by twisting the cellular $\mathbb{Z}G$-chain complex with the infinite dimensional G-representation $\phi^*\mathbb{R}\mathbb{Z}$ the *(μ, ϕ)-L^2-Euler characteristic* $\chi^{(2)}(X; \mu, \phi)$ which is an integer. We will not give the precise definition of (μ, ϕ)-L^2-Euler characteristic in the introduction but we refer to Section 2 for details and a summary of the key properties.

The (μ, ϕ)-L^2-Euler characteristic can be employed in many different contexts. For example it is used by Funke–Kielak [18] to study descending HNN-extensions of free groups and it is at least implicitly used by the authors and Tillmann [15] to study one-relator groups.

0.2. **The ϕ-L^2-Euler characteristic of 3-manifolds.** In this paper our main application of the (μ, ϕ)-L^2-Euler characteristic lies in the study of 3-manifolds to which we restrict ourselves in the remainder of the introduction. More precisely, our main focus will be on the following class of 3-manifolds.

Definition 0.1 (Admissible 3-manifold)**.** A 3-manifolds is called *admissible* if it is connected, orientable, and irreducible, its boundary is empty or a disjoint union of tori, and its fundamental group is infinite.

Let M be an admissible 3-manifold and let $\phi\colon \pi_1(M) \to \mathbb{Z}$ be a group homomorphism. Then all the conditions listed in Section 0.1 are satisfied for the triple $(X, \mathrm{id}_{\pi_1(M)}, \phi)$ and the corresponding L^2-Euler characteristic $\chi^{(2)}(M; \mathrm{id}_{\pi_1(M)}, \phi)$ is defined. We denote by $x_M(\phi)$ the Thurston norm of ϕ which is loosely speaking defined as the minimal complexity of a surface dual to ϕ. (We recall the precise definition of the Thurston norm in Section 1.) The following is one of our main theorems.

Date: JSeptember, 2016.

2010 *Mathematics Subject Classification.* 57M27, 12E15, 22D25, 58J52.

Key words and phrases. L^2-Euler characteristics, twisting with a cocycle, Thurston norm.

Theorem 0.2 (Equality of (μ,ϕ)-L^2-Euler characteristic and the Thurston norm for universal coverings)**.** *Let $M \neq S^1 \times D^2$ be an admissible 3-manifold.*

Then we get for any $\phi \in H^1(M;\mathbb{Z})$

$$-\chi^{(2)}(M;\mathrm{id}_{\pi_1(M)},\phi) \;=\; x_M(\phi).$$

If M is not a closed graph manifold, then Theorem 0.2 is a direct consequence of the subsequent Theorem 0.4 together with the fact that in this case the fundamental groups of M satisfies the Atiyah Conjecture, see Theorem 3.2. If M is a (closed) graph manifold Theorem 0.2 follows from Theorem 2.19.

It is also interesting to consider group homomorphisms $\pi_1(M) \to G$ that are not the identity. For example in Section 8 we will see that if G is a torsion-free elementary amenable group, then the (μ,ϕ)-L^2-Euler characteristic $\chi^{(2)}(M;\mu,\phi)$ is basically the same as the degrees of the non-commutative Alexander polynomials studied by Cochran [6], Harvey [21] and the first author [12]. In these three papers it was shown that the degrees of non-commutative Alexander polynomials give lower bounds on the Thurston norm. The following theorem can be viewed as a generalization of these results.

Theorem 0.3 (The negative of the (μ,ϕ)-L^2-Euler characteristic is a lower bound for the Thurston norm)**.** *Let $M \neq S^1 \times D^2$ be an admissible 3-manifold and let (μ,ϕ) be an L^2-acyclic Atiyah-pair.*

Then we get

$$-\chi^{(2)}(M;\mu,\phi) \;\le\; x_M(\phi\circ\mu).$$

In general the inequality of the above theorem is not an equality. But the following theorem shows that for any 'sufficiently large epimorphism', the inequality does become an equality.

Theorem 0.4 (Equality of (μ,ϕ)-L^2-Euler characteristic and the Thurston norm)**.** *Let $M \neq S^1 \times D^2$ be an admissible 3-manifold which is not a closed graph manifold.*

Then there exists a virtually abelian group Γ and a factorization

$$\mathrm{pr}_M\colon \pi_1(M) \xrightarrow{\alpha} \Gamma \xrightarrow{\beta} H_1(M)_f := H_1(M)/\operatorname{tors}(H_1(M))$$

of the canonical projection $\mathrm{pr}_M\colon \pi_1(M) \to H_1(M)_f$ into epimorphisms such that the following holds:

Given a group G satisfying the Atiyah Conjecture, a factorization of $\alpha\colon \pi \to \Gamma$ into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$, and a group homomorphism $\phi\colon H_1(M)_f \to \mathbb{Z}$, the pair $(\mu, \phi\circ\beta\circ\nu)$ is an L^2-acyclic Atiyah-pair, and we get

$$-\chi^{(2)}(M;\mu,\phi\circ\beta\circ\nu) \;= x_M(\phi).$$

With a little bit of extra effort one can use Theorem 0.4 to show that one can use epimorphisms onto torsion-free elementary amenable groups to detect the Thurston norm. Put differently, one can show that the aforementioned non-commutative Alexander polynomials detect the Thurston norm. We refer to Corollary 8.7 for the precise statement.

0.3. **Inequality of the Thurston norm.** One of the key motivations for developing the theory of L^2-Euler characteristics is the following question by Simon [25, Problem 1.12].

Question 0.5. *Let K and K' be two knots. If there is an epimorphism from the knot group of K to the knot group of K', does this imply that the genus of K is greater than or equal to the genus of K'?*

We propose the following conjecture.

Conjecture 0.6 (Inequality of the Thurston norm)**.** *Let $f\colon M \to N$ be a map of admissible 3-manifolds which is surjective on $\pi_1(N)$ and induces an isomorphism $f_*\colon H_n(M;\mathbb{Q}) \to H_n(N;\mathbb{Q})$ for $n \geq 0$.*

Then we get for any $\phi \in H^1(N;\mathbb{R})$ that

$$x_M(f^*\phi) \;\geq\; x_N(\phi).$$

Remark 0.7.
(1) In Section 1 we will recall that the Thurston norm can be viewed as a generalization of the knot genus. In particular a proof of Conjecture 0.6 would give an affirmative answer to Simon's question.
(2) If M and N are closed 3-manifolds, then the conclusion of Conjecture 0.6 follows from [19, Corollary 6.18].
(3) The condition on the induced map on rational homology cannot be dropped. For example, suppose that $M = S^1 \times \Sigma$ with Σ a surface of genus $g \geq 2$ with boundary. Let N be the exterior of a torus knot. Then $\pi_1(N)$ is generated by two elements, therefore there exists an epimorphism $f_*\colon \pi_1(S^1 \times \Sigma) \to \pi_1(N)$ which factors through the projection $\pi_1(S^1 \times \Sigma) \to \pi_1(\Sigma)$. If ϕ is a generator of $H^1(N;\mathbb{Z})$, then $x_N(\phi) \neq 0$ but it is straightforward to see that $x_M(f^*\phi) = 0$. We are grateful to Yi Liu for pointing out this example.

Before we state our main contribution to Conjecture 0.6 we need to recall one more definition. A group G is called *locally indicable* if any finitely generated nontrivial subgroup of G admits an epimorphism onto $\mathbb{Z}$. For example Howie [23] showed that the fundamental group of any admissible 3-manifold with non-trivial boundary is locally indicable.

Our main result is Theorem 0.8.

Theorem 0.8 (Inequality of the Thurston norm)**.** *Let $f\colon M \to N$ be a map of admissible 3-manifolds which is surjective on $\pi_1(N)$ and induces an isomorphism $f_*\colon H_n(M;\mathbb{Q}) \to H_n(N;\mathbb{Q})$ for $n \geq 0$. Suppose that $\pi_1(N)$ is residually locally indicable elementary amenable.*

Then we get for any $\phi \in H^1(N;\mathbb{R})$ that

$$x_M(f^*\phi) \;\geq\; x_N(\phi).$$

By Lemma 7.5 the fundamental group of any fibered 3-manifold is residually locally indicable elementary amenable. Thus we have proved Conjecture 0.6 in particular in the case that N is fibered. The conclusion of Conjecture 0.6 can be proved relatively easily for *fibered classes in* $H^1(N;\mathbb{R})$, but it seems to us that if N is fibered, then there is no immediate reason why the inequality should hold for *non-fibered* classes in $H^1(N;\mathbb{R})$.

We propose the following

Conjecture 0.9. *The fundamental group of any admissible 3-manifold M with $b_1(M) \geq 1$ is residually locally indicable elementary amenable.*

A proof of Conjecture 0.9 together with Theorem 0.8 implies Conjecture 0.6 and in particular an affirmative answer to Simon's Question 0.5.

0.4. **The (μ,ϕ)-L^2-Euler characteristic and the degree of the L^2-torsion function.** We briefly discuss a relation of the (μ,ϕ)-L^2-Euler characteristic to the degree of the L^2-torsion function in Section 9.

0.5. **Methods of proof.** One key ingredient in this paper is to replace the Ore localization of a group ring $\mathbb{Z}G$, which is known to exist for torsion-free elementary amenable groups and is definitely not available, if G contains a non-abelian free subgroup, by the division closure $\mathcal{D}(G)$ of $\mathbb{Z}G$ in the algebra $\mathcal{U}(G)$ of operators affiliated to the group von Neumann algebra $\mathcal{N}(G)$. This is a well-defined skew field containing $\mathbb{Z}G$ if and only if G is torsion-free and satisfies the Atiyah Conjecture with rational coefficients. This is known to be true in many interesting cases.

We will also take advantage of the recent proof by Agol and others of the Virtual Fibering Conjecture.

Acknowledgments. The first author gratefully acknowledges the support provided by the SFB 1085 "Higher Invariants" at the University of Regensburg, funded by the Deutsche Forschungsgemeinschaft DFG. The paper is financially supported by the Leibniz-Preis of the second author granted by the DFG and the ERC Advanced Grant "KL2MG-interactions" (no. 662400) of the second author granted by the European Research Council.

Contents

1. Brief review of the Thurston norm

Recall the definition in [42] of the *Thurston norm* $x_M(\phi)$ of a compact connected orientable 3-manifold M with empty or non-empty boundary and an element $\phi \in H^1(M;\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$

$$x_M(\phi) \ := \ \min\{\chi_-(F) \,|\, F \subset M \text{ properly embedded surface dual to } \phi\},$$

where, given a surface F with connected components $F_1, F_2, \ldots, F_k$, we define

$$\chi_-(F) \ := \ \sum_{i=1}^{k} \max\{-\chi(F_i), 0\}.$$

Thurston [42] showed that this defines a seminorm on $H^1(M;\mathbb{Z})$ which can be extended to a seminorm on $H^1(M;\mathbb{R})$ which we denote by x_M again. In particular we get for $r \in \mathbb{R}$ and $\phi \in H^1(M;\mathbb{R})$

$$x_M(r\cdot\phi) \quad = \quad |r| \cdot x_M(\phi). \tag{1.1}$$

If $K \subset S^3$ is a knot, then we denote by νK an open tubular neighborhood of K and we refer to $X_K = S^3 \setminus \nu K$ as the exterior of K. We refer to the minimal genus of a Seifert surface of K as the *genus* $g(K)$ *of* K. We have $H^1(X_K;\mathbb{Z}) \cong \mathbb{Z}$ and an elementary exercise shows that for any generator ϕ of $H^1(X_K;\mathbb{Z}) \cong \mathbb{Z}$ we have

$$x_{X_K}(\phi) \ = \ \max\{2g(K)-1, 0\}. \tag{1.2}$$

If $p\colon M' \to M$ is a finite covering with n sheets, then Gabai [19, Corollary 6.13] showed

$$x_{M'}(p^*\phi) \quad = \quad n \cdot x_M(\phi). \tag{1.3}$$

If $F \to M \xrightarrow{p} S^1$ is a fiber bundle for a compact connected orientable 3-manifold M and compact surface F and $\phi \in H^1(M;\mathbb{Z})$ is given by $H_1(p)\colon H_1(M) \to H_1(S^1)$, then

$$x_M(\phi) \quad = \quad \begin{cases} -\chi(F) & \text{if } \chi(F) \le 0; \\ 0 & \text{if } \chi(F) \ge 0. \end{cases} \tag{1.4}$$

2. Twisting the L^2-Euler characteristic with a cocycle in the first cohomology

In this section we introduce our main invariant on the L^2-side, namely certain variations of the L^2-Euler characteristic which are obtained in the special case of the universal covering $\widetilde{X} \to X$ with by twisting with an element $\phi \in H^1(X;\mathbb{Z})$. More generally, we will consider G-CW-complexes and twist with a group homomorphism $\phi\colon G \to \mathbb{Z}$.

2.1. **Review of the L^2-Euler characteristic.** Let G be a group. Denote by $\mathcal{N}(G)$ the group von Neumann algebra which can be identified with the algebra $B(L^2(G), L^2(G)^G$ of bounded left G-equivariant operators $L^2(G) \to L^2(G)$. Let C_* be a finitely generated based free left $\mathbb{Z}G$-chain complex. Then we can consider the chain complex of finitely generated Hilbert $\mathcal{N}(G)$-chain complexes $L^2(G) \otimes_{\mathbb{Z}G} C_*$. Its L^2-Betti numbers $b_n^{(2)}(L^2(G)\otimes_{\mathbb{Z}G} C_*)$ are defined as the von Neumann dimension of its L^2-homology, see [32, Section 1.1].

One can also work entirely algebraically by applying $\mathcal{N}(G) \otimes_{\mathbb{Z}G} -$ which yields a chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*$ of $\mathcal{N}(G)$-modules, where we consider $\mathcal{N}(G)$ as a ring and forget the topology. There is a dimension function defined for all $\mathcal{N}(G)$-modules, see [32, Section 6.1]. So one gets another definition of L^2-Betti numbers by taking this dimension of the $\mathcal{N}(G)$-module $H_n(\mathcal{N}(G)\otimes_{\mathbb{Z}G} C_*)$. These two definitions agree by [32, Section 6.2].

The advantage of the algebraic approach is that it works and often still gives finite L^2-Betti numbers also in the case where we drop the condition that C_* is a finitely generated free $\mathbb{Z}G$-chain complex and consider any $\mathbb{Z}G$-chain complex C_*. This is explained in detail in [32, Chapter 6]. We recall that for any chain complex of free left $\mathbb{Z}G$-chain modules C_* we can define its nth L^2-Betti number a

$$b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*) \quad := \quad \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*)\big) \quad \in [0,\infty]. \tag{2.1}$$

In particular given a G-CW-complex X we can define its nth L^2-Betti number as

$$b_n^{(2)}(X;\mathcal{N}(G)) \quad := \quad \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(X))\big) \quad \in [0,\infty]. \tag{2.2}$$

We leave the superscript in the notation $b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*)$ and $b_n^{(2)}(X;\mathcal{N}(G))$, although the definition is purely algebraic, in order to remind the reader that it is related to the classical notion of L^2-Betti numbers.

Definition 2.3 (L^2-Euler characteristic)**.** Let X be a G-CW-complex. Define

$$\begin{aligned} h^{(2)}(X;\mathcal{N}(G)) &:= \sum_{p\geq 0} b_p^{(2)}(X;\mathcal{N}(G)) \in [0,\infty]; \\ \chi^{(2)}(X;\mathcal{N}(G)) &:= \sum_{p\geq 0} (-1)^p \cdot b_p^{(2)}(X;\mathcal{N}(G)) \in \mathbb{R}, \quad \text{if } h^{(2)}(X;\mathcal{N}(G)) < \infty. \end{aligned}$$

We call $\chi^{(2)}(X;\mathcal{N}(G))$ the *L^2-Euler characteristic* of X.

The condition $h^{(2)}(X;\mathcal{N}(G)) < \infty$ ensures that the sum which appears in the definition of $\chi^{(2)}(X;\mathcal{N}(G))$ converges absolutely. In the sequel we assume that the reader is familiar with the notion of the L^2-Euler characteristic and its basic properties, as presented in [32, Section 6.6.1]. Another approach to L^2-Betti numbers for not necessarily finite G-CW-complexes is given by Cheeger-Gromov [4].

2.2. **The ϕ-twisted L^2-Euler characteristic.** We will be interested in the following version of an L^2-Euler characteristic.

Definition 2.4 (ϕ-twisted L^2-Betti number and L^2-Euler characteristic)**.** Let X be a G-CW-complex. Let $\phi\colon G \to \mathbb{Z}$ be a group homomorphism. Let $\phi^*\mathbb{Z}[\mathbb{Z}]$ be the $\mathbb{Z}G$-module obtained from $\mathbb{Z}[\mathbb{Z}]$ regarded as module over itself by restriction with ϕ. If $C_*(X)$ is the cellular $\mathbb{Z}G$-chain complex, denote by $C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}]$ the $\mathbb{Z}G$-chain complex obtained by the diagonal G-action. Define

$$\begin{aligned} b_n^{(2)}(X;\mathcal{N}(G),\phi) &:= \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G) \otimes_{\mathbb{Z}G} (C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}]))\big) \in [0,\infty]; \\ h^{(2)}(X;\mathcal{N}(G),\phi) &:= \sum_{p\geq 0} b_p^{(2)}(X;\mathcal{N}(G),\phi) \in [0,\infty]; \\ \chi^{(2)}(X;\mathcal{N}(G);\phi) &:= \sum_{p\geq 0} (-1)^p \cdot b_p^{(2)}(X;\mathcal{N}(G),\phi) \in \mathbb{R}, \ \text{if } h^{(2)}(X;\mathcal{N}(G),\phi) < \infty. \end{aligned}$$

We say that X is *ϕ-L^2-finite* if $h^{(2)}(X;\mathcal{N}(G),\phi) < \infty$ holds. If this the case, we call the real number $\chi^{(2)}(X;\mathcal{N}(G),\phi)$ the *ϕ-twisted L^2-Euler characteristic* of X.

Notice that so far we are not requiring that the G-CW-complex X is free or finite.

We collect the basic properties of this invariant.

Theorem 2.5 (Basic properties of the ϕ-twisted L^2-Euler characteristic)**.** *Let X be a G-CW-complex. Let $\phi\colon G \to \mathbb{Z}$ be a group homomorphism.*

(1) G-homotopy invariance
Let X and Y be G-CW-complexes which are G-homotopy equivalent. Then X is ϕ-L^2-finite if and only if Y is ϕ-L^2-finite, and in this case we get
$$\chi^{(2)}(X;\mathcal{N}(G),\phi) = \chi^{(2)}(Y;\mathcal{N}(G),\phi);$$

(2) Sum formula
Consider a G-pushout of G-CW-complexes
$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$
where the upper horizontal arrow is cellular, the left vertical arrow is an inclusion of G-CW-complexes and X has the obvious G-CW-structure coming from the ones on X_0, X_1 and X_2. Suppose that X_0, X_1 and X_2 are ϕ-L^2-finite. Then X is ϕ-L^2-finite and we get
$$\chi^{(2)}(X;\mathcal{N}(G),\phi) = \chi^{(2)}(X_1;\mathcal{N}(G),\phi) + \chi^{(2)}(X_2;\mathcal{N}(G),\phi) - \chi^{(2)}(X_0;\mathcal{N}(G),\phi);$$

(3) Induction
Let $i\colon H \to G$ be the inclusion of a subgroup of G. Then X is $(\phi\circ i)$-L^2-finite if and only if $G\times_H X$ is ϕ-L^2-finite. If this is the case, we get
$$\chi^{(2)}(G\times_H X;\mathcal{N}(G),\phi) = \chi^{(2)}(X;\mathcal{N}(H),\phi\circ i);$$

(4) Restriction
Let $i\colon H \to G$ be the inclusion of a subgroup H of G with $[G:H]<\infty$. Let X be a G-CW-complex. Denote by i^*X the H-CW-complex obtained from X by restriction with i. Then i^*X is $\phi\circ i$-L^2-finite if and only if X is ϕ-L^2-finite, and in this case we get
$$\chi^{(2)}(i^*X;\mathcal{N}(H),\phi\circ i) = [G:H]\cdot\chi^{(2)}(X;\mathcal{N}(G),\phi);$$

(5) Scaling ϕ
We get for every integer $k\geq 1$ that X is ϕ-L^2-finite if and only if X is $(k\cdot\phi)$-L^2-finite, and in this case we get
$$\chi^{(2)}(X;\mathcal{N}(G),k\cdot\phi) = k\cdot\chi^{(2)}(X;\mathcal{N}(G),\phi);$$

(6) Trivial ϕ
Suppose that ϕ is trivial. Then X is ϕ-L^2-finite if and only if we have $b_n^{(2)}(X;\mathcal{N}(G))=0$ for all $n\geq 0$. If this is the case, then
$$\chi^{(2)}(X;\mathcal{N}(G),\phi) = 0.$$

Proof. (1) This follows from the homotopy invariance of L^2-Betti numbers.

(2) The proof is analogous to the one of [32, Theorem 6.80 (2) on page 277], just replace the short split exact sequence of $\mathbb{Z}G$-chain complexes $0\to C_*(X_0)\to C_*(X_1)\oplus C_*(X_2)\to C_*(X)\to 0$ by the induced short split exact sequence of $\mathbb{Z}G$-chain complexes
$$\begin{aligned} 0 \to C_*(X_0)\otimes_{\mathbb{Z}}\phi^*\mathbb{Z}[\mathbb{Z}] \to C_*(X_1)\otimes_{\mathbb{Z}}\phi^*\mathbb{Z}[\mathbb{Z}] \oplus C_*(X_2)\otimes_{\mathbb{Z}}\phi^*\mathbb{Z}[\mathbb{Z}] \\ \to C_*(X)\otimes_{\mathbb{Z}}\phi^*\mathbb{Z}[\mathbb{Z}] \to 0. \end{aligned}$$

(3) The proof is analogous to the one of [32, Theorem 6.80 (8) on page 279] using the isomorphism of $\mathbb{Z}G$-chain complexes
$$\begin{aligned} C_*(G\times_H X)\otimes_{\mathbb{Z}}\phi^*\mathbb{Z}[\mathbb{Z}] &\cong \big(\mathbb{Z}G\otimes_{\mathbb{Z}H} C_*(X)\big)\otimes_{\mathbb{Z}}\phi^*\mathbb{Z}[\mathbb{Z}]) \\ &\cong \mathbb{Z}G\otimes_{\mathbb{Z}H}\big(C_*(X)\otimes_{\mathbb{Z}}(\phi\circ i)^*\mathbb{Z}[\mathbb{Z}]\big), \end{aligned}$$

where the second isomorphism is given by $(g \otimes u) \otimes v \mapsto g \otimes u \otimes g^{-1}v$.

(4) The proof is analogous to the one of [32, Theorem 6.80 (7) on page 279] using the obvious identification of $\mathbb{Z}H$-chain complexes $i^*\big(C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}]\big) = C_*(i^*X) \otimes_{\mathbb{Z}} i^*\phi^*\mathbb{Z}[\mathbb{Z}]$.

