

APPROXIMATELY FIBERING A MANIFOLD OVER AN ASPHERICAL ONE

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ABSTRACT. The paper is devoted to the problem when a map from some closed connected manifold to an aspherical closed manifold approximately fibers, i.e., is homotopic to Manifold Approximate Fibration. We define obstructions in algebraic K -theory. Their vanishing is necessary and under certain conditions sufficient. Basic ingredients are Quinn’s Thin h -Cobordism Theorem and End Theorem, and knowledge about the Farrell-Jones Conjectures in algebraic K - and L -theory and the MAF-Rigidity Conjecture by Hughes-Taylor-Williams.

INTRODUCTION

0.1. Manifold Approximate Fibrations. A Manifold Approximate Fibration (MAF) is a map between closed topological manifolds which is an approximate fibration in the sense that it has an “approximate lifting property”, generalizing the usual lifting property of Hurewicz fibrations, see Definition 1.3. This notion of an approximate fibration was introduced by Coram-Duvall [18]. The theory of MAFs is a “bundle theory”, see [42], which plays a prominent role in the study of topological manifolds and controlled topology. For instance, Edwards [21] proved in the early 1970s that a locally flat submanifold of a topological manifold of dimension greater than five has a mapping cylinder neighborhood, where the projection map is a MAF.

The investigations of group actions on topological manifold of Quinn [65, 66, 67] rely on the theory of approximate fibrations. Further papers dealing with approximate fibrations are [17, 32, 37, 38, 39, 40, 41, 43, 44, 50].

A map $p: M \rightarrow B$ between closed manifolds is a MAF if and only if the preimage of any open ball in B has the weak homotopy type of the homotopy fiber of p . It is also true that the restriction of a MAF to an open subset of the base space is again an approximate fibration, see [42, Corollary 12.14 and Theorem 12.15].

In this paper we treat the following question:

Given a closed aspherical manifold B and a map $p: M \rightarrow B$ from some closed connected manifold, when is p homotopic to MAF?

We emphasize that we do not want to change the source or target, only the map within its homotopy class.

0.2. Two appetizers. As an illustration we present two easy to formulate prototypes of more general results we will later prove.

Theorem 0.1 (Special case I). *Let B be a closed connected aspherical PL manifold with hyperbolic fundamental group, e.g., a closed connected smooth manifold with Riemannian metric of negative sectional curvature. Let M be a closed connected manifold of dimension $\neq 4$. Suppose $\pi_1(M)$ is torsionfree. Moreover, assume that $\pi_1(M)$ is a hyperbolic group, CAT(0)-group, a solvable group, arithmetic group, or*

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a lattice in an almost connected Lie group. (Actually, we only need that $\pi_1(M)$ satisfies the K - and L -theoretic Farrell-Jones Conjecture.)

Then a map $M \rightarrow B$ is homotopic to a MAF if and only if the homotopy fiber of p is finitely dominated.

Theorem 0.2 (Special case II). *Let B be an aspherical closed connected PL manifold with hyperbolic fundamental group, e.g., a closed connected smooth manifold with Riemannian metric of negative sectional curvature. Let M be a closed aspherical manifold of dimension $\neq 4$. Let $p: M \rightarrow B$ be a map such that the kernel of $\pi_1(p): \pi_1(M) \rightarrow \pi_1(B)$ is poly-cyclic and its image has finite index.*

Then $p: M \rightarrow B$ is homotopic to a MAF.

0.3. The Naive Conjecture. *Throughout this introduction, we will make the following standing assumptions:*

$p: M \rightarrow B$ is a continuous map, where

- (i) M is a connected closed (topological) manifold;
- (ii) B is an aspherical closed (topological) manifold which admits a PL structure (e.g., is smooth);
- (iii) The homotopy fiber of p is homotopy finite, i.e., has the homotopy type of a finite CW complex;
- (iv) p induces a surjection on fundamental groups.

(But see also Section 9 for a discussion of the case of a non-finite fiber, in particular see Theorem 9.5.)

In this situation, we will show in Theorem 4.1 that there exists a factorization up to homotopy

$$(0.3) \quad \begin{array}{ccc} M & \xrightarrow{f} & E \\ & \searrow p & \swarrow q \\ & & B \end{array}$$

where f is a homotopy equivalence, q is an approximate fibration, and E is a compact ENR.

Given such a factorization (0.3), denote by $\tau(f) \in \text{Wh}(\pi_1(E))$ the Whitehead torsion of the homotopy equivalence f . Its image in the cokernel $\text{NWh}(p)$ of the assembly map

$$H_1^T(\tilde{B}, \text{Wh}(p)) \rightarrow \pi_1(\mathbf{Wh}(M)) = \text{Wh}(\pi_1(M)),$$

see Section 3, depends only on the homotopy class of p . We call it *tight torsion* and denote it by

$$N\tau(p) \in \text{NWh}(p).$$

See Section 4 for more details and for the relation to the fibering obstruction $\tau_{\text{fib}}(p)$ defined by the authors in [31].

Clearly, if p is homotopic to a MAF, then $N\tau(p) = 0$. It is implicit in [23] that if $B = S^1$ and $\dim(M) \geq 6$, then the converse is also true. From this example, one may be tempted to propose the following

Conjecture 0.4 (Naive Conjecture). *Under the above assumptions (i) to (iv), the map p is homotopic to a MAF if and only if $N\tau(p) = 0$.*

0.4. The main results. This Naive Conjecture is wrong in general: In Section 11 we give a counterexample where B is a product of the Klein bottle with the circle S^1 . However, the Naive Conjecture holds in remarkable generality:

Theorem 0.5 (Hyperbolic or CAT(0)-fundamental group and the Klein bottle condition). *In addition to the standing assumptions above, assume the following conditions about $\pi_1(B)$:*

- (i) *The group $\pi_1(B)$ is hyperbolic. (This is the case if B is a connected smooth manifold with Riemannian metric of negative sectional curvature);*
- (ii) *The group $\pi_1(B)$ is a CAT(0)-group. (This is the case if B is a connected closed smooth manifold with Riemannian metric of non-positive sectional curvature). Moreover, $\pi_1(B)$ satisfies the “Klein bottle condition” that it does not contain the fundamental group $K = \mathbb{Z} \rtimes \mathbb{Z}$ of the Klein bottle.*

Suppose $\dim(M) \neq 4$.