(5) Since there is an obvious isomorphism of $\mathbb{Z}G$-modules $(k\cdot\phi)^*\mathbb{Z}[\mathbb{Z}] \cong \bigoplus_{i=1}^k \phi^*\mathbb{Z}[\mathbb{Z}]$, we get

$$b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} (C_*(X) \otimes_{\mathbb{Z}} (k\cdot\phi)^*\mathbb{Z}[\mathbb{Z}]\big) = k \cdot b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} (C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}])\big).$$

(6) Since the triviality of ϕ implies that $C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}]$ is $\mathbb{Z}G$-isomorphic to $\bigoplus_{\mathbb{Z}} C_*(X)$, we get

$$b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} (C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}]\big) = \begin{cases} 0 & \text{if } b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(X)\big) = 0; \\ \infty & \text{otherwise.} \end{cases}$$

This finishes the proof of Theorem 2.5. □

We can interpret the ϕ-twisted L^2-Euler characteristic also as an L^2-Euler characteristic for surjective ϕ as follows.

Lemma 2.6. *Let X be a G-CW-complex. Let $\phi\colon G \to \mathbb{Z}$ be a surjective group homomorphism. Denote by K the kernel of ϕ and by $i\colon K \to G$ the inclusion.*

Then X is ϕ-L^2-finite if and only if $h_n^{(2)}(i^*X;\mathcal{N}(K)) < \infty$ hold. If this is the case, then

$$\chi^{(2)}(X;\mathcal{N}(G),\phi) \;=\; \chi^{(2)}(i^*X;\mathcal{N}(K)).$$

Proof. We have the isomorphism of $\mathbb{Z}G$-chain complexes

$$\mathbb{Z}G \otimes_{\mathbb{Z}K} i^*C_*(X) \xrightarrow{\cong} C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}], \quad g \otimes x \mapsto gx \otimes \phi(g).$$

The inverse sends $y \otimes q$ to $g \otimes g^{-1}y$ for any choice of $g \in \phi^{-1}(q)$. Since $\mathcal{N}(G)$ is flat as an $\mathcal{N}(K)$-module by [32, Theorem 6.29 (1) on page 253], we obtain a sequence of obvious isomorphisms of $\mathcal{N}(G)$-modules

$$\begin{aligned} &\mathcal{N}(G) \otimes_{\mathcal{N}(K)} H_n(\mathcal{N}(K) \otimes_{\mathbb{Z}K} C_*(i^*X)) \cong \mathcal{N}(G) \otimes_{\mathcal{N}(K)} H_n(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*(X)) \\ &\cong H_n\big(\mathcal{N}(G) \otimes_{\mathcal{N}(K)} \mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*(X)\big) \cong H_n\big(\mathcal{N}(G) \otimes_{\mathbb{Z}K} i^*C_*(X)\big) \\ &\cong H_n\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} \mathbb{Z}G \otimes_{\mathbb{Z}K} i^*C_*(X))\big) \quad \cong H_n\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} (C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}])\big). \end{aligned}$$

Since $\dim_{\mathcal{N}(G)}(\mathcal{N}(G) \otimes_{\mathcal{N}(K)} M) = \dim_{\mathcal{N}(K)}(M)$ holds for every $\mathcal{N}(K)$-module M by [32, Theorem 6.29 (2) on page 253], we conclude for every $n \geq 0$

$$b_n^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} C_*(i^*X);\mathcal{N}(K)\big) = b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} (C_*(X) \otimes_{\mathbb{Z}} \phi^*\mathbb{Z}[\mathbb{Z}]);\mathcal{N}(G)\big).$$

□

2.3. **The (μ,ϕ)-L^2-Euler characteristic.** We will be interested in this paper mainly in the case, where the G-CW-complex X is free. If we put $Y = X/G$, then X is the disjoint union of the preimages of the components of Y. Therefore it suffices to study a connected CW-complex Y and G-coverings $\overline{Y} \to Y$. Any such G-covering is obtained from the universal covering $\widetilde{Y} \to Y$ and a group homomorphism $\mu\colon \pi = \pi_1(Y) \to G$ as the projection $G \times_\mu \widetilde{Y} \to Y$. Therefore we introduce the following notation:

Definition 2.7 ((μ,ϕ)-L^2-Euler characteristic)**.** Let X be a connected CW-complex. Let $\mu\colon \pi_1(X) \to G$ and $\phi\colon G \to \mathbb{Z}$ be a group homomorphisms. Let $\overline{X} \to X$ be the G-covering associated to μ. We call X *(μ,ϕ)-L^2-finite* if $\overline{X}$ is ϕ-L^2-finite, and in this case we define the (μ,ϕ)-*L^2-Euler characteristic* $\chi^{(2)}(X;\mu,\phi)$ to be $\chi^{(2)}(\overline{X};\mathcal{N}(G);\phi)$, see Definition 2.4.

The next lemma essentially reduces the general case (μ,ϕ) to the special case, where μ and ϕ are surjective or $\phi\circ\mu$ is trivial.

Lemma 2.8. *Let X be a connected CW-complex. Let $\mu\colon \pi_1(X)\to G$ and $\phi\colon G\to \mathbb{Z}$ be group homomorphisms. Let G' be the image of μ. Let $\mu'\colon \pi_1(X)\to G'$ be the epimorphism induced by μ and let $\phi'\colon G'\to\mathbb{Z}$ be obtained by restricting ϕ to G'.*

(1) *Then X is (μ,ϕ)-L^2-finite if and only if X is (μ',ϕ')-L^2-finite. If this is the case, we get*
$$\chi^{(2)}(X;\mu,\phi) \;=\; \chi^{(2)}(X;\mu',\phi');$$
(2) *Suppose that $\mu\circ\phi\neq 0$. Let $k\geq 1$ be the natural number such that the image of ϕ' is $k\cdot\mathbb{Z}$ and let $\phi''\colon G'\to\mathbb{Z}$ be the epimorphism uniquely determined by $k\cdot\phi''=\phi'$. Then X is (μ,ϕ)-L^2-finite, if and only if X is (μ',ϕ'')-L^2-finite. If this is the case, we get*
$$\chi^{(2)}(X;\mu,\phi) \;=\; \frac{1}{k}\cdot\chi^{(2)}(X;\mu',\phi'');$$
(3) *Suppose that $\phi\circ\mu=0$. Then X is (μ,ϕ)-L^2-finite if and only if $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanishes for the G-covering $\overline{X}$ associated to μ and every $n\geq 0$. If this is the case, then*
$$\chi^{(2)}(X;\mu,\phi) \;=\; 0.$$

Proof. The first statement follows from Theorem 2.5. The second and third statement follow from (1) and Theorem 2.5 (5) and (6). □

Example 2.9 (Mapping torus)**.** Let Y be a connected finite CW-complex and $f\colon Y\to Y$ be a self-map. Let T_f be its mapping torus. Consider any factorization $\pi_1(T_f)\xrightarrow{\mu} G\xrightarrow{\phi}\mathbb{Z}$ of the epimorphism $\pi_1(T_f)\to\pi_1(S^1)=\mathbb{Z}$ induced by the obvious projection $T_f\to S^1$. Then T_f is (μ,ϕ)-L^2-finite and we get
$$\chi^{(2)}(T_f;\mu,\phi) \;=\; \chi(Y)$$
by the following argument. Let $\overline{T_f}$ be the G-covering associated to $\mu\colon\pi_1(T_f)\to G$. Let K be the kernel of ϕ and $i\colon K\to G$ be the inclusion. The image of the composite $\pi_1(Y)\to\pi_1(T_f)\xrightarrow{\mu} G$ is contained in K and we can consider the K-covering $\widehat{Y}\to Y$ associated to it. The K-CW-complex $i^*\overline{T_f}$ is K-homotopy equivalent to the K-CW-complex $\widehat{Y}$, see [30, Section 2]. Hence we conclude $\chi^{(2)}(T_f;\mu,\phi)=\chi^{(2)}(\widehat{Y};\mathcal{N}(K))$ from Lemma 2.6 and the K-homotopy invariance of L^2-Betti numbers. Since Y is a finite CW-complex, we have $\chi^{(2)}(\widehat{Y};\mathcal{N}(K))=\chi(Y)$.

Notice that the L^2-Betti numbers $b_n^{(2)}(\overline{T_f};\mathcal{N}(G))$ are all trivial by [30, Theorem 2.1] and hence the L^2-Euler characteristic $\chi^{(2)}(\overline{T_f};\mathcal{N}(G))$ is trivial. So the passage to the subgroup of infinite index K or, equivalently, the twisting with the $\mathbb{Z}G$-module $\phi^*\mathbb{Z}[\mathbb{Z}]$ which is not finitely generated as abelian group, ensures that we get an interesting invariant by the (μ,ϕ)-L^2-Euler characteristic.

Lemma 2.10. *Let T^n be the n-dimensional torus for $n\geq 1$. Consider homomorphisms $\mu\colon\pi_1(T^n)\to G$ and $\phi\colon G\to\mathbb{Z}$ such that the image of μ is infinite.*

Then T^n is (μ,ϕ)-L^2-finite and we get
$$\chi^{(2)}(T^n;\mu,\phi) = \begin{cases}[\mathbb{Z}:\operatorname{im}(\phi\circ\mu)] & \text{if } n=1, \phi\circ\mu\neq 0;\\ 0 & \text{otherwise.}\end{cases}$$

Proof. Because of Lemma 2.8 (1) we can assume without loss of generality that μ is surjective. Suppose that $\phi \circ \mu$ is non-trivial. Because of Lemma 2.8 (2) it suffices to consider the case, where μ and ϕ are surjective. Then the claim follows from Example 2.9 since there is a homeomorphism $h\colon T^n \xrightarrow{\cong} T^{n-1} \times S^1$ that that the composite of the map of $\pi_1(T^{n-1} \times S^1) \to \pi_1(S^1)$ induced by the projection onto S^1 composed with $\pi_1(h)$ is ϕ. Suppose that ϕ is trivial. Since μ has infinite image, one can show using [30, Theorem 2.1] that $b_m^{(2)}(\overline{T^n};\mathcal{N}(G))$ vanishes for the G-covering $\overline{T^n} \to T^n$ associated to μ for all $m \geq 0$. Hence the claim follows from Lemma 2.8 (3). □

Theorem 2.11 (The (μ,ϕ)-L^2-Euler characteristic for S^1-CW-complexes)**.** *Let X be a connected finite S^1-CW-complex. Let $\mu\colon \pi_1(X) \to G$ and $\phi\colon G \to \mathbb{Z}$ be group homomorphisms. Suppose that for one and hence all $x \in X$ the composite*

$$\eta\colon \pi_1(S^1, 1) \xrightarrow{\pi_1(\mathrm{ev}_x, 1)} \pi_1(X, x) \xrightarrow{\mu} G \xrightarrow{\phi} \mathbb{Z}$$

is injective, where $\mathrm{ev}_x\colon S^1 \to X$ sends z to $z \cdot x$. Define the S^1-orbifold Euler characteristic of X by

$$\chi_{\mathrm{orb}}^{S^1}(X) \;=\; \sum_{n \geq 0} (-1)^n \cdot \sum_{e \in I_n} \frac{1}{|S_e^1|},$$

where I_n is the set of open n-dimensional S^1-cells of X and for $e \in I_n$ we denote by S_e^1 the isotropy group of any point in e. Then X is (μ,ϕ)-L^2-finite and we get

$$\chi^{(2)}(X;\mu,\phi) \;=\; \chi_{\mathrm{orb}}^{S^1}(X) \cdot [\mathbb{Z} : \mathrm{im}(\eta)].$$

Proof. The strategy of proof is the same as the the one of [32, Theorem 1.40 on page 43], where one considers more general finite S^1-CW-complex Y together with a S^1-map $f\colon Y \to X$ and does induction over the number of S^1-equivariant cells. A basic ingredient is the additivity of the two terms appearing in the desired equation in Theorem 2.11. □

We mention that the condition about the injectivity of the map η appearing in Theorem 2.11 is necessary.

For the reader's convenience we record the next result which we will not need in this paper and whose proof is a variation of the one of Theorem 2.11.

Theorem 2.12 (The (μ,ϕ)-L^2-Euler characteristic for fibrations)**.** *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibration of connected CW-complexes. Suppose that B is a finite CW-complex. Consider group homomorphisms $\mu\colon \pi_1(E) \to G$ and $\phi\colon G \to \mathbb{Z}$. Suppose that $\overline{F}$ is $(\mu \circ \pi_1(i), \phi)$-$L^2$-finite.*

Then $\overline{E}$ is (μ,ϕ)-L^2-finite and we get

$$\chi^{(2)}(E;\mu,\phi) = \chi(B) \cdot \chi^{(2)}(F, \mu \circ \pi_1(i), \phi).$$

If M is a compact connected orientable 3-manifold with proper S^1-action, then M is a compact connected orientable Seifert manifold. The converse is not true general. Associated to a compact connected orientable Seifert manifold is an orbifold X and X has a orbifold Euler characteristic $\chi_{\mathrm{orb}}(X)$. For a basic introduction to these notions we refer for instance to [40]. If M is a compact 3-manifold with proper S^1-action, then X is given by M/S^1 and $\chi_{\mathrm{orb}}(X)$ is the S^1-orbifold Euler characteristic $\chi_{\mathrm{orb}}^{S^1}(M)$. We omit the proof of the next result since it is essentially a variation of the one of Theorem 2.11, the role of the cells $S^1/H \times D^n$ in Theorem 2.11 is now played by the typical neighborhoods of the Seifert fibers given by solid tori.

Theorem 2.13 (The (μ, ϕ)-L^2-Euler characteristic for Seifert manifolds)**.** *Let M be a compact connected orientable Seifert manifold. Let $\mu\colon \pi_1(M) \to G$ and $\phi\colon G \to \mathbb{Z}$ be group homomorphisms. Suppose that for one (and hence all) $x \in M$*

$$\eta\colon \pi_1(S^1, 1) \xrightarrow{\pi_1(\mathrm{ev},1)} \pi_1(M, x) \xrightarrow{\mu} G \xrightarrow{\phi} \mathbb{Z}$$

is injective, where $\mathrm{ev}\colon S^1 \to M$ is the inclusion of a regular fiber. Let X be the associated orbifold of M and denote by $\chi_{\mathrm{orb}}(X)$ its orbifold Euler characteristic. Then M is (μ, ϕ)-L^2-finite and we get

$$\chi^{(2)}(M; \mu, \phi) \;=\; \chi_{\mathrm{orb}}(X) \cdot [\mathbb{Z} : \mathrm{im}(\eta)].$$

Lemma 2.14. *Let M be a 3-manifold, which is admissible, see Definition 0.1. Let M_1, M_2, …, M_r be its pieces in the Jaco-Shalen-Johannson decomposition. Consider group homomorphisms $\mu\colon \pi_1(M) \to G$ and $\phi\colon G \to \mathbb{Z}$. Suppose that the composite of μ with $\pi_1(j)\colon \pi_1(T^2) \to \pi_1(M)$ has infinite image for the inclusion $j\colon T^2 \to M$ of any splitting torus appearing in the Jaco-Shalen-Johannson decomposition. Let $\mu_i\colon \pi_1(M_i) \to G$ be the composite of μ with the map $\pi_1(M_i) \to \pi_1(M)$ induced by the inclusion $M_i \to M$. Suppose that M_i is (μ_i, ϕ)-L^2-finite for $i = 1, 2, \dots, r$.*

Then M is (μ, ϕ)-L^2-finite and we have

$$\chi^{(2)}(M; \mu, \phi) \;=\; \sum_{i=1}^{r} \chi^{(2)}\bigl(\widetilde{M_i}; \mu_i, \phi\bigr).$$

Proof. This follows from Theorem 2.5 (2) and Lemma 2.10. □

Theorem 2.15 (The (ϕ, μ)-L^2-Euler characteristic and the Thurston norm for graph manifolds)**.** *Let M be an admissible 3-manifold, which is a graph manifold and not homeomorphic to $S^1 \times D^2$. Consider group homomorphisms $\mu\colon \pi_1(M) \to G$ and $\phi\colon G \to \mathbb{Z}$. Suppose that for each piece M_i in the Jaco-Shalen-Johannson decomposition the map $\pi_1(S^1) \xrightarrow{\pi_1(\mathrm{ev}_i)} \pi_1(M_i) \xrightarrow{j_i} \pi_1(M) \xrightarrow{\mu} G \xrightarrow{\phi} \mathbb{Z}$ is injective, where $\mathrm{ev}_i\colon S^1 \to M_i$ is the inclusion of the regular fiber and $j_i\colon M_i \to M$ is the inclusion.*

Then M is (μ, ϕ)-L^2-finite and we get

$$-\chi^{(2)}(M; \mu, \phi) \;=\; x_M(\phi \circ \mu).$$

Proof. In the situation and notation of Lemma 2.14 we conclude from [10, Proposition 3.5 on page 33]

$$x_M(\phi) \;=\; \sum_{i=1}^{r} x_{M_i}(\phi_i).$$

if $\phi_i \in H^1(M_i; \mathbb{Z})$ is the restriction of ϕ to M_i. Moreover, we get from Lemma 2.13 and from [22, Lemma A] for $i = 1, 2 \dots, r$

$$\chi^{(2)}(M_i; \mu_i, \phi) \;=\; -x_M(\phi_i),$$

if μ_i is the composite of μ with the homomorphism $\pi_1(M_i) \to \pi_1(M)$ induced by the inclusion. Now the claim follows from Lemma 2.14 since any splitting torus appearing in the Jaco-Shalen-Johannson decomposition contains a regular fiber of one of the pieces M_i. □

2.4. **The ϕ-L^2-Euler characteristic for universal coverings.** In this section we consider the special case of the universal covering and of a group homomorphism $\phi\colon \pi_1(X) \to \mathbb{Z}$. This is in some sense the most canonical and important covering and in this case the formulations of the main results simplify in a convenient way.

Definition 2.16 (The ϕ-L^2-Euler characteristic for $\phi \in H^1(X;\mathbb{Z})$)**.** Let X be a connected CW-complex with fundamental group π. Let ϕ be an element in $H^1(X;\mathbb{Z})$, or, equivalently, let $\phi\colon \pi \to \mathbb{Z}$ be a group homomorphism. We say that the universal covering $\widetilde{X}$ of X is *ϕ-L^2-finite*, if X is (id_π, ϕ)-L^2-finite in the sense of Definition 2.7. If this is the case, we define its *ϕ-L^2-Euler characteristic*

$$\chi(\widetilde{X};\phi) \;:=\; \chi^{(2)}(X;\mathrm{id}_\pi,\phi)$$

where $\chi^{(2)}(X;\mathrm{id}_\pi,\phi)$ has been introduced in Definition 2.7.

If X is a (not necessarily connected) finite CW-complex and $\phi \in H^1(X;\mathbb{Z})$, we say that $\widetilde{X}$ is ϕ-L^2-finite if for each component $C \in X$ the universal covering $\widetilde{C} \to C$ is $\phi|_C$-L^2-finite and we put

$$\chi^{(2)}(\widetilde{X};\phi) \;=\; \sum_{C\in\pi_0(X)} \chi^{(2)}(\widetilde{C},\phi|_C).$$

For the reader's convenience we record the basic properties of the ϕ-L^2-Euler characteristic.

Theorem 2.17 (Basic properties of the ϕ-L^2-Euler characteristic for universal coverings)**.**

(1) Homotopy invariance

Let $f\colon X \to Y$ be a homotopy equivalence of CW-complexes. Consider $\phi \in H^1(Y;\mathbb{Z})$. Let $f^*\phi \in H^1(X;\mathbb{Z})$ be its pullback with f.

Then $\widetilde{X}$ is $f^*\phi$-L^2-finite if and only if $\widetilde{Y}$ is ϕ-L^2-finite, and in this case we get

$$\chi^{(2)}(\widetilde{X};f^*\phi) \;=\; \chi^{(2)}(\widetilde{Y};\phi);$$

(2) Sum formula

Consider a pushout of CW-complexes

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X \end{array}$$

where the upper horizontal arrow is cellular, the left vertical arrow is an inclusion of CW-complexes and X has the obvious CW-structure coming from the ones on X_0, X_1 and X_2. Consider $\phi \in H^1(X;\mathbb{Z})$. For every $i \in \{0,1,2\}$ suppose that for each base point $x_i \in X_i$ the map $\pi_1(j_i,x_i)\colon \pi_1(X_i,x_i) \to \pi_1(X,j_i(x_i))$ induced by the inclusion $j_i\colon X_i \to X$ is injective and that $\widetilde{X_i}$ is $j_i^*\phi$-L^2-finite.

Then $\widetilde{X}$ is ϕ-L^2-finite and we get

$$\chi^{(2)}(\widetilde{X};\phi) \;=\; \chi^{(2)}(\widetilde{X_1};j_1^*\phi) + \chi^{(2)}(\widetilde{X_2};j_2^*\phi) - \chi^{(2)}(\widetilde{X_0};j_0^*\phi);$$

(3) Finite coverings

Let $p\colon X \to Y$ be a finite d-sheeted covering of connected CW-complexes, ϕ be an element in $H^1(Y;\mathbb{Z})$ and $p^*\phi \in H^1(X;\mathbb{Z})$ be its pullback with p.

Then Y is ϕ-L^2-finite if and only if X is $p^*\phi$-L^2-finite, and in this case

$$\chi^{(2)}(\widetilde{X};p^*\phi) \;:=\; d\cdot\chi^{(2)}(\widetilde{Y};\phi);$$

(4) Scaling ϕ

Let X be CW-complex and ϕ be an element in $H^1(X;\mathbb{Z})$ Consider any integer $k \neq 0$. Then $\widetilde{X}$ is ϕ-L^2-finite if and only if $\widetilde{X}$ is $(k\cdot\phi)$-L^2-finite, and in this case we get

$$\chi^{(2)}(\widetilde{X};k\cdot\phi) \;=\; |k|\cdot\chi^{(2)}(\widetilde{X};\phi);$$

(5) Trivial ϕ
Let X be a CW-complex. Let ϕ be trivial. Then $\widetilde{X}$ is ϕ-L^2-finite if and only if $b_n^{(2)}(\widetilde{X};\mathcal{N}(\pi_1(X)) = 0$ holds for all $n \geq 0$. If this is the case, we get

$$\chi^{(2)}(\widetilde{X};\phi) \;=\; 0;$$

(6) Tori
Let T^n be the n-dimensional torus for $n \geq 1$. Consider any $\phi \in H^1(T^n;\mathbb{Z})$. Then $\widetilde{T^n}$ is ϕ-L^2-finite and we get

$$\chi^{(2)}(\widetilde{T^n};\phi) \;=\; \begin{cases} [\mathbb{Z} : \operatorname{im}(\phi)] & \text{if } n = 1, \phi \neq 0; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The second statement follows from Theorem 2.5 (2) and (3) using the fact that for every $i \in \{0,1,2\}$ and every base point $x_i \in X_i$ the total space of the pullback of the universal covering of the component D of X containing $j_i(x_i)$ with $j_i|_C \colon C \to D$ for C the component of X_i containing x_i is $\pi_1(D)$-homeomorphic to $\pi_1(D) \times_{\pi_1(j_i)} \widetilde{C}$ for the universal covering $\pi_1(C, x_i) \to \widetilde{C} \to C$ of C.

The last statement is a special case of Lemma 2.10. Finally all other statements follow from the corresponding statements of Theorem 2.5. □

Lemma 2.18. *Let M be an admissible 3-manifold. Let $M_1, M_2, \ldots, M_r$ be its pieces in the Jaco-Shalen-Johannson decomposition. Consider $\phi \in H^1(M;\mathbb{Z})$. Let $\phi_i \in H^1(M_i;\mathbb{Z})$ be the pullback of ϕ with the inclusion $M_i \to M$ for $i = 1, 2, \ldots, r$.*

Then M_i is ϕ_i-L^2-finite for $i = 1, 2, \ldots, r$ and M is ϕ-L^2-finite and we have

$$\chi^{(2)}(\widetilde{M};\phi) \;=\; \sum_{i=1}^{r} \chi^{(2)}(\widetilde{M_i};\phi_i).$$

Proof. If M_i is Seifert, then it is ϕ_i-L^2-finite by Theorem 2.13. If M_i is hyperbolic, $b_n^{(2)}(\widetilde{M_i})$ vanishes for all $n \geq 0$ by [29, Theorem 0,1], and hence M_i is ϕ_i-L^2-finite by Theorem 3.2 (3) and Theorem 3.4. Now the claim follows from Theorem 2.17 (2) and (6) using the fact that the splitting tori in the Jaco-Shalen-Johannson decomposition are incompressible. □

Theorem 2.19 (The ϕ-L^2-Euler characteristic and the Thurston norm for graph manifolds)**.** *Let M be an admissible 3-manifold, which is a graph manifold and not homeomorphic to $S^1 \times D^2$. Consider $\phi \in H^1(M;\mathbb{Z})$. Then $\widetilde{M}$ is ϕ-L^2-finite and we get*

$$-\chi^{(2)}(\widetilde{M};\phi) \;=\; x_M(\phi).$$

Proof. This follows from Theorem 2.15. □

Example 2.20 ($S^1 \times D^2$ and $S^1 \times S^2$)**.** Consider a homomorphism $\phi \colon H_1(S^1 \times D^2) \xrightarrow{\cong} \mathbb{Z}$. Let k be the index $[\mathbb{Z} : \operatorname{im}(\phi)]$ if ϕ is non-trivial, and let $k = 0$ if ϕ is trivial. Then we conclude from (1.1), (1.4), Lemma 2.8 and Example 2.9

$$x_{S^1 \times D^2}(\phi) \;=\; 0 \quad \text{and} \quad -\chi^{(2)}(\widetilde{S^1 \times D^2};\phi) \;=\; k.$$

Hence we have to exclude $S^1 \times D^2$ in Theorem 2.19 and thus also in Theorem 0.2. Analogously we get

$$x_{S^1 \times S^2}(\phi) \;=\; 0 \quad \text{and} \quad -\chi^{(2)}(\widetilde{S^1 \times S^2};\phi) \;=\; 2 \cdot k,$$

so that we cannot replace "irreducible" by "prime" in Theorem 0.2.

3. About the Atiyah Conjecture

So far the definition and the analysis of the ϕ-twisted L^2-Euler characteristic has been performed on an abstract level. In order to ensure that the condition (μ, ϕ)-L^2-finite is satisfied and that the (μ, ϕ)-L^2-Euler characteristic contains interesting information, we will need further input, namely, the following Atiyah Conjecture.

3.1. **The Atiyah Conjecture.**

Definition 3.1 (Atiyah Conjecture)**.** We say that a torsion-free group G satisfies the *Atiyah Conjecture* if for any matrix $A \in M_{m,n}(\mathbb{Q}G)$ the von Neumann dimension $\dim_{\mathcal{N}(G)}(\ker(r_A))$ of the kernel of the $\mathcal{N}(G)$-homomorphism $r_A \colon \mathcal{N}(G)^m \to \mathcal{N}(G)^n$ given by right multiplication with A is an integer.

The Atiyah Conjecture can also be formulated for any field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ and matrices $A \in M_{m,n}(FG)$ and for any group with a bound on the order of its finite subgroups. However, we only need and therefore consider in this paper the case, where $F = \mathbb{Q}$ and G is torsion-free.

Theorem 3.2 (Status of the Atiyah Conjecture)**.**
(1) *If the torsion-free group G satisfies the Atiyah Conjecture, then also each of its subgroups satisfies the Atiyah Conjecture;*
(2) *Let $\mathcal{C}$ be the smallest class of groups which contains all free groups and is closed under directed unions and extensions with elementary amenable quotients. Suppose that G is a torsion-free group which belongs to $\mathcal{C}$.*
Then G satisfies the Atiyah Conjecture;
(3) *Let G be an infinite group which is the fundamental group of an admissible 3-manifold M. Suppose that one of the following conditions is satisfied:*
   - *M is not a closed graph manifold;*
   - *M is a closed graph manifold which admits a Riemannian metric of non-positive sectional curvature.*

   Then G is torsion-free and belongs to $\mathcal{C}$. In particular G satisfies the Atiyah Conjecture.
(4) *Let $\mathcal{D}$ be the smallest class of groups such that*
   - *The trivial group belongs to $\mathcal{D}$;*
   - *If $p \colon G \to A$ is an epimorphism of a torsion-free group G onto an elementary amenable group A and if $p^{-1}(B) \in \mathcal{D}$ for every finite group $B \subset A$, then $G \in \mathcal{D}$;*
   - *$\mathcal{D}$ is closed under taking subgroups;*
   - *$\mathcal{D}$ is closed under colimits and inverse limits over directed systems.*

   If the group G belongs to $\mathcal{D}$, then G is torsion-free and the Atiyah Conjecture holds for G.

   The class $\mathcal{D}$ is closed under direct sums, direct products and free products. Every residually torsion-free elementary amenable group belongs to $\mathcal{D}$;

Proof. (1) This follows from [32, Theorem 6.29 (2) on page 253].

(2) This is due to Linnell, see for instance [26] or [32, Theorem 10.19 on page 378].

(3) It suffices to show that $G = \pi_1(M)$ belongs to the class $\mathcal{C}$ appearing in assertion (2). By the proof of the Virtual Fibering Theorem due to Agol, Liu, Przytycki-Wise, and Wise [1, 2, 28, 35, 36, 44, 45] there exists a finite normal covering $p \colon \overline{M} \to M$ and a fiber bundle $F \to \overline{M} \to S^1$ for some compact connected orientable surface F. Hence it suffices to show that $\pi_1(F)$ belongs to $\mathcal{C}$. If F has non-empty boundary, this follows from the fact that $\pi_1(F)$ is free. If M is closed, the commutator subgroup of $\pi_1(F)$ is free and hence $\pi_1(F)$ belongs to $\mathcal{C}$. Now assertion (3) follows from assertion (2).

(4) This result is due to Schick, see for instance [39] or [32, Theorem 10.22 on page 379]. □

3.2. L^2-acyclic Atiyah pair.

Definition 3.3 (L^2-acyclic Atiyah-pair)**.** An *L^2-acyclic Atiyah-pair* (μ, ϕ) for a finite connected CW-complex X consists of group homomorphisms $\mu\colon \pi_1(X) \to G$ and $\phi\colon G \to \mathbb{Z}$ such that the G-covering $\overline{X} \to X$ associated to μ is L^2-acyclic, i.e., the nth L^2-Betti number $b_n^{(2)}(\overline{X}; \mathcal{N}(G))$ vanish for every $n \geq 0$, and G is torsion-free and satisfies the Atiyah Conjecture.

Notice that the conditions appearing in Definition 3.3 only concern G and μ but not ϕ. The Atiyah Conjecture enters in this paper because of the following theorem.

Theorem 3.4 (The Atiyah Conjecture and the (μ, ϕ)-L^2-Euler characteristic)**.** *Let X be a connected finite CW-complex. Suppose that (μ, ϕ) is an L^2-acyclic Atiyah-pair. Then X is (μ, ϕ)-L^2-finite, and the (μ, ϕ)-L^2-Euler characteristic $\chi^{(2)}(X; \mu, \phi)$ is an integer.*

Theorem 3.4 will be a direct consequence of Lemma 2.6, Lemma 2.8 and the following Theorem 3.6 (4), whose formulation requires some preparation.

3.3. The division closure $\mathcal{D}(G)$ of $\mathbb{Q}G$ in $\mathcal{U}(G)$.

Let S be a ring with subring $R \subset S$. The *division closure* $\mathcal{D}(R \subset S)$ is the smallest subring of S which contains R and is division closed, i.e., every element in $\mathcal{D}(R \subset S)$ which is a unit in S is already a unit in $\mathcal{D}(R \subset S)$.

Notation 3.5 ($\mathcal{D}(G)$)**.** Let G be a group. Denote by $\mathcal{U}(G)$ the algebra of operators affiliated to the (complex) group von Neumann algebra $\mathcal{N}(G)$, see [32, Section 8.2]. (This is the Ore localization of $\mathcal{N}(G)$ with respect to the set of non-zero-divisors of $\mathcal{N}(G)$, see [32, Theorem 8.22 on page 327].) Denote by $\mathcal{D}(G)$ the division closure of $\mathbb{Q}G$ considered as a subring of $\mathcal{U}(G)$.

The proof of Theorem 3.6 will be based on ideas of Peter Linnell from [26] which have been explained in detail and a little bit extended in [32, Chapter 10] and [38].

Theorem 3.6 (Main properties of $\mathcal{D}(G)$)**.** *Let G be a torsion-free group.*

1. *The group G satisfies the Atiyah Conjecture if and only if $\mathcal{D}(G)$ is a skew field;*
2. *Suppose that G satisfies the Atiyah Conjecture. Let C_* be a projective $\mathbb{Q}G$-chain complex. Then we get for all $n \geq 0$*
$$b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*\big) = \dim_{\mathcal{D}(G)}\big(H_n(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*)\big).$$
In particular $b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*\big)$ is either infinite or an integer;
3. *Suppose that G satisfies the Atiyah Conjecture. Let $\phi\colon G \to \mathbb{Z}$ be a surjective group homomorphism. Let $K \subseteq G$ be the kernel of ϕ. Fix an element $\gamma \in G$ with $\phi(\gamma) = 1$. If we define the semi-direct product $K \rtimes \mathbb{Z}$ with respect to the conjugation automorphism $c_\gamma\colon K \to K$ of γ on K, we can identify G with $K \rtimes \mathbb{Z}$ and ϕ becomes the canonical projection $G = K \rtimes \mathbb{Z} \to \mathbb{Z}$. Let $\mathcal{D}(K)_t[u^{\pm 1}]$ be the ring of twisted Laurent polynomials with respect to the automorphism $t\colon \mathcal{D}(K) \xrightarrow{\cong} \mathcal{D}(K)$ coming from $c_\gamma\colon K \to K$.*

   Then $\mathcal{D}(K_t[u^{\pm 1}]$ is a non-commutative principal ideal domain, i.e., it has no non-trivial zero-divisor and every left ideal is a principal left ideal and every right ideal is a principal right ideal. Furthermore the set T of non-zero elements in $\mathcal{D}(K)_t[u^{\pm 1}]$ satisfies the Ore condition and there is a canonical isomorphism of skew fields
$$T^{-1}\mathcal{D}(K)_t[u^{\pm 1}] \xrightarrow{\cong} \mathcal{D}(G);$$

(4) *Let G be a torsion-free group which satisfies the Atiyah Conjecture. Let $\phi\colon G \to \mathbb{Z}$ be a surjective group homomorphism. Denote by $i\colon K \to G$ the inclusion of the kernel K of ϕ. Let C_* be a finitely generated projective $\mathbb{Q}G$-chain complex such that $b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*)$ vanishes for all $n \geq 0$. Denote by i^*C_* the restriction of the $\mathbb{Q}G$-chain complex C_* to a $\mathbb{Q}K$-chain complex.*

Then $H_n(\mathcal{D}(K) \otimes_{\mathbb{Q}K} i^*C_*)$ and $(H_n(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*)$ are finitely generated free as $\mathcal{D}(K)$-modules, $b_n^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Q}K} i^*C_*\big)$ is finite, and we have

$$\begin{aligned} b_n^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Q}K} i^*C_*\big) &= \dim_{\mathcal{D}(K)}\big(H_n(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*)\big). \\ &= \dim_{\mathcal{D}(K)}\big(H_n(\mathcal{D}(K) \otimes_{\mathbb{Q}K} i^*C_*)\big) \end{aligned}$$

for all $n \geq 0$.

Proof. (1) This is proved in the case $F = \mathbb{C}$ in [32, Lemma 10.39 on page 388]. The proof goes through for an arbitrary field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ without modifications.

(2) We have the following commutative diagram of inclusion of rings

$$\begin{array}{ccc} \mathbb{Q}G & \longrightarrow & \mathcal{N}(G) \\ \downarrow & & \downarrow \\ \mathcal{D}(G) & \longrightarrow & \mathcal{U}(G). \end{array}$$

There is a dimension function $\dim_{\mathcal{U}G}$ for arbitrary (algebraic) $\mathcal{U}(G)$-modules such that for any $\mathcal{N}(G)$-module M we have $\dim_{\mathcal{U}G}(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} M) = \dim_{\mathcal{N}(G)}(M)$ and basic features like additivity and continuity and cofinality are still satisfied, see [32, Theorem 8.29 on page 330]. Moreover, $\mathcal{U}(G)$ is flat over $\mathcal{N}(G)$, see [32, Theorem 8.22 (2) on page 327]. Since $\mathcal{D}(G)$ is a skew field by assertion (1), $\mathcal{U}(G)$ is also flat as a $\mathcal{D}(G)$-module and we have for any $\mathcal{D}(G)$-module M the equality $\dim_{\mathcal{U}(G)}(\mathcal{U}(G) \otimes_{\mathcal{D}(G)} M) = \dim_{\mathcal{D}(G)}(M)$. We conclude

$$\begin{aligned} b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*) &= \dim_{\mathcal{N}(G)}\big(H_n(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*)\big) \\ &= \dim_{\mathcal{U}(G)}\big(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} H_n(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*)\big) \\ &= \dim_{\mathcal{U}(G)}\big(H_n(\mathcal{U}(G) \otimes_{\mathcal{N}(G)} \mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*)\big) \\ &= \dim_{\mathcal{U}(G)}\big(H_n(\mathcal{U}(G) \otimes_{\mathbb{Q}G} C_*)\big) \\ &= \dim_{\mathcal{U}(G)}\big(H_n(\mathcal{U}(G) \otimes_{\mathcal{D}(G)} \mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*)\big) \\ &= \dim_{\mathcal{U}(G)}\big(\mathcal{U}(G) \otimes_{\mathcal{D}(G)} H_n(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*)\big) \\ &= \dim_{\mathcal{D}(G)}\big(H_n(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*)\big). \end{aligned}$$

This finishes the proof of assertion (2).

(3) Since $G = K \rtimes \mathbb{Z}$ satisfies the Atiyah Conjecture by assumption, the same is true for K by Theorem 3.2 (1). We know already from assertion (1) that $\mathcal{D}(K)$ and $\mathcal{D}(G)$ are skew fields. The ring $\mathcal{D}(K)_t[u^{\pm 1}]$ is a non-commutative principal ideal domain, see [8, 2.1.1 on page 49] or [6, Proposition 4.5]. The claim that the Ore localization $T^{-1}\mathcal{D}(K)_t[u^{\pm 1}]$ exists and is isomorphic to $\mathcal{D}(G)$ is proved in the case $F = \mathbb{C}$ in [32, Lemma 10.60 on page 399]. The proof goes through for an arbitrary field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ without modifications.

(4) We write the group ring $\mathbb{Q}G$ as the ring $\mathbb{Q}K_t[u^{\pm 1}]$ of twisted Laurent polynomials with coefficients in $\mathbb{Q}K$. We get a commutative diagram of inclusions of rings, where $\mathcal{D}(K)_t[u^{\pm 1}]$ is a (non-commutative) principal ideal domain and $\mathcal{D}(K)$ and $\mathcal{D}(G)$ are skew fields

$$\begin{array}{ccc}
\mathbb{Q}K & \longrightarrow & \mathbb{Q}G = \mathbb{Q}K_t[u^{\pm 1}] \\
\downarrow & & \\
\mathcal{D}(K) & \swarrow & \downarrow \\
\downarrow & & \\
\mathcal{D}(K)_t[u^{\pm 1}] & \longrightarrow & \mathcal{D}(G) \\
\downarrow & & \downarrow \mathrm{id} \\
T^{-1}\mathcal{D}(K)_t[u^{\pm 1}] & \xrightarrow{\cong} & \mathcal{D}(G).
\end{array}$$

Since C_* is a finitely generated projective $\mathbb{Q}G$-chain complex by assumption, the $\mathcal{D}(K)_t[u^{\pm 1}]$-chain complex $\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*$ is finitely generated projective. Since $\mathcal{D}(K \rtimes \mathbb{Z})$ is a (non-commutative) principal ideal domain, it follows from [7, p. 494], that there exist integers $r, s \geq 0$ and non-zero elements $p_1, p_2, \ldots, p_s \in \mathcal{D}(K)_t[u^{\pm 1}]$ such that we get an isomorphism of $\mathcal{D}(K)_t[u^{\pm 1}]$-modules

$$H_n\big(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*\big) \cong \mathcal{D}(K)_t[u^{\pm 1}]^r \oplus \bigoplus_{i=1}^{s} \mathcal{D}(K)_t[u^{\pm 1}]/(p_i).$$

Since $\mathcal{D}(G) = T^{-1}\mathcal{D}(K)_t[u^{\pm 1}]$ is flat over $\mathcal{D}(K)_t[u^{\pm 1}]$, we conclude using assertion (2)

$$\begin{aligned}
r &= \dim_{\mathcal{D}(G)}\big(\mathcal{D}(G) \otimes_{\mathcal{D}(K)_t[u^{\pm 1}]} H_n(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*)\big) \\
&= \dim_{\mathcal{D}(G)}\big(H_n\big(\mathcal{D}(G) \otimes_{\mathcal{D}(K)_t[u^{\pm 1}]} \mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*\big)\big) \\
&= \dim_{\mathcal{D}(G)}\big(H_n\big(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*\big)\big) \\
&= b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*).
\end{aligned}$$

Since by assumption $b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*) = 0$ holds, we conclude

$$H_n(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*) \ \cong\ \bigoplus_{i=1}^{s} \mathcal{D}(K)_t[u^{\pm 1}]/(p_i).$$

Lemma 4.3 implies that $\mathcal{D}(K)_t[u^{\pm 1}]/(p_i)$ considered as $\mathcal{D}(K)$-module is finitely generated free. This implies that $H_n\big(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*\big)$ considered as $\mathcal{D}(K)$-module is finitely generated free. Assertion (2) applied to K instead of G implies

$$b_n^{(2)}\big(\mathcal{N}(K) \otimes_{FK} i^*C_*\big) = \dim_{\mathcal{D}(K)}\big(H_n\big(\mathcal{D}(K) \otimes_{\mathbb{Z}K} i^*C_*\big)\big).$$

There is an obvious isomorphism of $\mathcal{D}(K)$-chain complexes

$$\mathcal{D}(K) \otimes_{\mathbb{Z}K} i^*C_* \xrightarrow{\cong} \mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Z}G} C_*, \quad x \otimes_{\mathbb{Z}K} y \mapsto x \otimes_{\mathbb{Z}G} y$$

which induces an isomorphism of $\mathcal{D}(K)$-modules

$$H_n\big(\mathcal{D}(K) \otimes_{\mathbb{Q}K} i^*C_*\big) \xrightarrow{\cong} H_n\big(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*\big).$$

Hence we get

$$\dim_{\mathcal{D}(K)}\big(H_n\big(\mathcal{D}(K) \otimes_{FK} i^*C_*\big)\big) \ = \ \dim_{\mathcal{D}(K)}\big(H_n\big(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Q}G} C_*\big)\big).$$

This finishes the proof of Theorem 3.6 and hence also of Theorem 3.4. □

4. The negative of the (μ, ϕ)-L^2-Euler characteristic is a lower bound for the Thurston norm

Theorem 4.1 (The negative of the (μ, ϕ)-L^2-Euler characteristic is a lower bound for the Thurston norm)**.** *Let $M \neq S^1 \times D^2$ be an admissible 3-manifold and let (μ, ϕ) be an L^2-acyclic Atiyah-pair. Then M is (μ, ϕ)-L^2-finite and we get*

$$-\chi^{(2)}(M; \mu, \phi) \ \leq \ x_M(\phi \circ \mu).$$

Its proof needs some preparation.

4.1. **Some preliminaries about twisted Laurent polynomials over skew fields.** In this subsection we consider a skew field $\mathcal{D}$ together with an automorphism $t\colon \mathcal{D} \to \mathcal{D}$ of skew fields. Let $\mathcal{D}_t[u^{\pm 1}]$ be the ring of twisted Laurent polynomials over $\mathcal{D}$. For a non-trivial element $x = \sum_{i\in\mathbb{Z}} d_i \cdot u^i$ in $\mathcal{D}_t[u^{\pm 1}]$ we define its degree to be the natural number

$$\deg(x) \quad := \quad n_+ - n_- \tag{4.2}$$

where n_- the smallest integer such that d_{n_-} does not vanish, and n_+ is largest integer such that d_{n_+} does not vanish.

Lemma 4.3. *Consider a non-trivial element x in $\mathcal{D}_t[u^{\pm 1}]$. Then the $\mathcal{D}_t[u^{\pm 1}]$-homomorphism $r_x\colon \mathcal{D}_t[u^{\pm 1}] \to \mathcal{D}_t[u^{\pm 1}]$ given by right multiplication with x is injective and its cokernel has finite dimension over $\mathcal{D}$, namely,*

$$\dim_{\mathcal{D}}(\operatorname{coker}(r_x)) = \deg(x).$$

Proof. Notice that for two non-trivial elements x and y we have $n_-(xy) = n_-(x) + n_-(y)$, $n_+(xy) = n_+(x) + n_+(y)$, and $\deg(xy) = \deg(x) + \deg(y)$. Now one easily checks that r_x is injective and that a $\mathcal{D}$-basis for the cokernel of r_x is given by the image of the subset $\{u^0, u^1, \ldots, u^{\deg(x)-1}\}$ of $\mathcal{D}_t[u^{\pm 1}]$ under the canonical projection $\mathcal{D}_t[u^{\pm 1}] \to \operatorname{coker}(r_x)$.

□

Lemma 4.4. *Consider integers k, n with $0 \le k$, $1 \le n$ and $k \le n$. Denote by I_k^n the (n,n)-matrix whose first k entries on the diagonal are 1 and all of whose other entries are zero. Let A be an (n,n)-matrix over $\mathcal{D}$. Suppose that the $\mathcal{D}_t[u^{\pm 1}]$-map*

$$r_{A+u\cdot I_k^n}\colon \mathcal{D}_t[u^{\pm 1}]^n \to \mathcal{D}_t[u^{\pm 1}]^n$$

given by right multiplication with $A + u \cdot I_k^n$ is injective.

Then the dimension over $\mathcal{D}$ of its cokernel is finite and satisfies

$$\dim_{\mathcal{D}}\bigl(\operatorname{coker}\bigl(r_{A+u\cdot I_k^n}\colon \mathcal{D}_t[u^{\pm 1}]^n \to \mathcal{D}_t[u^{\pm 1}]^n\bigr)\bigr) \ \le\ k.$$

Proof. We use induction over the size n. If $k = n$, the claim has already been proved in [21, Proposition 9.1]. So we can assume in the sequel $k < n$. We perform certain row and column operations on matrices $B \in M_{n,n}(\mathcal{D}_t[u^{\pm 1}])$ and it will be obvious that they will respect the property that $r_B\colon \mathcal{D}_t[u^{\pm 1}]^n \to \mathcal{D}_t[u^{\pm 1}]^n$ is injective and will not change $\dim_{\mathcal{D}}\bigl(\operatorname{coker}\bigl(r_B\colon \mathcal{D}_t[u^{\pm 1}]^n \to \mathcal{D}_t[u^{\pm 1}]^n\bigr)\bigr)$. With the help of these operations we will reduce the size of A by 1 and this will finish the induction step. To keep the exposition easy, we explain the induction step from $(n-1)$ to n in the special case $k = 3$ and $n = 5$. The general induction step is completely analogous.