Then the Naive Conjecture is true.¹

We show in Proposition 8.4 that if B is a non-positively curved locally symmetric space, then up to passage to a finite cover, the Klein bottle condition is also satisfied.

In the case where the Klein bottle condition fails, we are able to identify additional obstructions. Recall that the Whitehead group $\text{Wh}(\pi_1(M))$ carries a $w_1(M)$ -twisted involution $a \mapsto \bar{a}$. It induces an involution on $\text{NWh}(p)$. Recall also that if A is an abelian group with involution, and $s \in \mathbb{Z}$, then the s -th Tate cohomology group of $\mathbb{Z}/2$ with coefficients in A is defined by

$$\hat{H}^s(\mathbb{Z}/2; A) = \frac{\{x \in A \mid x = (-1)^s \cdot \bar{x}\}}{\{x + (-1)^s \cdot \bar{x} \mid x \in A\}}.$$

Theorem 0.6 (Non-positive sectional curvature in general). *In addition to the standing assumptions above, assume that B is smooth and admits a Riemannian metric with non-positive sectional curvature. Suppose $n = \dim M \neq 4$ and $N\tau(p) = 0$.*

Then there is an integer s and a sequence of obstructions

$$\kappa_i \in \hat{H}^{n+i}(\mathbb{Z}/2; \text{NWh}(p \times \text{id}_{T^i})), \quad (i = 0, \dots, s)$$

where κ_{i-1} is defined whenever κ_i vanishes. The map p is homotopic to a MAF if and only if all the κ_i vanish.

Notice that there are many more closed aspherical manifolds whose fundamental groups are hyperbolic or CAT(0) respectively than there are closed connected smooth manifolds with Riemannian metric of negative or non-positive respectively sectional curvature, see [20]. The reason why in Theorem 0.6 the existence of a smooth structure together with a Riemannian metric of non-negative sectional curvature occurs is that we need for its proof to know the MAF-Rigidity Conjecture, see Section 7.

Weinberger [85, 14.4.4 on page 263] conjectured that if the homotopy fiber of p is homotopy finite, the map p approximately fibers “modulo a Nil-obstruction”. Theorem 0.6 actually shows that there is a number of different Nil-obstructions whose vanishing, in the non-positively curved case, is necessary and sufficient for p to approximately fiber. In many situations, the number s can be specified, see Remark 5.6.

The reason why the κ_i -obstructions do not show up in Theorem 0.5 is that the Klein bottle condition implies the vanishing of the corresponding Tate cohomology groups. On the other hand Theorem 0.5 is a consequence of a more general statement. To formulate it, we introduce the following terminology:

¹In the special case where B is a (real) hyperbolic manifold, $\dim M > \dim B + 4$, and $\text{Wh}(\pi_1(M) \times \mathbb{Z}^n) = 0$ for all $n \geq 0$, the conclusion of Theorem 0.5 was proven by Farrell-Jones in [29, Theorem 10.7].

Definition 0.7 (Orientable cyclic subgroups). *Given a torsionfree group G , we say that the cyclic subgroups of G are orientable if there is a choice g_C of a generator for each non-trivial cyclic subgroup C , such that whenever $f: C \rightarrow C'$ is an inclusion or a conjugation by some element of G , the element $f(g_C)$ is a positive power of $g_{C'}$.*

Remark 0.8 (FJC). We will say that a group G satisfies FJC if it satisfies both the K - and L -theoretic Farrell-Jones Conjecture with coefficients in additive categories. A discussion of the FJC and a description of groups for which it has been proven is given in Section 2. These include hyperbolic groups, CAT(0)-groups, solvable groups, arithmetic groups, and lattices in almost connected Lie groups.

Theorem 0.9 (Aspherical base manifold, FJC, and orientable cyclic subgroups). *If $\dim M \neq 4$, the Naive Conjecture is true, provided $\pi_1(B)$ satisfies FJC and the cyclic subgroups of $\pi_1(B)$ are orientable.*

Since the Naive Conjecture is wrong in general, an immediate question is whether there is a geometric interpretation of the vanishing of the tight torsion $N\tau(p)$. We are grateful to Bruce Williams for suggesting to us an interpretation in terms of Q -manifold theory. In fact the following holds, where $Q = \prod_{i=1}^{\infty} I$ is the Hilbert cube:

Theorem 0.10 (Hilbert cube version). *Under the standing assumptions above, assume that $N\tau(p) = 0$. Then the composite $M \times Q \rightarrow B$ of the projection map $M \times Q \rightarrow M$ with p is homotopic to an approximate fibration.*

0.5. Block bundles. Every block bundle is homotopic to a MAF, see Quinn [64, Lemma 3.3.1]. The converse is not true: In Section 12 we give an example of a map to a 2-torus which satisfies the conditions of the Naive Conjecture and therefore approximately fibers, but does not block fiber. However, if certain middle and lower K -groups vanish, then by [64, Theorem 3.3.2] a MAF is homotopic to a block bundle, provided the difference $\dim(M) - \dim(B)$ is greater or equal to five.

The lower K -groups under consideration vanish in the situation of Theorem 0.1 and Theorem 0.2, essentially since there $\pi_1(M)$ is torsionfree and satisfies FJC by assumption. Hence we can replace in Theorem 0.1 and Theorem 0.2 MAF by the stronger conclusion that p is homotopic to a block bundle, provided the difference $\dim(M) - \dim(B)$ is greater or equal to five.

0.6. Locally trivial bundles. Of course it would be desirable if p is homotopic to the projection of a locally trivial bundle which is a very restrictive condition. This problem leads to new obstructions in the *higher* Whitehead groups and one can hope for a satisfactory answer under certain conditions on the dimensions of B and M .

There is one favorite case. Namely, if $f: M \rightarrow B$ is a homotopy equivalence of closed aspherical manifolds of dimension $\neq 4$ and $\pi_1(B)$ satisfies FJC, then the Borel Conjecture is known to be true that predicts that f is homotopic to a homeomorphism and in particular to the projection of a fiber bundle. Modulo the orientability assumption and in the high-dimensional case, this also follows from Theorem 0.9, because a manifold approximate fibration which is a homotopy equivalence is a CE map, which was shown to be homotopic to a homeomorphism by Siebenmann [74] in dimensions at least 5.