So we start with

$$A + u \cdot I_3^5 = \begin{pmatrix} u+* & * & * & * & * \\ * & u+* & * & * & * \\ * & * & u+* & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

where here and in the sequel $*$ denotes some element in $\mathcal{D}$. We first treat the case, where the $(2,2)$-submatrix sitting in the right lower corner is non-trivial. By interchanging rows and columns and right multiplying a row with a non-trivial element in $\mathcal{D}$, we can achieve

$$\begin{pmatrix} u+* & * & * & * & * \\ * & u+* & * & * & * \\ * & * & u+* & * & * \\ * & * & * & * & * \\ * & * & * & * & 1 \end{pmatrix}$$

By subtracting appropriate right $\mathcal{D}$-multiplies of the lowermost row from the other rows and subtracting appropriate left $\mathcal{D}$-multiplies of the rightmost column from the other columns, we achieve

$$\begin{pmatrix} u+* & * & * & * & 0 \\ * & u+* & * & * & 0 \\ * & * & u+* & * & 0 \\ * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For this matrix the claim follows from the induction hypothesis applied to the $(4,4)$-matrix obtained by deleting the lowermost row and the rightmost column.

It remains to treat the case, where the matrix looks like

$$\begin{pmatrix} u+* & * & * & * & * \\ * & u+* & * & * & * \\ * & * & u+* & * & * \\ * & * & * & 0 & 0 \\ * & * & * & 0 & 0 \end{pmatrix}$$

At least one of the entries in the lowermost row must be non-trivial since the map induced by right multiplication with it is assumed to be injective. By interchanging rows and columns and right multiplying a row with a non-trivial element in $\mathcal{D}$, we can achieve

$$\begin{pmatrix} u+* & * & * & * & * \\ * & u+* & * & * & * \\ * & * & u+* & * & * \\ * & * & * & 0 & 0 \\ 1 & * & * & 0 & 0 \end{pmatrix}$$

By subtracting appropriate $\mathcal{D}$-multiples of first column from the second and third column, we can arrange

$$\begin{pmatrix} u+* & *\cdot u+* & *\cdot u+* & * & * \\ * & u+* & * & * & * \\ * & * & u+* & * & * \\ * & * & * & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By subtracting appropriate right $\mathcal{D}$-multiples of last row from the other rows we can achieve

$$\begin{pmatrix} 0 & *\cdot u+* & *\cdot u+* & * & * \\ 0 & u+* & * & * & * \\ 0 & * & u+* & * & * \\ 0 & * & * & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By subtracting right $\mathcal{D}$-multiples of the second and the third row from the first row and then interchanging rows we can arrange

$$\begin{pmatrix} 0 & u+* & * & * & * \\ 0 & * & u+* & * & * \\ 0 & * & * & * & * \\ 0 & * & * & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the induction hypothesis applies to the matrix obtained by deleting the first column and the lowermost row, we have finished the induction step, and hence the proof of Lemma 4.4. □

4.2. **Proof of Theorem 4.1.**

Proof of Theorem 4.1. We have already shown in Theorem 3.4 that M is (μ,ϕ)-L^2-finite.

Because of (1.1) and Lemma 2.8 we can assume without loss of generality that ϕ is surjective. Let K be the kernel of ϕ and γ be an element in G with $\phi(\gamma) = 1$. It follows easily from [43, Section 1] that we can find an oriented surface $\Sigma \subset M$ with components $\Sigma_1, \dots, \Sigma_l$ and non-zero $r_1, \dots, r_l \in \mathbb{N}$ with the following properties:

- $r_1[\Sigma_1] + \dots + r_l[\Sigma_l]$ is dual to ϕ,
- $\sum_{i=1}^{l} -r_i\chi(\Sigma_i) \le x_M(\phi)$,
- $M \setminus \Sigma$ is connected.

(Here we use that $M \neq S^1 \times D^2$ and that N is irreducible.) For $i = 1, \dots, l$ we pick disjoint oriented tubular neighborhoods $\Sigma_i \times [0,1]$ and we identify Σ_i with $\Sigma_i \times \{0\}$. We write $M' := M \setminus \cup_{i=1}^{l} \Sigma_i \times [0,1]$. We pick once and for all a base point p in M' and we denote by $\widetilde{M}$ the universal cover of M. We write $\pi = \pi_1(M,p)$. For $i = 1, \dots, l$ we also pick a curve ν_i' based at p which intersects Σ_i precisely once in a positive direction and does not intersect any other component of Σ. Put $\nu_i = \mu(\nu_i')$. Note that $\phi(\nu_i) = r_i$. Finally for $i = 1, \dots, l$ we put

$$n_i := -\chi(\Sigma_i) + 2.$$

Following [13, Section 4] we now pick an appropriate CW-structure for M and we pick appropriate orientations and lifts of the cells to the universal cover. The resulting boundary maps are described in detail [13, Section 4]. In order to keep the notation manageable we now restrict to the case $l = 2$, the general case is completely analogous.

Let $\overline{M} \to M$ be the G-covering associated to μ. It follows from the discussion in [13, Section 4] and the definitions that the cellular $\mathbb{Z}G$-chain complex $C_*(\overline{M})$ of $\overline{M}$ looks like

$$0 \longrightarrow \mathbb{Z}[G]^4 \xrightarrow{B_3} \mathbb{Z}[G]^{4+2n_1+2n_2+s} \xrightarrow{B_2} \mathbb{Z}[G]^{4+2n_1+2n_2+s} \xrightarrow{B_1} \mathbb{Z}[G]^4 \longrightarrow 0$$

where s is a natural number and the matrices B_3, B_2, B_1 are matrices of the form

$$B_3 \;=\; \begin{array}{ccccccc|c} n_1 & n_2 & 1 & 1 & 1 & 1 & s+n_1+n_2 & \\ \hline * & 0 & 1 & -\nu_1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -z_1 & 0 & 0 & * & 1 \\ 0 & * & 0 & 0 & 1 & -\nu_2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -z_2 & * & 1 \end{array}$$

$$B_2 \;=\; \begin{array}{ccccccccc|c} 1 & 1 & n_1 & n_1 & n_2 & n_2 & 1 & 1 & s & \\ \hline * & 0 & \mathrm{id}_{n_1} & -\nu_1\,\mathrm{id}_{n_1} & 0 & 0 & 0 & 0 & 0 & n_1 \\ 0 & * & 0 & 0 & \mathrm{id}_{n_2} & -\nu_2\,\mathrm{id}_{n_2} & 0 & 0 & 0 & n_2 \\ 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 1 \\ 0 & 0 & * & * & * & * & * & * & * & s+n_1+n_2 \end{array}$$

$$B_1 \;=\; \begin{array}{cccc|c} 1 & 1 & 1 & 1 & \\ \hline 1 & -\nu_1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\nu_2 & 1 \\ * & 0 & 0 & 0 & n_1 \\ 0 & * & 0 & 0 & n_1 \\ 0 & 0 & * & 0 & n_2 \\ 0 & 0 & 0 & * & n_2 \\ 1 & -x_1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -x_2 & 1 \\ * & * & * & * & s \end{array}$$

where $x_1, x_2, z_1, z_2 \in K$, all the entries of the matrices marked by $*$ lie in $\mathbb{Z}K$ and the entries above the horizontal line and right to the vertical arrow indicate the size of the blocks. (Note that in [13] we view the elements of $\mathbb{Z}[G]^n$ as column vectors whereas we now view them as row vectors.)

Define submatrices B_i' of B_i for $i = 1, 2, 3$ by

$$B_3' = \begin{pmatrix} 1 & -\nu_1 & 0 & 0 \\ 1 & -z_1 & 0 & 0 \\ 0 & 0 & 1 & -\nu_2 \\ 0 & 0 & 1 & -z_2 \end{pmatrix}$$

$$B_2' = \begin{array}{ccccc|c} n_1 & n_1 & n_2 & n_2 & s & \\ \hline \mathrm{id}_{n_1} & -\nu_1 \,\mathrm{id}_{n_1} & 0 & 0 & 0 & n_1 \\ 0 & 0 & \mathrm{id}_{n_2} & -\nu_2 \,\mathrm{id}_{n_2} & 0 & n_2 \\ * & * & * & * & * & s + n_1 + n_2 \end{array}$$

$$B_1' = \begin{pmatrix} 1 & -\nu_1 & 0 & 0 \\ 0 & 0 & 1 & -\nu_2 \\ 1 & -x_1 & 0 & 0 \\ 0 & 0 & 1 & -x_2 \end{pmatrix}$$

where B_1' and B_3' are $(4,4)$-matrices and B_2' is a $(2n_1+2n_2+s, 2n_1+2n_2+s)$-matrix. For $i = 1, 2, 3$ let $C_*[i]$ be the $\mathbb{Z}G$-chain complex concentrated in dimensions i and $i-1$ whose i-th differential is given by the matrix B_i'. There is an appropriate finitely generated free $\mathbb{Z}G$-chain complex C_*' concentrated in dimensions 2, 1 and 0 of the shape

$$\ldots \to 0 \to 0 \to \mathbb{Z}G^{2n_1+2n_2+s} \to \mathbb{Z}G^{4+2n_1+2n_2+s} \to \mathbb{Z}G^4$$

and obvious short based exact sequences of finitely generated based free $\mathbb{Z}G$-chain complexes

$$0 \to C_*' \to C_*(\overline{M}) \to C_*[3] \to 0$$

and

$$0 \to C_*[1] \to C_*' \to C_*[2] \to 0.$$

Consider $\nu' \in G$ with $\phi(\nu) \neq 0$ and $x' \in H$. The matrix $\begin{pmatrix} 1 & -\nu' \\ 1 & x' \end{pmatrix}$ can be transformed by subtracting the first row from the second row to the matrix $\begin{pmatrix} 1 & -\nu' \\ 0 & \nu' + x' \end{pmatrix}$. The map $r_{\nu'+x'} \colon \mathcal{D}(K)_t[u^{\pm 1}] \to \mathcal{D}(K)_t[u^{\pm 1}]$ is injective and its cokernel has dimension $|\phi(\nu')|$ over $\mathcal{D}(K)$ by Lemma 4.3. Hence the map $\mathcal{D}(K)_t[u^{\pm 1}]^2 \to \mathcal{D}(K)_t[u^{\pm 1}]^2$ given by right multiplication with $\begin{pmatrix} 1 & -\nu' \\ 0 & \nu' + x' \end{pmatrix}$ is injective and its cokernel has dimension $|\phi(\nu')|$ over $\mathcal{D}(K)$. We conclude from Theorem 3.6 for $l = 1, 3$ using notation (2.1) and defining for a skewfield $\mathcal{D}$ and a $\mathcal{D}$-chain complex D_* its Betti number $b_n(D_*)$ to be $\dim_{\mathcal{D}}(H_n(D_*))$.

$$\begin{aligned} b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*[l]\big) &= 0 \quad \text{for } n \geq 0; \\ b_n\big(\mathcal{D}(K) \otimes_{\mathbb{Z}K} i^*C_*[l]\big) &= \begin{cases} 0 & \text{if } n \neq l-1; \\ r_1 + r_2 & \text{if } n = l-1; \end{cases} \\ \chi^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*[l]\big) &= r_1 + r_2. \end{aligned}$$

Since $b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(\overline{M})\big) = 0$ holds for all $n \geq 0$ by assumption, we get

$$b_n^{(2)}\big(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*[2]\big) = 0 \quad \text{for } n \geq 0.$$

We conclude from from Theorem 3.6 (2)

$$\dim_{\mathcal{D}(G)}\Big(\ker\big(r_{B_2'} \colon \mathcal{D}(G)^{1+2n_1+2n_2+s} \to \mathcal{D}(G)^{1+2n_1+2n_2+s}\big)\Big) = 0.$$

Theorem 3.6 (3) implies that

$$r_{B_2'} \colon \mathcal{D}(K)_t[u^{\pm 1}]^{1+2n_1+2n_2+s} \to \mathcal{D}(K)_t[u^{\pm 1}]^{1+2n_1+2n_2+s}$$

is injective. We get using Theorem 3.6 (2)

(4.5)

$$\begin{aligned}
& \chi^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*(\overline{M})\big) \;=\; \sum_{l=1}^{3} \chi^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*[l]\big) \\
= \;& 2r_1 + 2r_2 - \chi^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*[2]\big) \\
= \;& 2r_1 + 2r_2 - \chi\big(\mathcal{D}(K) \otimes_{\mathbb{Z}K} i^*C_*[2]\big) \\
= \;& 2r_1 + 2r_2 \\
& - \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_{B_2'} \colon \mathcal{D}(K)_t[u^{\pm 1}]^{1+2n_1+2n_2+s} \to \mathcal{D}(K)_t[u^{\pm 1}]^{1+2n_1+2n_2+s}\big)\big).
\end{aligned}$$

Since $\dim_{\mathcal{N}(G)}\big(\ker\big(\mathrm{id}_{\mathcal{N}(G)} \otimes_{\mathbb{Z}G} r_{B_2'}\big) \colon \mathcal{N}(G)^{1+2n_1+2n_2+s} \to \mathcal{N}(G)^{1+2n_1+2n_2+s}\big)\big)$ coincides with $b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*')$ and $b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*')$ is zero, one of the entries in the rightmost column of B_2' is non-trivial. By switching rows and multiplying a row with a non-trivial element in $\mathcal{D}(K)$, we can arrange that the entry in the lower right corner is 1. By elementary row and column operations we can arrange that the lowermost row and the rightmost column have all entries zero except the element in the lower right corner which is still equal to 1. By iterating this process we can transform B_2' by such row and column operations into a matrix of the shape

$$B_2'' \;=\; \begin{array}{ccccc|c}
n_1 & n_1 & n_2 & n_2 & s & \\ \hline
\mathrm{id}_{n_1} & -\nu_1\,\mathrm{id}_{n_1} & 0 & 0 & 0 & n_1 \\
0 & 0 & \mathrm{id}_{n_2} & -\nu_2\,\mathrm{id}_{n_2} & 0 & n_2 \\
* & * & * & * & * & n_1+n_2 \\
0 & 0 & 0 & 0 & \mathrm{id}_s & s
\end{array}$$

Let B_2''' be the $(2n_1+2n_2)$-submatrix of B_s'' given by

$$B_2''' \;=\; \begin{array}{cccc|c}
n_1 & n_1 & n_2 & n_2 & \\ \hline
\mathrm{id}_{n_1} & -\nu_1\,\mathrm{id}_{n_1} & 0 & 0 & n_1 \\
0 & 0 & \mathrm{id}_{n_2} & -\nu_2\,\mathrm{id}_{n_2} & n_2 \\
* & * & * & * & n_1+n_2
\end{array}$$

An inspection of the proof of [9, Lemma 9.3] shows that by elementary row and column operations and taking block sum with triangular matrices having id on each diagonal entry, we can transform B_2''' into a matrix of the shape

$$B_2'''' = A''' + u \cdot I^{2r_r \cdot n_1 + 2r_2 \cdot n_2}_{r_1 \cdot n_1 + r_2 \cdot n_2}$$

for some matrix $A'''' \in M_{2n_1+2n_2, 2n_1+2n_2}(\mathcal{D}(K))$ and $I^{2r_r \cdot n_1 + 2r_2 \cdot n_2}_{r_1 \cdot n_1 + r_2 \cdot n_2}$ as introduced in Lemma 4.4. Obviously we have

$$\begin{aligned}
& \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_{B_2'} \colon \mathcal{D}(K)_t[u^{\pm 1}]^{1+2n_1+2n_2+s} \to \mathcal{D}(K)_t[u^{\pm 1}]^{1+2n_1+2n_2+s}\big)\big) \\
& \qquad = \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_{B_2''''} \colon \mathcal{D}(K)_t[u^{\pm 1}]^{2n_1+2n_2} \to \mathcal{D}(K)_t[u^{\pm 1}]^{2n_1+2n_2}\big)\big).
\end{aligned}$$

We conclude from Lemma 4.4 applied to B_2''''

$$\begin{aligned}
& \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_{B_2''''} \colon \mathcal{D}(K)_t[u^{\pm 1}]^{2n_1+2n_2} \to \mathcal{D}(K)_t[u^{\pm 1}]^{2n_1+2n_2}\big)\big) \\
& \qquad \qquad \le r_1 \cdot n_1 + r_2 \cdot n_2.
\end{aligned}$$

Hence we get

$$\begin{aligned}
(4.6) \quad & \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_{B_2'} \colon \mathcal{D}(K)_t[u^{\pm 1}]^{1+2n_1+2n_2+s} \to \mathcal{D}(K)_t[u^{\pm 1}]^{1+2n_1+2n_2+s}\big)\big) \\
& \qquad \qquad \le r_1 \cdot n_1 + r_2 \cdot n_2.
\end{aligned}$$

We conclude from (4.5) and (4.6)

$$\chi^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*(\overline{M})\big) \ge 2r_1 + 2r_2 - r_1 \cdot n_1 - r_2 \cdot n_2.$$

This together with Lemma 2.6 implies

$$\begin{aligned} -\chi^{(2)}(M;\mu,\phi) &= -\chi^{(2)}\big(\mathcal{N}(K)\otimes_{\mathbb{Z}K} i^*C_*(\overline{M})\big) \\ &\leq r_1n_1 + r_2n_2 - 2r_1 - 2r_2 \\ &= r_1(n_1-2) + r_2(n_2-2) \;=\; -r_1\chi(\Sigma_1) - r_2\chi(\Sigma_2) \;\leq\; x_M(\phi). \end{aligned}$$

This finishes the proof of Theorem 4.1. □

5. Fox calculus and the (μ,ϕ)-L^2-Euler characteristic

The following calculations of the (μ,ϕ)-L^2-Euler characteristic from a presentation of the fundamental group is adapted to the corresponding calculation for the L^2-torsion function appearing in [14, Theorem 2.1] and will be used when we will compare these two invariants and the higher order Alexander polynomials. For information about the Fox matrix and the Fox calculus we refer for instance to [3, 9B on page 123], and [11].

Theorem 5.1 (Calculation of the (μ,ϕ)-L^2-Euler characteristic from a presentation of the fundamental group)**.** *Let M be an admissible 3-manifold with fundamental group π. Let*

$$\pi = \langle x_1, x_2, \ldots, x_a \mid R_1, R_2, \ldots, R_b\rangle$$

be a presentation of π. Let the (a,b)-matrix over $\mathbb{Z}\pi$

$$F = \begin{pmatrix} \frac{\partial R_1}{\partial x_1} & \cdots & \frac{\partial R_1}{\partial x_a} \\ \vdots & \ddots & \vdots \\ \frac{\partial R_b}{\partial x_1} & \cdots & \frac{\partial R_b}{\partial x_a} \end{pmatrix}$$

be the Fox matrix of the presentation.

Let G be a torsion-free group which satisfies the Atiyah Conjecture. Consider two group homomorphisms $\mu\colon \pi \to G$ and $\phi\colon G \to \mathbb{Z}$. Suppose that μ is non-trivial and that ϕ is surjective.

Let $i\colon K \to G$ be the inclusion of the kernel $K = \ker(\phi)$ of ϕ and let $t\colon \mathcal{D}(K) \xrightarrow{\cong} \mathcal{D}(K)$ be the automorphism introduced in Theorem 3.6 (3)*. Let $\overline{M} \to M$ be the G-covering associated to μ.*

1. *Suppose that ∂M is non-empty and and that $a = b+1$. Choose $i \in \{1,2,\ldots,r\}$ with $|\mu(x_i)| = \infty$. Let A be the $(a-1,a-1)$-matrix with entries in $\mathbb{Z}G$ obtained from the Fox matrix F by deleting the i-th column and then applying the homomorphism $M_{a-1,a-1}(\mathbb{Z}\pi) \to M_{a-1,a-1}(\mathbb{Z}G)$ induced by $\mu\colon \pi \to G$.*

   Then $b_n^{(2)}(\overline{M};\mathcal{N}(G))$ vanishes for all $n \geq 0$ if and only if we have $\dim_{\mathcal{N}(G)}\big(\ker(r_A\colon \mathcal{N}(G)^{a-1} \to \mathcal{N}(G)^{a-1})\big) = 0$. If this is the case, then (μ,ϕ) is an L^2-acyclic Atiyah-pair in the sense of Definition 3.3 and we get

$$\begin{aligned} &\chi^{(2)}(M;\mu,\phi) \\ &\quad = -\dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_A\colon \mathcal{D}(K)_t[u^{\pm 1}]^{a-1} \to \mathcal{D}(K)_t[u^{\pm 1}]^{a-1}\big)\big) \\ &\qquad\qquad + |\phi\circ\mu(x_i)|; \end{aligned}$$

2. *Suppose ∂M is empty. We make the assumption that the given presentation comes from a Heegaard decomposition as described in* [34, Proof of Theorem 5.1]*. Then $a = b$ and there is another set of dual generators $\{x_1', x_2', \ldots, x_a'\}$ coming from the Heegaard decomposition as described in* [34, Proof of Theorem 5.1]*. Choose $i,j \in \{1,2,\ldots,a\}$ with $|\mu(x_i)| = \infty$ and $|\mu(x_j')| = \infty$. Let A be the $(a-1,a-1)$-matrix with entries in $\mathbb{Z}G$ obtained from the Fox matrix F by deleting the ith column and the jth row and*

then applying the homomorphism $M_{a-1,a-1}(\mathbb{Z}\pi) \to M_{a-1,a-1}(\mathbb{Z}G)$ induced by $\mu\colon \pi \to G$.

Then $b_n^{(2)}(\overline{M};\mathcal{N}(G))$ vanishes for all $n \geq 0$ if and only if we have $\dim_{\mathcal{N}(G)}\big(\ker(r_A\colon \mathcal{N}(G)^{a-1} \to \mathcal{N}(G)^{a-1})\big) = 0$. If this is the case, then (μ,ϕ) is an L^2-acyclic Atiyah-pair and we get

$$\chi^{(2)}(M;\mu,\phi) = -\dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_A\colon \mathcal{D}(K)_t[u^{\pm 1}]^{a-1} \to \mathcal{D}(K)_t[u^{\pm 1}]^{a-1}\big)\big) + |\phi \circ \mu(x_i)| + |\phi \circ \mu(x_j')|.$$

Proof. We only treat the case, where ∂M is empty, and leave it to the reader to figure out the details for the case of a non-empty boundary using the proof of [31, Theorem 2.4].

From [34, Proof of Theorem 5.1] we obtain a compact 3-dimensional CW-complex X together with a homotopy equivalence $f\colon X \to M$ and a set of generators $\{x_1',\ldots,x_a'\}$ such that the $\mathbb{Z}G$-chain complex $C_*(\overline{X})$ of $\overline{X}$ looks like

$$\mathbb{Z}G \xrightarrow{\prod_{i=1}^a r_{\mu(x_i')-1}} \bigoplus_{i=1}^a \mathbb{Z}G \xrightarrow{r_{\mu(F)}} \bigoplus_{i=1}^a \mathbb{Z}G \xrightarrow{\bigoplus_{i=1}^a r_{\mu(x_i)-1}} \mathbb{Z}G,$$

where $\mu(F)$ is the image of F under the map $M_{a,a}(\mathbb{Z}\pi) \to M_{a,a}(\mathbb{Z}G)$ induced by μ and $\overline{X} \to X$ is the pullback of $\overline{M} \to M$ with f.

For $g \in G$ with $g \neq 1$ let $D(g)_*$ be the 1-dimensional $\mathbb{Z}G$-chain complexes which has as first differential $r_{g-1}\colon \mathbb{Z}G \to \mathbb{Z}G$. Since g generates an infinite cyclic subgroup of G, we conclude $b_n^{(2)}(\mathcal{N}(G) \otimes_{|\mathbb{Z}G} D(g)_*) = 0$ for $n \geq 0$ from [32, Lemma 1.24 (4) on page 30 and Lemma 1.34 (1) on page 35]. Theorem 3.6 (4) and Lemma 4.3 imply

$$\chi^{(2)}\big(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*D_*(g)\big) \quad = \quad |\phi(g)|. \tag{5.2}$$

There is a surjective $\mathbb{Z}G$-chain map $p_*\colon C_*(\overline{X}) \to \Sigma^2 D(\mu(x_j'))_*$ which is the identity in degree 3 and the projection onto the summand corresponding to j in degree 2, and an injective $\mathbb{Z}G$-chain map $i_*\colon D(\mu(x_i))_* \to C_*(\overline{X})$ which is the identity in degree 0 and the inclusion to the summand corresponding to i in degree 1. Let P_* be the kernel of p_* and let Q_* be the cokernel of the injective map $j_*\colon D(\mu(x_1))_* \to P_*$ induced by $i_*\colon D(\mu(x_1))_* \to C_*(\overline{X})$. Then Q_* is concentrated in dimensions 1 and 2 and its second differential is $r_A\colon \mathbb{Z}G^{a-1} \to \mathbb{Z}G^{a-1}$. We conclude from the long exact homology sequence of a short exact sequence of Hilbert $\mathcal{N}(G)$-chain complexes, the homotopy invariance of L^2-Betti numbers and the additivity of the von Neumann dimension

$$b_n^{(2)}(\overline{X};\mathcal{N}(G)) = \begin{cases} \dim_{\mathcal{N}(G)}\big(\ker(r_A\colon \mathcal{N}(G)^{a-1} \to \mathcal{N}(G)^{a-1})\big)\big) & \text{if } n = 1,2; \\ 0 & \text{otherwise.} \end{cases}$$

This shows that $b_n^{(2)}(\overline{M};\mathcal{N}(G))$ vanishes for all $n \geq 0$ if and only if we have $\dim_{\mathcal{N}(G)}\big(\ker(r_A\colon \mathcal{N}(G)^{a-1} \to \mathcal{N}(G)^{a-1})\big)\big) = 0$.

Suppose that this is the case. Then (μ,ϕ) is an L^2-acyclic Atiyah-pair. We conclude from Theorem 3.6 that $r_A\colon \mathcal{D}(G)^{a-1} \to \mathcal{D}(G)^{a-1}$ is injective and hence $r_A\colon \mathcal{D}(K)_t[u^{\pm 1}]^{a-1} \to \mathcal{D}(K)_t[u^{\pm 1}]^{a-1}$ is injective and

$$\chi^{(2)}(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*Q_*) = \dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_A\colon \mathcal{D}(K)_t[u^{\pm 1}]^{a-1} \to \mathcal{D}(K)_t[u^{\pm 1}]^{a-1}\big)\big). \tag{5.3}$$

We get from Lemma 2.6 and the chain complex version of [32, Theorem 6.80 (2) on page 277] applied to the short exact sequences of $\mathcal{N}(K)$-chain complexes obtained by applying $\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*-$ to the short exact sequences of $\mathbb{Z}G$-chain complexes $0 \to P_* \to C_*(\overline{X}) \to \Sigma^2 D(\mu(x_j'))_* \to 0$ and $0 \to D(\mu(x_1))_* \to P_* \to Q_* \to 0$

$$\begin{aligned}
&\chi^{(2)}(\overline{M};\mu,\phi)\\
&\quad = \chi^{(2)}(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*C_*(\overline{X}))\\
&\quad = \chi^{(2)}(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*Q_*) + \chi^{(2)}(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*D(\mu(x_j'))_*)\\
&\qquad\qquad + \chi^{(2)}(\mathcal{N}(K) \otimes_{\mathbb{Z}K} i^*D(\mu(x_i))_*)\\
&\quad \overset{(5.2)\text{ and }(5.3)}{=} -\dim_{\mathcal{D}(K)}\big(\operatorname{coker}\big(r_A\colon \mathcal{D}(K)_t[u^{\pm 1}]^{a-1} \to \mathcal{D}(K)_t[u^{\pm 1}]^{a-1}\big)\big)\\
&\qquad\qquad + |\phi \circ \mu(x_i) + |\phi \circ \mu(x_j')|.
\end{aligned}$$

This finishes the proof of Theorem 5.1. □

Lemma 5.4. (1) *Let M be an admissible 3-manifold. Consider an infinite group G and a G-covering $\overline{M} \to M$. Then we get $b_n^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ for all $n \geq 0$ if $b_1(\overline{M};\mathcal{N}(G)) = 1$;*

(2) *If M is an admissible 3-manifold, then we get $b_n^{(2)}(\widetilde{M};\mathcal{N}(\pi)) = 0$ for all $n \geq 0$.*

Proof. (1) Since G is infinite, we have $b_0^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ by [32, Theorem 1.35 (8) on page 38]. Using Poincaré duality in the closed case, see [32, Theorem 1.35 (3) on page 37] we conclude $b_m^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ for $m \geq 3$. Since $\chi(M) = 0$, we get $b_1^{(2)}(\overline{M};\mathcal{N}(G)) = b_2^{(2)}(\overline{M};\mathcal{N}(G))$. Hence the assumption $b_1^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ implies that we have $b_m^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ for all $m \geq 0$.

(2) This follows from [29, Theorem 0.1]. □

Theorem 5.5 (The (μ,ϕ)-L^2-Euler characteristic in terms of the first homology)**.** *Let M be an admissible 3-manifold. Let G be a torsion-free group which satisfies the Atiyah Conjecture. Consider group homomorphisms $\mu\colon \pi \to G$ and $\phi\colon G \to \mathbb{Z}$. Let K be the kernel of ϕ and $i\colon K \to G$ be the inclusion. Let $t\colon \mathcal{D}(K) \xrightarrow{\cong} \mathcal{D}(K)$ be the automorphism introduced in Theorem 3.6 (3). Assume that ϕ is surjective, $\phi \circ \mu$ is not trivial, and the intersection $\operatorname{im}(\mu) \cap \ker(\phi)$ is not trivial. Suppose that $b_1^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ holds for the G-covering $\overline{M} \to M$ associated to μ. Then:*

(1) *The pair (μ,ϕ) is an L^2-acyclic Atiyah-pair;*

(2) *We have*

$$\begin{aligned}
\chi^{(2)}(M;\mu,\phi) &= -b_1^{(2)}(i^*\overline{M};\mathcal{N}(K))\\
&= -\dim_{\mathcal{D}(K)}\big(H_1\big(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Z}G} C_*(\overline{M})\big)\big).
\end{aligned}$$

In particular $\chi^{(2)}(M;\mu,\phi)$ is an integer ≥ 0. Moreover, we have for $m \neq 1$

$$H_m\big(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Z}G} C_*(\overline{M})\big) \;=\; 0,$$

and

$$b_m^{(2)}(i^*\overline{M};\mathcal{N}(K)) \;=\; \dim_{\mathcal{D}(K)}\big(H_m\big(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Z}G} C_*(\overline{M})\big)\big) \;=\; 0.$$

Proof. (1) This follows from Lemma 5.4 (1).

(2) Consider any presentation

$$\pi = \langle x_1, x_2, \ldots, x_a \mid R_1, R_2, \ldots, R_b \rangle$$

of π. We want to modify it to another presentation

$$\pi = \langle \widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_a \mid \widehat{R}_1, \widehat{R}_2, \ldots, \widehat{R}_b' \rangle$$

by a sequence of Nielsen transformations, i.e., we replace the ordered set of generators $x_1, x_2, \ldots, x_a$ by $x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(a)}$ for a permutation $\sigma \in S_a$ or we replace $x_1, x_2, \ldots, x_a$ by $x_1, x_2, x_{i-1}, x_j^k x_i, x_{i+1}, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_a$ for some integers i, j, k with $1 \le i < j \le a$, and change the relations accordingly, such that $\mu(\widehat{x}_1) \neq 0$ and $\phi(\widehat{x}_1) = 0$ holds, provided that ϕ is surjective and $\operatorname{im}(\phi) \cap \ker(\phi)$ is non-trivial. We use induction over $n = |\{i \in \{1, 2, \ldots, a\} \mid \mu(x_i) \neq 0\}|$. The induction beginning $n = 0, 1$ is obvious since the case $n = 0$ cannot occur, since $\phi \circ \mu \neq 0$ and ϕ is surjective, and in the case $n = 1$ we must have $\phi \circ \mu(x_i) = 0$ for the only index i with $\mu(x_i) \neq 0$, since $\operatorname{im}(\mu) \cap \ker(\phi) \neq \{0\}$. The induction step from $(n-1)$ to $n \geq 2$ is done as follows. We use subinduction over $m = \min\{|\phi(x_i)| \mid i = 1, 2, \ldots, a, \mu(x_i) \neq 0\}$. If $m = 0$, then $\mu(x_i) \neq 0$ and $\phi(x_i) = 0$ for some $i \in \{1, 2, \ldots, a\}$, and we can achieve our goal by enumeration. The induction step from $(m-1)$ to $m \geq 1$ is done as follows. Choose $j \in \{1, 2, \ldots, a\}$ such that $\mu(x_j) \neq 0$ and $\phi(x_j) = m$. Since $n \geq 2$, there must be an index $i \in \{1, 2, \ldots, a\}$ with $i \neq j$ and $\mu(x_i) \neq 0$. By changing the enumeration we can arrange $i < j$. Choose integers k, l with $0 \le l < m$ and $\phi(x_i) = k \cdot \phi(x_j) + l$. If we replace x_i by $x_i \cdot x_j^{-k}$ and leave the other elements in $\{x_1, x_2, \ldots, x_a\}$ fixed, we get a new set of generators which is part of a finite presentation with a generators and b relations of π for which the induction hypothesis applies since either $\mu(x_i \cdot x_j^{-k}) = 0$ holds or we have $\mu(x_i \cdot x_j^{-k}) \neq 0$ and $|\phi(x_i \cdot x_j^{-k})| = l \le m - 1$.

We have the following equations in $\mathbb{Z}\pi$ for $u \in \mathbb{Z}\pi$ and integers i, j, k with $1 \le i < j \le a$ and $k \geq 1$:

$$\begin{aligned} r_{x_j^k x_i - 1}(u) &= u(x_j^k x_i - 1) = u x_j^k (x_i - 1) + u(x_j^k - 1) = u x_j^k (x_i - 1) + u \left(\sum_{i=0}^{k-1} x_j^i \right) \cdot (x_j - 1) \\ &= u x_j^k (x_i - 1) + \left(\sum_{i=0}^{k-1} u x_j^i \right) \cdot (x_j - 1) = r_{x_i - 1}(u x_j^k) - r_{x_j - 1}\left(\sum_{i=0}^{k-1} u x_j^i \right), \end{aligned}$$

and

$$\begin{aligned} r_{x_j^{-k} x_i - 1}(u) &= u(x_j^{-k} x_i - 1) = u x_j^{-k}(x_i - 1) - u x_j^{-k}(x_j^k - 1) \\ &= u x_j^{-k}(x_i - 1) - u x_j^{-k} \left(\sum_{i=0}^{k-1} x_j^i \right) \cdot (x_j - 1) \\ &= u x_j^{-k}(x_i - 1) - \left(\sum_{i=0}^{k-1} u x_j^{i-k} \right) \cdot (x_j - 1) = r_{x_i - 1}(u x_j^{-k}) - r_{x_j - 1}\left(\sum_{i=0}^{k-1} u x_j^{i-k} \right). \end{aligned}$$

This implies that we can find $\mathbb{Z}\pi$-isomorphisms $f_1 \colon \mathbb{Z}\pi^a \to \mathbb{Z}\pi^a$ and $f_2 \colon \mathbb{Z}\pi^a \to \mathbb{Z}\pi^a$ such that the following diagrams commute

$$\begin{array}{ccc} \mathbb{Z}\pi^a & \xrightarrow{\bigoplus_{i=1}^a r_{x_i - 1}} & \mathbb{Z}\pi \\ f_1 \downarrow \cong & & \| \\ \mathbb{Z}\pi^a & \xrightarrow{\bigoplus_{i=1}^a r_{\widehat{x}_i - 1}} & \mathbb{Z}\pi \end{array} \qquad \begin{array}{ccc} \mathbb{Z}\pi & \xrightarrow{\prod_{i=1}^a r_{x'_i - 1}} & \mathbb{Z}\pi^a \\ \| & & f_2 \downarrow \cong \\ \mathbb{Z}\pi & \xrightarrow{\prod_{i=1}^a r_{\widehat{x}'_i - 1}} & \mathbb{Z}\pi^a \end{array}$$

Now we proceed as in the proof of Theorem 5.1 (2), explaining only the case where the boundary is empty. There we have we constructed a compact 3-dimensional CW-complex X together with a homotopy equivalence $f \colon X \to M$ and set of generators $\{x_1, \ldots, x_a\}$ and $\{x'_1, \ldots, x'_a\}$ such that the $\mathbb{Z}G$-chain complex $C_*(\overline{X})$ of $\overline{X}$ looks like

$$\mathbb{Z}G \xrightarrow{\prod_{j=1}^a r_{\mu(x'_i)-1}} \bigoplus_{i=1}^a \mathbb{Z}G \xrightarrow{r_{\mu(F)}} \bigoplus_{i=1}^a \mathbb{Z}G \xrightarrow{\bigoplus_{i=1}^a r_{\mu(x_i)-1}} \mathbb{Z}G.$$

Applying the construction above yields new set of generators $\{\widehat{x}_1, \ldots, \widehat{x}_a\}$ and $\{\widehat{x}'_1, \ldots, \widehat{x}'_a\}$ such that $|\mu(\widehat{x}_1)| = |\mu(\widehat{x}'_1)| = \infty$ and $\phi \circ \mu(\widehat{x_1}) = \phi \circ \mu(\widehat{x}'_1) = 0$ and we can find a matrix $B \in M_{a,a}(\mathbb{Z}G)$ and isomorphisms f_2 and f_1 making the following diagram of $\mathbb{Z}G$-modules commute.

$$\begin{array}{ccccccc}
\mathbb{Z}G & \xrightarrow{\prod_{j=1}^a r_{\mu(x'_i)-1}} & \bigoplus_{i=1}^a \mathbb{Z}G & \xrightarrow{r_{\mu(F)}} & \bigoplus_{i=1}^a \mathbb{Z}G & \xrightarrow{\bigoplus_{i=1}^a r_{\mu(x_i)-1}} & \mathbb{Z}G \\
\downarrow \mathrm{id} & & \cong \downarrow f_2 & & \cong \downarrow f_1 & & \downarrow \mathrm{id} \\
\mathbb{Z}G & \xrightarrow[\prod_{j=1}^a r_{\mu(\widehat{x}'_i)-1}]{} & \bigoplus_{i=1}^a \mathbb{Z}G & \xrightarrow[r_B]{} & \bigoplus_{i=1}^a \mathbb{Z}G & \xrightarrow[\bigoplus_{i=1}^a r_{\mu(\widehat{x}_i)-1}]{} & \mathbb{Z}G
\end{array}$$

Now proceed as in the proof of Theorem 5.1 (2) using the chain complex given by the lower row instead of the chain complex given by the upper row in the diagram above. If A is the square matrix over $\mathbb{Z}G$ obtained from B by deleting the first row and the first column and then applying the ring homomorphism $\mathbb{Z}\pi \to \mathbb{Z}G$ induced by μ to each entry, then A is invertible over $\mathcal{D}(G)$ and we get because of $\phi \circ \mu(\widehat{x_1}) = \phi \circ \mu(\widehat{x}'_1) = 0$ and Lemma 4.1

$$\begin{aligned}\chi^{(2)}(M; \mu, \phi) &= -\dim_{\mathcal{D}(K)}\bigl(\operatorname{coker}\bigl(r_A \colon \mathcal{D}(K)_t[u^{\pm 1}]^{a-1} \to \mathcal{D}(K)_t[u^{\pm 1}]^{a-1}\bigr)\bigr) \\ &= \dim_{\mathcal{D}(K)}\bigl(H_1\bigl(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Z}G} C_*(\overline{M})\bigr)\bigr),\end{aligned}$$

and

$$\begin{aligned}0 &= \dim_{\mathcal{D}(K)}\bigl(\ker\bigl(r_A \colon \mathcal{D}(K)_t[u^{\pm 1}]^{a-1} \to \mathcal{D}(K)_t[u^{\pm 1}]^{a-1}\bigr)\bigr). \\ &= \dim_{\mathcal{D}(K)}\bigl(H_2\bigl(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Z}G} C_*(\overline{M})\bigr)\bigr),\end{aligned}$$

and

$$0 = \dim_{\mathcal{D}(K)}\bigl(H_m\bigl(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Z}G} C_*(\overline{M})\bigr)\bigr) \quad \text{for } m \neq 1, 2.$$

We conclude from Theorem 3.6 (4) for all $m \geq 0$

$$b_m^{(2)}(i^*\overline{M}; \mathcal{N}(K)) = \dim_{\mathcal{D}(K)}\bigl(H_m\bigl(\mathcal{D}(K)_t[u^{\pm 1}] \otimes_{\mathbb{Z}G} C_*(\overline{M})\bigr)\bigr).$$

This finishes the proof of Theorem 5.5. □

In view of Lemma 2.8 (2), the next example covers the case $\operatorname{im}(\mu) \cap \ker(\phi) = \{1\}$ which is not treated in Theorem 5.5.

Example 5.6 (Infinite cyclic covering)**.** Let M be an admissible 3-manifold. Consider an epimorphism $\mu \colon \pi \to \mathbb{Z}$ and a non-trivial homomorphism $\phi \colon \mathbb{Z} \to \mathbb{Z}$. Let k be the index of $\operatorname{im}(\phi)$ in $\mathbb{Z}$. Let $\overline{M} \to M$ be the infinite cyclic covering associated to μ.

Then (μ, ϕ) is an L^2-acyclic pair if and only if $b_1^{(2)}(\overline{M}; \mathcal{N}(\mathbb{Z})) = 0$. This follows from Lemma 5.4 and the fact that $\mathbb{Z}$ satisfies the Atiyah Conjecture by Theorem 3.2 (2).

Suppose $b_1^{(2)}(\overline{M}; \mathcal{N}(\mathbb{Z})) = 0$. Lemma 2.6, Lemma 2.8 (2) and Theorem 3.6 (4) imply that $\dim_{\mathbb{Q}}(H_n(M; \mathbb{Q})) < \infty$ holds for $n \geq 0$ and we get

$$\chi^{(2)}(M; \mu, \phi) = k \cdot \chi(\overline{M}). \tag{5.7}$$

Next we show

$$\chi(\overline{M}) \;=\; \begin{cases} 1 - \dim_Q(H_1(\overline{M}; \mathbb{Q})) & \text{if } \partial M \neq \emptyset; \\ 2 - \dim_Q(H_1(\overline{M}; \mathbb{Q})) & \text{if } \partial M = \emptyset. \end{cases} \tag{5.8}$$

Since $H_n(\overline{M};\mathbb{Q}) = 0$ for $n \geq 3$ holds because $\overline{M}$ is a non-compact 3-manifold, this will follow if we can show

$$\dim_Q(H_2(\overline{M};\mathbb{Q})) \; = \; \begin{cases} 0 & \text{if } \partial M \neq \emptyset; \\ \mathbb{Q} & \text{if } \partial M = \emptyset. \end{cases}$$

We begin with the case that M has non-empty boundary. Then M is homotopy equivalent to a 2-dimensional complex X. Since $H_2(\overline{M};\mathbb{Q}) \cong_{\mathbb{Q}[\mathbb{Z}]} H_2(\overline{X};\mathbb{Q})$ is a $\mathbb{Q}[\mathbb{Z}]$-submodule of the free $\mathbb{Q}[\mathbb{Z}]$-module $C_2(\overline{X}) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the pull back $\overline{X} \to X$ of $\overline{M} \to M$ with a homotopy equivalence $X \to M$, the $\mathbb{Q}[\mathbb{Z}]$-module $H_2(\overline{M};\mathbb{Q})$ is free. Since $\dim_{\mathbb{Q}}(H_2(\overline{M};\mathbb{Q})) < \infty$ holds, this implies $H_2(\overline{M};\mathbb{Q}) = 0$.

Now suppose that ∂M is empty. Then we get a Poincaré $\mathbb{Z}\pi$-chain homotopy equivalence of $\mathbb{Z}\pi$-chain complexes $\operatorname{Hom}_{\mathbb{Z}\pi}(C_{3-*}(\widetilde{M}), \mathbb{Z}\pi) \to C_*(\widetilde{M})$. It induces a $\mathbb{Q}[\mathbb{Z}]$-chain homotopy equivalence $\operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}]}(\mathbb{Q} \otimes_{\mathbb{Z}} C_{3-*}(\overline{M}), \mathbb{Q}[\mathbb{Z}]) \to \mathbb{Q} \otimes_{\mathbb{Z}} C_*(\overline{M})$ and hence a $\mathbb{Q}[\mathbb{Z}]$-isomorphism

$$H^1\big(\operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}]}(\mathbb{Q} \otimes_{\mathbb{Z}} C_{3-*}(\overline{M}), \mathbb{Q}[\mathbb{Z}])\big) \xrightarrow{\cong} H_2(\overline{M}).$$

Since $\mathbb{Q}[\mathbb{Z}[$ is a principal ideal domain, we get from the Universal Coefficient Theorem for cohomology an exact sequence of $\mathbb{Q}[\mathbb{Z}]$-modules

$$0 \to \operatorname{Ext}^1_{\mathbb{Q}[\mathbb{Z}|}\big(H_0(C_*(\overline{M})), \mathbb{Q}[\mathbb{Z}])\big) \to H^1\big(\operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}]}(\mathbb{Q} \otimes_{\mathbb{Z}} C_{3-*}(\overline{M}), \mathbb{Q}[\mathbb{Z}])\big) \\ \to \operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}]}\big(H_1(\overline{M};\mathbb{Q}), \mathbb{Q}[\mathbb{Z}]\big) \to 0.$$

From $\dim_{\mathbb{Q}}(H_1(\overline{M};\mathbb{Q})) < \infty$, we conclude $\operatorname{Hom}_{\mathbb{Q}[\mathbb{Z}]}\big(H_1(\overline{M};\mathbb{Q}), \mathbb{Q}[\mathbb{Z}]\big) = 0$. Since $H_0(C_*(\overline{M}))$ is the trivial $\mathbb{Q}[\mathbb{Z}]$-module $\mathbb{Q}$, we conclude that $\operatorname{Ext}^1_{\mathbb{Q}[\mathbb{Z}|}\big(H_0(C_*(\overline{M})), \mathbb{Q}[\mathbb{Z}]\big)$ is the trivial $\mathbb{Q}[\mathbb{Z}]$-module $\mathbb{Q}$ and (5.8) follows.