More generally, if M is a closed aspherical manifold of dimension ≥ 5 such that $\pi_1(M)$ is a direct product $G_1 \times G_2$ and the cohomological dimension of G_i is not equal to 3, 4, 5 for $i = 1, 2$, then there are closed aspherical manifolds M_1 and M_2 with $G_i = \pi_1(M_i)$ together with a homeomorphism $f: M \xrightarrow{\cong} M_1 \times M_2$ implementing on fundamental groups the given isomorphism $\pi_1(M) \cong G_1 \times G_2$, see [53,

Theorem 6.1]. The manifolds M_i are unique up to homeomorphism. This implies (by inspecting the proof to make sure that $\dim(N) = 3, 4, 5$ is not needed) for a π_1 -surjective map $p: M \rightarrow N$ of closed aspherical manifolds that p is homotopic to the projection of a trivial bundle, if $\dim(M) \geq 5$ and $\dim(M) - \dim(N) \neq 3, 4, 5$, the inclusion of the kernel of $\pi_1(p): \pi_1(M) \rightarrow \pi_1(N)$ into $\pi_1(M)$ has a retraction, and $\pi_1(M)$ satisfies FJC.

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1. ORGANIZATION OF THE PROOFS

In this section, after recalling some definitions, we describe our main stabilization-destabilization technique. This technique is essentially the combination of Theorem 1.5 and Theorem 1.6 whose proof will be given in later sections. In this section we explain how to derive the results stated in the introduction from this technique.

From a logical perspective, parts of this section should be read at the very end of the article. We decided to place it here because it summarizes the main strategy.

1.1. Review of MAF. We first recall the definition of an ε -fibration, approximate fibration and MAF.

Definition 1.1 (ε -fibration). *Let E be a topological space, B be a metric space, and $p: E \rightarrow B$ be a continuous map. We call p an ε -fibration, if for any homotopy $h: X \times [0, 1] \rightarrow B$ and map $f: X \rightarrow E$ with $p \circ f = h_0$ for $h_0(x) := h(x, 0)$ there is a map $H: X \times [0, 1] \rightarrow E$ such that $H_0 = f$ and for all $(x, t) \in X \times [0, 1]$ we have $d(p \circ H(x, t), h(x, t)) \leq \varepsilon$, in other words, we can solve the lifting problem below up to an error bounded by ε when measured in B .*

$$\begin{array}{ccc}
 X & \xrightarrow{f} & E \\
 \downarrow i_0 & \nearrow H & \downarrow p \\
 X \times [0, 1] & \xrightarrow{h} & B
 \end{array}$$

An ε -fibration for $\varepsilon = 0$ is the same as a (Hurewicz) fibration in the classical sense.

Definition 1.2 (Approximate fibration). *Let $p: E \rightarrow B$ be a continuous map of topological spaces with B compact metric. We call p an approximate fibration if it is an ε -fibration for every $\varepsilon > 0$.*

This definition is independent of the choice of metric on B . If one wants to consider non-compact base spaces, one should use a slightly more complicated definition in terms of open coverings.

Definition 1.3 (MAF). *A Manifold Approximate Fibration (MAF) is an approximate fibration with closed manifolds as source and target.*

1.2. Low dimensions. The results of the introduction are true by inspection if the dimension of M is 1 or 2. In Section 10, which is independent of the rest of the paper, we treat the case where M has dimension three.

1.3. Stabilization-Destabilization Strategy. The following line of argument applies in the high-dimensional case $\dim(M) \geq 5$.

Notation 1.4. Consider a map $p: M \rightarrow B$. Let T^s denote the s -torus, i.e., $T^s = S^1 \times \cdots \times S^1$ (s factors), and

$$p_s: M \times T^s \rightarrow B \times T^s$$

be the map $p \times \text{id}_{T^s}$.

The following is the starting point for our results. It will be proven in Section 5 for homotopy finite homotopy fiber and finally for finitely dominated homotopy fiber in Subsection 9.3.

Theorem 1.5 (Stabilization Theorem). *Let $p: M \rightarrow B$ be a π_1 -surjective map of closed manifolds such that B is PL and aspherical and the homotopy fiber F_p is finitely dominated. Suppose that the L -theoretic FJC is true for the group $\pi_1(B)$.*

Then there exists a natural number s such that $p_s: M \times T^s \rightarrow B \times T^s$ is homotopic to a MAF.

Theorem 1.5 suggests the following strategy to obtain an approximate fibration $M \rightarrow B$ homotopic to p : First choose an approximate fibration $M \times T^s \rightarrow B \times T^s$ homotopic to p_s and try to “split”. The next theorem yields a sufficient condition to get from s one step down to $(s - 1)$. It will be proven in Section 6.

Theorem 1.6 (Splitting Theorem). *Let $p: M \rightarrow B$ be a π_1 -surjective map of closed manifolds such that B is PL and aspherical.*

(i) *Suppose that $n = \dim M \geq 5$, the K -theoretic FJC is true for the group $\pi_1(B)$ and $p_1: M \times S^1 \rightarrow B \times S^1$ is homotopic to a MAF.*

Then p is h -cobordant to $q: N \rightarrow B$ where q is a MAF.

(ii) *If, additionally, $N\tau(p) = 0$ and $\widehat{H}^n(\mathbb{Z}/2; \text{NWh}(\pi_1 M)) = 0$, then $p: M \rightarrow B$ is homotopic to a MAF.*

1.4. Proof of Theorem 0.9. We need to show that if $N\tau(p) = 0$ and $\dim(M) \geq 5$, then p is homotopic to a MAF.

By Theorem 1.5 the map

$$p_s: M \times T^s \rightarrow B \times T^s$$

is homotopic to a MAF for some $s \geq 1$. Now we want to apply Theorem 1.6 in the case, where M , p and p_1 are specialized to be $M \times T^{s-1}$, p_{s-1} and p_s respectively.

The orientability assumption for $\pi_1(B)$ implies the orientability assumption for $\pi_1(B \times T^{n-1})$, see Lemma 8.2. As mentioned in the introduction, this orientability assumption implies that the Tate cohomology groups $\widehat{H}^{n+s-1}(\mathbb{Z}/2; \text{NWh}(\pi_1(M \times T^{s-1})))$ vanish, see Theorem 3.12.