6. Equality of (μ,ϕ)-L^2-Euler characteristic and the Thurston norm

The section is devoted to the proof of Theorem 0.4 which needs some preparation.

For the remainder of this section G is a torsion-free group satisfying the Atiyah Conjecture, H is a finitely generated abelian group and $\nu\colon G \to H$ a surjective group homomorphism.

6.1. **The non-commutative Newton polytope.** Choose a set-theoretic section s of ν, i.e., a map of sets $s\colon H \to G$ with $\nu \circ s = \operatorname{id}_H$. Notice that we do *not* require that s is a group homomorphism. Let K_ν be the kernel of ν. Then K_ν also satisfies the Atiyah Conjecture by Theorem 3.2 (1), and $\mathcal{D}(K_\nu)$ and $\mathcal{D}(G)$ are skew fields by Theorem 3.6 (1). There are obvious inclusions $\mathbb{Z}K_\nu \subseteq \mathbb{Z}G$ and $\mathcal{D}(K_\nu) \subseteq \mathcal{D}(G)$ coming from the inclusion $j\colon K_\nu \to G$. Let $c\colon H \to \operatorname{aut}(\mathcal{D}(K_\nu))$ be the map sending h to the automorphisms $c(h)\colon \mathcal{D}(K_\nu) \xrightarrow{\cong} \mathcal{D}(K_\nu)$ induced by the conjugation automorphism $K_\nu \to K_\nu$, $k \mapsto s(h) \cdot k \cdot s(h)^{-1}$. Define $c\colon H \to \operatorname{aut}(\mathbb{Z}K_\nu)$ analogously. Define $\tau\colon H \times H \to (\mathbb{Z}K_\nu)^\times$ and $\tau\colon H \times H \to \mathcal{D}(K_\nu)^\times$ by sending (h,h') to $s(h) \cdot s(h') \cdot s(hh')^{-1}$. Let $\mathbb{Z}K_\nu *_s H$ and $\mathcal{D}(K_\nu) *_s H$ be the crossed product rings associated to the pair (c,τ). Elements in $\mathbb{Z}K_\nu *_s H$ and $\mathcal{D}(K_\nu) *_s H$ respectively are finite formal sums $\sum_{h \in H} x_h \cdot h$ for x_h in $\mathbb{Z}K_\nu$ and $\mathcal{D}(K_\nu)$ respectively. Addition is given by adding the coefficients. Multiplication is given by the formula

$$\Big(\sum_{h \in H} x_h \cdot h\Big) \cdot \Big(\sum_{h \in H} y_h \cdot h\Big) \; = \; \sum_{h \in H} \Big(\sum_{\substack{h',h'' \in H,\\ h'h''=h}} x_{h'} c_{h'}(y_{h''}) \tau(h',h'')\Big) \cdot h.$$

This multiplication is uniquely determined by the properties $h \cdot x = c(h)(x) \cdot h$ for x in $\mathbb{Z}K_\nu$ and $\mathcal{D}(K_\nu)$ respectively, and $h \cdot h' = \tau(h,h') \cdot (hh')$ for $h, h' \in H$. We obtain an isomorphism of rings

$$j_s'\colon \mathbb{Z}K_\nu *_s H \xrightarrow{\cong} \mathbb{Z}G, \quad \sum_{h \in H} x_h \cdot h \mapsto \sum_{h \in H} x_h \cdot s(h),$$

see [32, Example 10.53 on page 396], and an injective ring homomorphism

$$j_s\colon \mathcal{D}(K_\nu) *_s H \to \mathcal{D}(G), \quad \sum_{h \in H} x_h \cdot h \mapsto \sum_{h \in H} x_h \cdot s(h).$$

Moreover, the set T of non-trivial elements in $\mathcal{D}(K_\nu) *_s H$ satisfies the Ore condition and j_s induces an isomorphism of skew fields

$$\widehat{j_s}\colon T^{-1}(\mathcal{D}(K_\nu) *_s H) \xrightarrow{\cong} \mathcal{D}(G). \tag{6.1}$$

This is proved for $F = \mathbb{C}$ in [32, Lemma 10.68 on page 403], the proof carries over for any field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$.

Fix an element $u = \sum_{h \in H} x_h \cdot h$ in $\mathcal{D}(K_\nu) *_s H$ with $u \neq 0$. Now we introduce a non-commutative analogue of the Newton polytope. A *polytope* in a finite dimensional real vector space is a subset which is the convex hull of a finite subset. An element p in a polytope is called *extreme* if the implication $p = \frac{q_1}{2} + \frac{q_2}{2} \implies q_1 = q_2 = p$ holds for all elements q_1 and q_2 in the polytope. Denote by $\operatorname{Ext}(P)$ the set of extreme points of P. If P is the convex hull of the finite set S, then $\operatorname{Ext}(P) \subseteq S$ and P is the convex hull of $\operatorname{Ext}(P)$. The *Minkowski sum* of two polytopes P_1 and P_2 is defined to be the polytope

$$P_1 + P_2 := \{p_1 + p_2 \mid p_1 \in P_1, p \in P_2\}.$$

It is the convex hull of the set $\{p_1 + p_2 \mid p_1 \in \operatorname{Ext}(P_1), p_2 \in \operatorname{Ext}(P_2)\}$. Define the *non-commutative Newton polytope* of u

$$P(u) \subseteq \mathbb{R} \otimes_{\mathbb{Z}} H \tag{6.2}$$

to be the polytope given by the convex hull of the finite set $\{1 \otimes h \mid x_h \neq 0\}$. The definition of $P(u)$ is independent of the choice of the section s by the following argument. Consider two set-theoretic sections s and s' of $\nu\colon G \to H$. Then we get for $u \in \mathcal{D}(K_\nu) *_s H$ and $u' \in \mathcal{D}(K_\nu) *_{s'} H$

$$\widehat{j_s}(u) = \widehat{j_{s'}}(u') \quad \implies \quad P(u; s) = P(u'; s') \tag{6.3}$$

by the following argument. If we write $u = \sum_{h \in H} x_h \cdot h$ and $u' = \sum_{h \in H} y_h \cdot h$, then $\widehat{j_s}(u) = \widehat{j_{s'}}(u')$ implies

$$u = \sum_{h \in H} x_h \cdot h = \sum_{h \in H} \bigl(y_h \cdot s'(h) s(h)^{-1}\bigr) \cdot h,$$

and hence $x_h \neq 0 \Leftrightarrow y_h \neq 0$.

The following lemma is well-known in the commutative setting and we explain its proof since we could not find a reference for it in the literature.

Lemma 6.4. *For $u, v \in \mathcal{D}(K_\nu) *_s H$ with $u, v \neq 0$, we have*

$$P(uv) = P(u) + P(v).$$

Proof. Consider an extreme point $p \in P(u; s) + P(v; s)$. Then we can find points $q_1 \in P(u; s)$ and $q_2 \in P(v; s)$ with $p = q_1 + q_2$. We want to show that q_1 and q_2 are extreme. Consider $q_1', q_1'' \in P(u; s)$ and $q_2', q_2'' \in P(v; s)$ with $q_1 = (q_1' + q_1'')/2$ and $q_2 = (q_2' + q_1'')/2$. Then $(q_1' + q_2')$, $(q_1'' + q_2')$, $(q_1' + q_2'')$, and $(q_1'' + q_2'')$ belong to $P(u; s) + P(v; s)$ and satisfy

$$p = \frac{q_1' + q_2'}{2} + \frac{q_1'' + q_2''}{2} = \frac{q_1' + q_2''}{2} + \frac{q_1'' + q_2'}{2}.$$

Since $p \in P(u; s) + P(v; s)$ is extreme, we conclude $q_1' + q_2' = q_1'' + q_2''$ and $q_1' + q_2'' = q_1'' + q_2'$. If we subtract the second equation from the first, we obtain $q_2' - q_2'' = q_2'' - q_2'$

and hence $q_2' = q_2''$. This implies also $q_1' = q_1''$. This shows that $q_1 \in P(u;s)$ and $q_2 \in P(v;s)$ are extreme.

Suppose that we have other points $q_1' \in P(u;s)$ and $q_2' \in P(v;s)$ with $p = q_1'+q_2'$. Then $q_1 + q_2'$ and $q_1' + q_2$ belong to $P(u;s) + P(v;s)$ and satisfy $p = \frac{q_1+q_2'}{2} + \frac{q_1'+q_2}{2}$. Since p is extreme, this implies $p = q_1 + q_2' = q_1' + q_2$. Since we also have $p = q_1 + q_2 = q_1' + q_2'$, we conclude $q_1 = q_1'$ and $q_2 = q_2'$.

Now write $u = \sum_{h\in H} x_h \cdot h$, $v = \sum_{h \in H} y_h \cdot h$, and $uv = \sum_{h\in H} z_h \cdot h$. Since $p \in P(u;s) + P(v;s)$ is extreme, there is $h \in H$ with $p = 1 \otimes h$. If we write $p = q_1 + q_2$ for $q_1 \in P(u;s)$ and $q_2 \in P(v;s)$, then we have already seen that q_1 and q_2 are extreme and hence there are $h_1, h_2 \in H$ with $q_1 = 1 \otimes h_1$, $q_2 = 1 \otimes h_2$, $x_{h_1} \neq 0$ and $y_{h_2} \neq 0$. The equation $p = q_1 + q_2$ implies $h = h_1 + h_2$. Now consider elements $h_1', h_2' \in H$ with $h = h_1' + h_2'$, $x_{h_1'} \neq 0$ and $y_{h_2'} \neq 0$. Put $q_1' = 1 \otimes h_1'$ and $q_2' = 1 \otimes h_2'$. Then $q_1' \in P(u;s)$ and $q_2' \in P(v;s)$ and we have $p = q_1' + q_2'$. We have already explained that this implies $q_1 = q_1'$ and $q_2 = q_2'$ and hence $h_1 = h_1'$ and $h_2 = h_2'$. Therefore we get $z_h = x_{h_1} c(h_1)(y_{h_2})\tau(h_1,h_2)$. Since x_{h_1} and y_{h_2} are non-trivial, we conclude $z_h \neq 0$ and hence $p \in P(uv;s)$. Hence every extreme point in $P(u;s) + P(v;s)$ belongs to $P(uv;s)$ which implies $P(u;s) + P(v;s) \subseteq P(uv;s)$.

One easily checks that any point of the shape $1 \otimes h$ for $z_h \neq 0$ belongs to $P(u;s) + P(v;s)$ since $z_h \neq 0$ implies the existence of h_1 and h_2 with $x_{h_1}, y_{h_2} \neq 0$ and $h = h_1 + h_2$. We conclude $P(uv;s) \subseteq P(u;s) + P(v;s)$. This finishes the proof of Lemma 6.4. □

6.2. **The polytope homomorphism.** We obtain a finite dimensional real vector space $\mathbb{R} \otimes_{\mathbb{Z}} H$. An integral polytope in $\mathbb{R} \otimes_{\mathbb{Z}} H$ is a polytope such that $\operatorname{Ext}(P)$ is contained in H, where we consider H as a lattice in $\mathbb{R} \otimes_{\mathbb{Z}} H$ by the standard embedding $H \to \mathbb{R} \otimes_{\mathbb{Z}} H$, $h \mapsto 1 \otimes h$. The Minkowski sum of two integral polytopes is again an integral polytope. Hence the integral polytopes form an abelian monoid under the Minkowski sum with the integral polytope $\{0\}$ as neutral element.

Definition 6.5 (Grothendieck group of integral polytopes)**.** Let $\mathcal{P}_{\mathbb{Z}}(H)$ be the abelian group given by the Grothendieck construction applied to the abelian monoid of integral polytopes in $\mathbb{R} \otimes_{\mathbb{Z}} H$ under the Minkowski sum.

Notice that for polytopes P_0, P_1 and Q in a finite dimensional real vector space we have the implication $P_0 + Q = P_1 + Q \Longrightarrow P_0 = P_1$, see [37, Lemma 2]. Hence elements in $\mathcal{P}_{\mathbb{Z}}(H)$ are given by formal differences $[P] - [Q]$ for integral polytopes P and Q in $\mathbb{R} \otimes_{\mathbb{Z}} H$ and we have $[P_0] - [Q_0] = [P_1] - [Q_1] \Longleftrightarrow P_0 + Q_1 = P_1 + Q_0$.

The polytope group $\mathcal{P}_{\mathbb{Z}}(H)$ is a free abelian group by Funke [17].

In the sequel we denote by G_{abel} the abelianization $G/[G,G]$ of a group G. Define the *polytope homomorphism* for a surjective homomorphism $\nu\colon G \to H$ onto a finitely generated free abelian group H

$$\mathbb{P}_\nu'\colon \mathcal{D}(G)^\times_{\text{abel}} \to \mathcal{P}_{\mathbb{Z}}(H). \tag{6.6}$$

as follows. Choose a set-theoretic section s of ν. Consider an element $z \in \mathcal{D}(G)$ with $z \neq 0$. Choose $u, v \in \mathcal{D}(K_\nu) *_s H)$ such that $\widehat{j_s}(uv^{-1}) = z$, where the isomorphism $\widehat{j_s}$ has been introduced in (6.1). Then we define the image of the class $[z]$ in $\mathcal{D}(G)^\times_{\text{abel}}$ represented by z under $\mathbb{P}_\nu'$ to be $[P(u)] - [P(v)]$.

We have to show that this is independent of the choices of s,u and v. Suppose that we have another set-theoretic section $s'\colon H \to G$ of ν and $u', v' \in \mathcal{D}(K_\nu) *_{s'} H$ with $u', v' \neq 0$ and $z = \widehat{j_{s'}}(u'v'^{-1})$. Choose $u'', v'' \in \mathcal{D}(K_\nu) *_{s'} H$ with $j_s(u) = j_{s'}(u'')$ and $j_s(v) = j_{s'}(v'')$. From $\widehat{j_s}(uv^{-1}) = z = \widehat{j_{s'}}(u'v'^{-1})$ we conclude $u'v'^{-1} = u''v''^{-1}$ in $T^{-1}\mathcal{D}(K_\nu) *_{s'} H$. Hence there exist $w', w'' \in \mathcal{D}(K_\nu) *_{s'} H$ with $w', w'' \neq 0$,

$u'w' = u''w''$ and $v'w' = v''w''$. We conclude

$$\begin{aligned}
P(u) - P(v) &\stackrel{(6.3)}{=} P(u'') - P(v'') \\
&= P(u'') + P(w'') - P(w') - P(v'') - P(w'') + P(w') \\
&\stackrel{\text{Lemma 6.4}}{=} P(u''w'') - P(w') - P(v''w'') + P(w') \\
&= P(u'w') - P(w') - P(v'w') + P(w) \\
&\stackrel{\text{Lemma 6.4}}{=} P(u') + P(w') - P(w') - P(v') - P(w') + P(w') \\
&= P(u') - P(v').
\end{aligned}$$

Hence $\mathbb{P}'_\nu\colon \mathcal{D}(G)^\times \to \mathcal{P}_\mathbb{Z}(H)$ is well-defined. We conclude from Lemma 6.4 that $\mathbb{P}'_\nu$ is a homomorphism of abelian groups.

6.3. **Semi-norms and matrices over $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$.** Let $P \subseteq \mathbb{R} \otimes_\mathbb{Z} H$ be a polytope. It defines a seminorm on $\operatorname{Hom}_\mathbb{Z}(H, \mathbb{R}) = \operatorname{Hom}_\mathbb{R}(\mathbb{R} \otimes_\mathbb{Z} H, \mathbb{R})$ by

$$\|\phi\|_P \quad := \quad \frac{1}{2} \sup\{\phi(p_0) - \phi(p_1) \mid p_0, p_1 \in P\}. \tag{6.7}$$

It is compatible with the Minkowski sum, namely, for two integral polytopes $P, Q \subseteq \mathbb{R} \otimes_\mathbb{Z} H$ we have

$$\|\phi\|_{P+Q} \quad = \quad \|\phi\|_P + \|\phi\|_Q. \tag{6.8}$$

Put

$$\begin{aligned}\mathcal{SN}(H) := \{f\colon \operatorname{Hom}_\mathbb{Z}(H;\mathbb{R}) \to \mathbb{R} \mid & \text{ there exists integral polytopes} \\ & P \text{ and } Q \text{ in } \mathbb{R} \otimes_\mathbb{Z} H \text{ with } f = \|\ \|_P - \|\ \|_Q\}.\end{aligned} \tag{6.9}$$

This becomes an abelian group by $(f-g)(\phi) = f(\phi) - g(\phi)$ because of (6.8). Again because of (6.8) we obtain an epimorphism of abelian groups

$$\operatorname{sn}\colon \mathcal{P}_{\mathbb{Z},\mathrm{Wh}}(H) \to \mathcal{SN}(H) \tag{6.10}$$

by sending $[P] - [Q]$ for two polytopes $P, Q \subseteq \mathbb{R} \otimes_\mathbb{Z} H$ to the function

$$\operatorname{Hom}_\mathbb{Z}(H, \mathbb{R}) \to \mathbb{R}, \quad \phi \mapsto \|\phi\|_P - \|\phi\|_Q.$$

Consider any finitely generated abelian group H and group homomorphisms $\nu\colon G \to H$ and $\phi\colon H \to \mathbb{Z}$ such that ϕ is surjective. Define a homomorphism

$$D_{\nu,\phi}\colon \mathcal{D}(G)^\times_{\mathrm{abel}} \xrightarrow{\mathbb{P}'_\nu} \mathcal{P}_\mathbb{Z}(H) \xrightarrow{\operatorname{sn}} \mathcal{SN}(H) \xrightarrow{\operatorname{ev}_\phi} \mathbb{R} \tag{6.11}$$

to be the composite of the homomorphism defined in (6.6) and (6.10) and the evaluation homomorphism ev_ϕ.

Lemma 6.12. *Consider finitely generated free abelian groups H and H' and surjective group homomorphisms $\nu\colon G \to H$, $\omega\colon H \to H'$ and a group homomorphism $\psi\colon H' \to \mathbb{Z}$. Then we get the following equality of homomorphisms $\mathcal{D}(G)^\times_{\mathrm{abel}} \to \mathbb{R}$*

$$D_{\mu,\psi\circ\omega} = D_{\omega\circ\nu,\psi}.$$

Proof. Choose a set-theoretic section $s\colon H \to G$ of ν and a set-theoretic section $t\colon H' \to H$ of ω. Then $s \circ t\colon H' \to G$ is a set-theoretic section of $\omega \circ \nu\colon G \to H'$. Let $K_\nu \subseteq G$ be the kernel of ν, $K_{\omega\circ\nu} \subseteq G$ be the kernel of $\omega \circ \nu$ and $K_\omega \subseteq H$ be the kernel of ω. Let $k\colon K_\nu \to K_{\omega\circ\nu}$ be the inclusion. We obtain an exact sequence $0 \to K_\nu \xrightarrow{k} K_{\omega\circ\nu} \xrightarrow{\nu|_{K_{\omega\circ\nu}}} K_\omega \to 0$ of groups such that K_ω is finitely generated free abelian. The section s induces a section $s|_{K_{\omega\circ\nu}}\colon K_\omega \to K_{\omega\circ\nu}$ of $\omega|_{K_{\omega\circ\nu}}\colon K_{\omega\circ\nu} \to K_\omega$. We also have the exact sequence $0 \to K_{\omega\circ\nu} \xrightarrow{l} G \xrightarrow{\omega\circ\nu} H' \to 0$ for l the inclusion and the set-theoretic section $s \circ t$ of $\omega \circ \nu$. Thus we get isomorphisms of skew fields

$$\begin{array}{rcl} \widehat{j_s}\colon T^{-1}\mathcal{D}(K_\nu) *_s H & \xrightarrow{\cong} & \mathcal{D}(G); \\ \widehat{k_{s|_{K_\omega}}}\colon T^{-1}\mathcal{D}(K_\nu) *_{s|_{K_\omega}} K_\omega & \xrightarrow{\cong} & \mathcal{D}(K_{\omega\circ\nu}); \\ \widehat{l_t}\colon T^{-1}\mathcal{D}(K_{\omega\circ\nu}) *_{s\circ t} H' & \xrightarrow{\cong} & \mathcal{D}(G), \end{array}$$

where T^{-1} always indicates the Ore localization with respect to the non-trivial elements. Consider $u = \sum_{h\in H} x_h \cdot h$ in $\mathcal{D}(K_\nu) *_s H$. For $h' \in H'$ define an element in $\mathcal{D}(K_\nu) *_{s|_{K_\omega}} K_\omega$ by

$$u_{h'} = \sum_{h\in K_\omega} \bigl(x_{h\cdot t(h')} \cdot s(h\cdot t(h')) \cdot s\circ t(h')^{-1} \cdot s(h)^{-1}\bigr) \cdot h.$$

It is well-defined since $s(h\cdot t(h'))\cdot s\circ t(h')^{-1}\cdot s(h)^{-1} \in K_\nu$ holds. Define an element in $\mathcal{D}(K_{\omega\circ\nu}) *_{s\circ t} H'$ by

$$v = \sum_{h'\in H'} \widehat{k_{s|_{\ker_\omega}}}(u_{h'}) \cdot h'.$$

Then we compute in $\mathcal{D}(G)$

$$\begin{aligned} \widehat{j_s}(u) &= \sum_{h\in H} x_h \cdot s(h) \;=\; \sum_{h'\in H'}\sum_{h\in\omega^{-1}(h')} x_h \cdot s(h) \\ &= \sum_{h'\in H'} \Biggl(\sum_{h\in\omega^{-1}(h')} x_h \cdot s(h)\cdot s\circ t(h')^{-1}\Biggr) \cdot s\circ t(h') \\ &= \sum_{h'\in H'} \Biggl(\sum_{h\in K_\omega} x_{h\cdot t(h')} \cdot s(h\cdot t(h'))\cdot s\circ t(h')^{-1}\Biggr) \cdot s\circ t(h') \\ &= \sum_{h'\in H'} \Biggl(\sum_{h\in K_\omega} \bigl(x_{h\cdot t(h')} \cdot s(h\cdot t(h'))\cdot s\circ t(h')^{-1}\cdot s(h)^{-1}\bigr)\cdot s(h)\Biggr) \cdot s\circ t(h') \\ &= \sum_{h'\in H'} \widehat{k_{s|_{\ker(\omega)}}}(u_{h'}) \cdot s\circ t(h') \;=\; \widehat{l_t}\Biggl(\sum_{h'\in H'} \widehat{k_{s|_{\ker(\omega)}}}(u_{h'})\cdot h'\Biggr) \;=\; \widehat{l_t}(v). \end{aligned}$$

Obviously we get for $h' \in H'$

$$u_{h'} \neq 0 \Leftrightarrow \exists h \in \omega^{-1}(h') \text{ with } x_h \neq 0.$$

This implies

$$\begin{aligned} &\sup\{\psi(h') - \psi(k') \mid h',k' \in H', u_{h'} \neq 0, u_{k'} \neq 0\} \\ &\qquad = \sup\{\psi\circ\omega(h) - \psi\circ\omega(k) \mid h,k \in H, x_h \neq 0, x_k \neq 0\}. \end{aligned}$$

Hence we get for the element $z \in \mathcal{D}(G)$ given by $z = \widehat{j_s}(u) = \widehat{l_t}(v)$

$$D_{\omega\circ\nu,\psi}(z) = D_{\nu,\psi\circ\omega}(z).$$

Since $D_{\omega\circ\nu,\psi}$ and $D_{\nu,\psi\circ\omega}$ are homomorphisms, Lemma 6.12 follows. □

Recall from Theorem 3.6 (3) that $\mathcal{D}(G)$ is the Ore localization of with respect to the set of non-zero elements of the ring $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ of twisted Laurent polynomials in the variable u with coefficients in the skew-field $\mathcal{D}(K_{\phi\circ\nu})$. Hence $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ is contained in $\mathcal{D}(G)$ and we can consider for any $x \in \mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ with $x \neq 0$ its image $D_{\nu,\phi}([x]) \in \mathbb{R}$ under the homomorphism $D_{\nu,\phi}$ defined in (6.11).