Let $p \simeq q \circ f$ be a factorization as provided by Theorem 4.1. Then

$$q_{s-1} := q \times \text{id}_{T^{s-1}}: E \times T^{s-1} \rightarrow B \times T^{s-1}$$

is also an approximate fibration. Also

$$f_{s-1} := f \times \text{id}_{T^{s-1}}: M \times T^{s-1} \rightarrow E \times T^{s-1}$$

is a homotopy equivalence and $p_{s-1} \simeq q_{s-1} \circ f_{s-1}$. So we can compute $N\tau(p_{s-1})$ from $\tau(f_{s-1}) = \chi(T^{s-1})\tau(f)$.

There are two cases to consider: $s - 1 > 0$ and $s - 1 = 0$. When $s - 1 > 0$, $\chi(T^{s-1}) = 0$ and hence $N\tau(p_{s-1}) = 0$. When $s - 1 = 0$, $p_{s-1} = p$ and $N\tau(p_{s-1}) = 0$ by assumption. Hence all conditions of Theorem 1.6 hold in both cases, and we conclude that p_{s-1} is homotopic to an approximate fibration. Continuing this argument inductively, we finally arrive at the desired result, namely $p: M \rightarrow B$ is homotopic to an approximate fibration. This finishes the proof of Theorem 0.9.

1.5. Proof of Theorem 0.5. In the situation of Theorem 0.5 the fundamental group $\pi_1(B)$ satisfies FJC by Theorem 2.3 and has orientable cyclic subgroups by Theorem 8.1. Hence Theorem 0.5 follows from Theorem 0.9.

1.6. Proof of Theorem 0.6. The statement of Theorem 1.6 becomes wrong if we drop the assumption on $\widehat{H}^n(\mathbb{Z}/2, \text{NWh}(p))$; see Section 11 for a counterexample. However, in the case where B is non-positively curved (or more generally, for all manifolds B that satisfy the MAF Rigidity Conjecture 7.1), we are able to identify a specific element

$$\kappa_0 \in \widehat{H}^n(\mathbb{Z}/2, \text{NWh}(p)),$$

the *splitting obstruction*, such that:

Theorem 1.7 (MAF-Rigidity Conjecture and the κ_0 -obstruction). *Suppose that the MAF Rigidity Conjecture holds for B . In the situation of Theorem 1.6, suppose that the conditions of (i) hold, and that $N\tau(p) = 0$. Then $\kappa_0 = 0$ if and only if p is homotopic to an approximate fibration.*

See Section 7 for a proof of this result. It implies Theorem 0.6 in the same way as Theorem 0.9 followed from Theorem 1.6 (and recalling that the FJC is known to hold in this situation).

Remark 1.8. In practice, the π_1 -surjectivity condition on $p: M \rightarrow B$ is not restrictive. Indeed, if the homotopy fiber is finitely dominated, then the image of $\pi_1(M)$ in $\pi_1(B)$ has finite index so p lifts along a finite covering $\overline{B} \rightarrow B$ of B , such that the new map $p': M \rightarrow \overline{B}$ is π_1 -surjective. It follows from the definitions that p is a MAF if and only if p' is a MAF.

1.7. Proof of Theorem 0.1. Using Remark 1.8, we can reduce to the special case where p is π_1 -surjective. We want to apply Theorem 0.5 and therefore have to show that all conditions are satisfied. By Theorem 2.3 the K -theoretic FJC holds for $\pi_1(M)$, for $\pi_1(M) \times \mathbb{Z}$, and for $\pi_1(F_p)$. Notice that the Farrell-Jones Conjecture with coefficients in an additive category implies the original Farrell-Jones Conjecture. By assumption $\pi_1(M)$ and therefore also $\pi_1(M) \times \mathbb{Z}$ and $\pi_1(F_p)$ are torsionfree. Hence the Whitehead groups $\text{Wh}(\pi_1(M))$ and $\text{Wh}(\pi_1(M \times S^1))$ and the reduced projective class group $\widetilde{K}_0(\mathbb{Z}[\pi_1(F_p)])$ vanish, see for instance [54, Conjecture 1.1 on page 652, Conjecture 1.3 on page 653, Corollary 2.3 on page 685]. Therefore the finiteness obstruction of F_p is zero and hence F_p is homotopy equivalent to a finite CW -complex. Moreover $N\tau(p)$ vanishes. We conclude $K \not\subseteq \pi_1(B)$ from the fact that a hyperbolic group does not contain $\mathbb{Z} \oplus \mathbb{Z}$ as subgroup. Hence, by Theorem 0.5, p is homotopic to a MAF. This finishes the proof of Theorem 0.1.

1.8. Proof of Theorem 0.2. Again by Remark 1.8, we only need to consider the case where p is π_1 -surjective. Since M and B are aspherical, F_p is aspherical with a torsionfree poly-cyclic fundamental group. Any torsionfree poly-cyclic group has a finite model for its classifying space since it is an iterated extension by an infinite cyclic group. Hence F_p is homotopy equivalent to a finite CW -complex. Since an extension of a poly-cyclic group by an infinite cyclic group is again poly-cyclic, we conclude from Theorem 2.3 that $\pi_1(M)$ and $\pi_1(M) \times \mathbb{Z}$ satisfies FJC. Now proceed as in the proof of Theorem 0.1.

1.9. Proof of Theorem 0.10. The proof of Theorem 0.10, finally, will be given Subsection 4.5. It does not depend on the stabilization-destabilization technique.

To summarize, we still have to prove Theorem 1.5, Theorem 1.6, Theorem 1.7, and Theorem 0.10.

2. BRIEF REVIEW OF THE FARRELL-JONES CONJECTURE WITH COEFFICIENTS IN ADDITIVE CATEGORIES

The original statement of the Farrell-Jones Conjecture can be found in [30, 1.6 on page 257]. We will focus on the formulation with coefficients in additive categories as it is for instance given for K -theory in [9, Conjecture 2.3] and for L -theory in [5, Definition 0.2].

Definition 2.1 (*K -theoretic Farrell-Jones Conjecture*). *A group G satisfies the K -theoretic Farrell-Jones Conjecture with coefficients in additive categories if for any additive G -category \mathcal{A} the assembly map*

$$\text{asmb}_n^{G,\mathcal{A}}: H_n^G(E_{\mathcal{V}C_{yc}}(G); \mathbf{K}_{\mathcal{A}}) \rightarrow H_n^G(\text{pt}; \mathbf{K}_{\mathcal{A}}) = K_n(\int_G \mathcal{A})$$

induced by the projection $E_{\mathcal{V}C_{yc}}(G) \rightarrow \text{pt}$ is bijective for all $n \in \mathbb{Z}$.

Definition 2.2 (*L -theoretic Farrell-Jones Conjecture*). *A group G satisfy the L -theoretic Farrell-Jones Conjecture with coefficients in additive categories if for any additive G -category with involution \mathcal{A} the assembly map*

$$\text{asmb}_n^{G,\mathcal{A}}: H_n^G(E_{\mathcal{V}C_{yc}}(G); \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) \rightarrow H_n^G(\text{pt}; \mathbf{L}_{\mathcal{A}}^{\langle -\infty \rangle}) = L_n^{\langle -\infty \rangle}(\int_G \mathcal{A})$$

induced by the projection $E_{\mathcal{V}C_{yc}}(G) \rightarrow \text{pt}$ is bijective for all $n \in \mathbb{Z}$.

For more information about these conjecture we refer for instance to the survey article [54]. In this paper we will use it as a black box and it is not necessary to understand the details of the construction of the assembly maps for equivariant additive categories (with involutions). It is more important to know for which groups these conjectures are known.

Theorem 2.3 (*Status of the Farrell-Jones Conjecture*). *The class of groups for which both the K -theoretic and the L -theoretic Farrell-Jones Conjectures 2.1 and 2.2 are known has the following properties:*

- (i) *It contains all hyperbolic groups;*
- (ii) *It contains all $CAT(0)$ -groups;*
- (iii) *It contains all solvable groups;*
- (iv) *It contains all lattices in almost connected Lie groups;*
- (v) *It contains all arithmetic groups;*
- (vi) *It contains all fundamental groups of (not necessarily compact) 3-manifolds (possibly with boundary);*
- (vii) *It is closed under taking subgroups;*
- (viii) *It is closed under finite free products and finite direct products;*
- (ix) *It is closed under directed colimits over directed systems (with arbitrary structure maps);*
- (x) *Let $1 \rightarrow H \rightarrow G \xrightarrow{p} Q \rightarrow 1$ be an extension of groups. Suppose that Q and $p^{-1}(V)$ for any virtually cyclic subgroup $V \subseteq Q$ belong to this class of groups, then also G does.*

Proof. The proofs can be found in [3, 4, 6, 7, 8, 46, 70, 83, 84]. □

3. ASSEMBLY MAPS

3.1. Fibered assembly maps. We will use the setup for fibered assembly maps from [56, Section 11], which we recall here for the reader's convenience.

Let

$$\mathbf{E}: \text{Spaces} \rightarrow \text{Spectra}$$

be a covariant functor which is homotopy invariant, i.e., sends weak equivalences of spaces to weak equivalences of spectra. Our main examples are

$$\mathbf{K}_{\mathbb{Z}\Pi}, \mathbf{A}, \mathbf{Wh}, \mathbf{A}^{\%}: \text{Spaces} \rightarrow \text{Spectra}$$

which are defined as follows: Given a space Y , let $\mathbf{K}_{\mathbb{Z}\Pi}(Y)$ be the non-connective K -theory spectrum associated to the additive category of finitely generated projective $\mathbb{Z}\Pi(Y)$ -modules, let $\mathbf{A}(Y)$ be the non-connective A -theory spectrum associated to Y , let $\mathbf{Wh}(Y)$ be the associated non-connective PL Whitehead spectrum and let $\mathbf{A}^{\%}(Y)$ be $Y_+ \wedge \mathbf{A}(\text{pt})$. By definition there is a cofibration sequence of spectra, natural in Y

$$(3.1) \quad \mathbf{A}^{\%}(Y) \xrightarrow{\text{asmb}} \mathbf{A}(Y) \rightarrow \mathbf{Wh}(Y).$$

We emphasize that we use non-connective K -theory spectra. More information about these spectra can be found for instance in [51, 55, 61, 78, 79].

Let $p: X \rightarrow B$ be a map of path connected spaces which induces an epimorphism on fundamental groups. Suppose that B admits a universal covering $q: \tilde{B} \rightarrow B$.

Choose base points $x_0 \in X$, $b_0 \in B$ and $\tilde{b}_0 \in \tilde{B}$ satisfying $p(x_0) = b_0 = q(\tilde{b}_0)$. We will abbreviate $\Gamma = \pi_1(X, x_0)$ and $\pi = \pi_1(B, b_0)$. The free right proper π -action on \tilde{B} and q induces a homeomorphism $\tilde{B}/\pi \xrightarrow{\cong} B$. For a subgroup $H \subseteq \pi$ denote by $q(\pi/H): \tilde{B} \times_{\pi} \pi/H = \tilde{B}/H \rightarrow B$ the obvious covering induced by q . The pullback construction yields a commutative square of spaces

$$\begin{array}{ccc} X(\pi/H) & \xrightarrow{\bar{q}(\pi/H)} & X \\ \bar{p}(\pi/H) \downarrow & & \downarrow p \\ \tilde{B} \times_{\pi} \pi/H & \xrightarrow{q(\pi/H)} & B \end{array}$$

where $\bar{q}(\pi/H)$ is again a covering. This yields covariant functors from the orbit category of π to the category of topological spaces

$$\begin{aligned} \underline{B}: \text{Or}(\pi) &\rightarrow \text{Spaces}, & \pi/H &\mapsto \tilde{B} \times_{\pi} \pi/H; \\ \underline{X}: \text{Or}(\pi) &\rightarrow \text{Spaces}, & \pi/H &\mapsto X(\pi/H). \end{aligned}$$

Let

$$\mathbf{E}(p) := \mathbf{E} \circ \underline{X}: \text{Or}(\pi) \rightarrow \text{Spectra}.$$

Associated to this functor is a π -homology theory on the category of π -CW-complexes

$$(3.2) \quad H_*^{\pi}(-; \mathbf{E}(p))$$

such that

$$(3.3) \quad H_n^{\pi}(\pi/H; \mathbf{E}(p)) = \pi_n(\mathbf{E}(X(\pi/H)))$$

holds for any subgroup $H \subseteq \pi$ and $n \in \mathbb{Z}$, see [19, Sections 4 and 7].