Lemma 6.13. *Consider an element $x \in \mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ with $x \neq 0$. Then*

$$D_{\nu,\phi}([x]) \;=\; \tfrac{1}{2}\deg(x),$$

where $\deg(x)$ *has been defined to be* $k_+ - k_-$ *if we write* $x = \sum_{k=k_-}^{k_+} z_n \cdot u^k$ *with* $z_{k_+}, z_{k_-} \neq 0$.

Proof. Recall that $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ does depend on a choice of a preimage of 1 under $\phi \circ \nu \colon G \to \mathbb{Z}$ which is the same as a choice of a group homomorphism $t \colon \mathbb{Z} \to G$ with $\phi \circ \nu \circ t = \mathrm{id}_{\mathbb{Z}}$. Choose a set theoretic map $s \colon H \to G$ with $\nu \circ s = \mathrm{id}_H$. One easily checks that $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ agrees with $\mathcal{D}(K_{\phi\circ\nu}) *_t \mathbb{Z}$. We conclude $D_{\nu,\phi}(x) = D_{\nu\circ\phi,\mathrm{id}_{\mathbb{Z}}}(x)$ from Lemma 6.12. Now one easily checks $D_{\phi\circ\nu,\mathrm{id}_{\mathbb{Z}}}(x) = \frac{1}{2} \cdot \deg(x)$ by inspecting the definitions, since for a polynomial $\sum_{k=k_-}^{k_+} z_k u^k$ in one variable u with $z_{k_-}, z_{k_+} \neq 0$ its Newton polytop is the interval $[k_-, k_+] \subseteq \mathbb{R}$. (The factor $1/2$ comes from the factor $1/2$ in (6.7).) □

There is a Dieudonné determinant for invertible matrices over a skew field D which takes values in the abelianization of the group of units $D^\times_{\mathrm{abel}}$ and induces an isomorphism, see [41, Corollary 4.3 in page 133]

$$\det_D \colon K_1(D) \xrightarrow{\cong} D^\times_{\mathrm{abel}} \tag{6.14}$$

The inverse

$$J_D \colon D^\times_{\mathrm{abel}} \xrightarrow{\cong} K_1(D) \tag{6.15}$$

sends the class of a unit in D to the class of the corresponding $(1,1)$-matrix.

Lemma 6.16. *Let* $A \in \mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ *be a matrix which becomes invertible over* $\mathcal{D}(G)$. *Then the composite of the homomorphisms defined in* (6.11) *and* (6.14)

$$K_1(\mathcal{D}(G)) \xrightarrow{\det_{\mathcal{D}G)}} \mathcal{D}(G)^\times_{\mathrm{abel}} \xrightarrow{D_{\nu,\phi}} \mathbb{R}$$

sends the class $[A] \in K_1(\mathcal{D}(G))$ *of* A *to*

$$\tfrac{1}{2} \dim_{\mathcal{D}(K_{\phi\circ\nu})}\big(\mathrm{coker}\big(r_A \colon \mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]^n \to \mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]^n\big)\big).$$

Proof. The twisted polynomial ring $\mathcal{D}(K_{\phi\circ\nu})_t[u]$ has a Euclidean function given by the degree and hence there is a Euclidean algorithm with respect to it. This algorithm allows to transform A to a diagonal matrix over $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ by the following operations

(1) Permute rows or columns;
(2) Multiply a row on the right or a column on the left with an element of the shape yu^m for some $y \in \mathcal{D}(K_{\phi\circ\nu})$ with $y \neq 0$ and $m \in \mathbb{Z}$;
(3) Add a right $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$-multiple of a row to another row and analogously for columns;

These operations change the class $[A]$ of A in $K_1(\mathcal{D}(G))$ by adding an element of the shape $J_{\mathcal{D}(G)}([yu^m])$ for $y \in \mathcal{D}(K_{\phi\circ\nu})$ with $y \neq 0$ and $m \in \mathbb{Z}$ for the homomorphism $J_{\mathcal{D}(G)}$ of (6.15). Moreover, they do not change the kernel of the cokernel of r_A since yu^m is unit in $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$. Since $D_{\nu,\phi}[yu^m]) = 0$ follows from Lemma 6.13, it suffices to treat the special case, where A is a diagonal matrix over $\mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]$ with non-trivial entries on the diagonal.

Let x be the product of the diagonal entries of the diagonal matrix A. We get in $\mathcal{D}(G)^\times_{\mathrm{abel}}$

$$\det_{\mathcal{D}(G)^\times}([A]) = [x].$$

Using the obvious exact sequence for $\mathcal{D}(G)$-maps $f_1 \colon M_0 \to M_1$ and $f_2 \colon M_1 \to M_2$

$$0 \to \ker(f_1) \to \ker(f_2 \circ f_1) \to \ker(f_2) \to \mathrm{coker}(f_1) \to \mathrm{coker}(f_2 \circ f_1) \to \mathrm{coker}(f_2) \to 0,$$

we conclude

$$\dim_{\mathcal{D}(K_{\phi\circ\nu})}(\mathrm{coker}(r_A)) = \dim_{\mathcal{D}(K_{\phi\circ\nu})}\big(\mathrm{coker}(r_x \colon \mathcal{D}(G) \to \mathcal{D}(G))\big).$$

We conclude from Lemma 4.3 and Lemma 6.13.

$$\frac{1}{2}\dim_{\mathcal{D}(K_{\phi\circ\nu})}(\operatorname{coker}(r_x)) \;=\; \frac{1}{2}\deg(x) \;=\; D_{\nu,\phi}([x]) \;=\; D_{\nu,\phi}\circ\det_{\mathcal{D}(G)}([A]).$$

This finishes the proof of Lemma 6.16. □

Lemma 6.17. *Let M be an admissible 3-manifold. Let G be a torsion-free group which satisfies the Atiyah Conjecture. Consider any factorization $\pi \xrightarrow{\mu} G \xrightarrow{\nu} H_1(M)_f$ of the canonical projection $\pi \to H_1(M)_f$. Assume that $b_1^{(2)}(\overline{M};\mathcal{N}(G)) = 0$ holds for the G-covering $\overline{M} \to M$ associated to μ.*

Then there exist two seminorms s_1 and s_2 on $H^1(M;\mathbb{R})$ such that we get for every $\phi \in H^1(M;\mathbb{Z}) = \operatorname{Hom}_{\mathbb{Z}}(H_1(M)_f;\mathbb{Z})$

$$\chi^{(2)}(M;\mu,\phi\circ\nu) \;=\; s_1(\phi) - s_2(\phi).$$

Proof. We treat only the case, where ∂M is non-empty, the case of empty ∂M is completely analogous. Let $x_1, x_2, \ldots, x_a$ be the element in G and A be the $(a-1, a-1)$-matrix over $\mathbb{Z}G$ appearing Theorem 5.1 (1). (Notice that they are independent of ϕ.) We conclude from Theorem 5.1 (1) that for any surjective group homomorphism $\phi\colon G \to \mathbb{Z}$ we have

$$\begin{aligned}\chi^{(2)}(M;\mu,\phi) = {} & |\phi\circ\nu\circ\mu(x_i)| \\ & - \dim_{\mathcal{D}(K_{\phi\circ\nu})}\bigl(\operatorname{coker}\bigl(r_A\colon \mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]^n \to \mathcal{D}(K_{\phi\circ\nu})_t[u^{\pm 1}]^n\bigr)\bigr).\end{aligned}$$

Choose two seminorms s_1 and s_2 such that the image of the class $2\cdot[A]$ in $K_1(\mathcal{N}(G))$ under the composite

$$K_1(\mathcal{D}(G)) \xrightarrow{\det_{\mathcal{D}G)}} \mathcal{D}(G)^\times_{\text{abel}} \xrightarrow{\mathbb{P}'_\nu} \mathcal{P}_{\mathbb{Z}}(H_1(M)_f) \xrightarrow{\text{sn}} \mathcal{SN}(H_1(M)_f)$$

is $s_1 - s_2$. We get from the definitions that for any surjective group homomorphism $\phi\colon G \to \mathbb{Z}$ the image of $2\cdot[A]$ under the composite $K_1(\mathcal{D}(G)) \xrightarrow{\det_{\mathcal{D}G)}} \mathcal{D}(G)^\times_{\text{abel}} \xrightarrow{D_{\nu,\phi}} \mathbb{R}$ equals $s_1(\phi) - s_2(\phi)$. We conclude from Lemma 6.16

$$\chi^{(2)}(M;\mu,\phi) = s_1(\phi) + |\phi\circ\nu\circ\mu(s_i)| - s_2(\phi),$$

provided that ϕ is surjective. We conclude from Lemma 2.8 that the last equation holds for every group homomorphism $\phi\colon G \to \mathbb{Z}$. □

6.4. **The quasi-fibered case.**

Definition 6.18 (Fibered and quasi-fibered)**.** Let M be a 3-manifold and consider an element $\phi \in H^1(M;\mathbb{Q})$. We say that ϕ is *fibered* if there exists a locally trivial fiber bundle $p\colon M \to S^1$ and $k \in \mathbb{Q}$, $k > 0$ such that the induced map $p_*\colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ coincides with $k\cdot\phi$. We call an element $\phi \in H^1(M;\mathbb{R})$ *quasi-fibered*, if there exists a sequence of fibered elements $\phi_n \in H^1(M;\mathbb{Q})$ converging to ϕ in $H^1(M;\mathbb{R})$.

Theorem 6.19 (Equality of (μ,ϕ)-L^2-Euler characteristic and the Thurston norm in the quasi-fibered case)**.** *Let M be an admissible 3-manifold, which is not homeomorphic to $S^1\times D^2$. Let G be a torsion-free group which satisfies the Atiyah Conjecture. Consider any factorization $\operatorname{pr}_M\colon \pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} H_1(M)_f$ of the canonical projection pr_M. Let $\phi\colon H_1(M)_f \to \mathbb{Z}$ be a quasi-fibered homomorphism.*

Then $(\mu,\phi\circ\nu)$ is an L^2-acyclic Atiyah pair and we get

$$-\chi^{(2)}(M;\mu,\phi\circ\nu) \;=\; x_M(\phi).$$

Proof. Choose a sequence of fibered elements $\phi_n \in H^1(M;\mathbb{Q})$ converging to ϕ in $H^1(M;\mathbb{R})$. For each n choose a locally trivial fiber bundle $F_n \to M \xrightarrow{p_n} S^1$ and an $k_n \in \mathbb{Q}$, $k_n > 0$ such that the induced map $(p_n)_*\colon \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ coincides with $k_n\cdot\phi_n$. Since there is a finite covering $F_n \to F'_n$ for a connected surface

F'_n, a locally trivially fiber bundle $F'_n \to M \to S^1$, and M is not homeomorphic to $S^1 \times S^2$ or $S^1 \times D^2$, we have $\chi(F'_n) \le 0$ and hence $\chi(F_n) \le 0$. We conclude from (1.4), Example 2.9, Theorem 3.2 (3), and [30, Theorem 2.1] that $(\mu, k_n \cdot \phi_n)$ is an L^2-acyclic Atiyah-pair for M and

$$-\chi^{(2)}(M;\mu,(k_n \cdot \phi_n)\circ\nu) = -\chi(F_n) = x_M(k_n \cdot \phi_n). \tag{6.20}$$

Let s_1 and s_2 be the two seminorms appearing in Lemma 6.17. Recall that we have for every $\psi \in H^1(M;\mathbb{Z})$

$$\chi^{(2)}(M;\mu,\psi\circ\nu) \;=\; s_1(\psi) - s_2(\psi). \tag{6.21}$$

Since any seminorm on $H^1(M;\mathbb{R})$ is continuous, we get

$$\begin{aligned} x_M(\phi) &= \lim_{n\to\infty} x_M(\phi_n) = \lim_{n\to\infty} \frac{x_M(k_n\cdot\phi_n)}{k_n} \\ &\overset{(6.20)}{=} \lim_{n\to\infty} \frac{-\chi^{(2)}(M;\mu,(k_n\cdot\phi_n)\circ\nu)}{k_n} \overset{(6.21)}{=} \lim_{n\to\infty} \frac{-s_1(k_n\cdot\phi_n)+s_2(k_n\cdot\phi_n)}{k_n} \\ &= \lim_{n\to\infty} -s_1(\phi_n)+s_2(\phi_n) = -s_1(\phi)+s_2(\phi) \overset{(6.21)}{=} \chi^{(2)}(M;\mu,\phi). \quad \square \end{aligned}$$

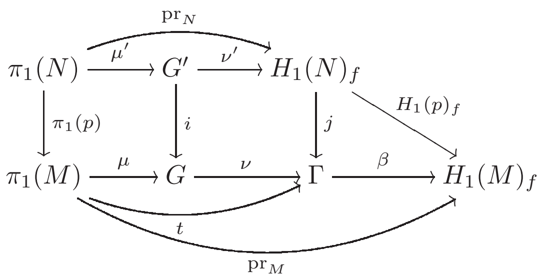
6.5. **Proof of Theorem 0.4.**

Proof of Theorem 0.4. This is a variation of the proof of [14, Theorem 5.1]. For the reader's convenience we give some details here as well.

As explained in [9, Section 10], we conclude from combining [1, 2, 28, 35, 36, 44, 45] that there exists a finite regular covering $p\colon N \to M$ such that for any $\phi \in H^1(M;\mathbb{R})$ its pullback $p^*\phi \in H^1(N;\mathbb{R})$ is quasi-fibered. Let k be the number of sheets of p. Let $\mathrm{pr}_N\colon \pi_1(N) \to H_1(N)_f$ be the canonical projection. Its kernel is a characteristic subgroup of $\pi_1(N)$. The regular finite covering p induces an injection $\pi_1(p)\colon \pi_1(N) \to \pi_1(M)$ such that the image of $\pi_1(p)$ is a normal subgroup of $\pi_1(M)$ of finite index. Let Γ be the quotient of $\pi_1(M)$ by the normal subgroup $\pi_1(p)(\ker(\mathrm{pr}_N))$. Let $\alpha\colon \pi_1(M) \to \Gamma$ be the projection. Since $\pi_1(p)(\ker(\mathrm{pr}_N))$ is contained in the kernel of the canonical projection $\mathrm{pr}_M\colon \pi_1(M) \to H_1(M)_f$ because of $H_1(p;\mathbb{Z})_f \circ \mathrm{pr}_N = \mathrm{pr}_M \circ \pi_1(p)$, there exists precisely one epimorphism $\beta\colon \Gamma \to H_1(M)_f$ satisfying $\mathrm{pr}_M = \beta\circ\alpha$. Moreover, $\alpha\circ\pi_1(p)$ factorizes over pr_M into an injective homomorphism $j\colon H_1(M)_f \to \Gamma$ with finite cokernel. Hence Γ is virtually finitely generated free abelian.

Consider a factorization of $\alpha\colon \pi_1(M) \to \Gamma$ into group homomorphisms $\pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ for a group G which satisfies the Atiyah Conjecture. Let G' be the quotient of $\pi_1(N)$ by the normal subgroup $\pi_1(p)^{-1}(\ker(\mu))$ and $\mu'\colon \pi_1(N) \to G'$ be the projection. Since the kernels of μ' and of $\mu\circ\pi_1(p)$ agree, there is precisely one injective group homomorphism $i\colon G' \to G$ satisfying $\mu\circ\pi_1(p) = i\circ\mu'$. The kernel of μ' is contained in the kernel of $\mathrm{pr}_N\colon \pi_1(N) \to H_1(N)_f$ since j is injective and we have $j\circ\mathrm{pr}_N = \nu\circ i\circ\mu'$. Hence there is precisely one group homomorphism $\nu'\colon G' \to H_1(N)_f$ satisfying $\nu'\circ\mu' = \mathrm{pr}_N$. One easily checks that the following diagram commutes, and all vertical arrows are injective, the indices $[\pi_1(N) : \mathrm{im}(\pi_1(p)]$ and

$[\Gamma : H_1(MN_f]$ are finite, and μ', ν' and β are surjective.



Since G satisfies the Atiyah Conjecture, the group G' satisfies the Atiyah Conjecture by Theorem 3.2 (1).

Since $\ker(\mu) \subseteq \ker(\alpha) \subseteq \operatorname{im}(\pi_1(p))$ holds, we get $[G : G'] = k$ conclude from (1.3) and from Lemma 2.17 (3)

$$-\chi^{(2)}(M; \mu, \phi \circ \beta \circ \nu) \;=\; -\frac{\chi^{(2)}(N; \mu', p^*\phi \circ \nu')}{k};$$
$$x_M(\phi) \;=\; \frac{x_N(p^*\phi)}{k}.$$

We get from Theorem 6.19 applied to N, μ', ν' and $p^*\phi$

$$-\chi^{(2)}(N; \mu', p^*\phi \circ \nu') \;=\; x_N(p^*\phi).$$

Hence we get

$$-\chi^{(2)}(M; \mu, \phi \circ \beta \circ \nu) \;=\; x_M(\phi).$$

This finishes the proof of Theorem 0.4. □

7. Epimorphism of fundamental groups and the Thurston norm

On several occasions we will use the following lemma.

Lemma 7.1. *Let G be a torsion-free elementary amenable group. Then the Ore localization $T^{-1}\mathbb{Q}G$ for T the set of non-trivial elements in $\mathbb{Q}G$ exists and agrees with the skew field $\mathcal{D}(G)$. In particular $\mathcal{D}(G)$ is flat over $\mathbb{Q}G$. Moreover, G satisfies the Atiyah Conjecture and we get for every finitely generated free $\mathbb{Q}G$-chain complex*

$$b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*) \;=\; \dim_{\mathcal{D}(G)}(H_n(\mathcal{D}(G) \otimes_{\mathbb{Q}G} C_*)).$$

Proof. This follows from Theorem 3.2 (2), Theorem 3.6 (2) and [32, Example 8.16 on page 324 and Lemma 10.16 on page 376]. The proofs there deal only with $\mathbb{C}$, but carry over without changes to any field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$. □

Theorem 7.2. *Let G be a group which is residually a locally indicable elementary amenable group. Let $f_* \colon C_* \to D_*$ be a $\mathbb{Q}G$-chain map of finitely generated free $\mathbb{Q}G$-chain complexes. Suppose that $\mathrm{id}_{\mathbb{Q}} \otimes_{\mathbb{Q}G} f_* \colon \mathbb{Q} \otimes_{\mathbb{Q}G} C_* \to \mathbb{Q} \otimes_{\mathbb{Q}G} D_*$ induces an isomorphism on homology. Then we get for $n \geq 0$*

$$b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*) \;=\; b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} D_*).$$

Proof. By considering the mapping cone, it suffices to show for a finitely generated free $\mathbb{Q}G$-chain complex C_* that $b_n^{(2)}(\mathcal{N}(G) \otimes_{\mathbb{Q}G} C_*)$ vanishes for all $n \geq 0$ if $H_n(\mathbb{Q} \otimes_{\mathbb{Q}G} C_*)$ vanishes for all $n \geq 0$.

Since G is residually a torsion-free locally indicable elementary amenable group, there exists a sequence of epimorphisms $G \to G_i$, $i \in \mathbb{N}$ onto torsion-free locally indicable elementary amenable groups such that the intersections of the kernels is trivial. We conclude from [5], [39, Theorem 1.14] or [32, Theorem 13.3 on page 454]

$$\begin{aligned} b_n^{(2)}(\mathcal{N}(G)\otimes_{\mathbb{Q}G} C_*) &= \lim_{i\in\mathbb{N}} b_n^{(2)}(\mathcal{N}(G_i)\otimes_{\mathbb{Q}G_i}\mathbb{Q}G_i\otimes_{\mathbb{Q}G} C_*);\\ \mathbb{Q}\otimes_{\mathbb{Q}G} C_* &\cong \mathbb{Q}\otimes_{\mathbb{Q}G_i}\mathbb{Q}G_i\otimes_{\mathbb{Q}G} C. \end{aligned}$$

It follows that without loss of generality we can assume that G itself is torsion-free locally indicable elementary amenable.

There is an involution of rings $\mathbb{Q}G\to\mathbb{Q}G$ sending $\sum_{g\in G}\lambda_g\cdot g$ to $\sum_{g\in G}\lambda_g\cdot g^{-1}$. In the sequel we equip each C_n with a $\mathbb{Q}G$-basis. With respect to this involution and $\mathbb{Q}G$-basis one can define the combinatorial Laplace operator $\Delta_n\colon C_n\to C_n$ which is the $\mathbb{Q}G$-linear map given by $c_{n+1}\circ c_n^*+c_{n-1}^*\circ c_{n-1}$. Since the augmentation homomorphism $\mathbb{Q}G\to\mathbb{Q}$ sending $\sum_{g\in G}\lambda_g\cdot g$ to $\sum_{g\in G}\lambda_g$ is compatible with the involution, $\mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Q}G}\Delta_n\colon \mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Q}G}C_n\to\mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Q}G}C_n$ is the combinatorial Laplace operator for $\mathbb{Q}\otimes_{\mathbb{Q}G}C_*$. We conclude from [32, Lemma 1.18 on page 24 and Theorem 6.25 on page 249]

$$\begin{aligned} b_n^{(2)}(\mathcal{N}(G)\otimes_{\mathbb{Q}G} C_*) &= \dim_{\mathcal{N}(G)}(\ker(\mathrm{id}_{\mathcal{N}(G)}\otimes_{\mathbb{Q}G}\Delta_n));\\ \dim_{\mathbb{Q}}(H_n(\mathbb{Q}\otimes_{\mathbb{Q}G} C_*) &= \dim_{\mathbb{Q}}(\ker(\mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Q}G}\Delta_n)). \end{aligned}$$

Since G is amenable, we conclude from [32, (6.74) on page 275,]

$$\dim_{\mathcal{N}(G)}(\ker(\mathrm{id}_{\mathcal{N}(G)}\otimes_{\mathbb{Q}G}\Delta_n)) = \dim_{\mathcal{N}(G)}(\mathcal{N}(G)\otimes_{\mathbb{Q}G}\ker(\Delta_n)).$$

Hence $b_n^{(2)}(\mathcal{N}(G)\otimes_{\mathbb{Q}G}C_*)$ vanishes if Δ_n is injective. The injectivity of Δ_n follows from the injectivity of $\mathrm{id}_{\mathbb{Q}}\otimes_{\mathbb{Q}G}\Delta_n$ by [24, Theorem 1] or [20, Theorem 4.1], since G is locally indicable. ☐

Theorem 7.3. *Let $f\colon M\to N$ be a map of admissible 3-manifolds. Suppose that $\pi_1(f)$ is surjective and f induces an isomorphism $H_n(f;\mathbb{Q})\colon H_n(M;\mathbb{Q})\to H_n(N;\mathbb{Q})$ for $n\ge 0$. Suppose that G is residually locally indicable elementary amenable. Let $\mu\colon\pi_1(N)\to G$, $\nu\colon G\to H_1(N)_f$ and $\phi\colon H_1(N)_f\to\mathbb{Z}$ be group homomorphisms. Let $\overline{N}\to N$ be the G-covering associated to μ and $\overline{M}\to M$ be the G-covering associated to $\mu\circ\pi_1(f)$. Suppose that $b_n^{(2)}(\overline{N};\mathcal{N}(G))$ vanishes for $n\ge 0$.*

Then $b_n^{(2)}(\overline{M};\mathcal{N}(G))$ vanishes for $n\ge 0$, M is $(\mu\circ\pi_1(f),\phi\circ\nu)$-$L^2$-finite, N is $(\mu,\phi\circ\nu)$-L^2-finite and we get

$$-\chi^{(2)}(M;\mu\circ\pi_1(f);\phi\circ\nu)\ \ge\ -\chi^{(2)}(N;\mu,\phi\circ\nu).$$

Proof. Since a locally indicable group is torsion-free, G is a residually torsion-free elementary amenable group and hence satisfies the Atiyah Conjecture by Theorem 3.2 (4). Because of Lemma 2.8 and Theorem 3.2 (1) we can assume without loss of generality that μ and $\phi\circ\nu$ are epimorphisms. Theorem 7.2 implies that $b_n^{(2)}(\overline{M};\mathcal{N}(G))$ vanishes for $n\ge 0$. We conclude from Theorem 3.4 that M is $(\mu\circ\pi_1(f),\phi\circ\nu)$-$L^2$-finite and N is $(\mu,\phi\circ\nu)$-L^2-finite.

Since $\pi_1(f)$ is surjective and hence the G-map $\overline{f}\colon\overline{M}\to\overline{N}$ induced by f is 1-connected, we get $b_1^{(2)}(i^*\overline{M};\mathcal{N}(K))\ge b_1^{(2)}(i^*\overline{N};\mathcal{N}(K))$ for the inclusion $i\colon K=\ker(\phi\circ\nu)\to G$. If $\phi\circ\nu\circ\mu=0$, we conclude $\chi^{(2)}(M;\mu\circ\pi_1(f);\phi\circ\nu)=\chi^{(2)}(N;\mu,\phi\circ\nu)=0$ from Lemma 2.8 (3) and the claim follows. Hence we can assume without loss of generality that $\phi\circ\nu\circ\mu$ is not trivial.

We begin with the case $\mathrm{im}(\mu)\cap\ker(\phi\circ\nu)\ne\{1\}$. Then also $\mathrm{im}(\mu\circ\pi_1(f))\cap\ker(\phi\circ\nu)\ne\{1\}$. We conclude from Theorem 5.5

$$\begin{aligned} -\chi^{(2)}(N;\mu,\phi\circ\nu) &= b_1^{(2)}(i^*\overline{N};\mathcal{N}(K));\\ -\chi^{(2)}(M;\mu\circ\pi_1(f);\phi\circ\nu) &= b_1^{(2)}(i^*\overline{M};\mathcal{N}(K)). \end{aligned}$$

and Theorem 7.3 follows.

It remains to treat the case, where $\operatorname{im}(\mu)\cap\ker(\phi\circ\nu) = \{1\}$. Then $\phi\circ\nu\colon G\to\mathbb{Z}$ is an injection and hence a bijection, $K = \{0\}$, and we get from Example 5.6

$$-\chi^{(2)}(N;\mu,\phi\circ\nu) \;=\; \begin{cases}\dim_Q\big(H_1(\overline{M};\mathbb{Q})\big)-1 & \text{if } \partial M\neq\emptyset;\\ \dim_Q\big(H_1(\overline{M};\mathbb{Q})\big)-2 & \text{if } \partial M=\emptyset;\end{cases}$$

$$-\chi^{(2)}(N;\mu\circ\pi_1(f),\phi\circ\nu) \;=\; \begin{cases}\dim_Q\big(H_1(\overline{N};\mathbb{Q})\big)-1 & \text{if } \partial N\neq\emptyset;\\ \dim_Q\big(H_1(\overline{N};\mathbb{Q})\big)-2 & \text{if } \partial N=\emptyset;\end{cases}$$

We already have shown $b_1^{(2)}(i^*\overline{N};\mathcal{N}(K)) \geq b_1^{(2)}(i^*\overline{M};\mathcal{N}(K))$ which boils down in this special case to $\dim_Q(H_1(\overline{M};\mathbb{Q}))\big) \geq \dim_Q(H_1(\overline{N};\mathbb{Q}))\big)$. We conclude from $H_3(M;\mathbb{Q})\cong H_3(N;\mathbb{Q})$ that ∂M is empty if and only if ∂N is empty. This finishes the proof of Theorem 7.3. □

Theorem 7.4 (Inequality of the Thurston norm)**.** *Let $f\colon M\to N$ be a map of admissible 3-manifolds which is surjective on $\pi_1(N)$ and induces an isomorphism $f_*\colon H_n(M;\mathbb{Q})\to H_n(N;\mathbb{Q})$ for $n\geq 0$. Suppose that $\pi_1(N)$ is residually locally indicable elementary amenable. Then we get for any $\phi\in H^1(N;\mathbb{R})$ that*

$$x_M(f^*\phi) \;\geq\; x_N(\phi).$$

Proof. Since seminorms are continuous and homogeneous it suffices to prove the statement for all primitive classes $\phi\in H^1(N;\mathbb{Z}) = \operatorname{Hom}(\pi_1(N),\mathbb{Z})$. The case $N = S^1\times D^2$ is trivial. Hence we can assume that $N\neq S^1\times D^2$. We conclude from Theorem 7.3 applied in the case $G=\pi_1(N)$ and $\mu=\operatorname{id}_{\pi_1(N)}$

$$-\chi^{(2)}(M;\pi_1(f),\phi) \;\geq\; -\chi^{(2)}(\widetilde{N};\phi)$$

Theorem 4.1 implies

$$x_M(\phi\circ\pi_1(f)) \;\geq\; -\chi^{(2)}(M;\pi_1(f),\phi)$$

and Theorem 0.2 implies that

$$-\chi^{(2)}(\widetilde{N};\phi) \;=\; x_N(\phi).$$

Now Theorem 7.4 follows. □

The following lemma shows that Theorem 7.4 applies in particular if the manifold N is fibered. Since it is well-known, thus we only provide a sketch of the proof.

Lemma 7.5. *The fundamental group of any fibered 3-manifold is residually locally indicable elementary amenable.*

Sketch of proof. Let N be a fibered 3-manifold. Then $\pi_1(N)\cong\mathbb{Z}\ltimes_\varphi\Gamma$ where Γ is a free group or a surface group and where $\varphi\colon\Gamma\to\Gamma$ is an automorphism. The derived series of Γ is defined by $\Gamma^{(0)}=\Gamma$ and inductively by $\Gamma^{(n+1)} = [\Gamma^{(n)},\Gamma^{(n)}]$. Since Γ is a free group or a surface group, each quotient $\Gamma^{(n)}/\Gamma^{(n+1)}$ is free abelian and $\bigcap_{n\geq 1}\Gamma^{(n)} = \{1\}$.

The subgroups $\Gamma^{(n)}$ are characteristic subgroups of Γ, in particular they are preserved by φ. Thus φ descends to an automorphism on $\Gamma/\Gamma^{(n)}$. It is now straightforward to see that the epimorphisms $\pi_1(N)=\mathbb{Z}\ltimes\Gamma\to\mathbb{Z}\ltimes\Gamma/\Gamma^{(n)}$, $n\in\mathbb{N}$ form a cofinal nested sequence of epimorphisms onto locally indicable elementary amenable groups. □

8. The (μ,ϕ)-L^2-Euler characteristic and the degree of non-commutative Alexander polynomials

Let M be an admissible 3-manifold. Regard group homomorphisms $\mu\colon \pi_1(M) \to G$, $\nu\colon G \to H_1(M)_f$ and $\phi\colon H_1(M)_f \to \mathbb{Z}$ such that $\nu \circ \mu$ is the projection $\pi_1(M) \to H_1(M)_f$ and G is torsion-free elementary amenable. For simplicity we discuss only the case, where ϕ is surjective. Let $\overline{M} \to M$ be the G-covering associated to $\mu\colon \pi_1(M) \to G$. Recall the following definition from Harvey [21] which extends ideas of Cochran [6]. (Actually they consider only certain solvable quotients G of $\pi_1(M)$ coming from the rational derived series, but their constructions apply directly to torsion-free elementary amenable groups.) Let T be the set of non-trivial elements in $\mathbb{Z}G$. As recorded already in Lemma 7.1, the Ore localization $T^{-1}\mathbb{Z}G$ is defined and is a skewfield. Define a natural number

$$r_n(M;\mu) := \dim_{T^{-1}\mathbb{Z}G}\big(H_n\big(T^{-1}\mathbb{Z}G \otimes_{\mathbb{Z}G} C_*(\overline{M})\big)\big). \tag{8.1}$$

Let $i\colon K \to G$ be the inclusion of the kernel of the composite $\phi \circ \nu\colon G \to \mathbb{Z}$. If T is the set of non-zero elements in $\mathbb{Z}K$, we can again consider its Ore localization $T^{-1}\mathbb{Z}K$ which is a skew field. We obtain an isomorphism for an appropriate automorphism t of $T^{-1}\mathbb{Z}K$, which comes from the conjugation automorphism of K associated to a lift of a generator of $\mathbb{Z}$ to G, an isomorphism of skew-fields

$$(T^{-1}\mathbb{Z}K)_t[u^{\pm 1}] \xrightarrow{\cong} T^{-1}\mathbb{Z}G. \tag{8.2}$$

If $r_1(M;\mu,\nu,\phi)$ vanishes for all $n \geq 0$, then we can define natural numbers

$$\delta_n(M;\mu,\nu,\phi) := \dim_{T^{-1}\mathbb{Z}K}\big(H_1(T^{-1}\mathbb{Z}G \otimes_{\mathbb{Z}G} C_*(\overline{M})\big). \tag{8.3}$$

This construction and the invariants above turn out to be special cases of the constructions defined in this paper. Namely, K and G satisfy the Atiyah Conjecture by Theorem 3.2 (2), and Lemma 7.1 shows that we get identifications $T^{-1}\mathbb{Z}K = \mathcal{D}(K)$ and $T^{-1}\mathbb{Z}G = \mathcal{D}(G)$ under which the isomorphism (8.2) corresponds to the isomorphism appearing in Theorem 3.6 (3). Moreover $r_n(M;\nu)$ agrees with $b_n^{(2)}(\overline{M};\mathcal{N}(G))$ by Theorem 3.6 (2). Hence Theorem 3.6 (4) and Lemma 5.4 (1) imply

Theorem 8.4 (The (μ,ϕ)-L^2-Euler characteristic and $\delta_n(M,\mu,\nu,\phi)$)**.** *Let M be an admissible 3-manifold. Consider group homomorphisms $\mu\colon \pi_1(M) \to G$, $\nu\colon G \to H_1(M)_f$ and $\phi\colon H_1(M)_f \to \mathbb{Z}$ such that $\nu \circ \mu$ is the projection $\pi_1(M) \to H_1(M)_f$ and ϕ is surjective.*

Then $r_1(M;\mu)$ vanishes if and only if $(\mu, \phi \circ \nu)$ is an L^2-Atiyah pair, and in this case we get

$$\chi^{(2)}(M;\mu,\phi\circ\nu) \;=\; -\delta(M;\mu,\nu,\phi).$$

Remark 8.5. Another way of interpreting Theorem 8.4 is to say that our L^2-Euler characteristic invariant extends the original invariant due to Cochran, Harvey and the third author [6, 21, 12] to other coverings, in particular to the universal covering or to a G-covering for residually torsion-free elementary amenable group G of an admissible 3-manifold.

The following lemma might also be of independent interest.

Lemma 8.6. *Let $\alpha\colon \pi_1(M) \to \Gamma$ be an epimorphism onto a group that is virtually torsion-free abelian. Then there exists a factorization of α into group homomorphisms $\pi \xrightarrow{\mu} G \xrightarrow{\nu} \Gamma$ such that G is torsion-free elementary amenable.*

Proof. Let $\alpha\colon \pi_1(M) \to \Gamma$ be an epimorphism onto a group Γ that admits a finite index subgroup F that is free abelian. After possibly going to the core of F we can assume that F is normal.

Since $\alpha^{-1}(F)$ is a finite-index subgroup of $\pi_1(M)$ we conclude from [16, Theorem 4.3] that there is a torsion-free elementary amenable group G' together with an epimorphism $\mu'\colon \pi_1(M) \to G'$ such that $\ker(\mu') \subset \alpha^{-1}(F)$. Define the epimorphism $\mu\colon \pi_1(M) \to G$ to be the projection onto the quotient $G = \pi_1(M)/\big(\ker(\mu') \cap \ker(\alpha)\big)$. Obviously there is an epimorphism $\nu\colon G \to \Gamma$ such that $\nu \circ \mu = \alpha$ since α is by construction the projection from $\pi_1(M)$ to the quotient $\Gamma = \pi_1(M)/\ker(\alpha)$. It remains to show that G is torsion-free elementary amenable. We have the obvious exact sequence

$$1 \to \ker(\mu')/\big(\ker(\mu') \cap \ker(\alpha)\big) \to G \to G' \to 1.$$

and obviously α defines an injection

$$\ker(\mu')/\big(\ker(\mu') \cap \ker(\alpha)\big) \hookrightarrow F.$$

Since F and G' are torsion-free elementary amenable, the same is true for G. □

Theorems 0.4 and 8.4 and Lemma 8.6 imply that the non-commutative Reidemeister torsions of [12] detect the Thurston norm of most 3-manifolds.

Corollary 8.7. *Let M be a 3-manifold, which is admissible, see Definition 0.1, is not a closed graph manifold and is not homeomorphic to $S^1 \times D^2$. Then there is a torsion-free elementary amenable group G and a factorization $\operatorname{pr}_M\colon \pi_1(M) \xrightarrow{\mu} G \xrightarrow{\nu} H_1(M)_f$ of the canonical projection pr_M into epimorphisms such that for any group homomorphism $\phi\colon H_1(M)_f \to \mathbb{Z}$, the pair $(\mu, \phi \circ \nu)$ is an L^2-acyclic Atiyah-pair and we get*

$$\delta(M; \mu, \nu, \phi) \;=\; -\chi^{(2)}(M; \mu, \phi \circ \nu) \;=\; x_M(\phi).$$

Remark 8.8. The invariant $\delta(M; \mu, \nu), \phi$ of [12] is essentially the same as the Cochran-Harvey invariant [6, 21], except that Cochran–Harvey only study solvable quotients of $\pi_1(M)$. But as pointed out in [6, Example 2.3], in general invariants coming from solvable quotients do not suffice to detect the knot genus respectively the Thurston norm. It is necessary to work with the extra flexibility given by torsion-free elementary amenable groups.

9. The degree of the L^2-torsion function

In [33] the ϕ-twisted L^2-torsion function has been introduced and analyzed for G-coverings of compact connected manifolds in all dimensions. In the sequel we will consider only G-coverings $\overline{M} \to M$ of admissible 3-manifolds M for countable residually finite G. Then all the necessary conditions such as det-L^2-acyclicity for $\overline{M}$ and the K-theoretic Farrell-Jones Conjecture for $\pi_1(M)$ are automatically satisfied and do not have to be discussed anymore. One can assign to the L^2-torsion function by considering its asymptotic behavior at infinity a real number called its degree and denoted by $\deg\big(\rho^{(2)}(M; \mu, \phi)\big)$. If $G = \pi_1(M)$ and $\mu \;=\; \operatorname{id}_{\pi_1(M)}$, i.e., for the universal covering $\widetilde{M}$, the equality

$$\deg\big(\rho^{(2)}(M; \mu, \phi)\big) = x_M(\phi \circ \mu)$$

was proved by the authors in [14, Theorem 0.1] and independently by Liu [27]. Actually many more instances of G-coverings are considered in [14, Theorem 5.1], where this equality holds.

We just mention without proof the following theorem.

Theorem 9.1 (The (μ, ϕ)-L^2-Euler characteristic is a lower bound for the degree of the L^2-torsion function)**.** *Let M be an admissible 3-manifold. Let $\mu\colon \pi \to G$ be a homomorphism to a torsion-free, elementary amenable, residually finite, countable group G and $\phi\colon G \to \mathbb{Z}$ be a group homomorphism. Let $\overline{M} \to M$ be the G-covering* *associated to* μ*. Suppose* $b_1^{(2)}(\overline{M};\mathcal{N}(G)) = 0$*. Then* (μ,ϕ) *is an* L^2*-acyclic Atiyah pair and we get*

$$\chi^{(2)}(M;\mu,\phi) \leq \deg\big(\overline{\rho}^{(2)}(M;\mu,\phi)\big).$$

References

[1] I. Agol. Criteria for virtual fibering. *J. Topol.*, 1(2):269–284, 2008.

[2] I. Agol. The virtual Haken conjecture. *Doc. Math.*, 18:1045–1087, 2013. With an appendix by Agol, Groves, and Manning.

[3] G. Burde and H. Zieschang. *Knots*. Walter de Gruyter & Co., Berlin, 1985.

[4] J. Cheeger and M. Gromov. L_2-cohomology and group cohomology. *Topology*, 25(2):189–215, 1986.

[5] B. Clair. Residual amenability and the approximation of L^2-invariants. *Michigan Math. J.*, 46(2):331–346, 1999.

[6] T. D. Cochran. Noncommutative knot theory. *Algebr. Geom. Topol.*, 4:347–398, 2004.

[7] P. M. Cohn. *Free rings and their relations*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], London, second edition, 1985.

[8] P. M. Cohn. *Skew fields*, volume 57 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. Theory of general division rings.

[9] J. Dubois, S. Friedl, and W. Lück. The L^2–Alexander torsion of 3-manifolds. Preprint, arXiv:1410.6918 [math.GT], to appear in in Journal of Topology, 2014.

[10] D. Eisenbud and W. Neumann. *Three-dimensional link theory and invariants of plane curve singularities*, volume 110 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[11] R. H. Fox. Free differential calculus. I. Derivation in the free group ring. *Ann. of Math. (2)*, 57:547–560, 1953.

[12] S. Friedl. Reidemeister torsion, the Thurston norm and Harvey's invariants. *Pacific J. Math.*, 230(2):271–296, 2007.

[13] S. Friedl. Twisted Reidemeister torsion, the Thurston norm and fibered manifolds. *Geom. Dedicata*, 172:135–145, 2014.

[14] S. Friedl and W. Lück. The L^2-torsion function and the Thurston norm of 3-manifolds. Preprint, arXiv:1510.00264 [math.GT], 2015.

[15] S. Friedl, W. Lück, and S. Tillman. Groups and polytopes. in preparation, 2016.

[16] S. Friedl, K. Schreve, and S. Tillmann. Thurston norm via Fox calculus. Preprint, arXiv:1507.05660 [math.GT], 2015.

[17] F. Funke. The integral polytope group. Preprint, arXiv:1605.01217 [math.MG], 2016.

[18] F. Funke and D. Kielak. Alexander and Thurston norms, and the Bier-Neumann-Strebel invariants for free-by-cyclic groups. Preprint, arXiv:1605.09069 [math.RA], 2016.

[19] D. Gabai. Foliations and the topology of 3-manifolds. *J. Differential Geom.*, 18(3):445–503, 1983.

[20] S. M. Gersten. Conservative groups, indicability, and a conjecture of Howie. *J. Pure Appl. Algebra*, 29(1):59–74, 1983.

[21] S. L. Harvey. Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm. *Topology*, 44(5):895–945, 2005.

[22] G. Herrmann. The L^2-Alexander torsion of Seifert fibered 3-manifolds. Preprint, arXiv:1602.08768 [math.GT], 2016.

[23] J. Howie. On locally indicable groups. *Math. Z.*, 180(4):445–461, 1982.

[24] J. Howie and H. R. Schneebeli. Homological and topological properties of locally indicable groups. *Manuscripta Math.*, 44(1-3):71–93, 1983.

[25] R. Kirby. Problems in low dimensional manifold theory. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 273–312. Amer. Math. Soc., Providence, R.I., 1978.

[26] P. A. Linnell. Division rings and group von Neumann algebras. *Forum Math.*, 5(6):561–576, 1993.

[27] X. Liu. Degree of L^2-Alexander torsion for 3-manifolds. Preprint, arXiv:1509.08866 [math.GT], 2015.

[28] Y. Liu. Virtual cubulation of nonpositively curved graph manifolds. *J. Topol.*, 6(4):793–822, 2013.

[29] J. Lott and W. Lück. L^2-topological invariants of 3-manifolds. *Invent. Math.*, 120(1):15–60, 1995.

[30] W. Lück. L^2-Betti numbers of mapping tori and groups. *Topology*, 33(2):203–214, 1994.

[31] W. Lück. L^2-torsion and 3-manifolds. In *Low-dimensional topology (Knoxville, TN, 1992)*, pages 75–107. Internat. Press, Cambridge, MA, 1994.

[32] W. Lück. *L^2-Invariants: Theory and Applications to Geometry and K-Theory*, volume 44 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2002.
[33] W. Lück. Twisting L^2-invariants with finite-dimensional representations. Preprint, arXiv:1510.00057 [math.GT], 2015.
[34] C. T. McMullen. The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology. *Ann. Sci. École Norm. Sup. (4)*, 35(2):153–171, 2002.
[35] P. Przytycki and D. T. Wise. Mixed 3-manifolds are virtually special. Preprint, arXiv:1205.6742 [math.GR], 2012.
[36] P. Przytycki and D. T. Wise. Graph manifolds with boundary are virtually special. *Journal of Topology*, 7:419–435, 2014.
[37] H. Rådström. An embedding theorem for spaces of convex sets. *Proc. Amer. Math. Soc.*, 3:165–169, 1952.
[38] H. Reich. L^2-Betti numbers, isomorphism conjectures and noncommutative localization. In *Non-commutative localization in algebra and topology*, volume 330 of *London Math. Soc. Lecture Note Ser.*, pages 103–142. Cambridge Univ. Press, Cambridge, 2006.
[39] T. Schick. L^2-determinant class and approximation of L^2-Betti numbers. *Trans. Amer. Math. Soc.*, 353(8):3247–3265 (electronic), 2001.
[40] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.
[41] J. R. Silvester. *Introduction to algebraic K-theory*. Chapman & Hall, London, 1981. Chapman and Hall Mathematics Series.
[42] W. P. Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 59(339):i–vi and 99–130, 1986.
[43] V. Turaev. A homological estimate for the thurston norm. Preprint, arXiv:math/0207267 [math.GT], 2002.
[44] D. T. Wise. *From riches to raags: 3-manifolds, right-angled Artin groups, and cubical geometry*, volume 117 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2012.
[45] D. T. Wise. The structure of groups with a quasiconvex hierarchy. Preprint, `http://www.math.mcgill.ca/wise/papers.html`, 2012.

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: `sfriedl@gmail.com`

Mathematisches Institut der Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany
E-mail address: `wolfgang.lueck@him.uni-bonn.de`
URL: `http://www.him.uni-bonn.de/lueck`