We will be interested in the two maps

$$(3.4) \quad H_n^{\pi}(\text{pr}; \mathbf{E}(p)): H_n^{\pi}(E\pi; \mathbf{E}(p)) \rightarrow H_n^{\pi}(\text{pt}; \mathbf{E}(p)) = \pi_n(\mathbf{E}(X));$$

$$(3.5) \quad H_n^{\pi}(\text{pr}; \mathbf{E}(p)): H_n^{\pi}(\tilde{B}; \mathbf{E}(p)) \rightarrow H_n^{\pi}(\text{pt}; \mathbf{E}(p)) = \pi_n(\mathbf{E}(X)),$$

where pr denotes the projection onto $\text{pt} = \pi/\pi$.

Definition 3.6 ($\text{NWh}(p)$). Let $p: X \rightarrow B$ be a π_1 -surjective map of path connected spaces. Suppose that B admits a universal covering $q: \tilde{B} \rightarrow B$. Define $\text{NWh}_n(p)$ to be the cokernel of the assembly map (3.5) with $\mathbf{E} = \mathbf{Wh}$,

$$H_n^\pi(\text{pr}; \mathbf{Wh}(p)): H_n^\pi(\tilde{B}; \mathbf{Wh}(p)) \rightarrow H_n^\pi(\text{pt}; \mathbf{Wh}(p)) = \pi_n(\mathbf{Wh}(X)).$$

We abbreviate $\text{NWh}(p) := \text{NWh}_1(p)$.

Remark 3.7. If B is aspherical, we can replace in Definition 3.6 of $\text{NWh}(p)$ the spectrum \mathbf{Wh} by either \mathbf{A} or $\mathbf{K}_{\mathbb{Z}\Pi}$ and replace the assembly map (3.5) by (3.4). The proof of this claim is contained in the proof of Theorem 3.12.

Remark 3.8 (Involutions). Notice that all four functors $\mathbf{K}_{\mathbb{Z}\Pi}$, \mathbf{A} , \mathbf{Wh} , $\mathbf{A}^\%$ take values in the category of spectra with (strict) involutions. See for instance [5, 76, 77, 86]. Hence all the homology groups above come with involutions and all maps are compatible with them. In particular $\text{NWh}_n(p)$ has an involution.

We finish this subsection by giving some naturality properties. Given a commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{f} & X_1 \\ & \searrow p_0 & \swarrow p_1 \\ & & B \end{array}$$

we obtain a natural transformation of homology theories

$$H_*^\pi(-, \mathbf{E}(f)): H_*^\pi(-, \mathbf{E}(p_0)) \rightarrow H_*^\pi(-, \mathbf{E}(p_1))$$

from the obvious transformation $\underline{f}: \underline{X}_0 \rightarrow \underline{X}_1$ of functors $\text{Or}(\pi) \rightarrow \text{Spaces}$.

Lemma 3.9. If f is weak homotopy equivalence, then $H_*^\pi(-, \mathbf{E}(f))$ is a natural equivalence of π -homology theories, i.e., $H_n^\pi(Y, \mathbf{E}(f)): H_n^\pi(Y, \mathbf{E}(p_0)) \rightarrow H_n^\pi(Y, \mathbf{E}(p_1))$ is bijective for any π -CW-complex Y and $n \in \mathbb{Z}$.

Proof. Each map $\underline{f}(\pi/H)$ is a weak homotopy equivalence since it is a homotopy pullback of the weak homotopy equivalence f . Hence by assumption each map $\mathbf{E}(f)(\pi/H): \mathbf{E}(p_0)(\pi/H) \rightarrow \mathbf{E}(p_1)(\pi/H)$ is a weak homotopy equivalence of spectra. Now apply [19, Lemma 4.6]. \square

Corollary 3.10. The groups $H_*^\pi(X, \mathbf{E}(p))$ only depend on the homotopy class of p .

Let $\mathbf{u}: \mathbf{E}_0 \rightarrow \mathbf{E}_1$ be a transformation of functors $\text{Spaces} \rightarrow \text{Spectra}$. It induces in the obvious way a natural transformation on homology theories

$$H_*^\pi(Y, \mathbf{u}(p)): H_*^\pi(Y, \mathbf{E}_0(p)) \rightarrow H_*^\pi(Y, \mathbf{E}_1(p)).$$

Lemma 3.11. Let k be an integer. Suppose that for every space Z the homomorphisms $\pi_m(\mathbf{u}(Z)): \pi_m(\mathbf{E}_0(Z)) \rightarrow \pi_m(\mathbf{E}_1(Z))$ is bijective for $m < k$ and surjective for $m = k$. Let Y be any π -CW-complex.

Then $H_n^\pi(Y, \mathbf{u}(p)): H_n^\pi(Y, \mathbf{E}_0(p)) \rightarrow H_n^\pi(Y, \mathbf{E}_1(p))$ is bijective for $n < k$ and surjective for $n = k$.

Proof. This follows from a standard spectral sequence comparison argument applied to the spectral sequence appearing in [19, Theorem 4.7] since \mathbf{u} induces a map between the spectral sequences associated to $H_*^\pi(Y, \mathbf{E}_0(p))$ and $H_*^\pi(Y, \mathbf{E}_1(p))$. \square

3.2. Fibered assembly maps and the Farrell-Jones Conjecture. The content of this subsection is a proof the following result:

Theorem 3.12. *Let $p: E \rightarrow B$ be a map of path connected spaces which is π_1 -surjective. Suppose that B is an aspherical CW-complex with fundamental group a torsionfree group π . Assume that the K-theoretic Farrell-Jones Conjecture of Definition 2.1 holds for π and that the cyclic subgroups of π are orientable in the sense of Definition 0.7.*

Then, for $n \leq 1$, the map

$$(3.13) \quad H_n^\pi(E\pi; \mathbf{Wh}(p)) \rightarrow H_n^\pi(\text{pt}; \mathbf{Wh}(p)) = \pi_n(\mathbf{Wh}(E))$$

induced by the projection $\text{pr}: E\pi \rightarrow \text{pt}$ is split injective, its cokernel is $\text{NWh}_n(p)$, and we get for the Tate cohomology

$$\widehat{H}^i(\mathbb{Z}/2; \text{NWh}_n(p)) = 0$$

for all $i \in \mathbb{Z}$.

For its proof we will need the following

Lemma 3.14. *If $u: Y \rightarrow Z$ is a π -map of π -CW-complexes which is a weak homotopy equivalence after forgetting the π -action, then the induced map*

$$H_n^\pi(u; \mathbf{A}^\%(p)): H_n^\pi(Y; \mathbf{A}^\%(p)) \xrightarrow{\cong} H_n^\pi(Z; \mathbf{A}^\%(p))$$

is bijective for all $n \in \mathbb{Z}$.

Proof. Define the contravariant functor

$$O(Y): \text{Or}(\pi) \rightarrow \text{Spaces}, \quad \pi/H \mapsto \text{map}_\pi(\pi/H, Y).$$

By unravelling the definitions, we see that $H_n^\pi(u; \mathbf{A}^\%(p))$ is the homomorphism on π_n coming from the map of spectra

$$(O(Y) \times_{\text{Or}(\pi)} \underline{X})_+ \wedge \mathbf{A}(\text{pt}) \rightarrow (O(Z) \times_{\text{Or}(\pi)} \underline{X})_+ \wedge \mathbf{A}(\text{pt}).$$

Hence it suffices to show that the map of spaces

$$O(Y) \times_{\text{Or}(\pi)} \underline{X} \rightarrow O(Z) \times_{\text{Or}(\pi)} \underline{X}$$

is a weak homotopy equivalence. If we put $\overline{X} := X(\pi/1)$, then \overline{X} carries a proper free right π -action and \underline{X} can be identified with the functor sending π/H to the space $\overline{X} \times_\pi \pi/H$. There is a homeomorphism

$$\alpha(Y): O(Y) \times_{\text{Or}(\pi)} \underline{X} \xrightarrow{\cong} \overline{X} \times_\pi Y$$

which is induced by the various maps

$$\text{map}_\pi(\pi/H, Y) \times (\overline{X} \times_\pi \pi/H) \rightarrow \overline{X} \times_\pi Y, \quad (\sigma, (\overline{x}, wH)) \mapsto (\overline{x}, \sigma(wH)).$$

The inverse of α sends (\overline{x}, y) to the element in $O(Y) \times_{\text{Or}(\pi)} \underline{X}$ given by (σ_y, \overline{x}) in $\text{map}_\pi(\pi, Y) \times_\pi \overline{X} = \text{map}_\pi(\pi/\{1\}, Y) \times_\pi \underline{X}(\pi/\{1\})$ for $\sigma_y: \pi \rightarrow Y$, $w \mapsto w \cdot y$. Hence it suffices to show that

$$\text{id}_{\overline{X}} \times_\pi u: \overline{X} \times_\pi Y \rightarrow \overline{X} \times_\pi Z$$

is a weak homotopy equivalence. Since $\overline{X} \rightarrow X$ is a covering, we obtain a commutative diagram whose rows are fibrations.

$$\begin{array}{ccccc} Y & \longrightarrow & \overline{X} \times_\pi Y & \longrightarrow & X \\ \downarrow u & & \downarrow \text{id}_{\overline{X}} \times_\pi u & & \downarrow \text{id}_X \\ Z & \longrightarrow & \overline{X} \times_\pi Z & \longrightarrow & X \end{array}$$

Now the long homotopy sequence associated to a fibration and the Five-Lemma prove that $\text{id}_{\overline{X}} \times_{\pi} u$ is a weak homotopy equivalence. This finishes the proof of Lemma 3.14. \square

Now we can give the

Proof of Theorem 3.12. Notice that we can use \tilde{B} as a model for $E\pi$ for $\pi = \pi_1(B)$, as B is aspherical. So the cokernel of the map (3.13) is $\text{NWh}_n(p)$ by its very definition.

Let $\underline{E}\pi$ be the classifying space for the family of virtually cyclic subgroups. By [56, Lemma 11.3], the K -theoretic Farrell-Jones Conjecture implies that the map induced by the projection $\underline{E}\pi \rightarrow \text{pt}$

$$H_n^\pi(\underline{E}\pi; \mathbf{K}_{\mathbb{Z}\Pi}(p)) \xrightarrow{\cong} H_n^\pi(\text{pt}; \mathbf{K}_{\mathbb{Z}\Pi}(p))$$

is an isomorphism for all $n \in \mathbb{Z}$. (Here we use the π_1 -surjectivity.) As π is assumed to be torsionfree, the π -space $E\pi$ is also a classifying space $\underline{E}\pi$ for π -actions with finite stabilizers. We conclude from [56, Theorem 0.1 and Theorem 0.2] that the map induced by the projection $\text{pr}: E\pi \rightarrow \text{pt}$

$$(3.15) \quad H_n^\pi(\text{pr}; \mathbf{K}_{\mathbb{Z}\Pi}(p)): H_n^\pi(E\pi; \mathbf{K}_{\mathbb{Z}\Pi}(p)) \rightarrow H_n^\pi(\text{pt}; \mathbf{K}_{\mathbb{Z}\Pi}(p))$$

is split injective (for all n) and that the Tate cohomology groups of its cokernel (considered as a $\mathbb{Z}/2$ -module under the canonical involution) vanish.

There exists a linearization map

$$\mathbf{L}(Y): \mathbf{A}(Y) \rightarrow \mathbf{K}_{\mathbb{Z}\Pi}(Y)$$

which is natural in Y and always 2-connected. It induces a transformation of functors $\text{Or}(\pi) \rightarrow \text{Spectra}$ from $\mathbf{A}(p)$ to $\mathbf{K}_{\mathbb{Z}\Pi}(p)$ whose evaluation at any object π/H is 2-connected. From Lemma 3.11 we obtain for every $n \leq 1$ and every π -CW-complex an isomorphism, natural in Y ,

$$(3.16) \quad H_n^\pi(Y; \mathbf{A}(p)) \xrightarrow{\cong} H_n^\pi(Y; \mathbf{K}_{\mathbb{Z}\Pi}(p)).$$

We conclude that the map

$$(3.17) \quad H_n^\pi(\text{pr}, \mathbf{A}(p)): H_n^\pi(E\pi; \mathbf{A}(p)) \rightarrow H_n^\pi(\text{pt}, \mathbf{A}(p))$$

is split injective at least for $n \leq 1$ with the same cokernel as the map (3.15).

Next we apply Lemma 3.14 for $Y = E\pi$ and $Z = \text{pt}$. The long exact sequence obtained from the cofiber sequence (3.1) implies that the natural transformation $\mathbf{A} \rightarrow \mathbf{Wh}$ induces an isomorphism

$$H_n^\pi(E\pi \rightarrow \text{pt}; \mathbf{A}(p)) \xrightarrow{\cong} H_n^\pi(E\pi \rightarrow \text{pt}; \mathbf{Wh}(p))$$

on the relative homology groups. We obtain a commutative diagram with exact columns whose horizontal arrows marked with \cong are bijective

$$\begin{array}{ccc}
\vdots & & \vdots \\
z_{n+1} \downarrow & & \downarrow z_{n+1} \\
H_{n+1}^\pi(E\pi \rightarrow \text{pt}; \mathbf{A}(p)) & \xrightarrow{\cong} & H_{n+1}^\pi(E\pi \rightarrow \text{pt}; \mathbf{Wh}(p)) \\
\delta_{n+1} \downarrow & & \downarrow \delta_{n+1} \\
H_n^\pi(E\pi; \mathbf{A}(p)) & \longrightarrow & H_n^\pi(E\pi; \mathbf{Wh}(p)) \\
H_n^\pi(\text{pr}; \mathbf{A}(p)) \downarrow & & \downarrow H_n^\pi(\text{pr}; \mathbf{Wh}(p)) \\
H_n^\pi(\text{pt}; \mathbf{A}(p)) & \longrightarrow & H_n^\pi(\text{pt}; \mathbf{Wh}(p)) \\
z_n \downarrow & & \downarrow z_n \\
H_n^\pi(E\pi \rightarrow \text{pt}; \mathbf{A}(p)) & \xrightarrow{\cong} & H_n^\pi(E\pi \rightarrow \text{pt}; \mathbf{Wh}(p)) \\
\delta_n \downarrow & & \downarrow \delta_n \\
\vdots & & \vdots
\end{array}$$

Since each of the maps $H_n^\pi(\text{pr}; \mathbf{A}(p))$ is split injective for $n \leq 1$, one easily checks that each of the maps $z_n: H_n^\pi(\text{pt}; \mathbf{A}(p)) \rightarrow H_n^\pi(E\pi \rightarrow \text{pt}; \mathbf{A}(p))$ is split surjective for $n \leq 1$. This implies that each of the maps $z_n: H_n^\pi(\text{pt}; \mathbf{Wh}(p)) \rightarrow H_n^\pi(E\pi \rightarrow \text{pt}; \mathbf{Wh}(p))$ is split surjective for $n \leq 1$. Finally we conclude for $n \leq 1$ that the map $H_n^\pi(\text{pr}; \mathbf{Wh}(p))$ is split injective and has the same cokernel as $H_n^\pi(\text{pr}; \mathbf{A}(p))$. Since we have already shown that $H_n^\pi(\text{pr}; \mathbf{A}(p))$ has the same cokernel as the map (3.15), Theorem 3.12 follows. \square

3.3. Comparison to Quinn's assembly maps. Quinn [65, 8. Appendix] defines for a stratified system of fibrations $p: X \rightarrow B$ over a simplicial complex B and a homotopy invariant functor $\mathbf{E}: \mathbf{Spaces} \rightarrow \mathbf{Spectra}$ an assembly map

$$(3.18) \quad \text{asmb}_n: \mathbb{H}_n(B; \mathbf{E}(p)) \rightarrow \pi_n(\mathbf{E}(X)).$$

We want to compare the three assembly maps (3.4), (3.5), and (3.18). Roughly speaking, the map (3.4) is best suited for calculations based on the Farrell-Jones Conjecture, (3.18) is best suited for geometric applications, and (3.5) interpolates between (3.4) and (3.18).

Lemma 3.19. *Let $p: X \rightarrow B$ be a π_1 -surjective map of path connected spaces such that B is a simplicial complex. Assume that p is a stratified system of fibrations.*

(i) *There is a natural homomorphism*

$$\mu_n: \mathbb{H}_n(B; \mathbf{E}(p)) \rightarrow H_n^\pi(\tilde{B}; \mathbf{E}(p))$$

from the group $\mathbb{H}_n(B; \mathbf{E}(p))$ appearing in (3.18) to the π -homology group $H_n^\pi(\tilde{B}; \mathbf{E}(p))$ of (3.2) such that the following diagram commutes

$$\begin{array}{ccc}
\mathbb{H}_n(B; \mathbf{E}(p)) & \xrightarrow{\mu_n} & H_n^\pi(\tilde{B}; \mathbf{E}(p)) \\
\text{asmb}_n \downarrow & & \downarrow H_n^\pi(\text{pr}; \mathbf{E}(p)) \\
\pi_n(\mathbf{E}(X)) & \xleftarrow[\nu_n]{\cong} & H_n^\pi(\text{pt}; \mathbf{E}(p))
\end{array}$$

where ν_n is the canonical isomorphism from (3.3) for the homogeneous space π/π ;

- (ii) Suppose that p is a (Hurewicz) fibration and B is aspherical. Then the natural homomorphism

$$\mu_n: \mathbb{H}_n(B; \mathbf{E}(p)) \rightarrow H_n^\pi(\tilde{B}; \mathbf{E}(p))$$

is bijective for $n \in \mathbb{Z}$;

- (iii) Suppose that B is aspherical. Then the assembly maps

$$\text{asmb}_n: \mathbb{H}_n(B; \mathbf{E}(p)) \rightarrow \pi_n(\mathbf{E}(X))$$

of (3.18) and the assembly maps of (3.5)

$$H_n^\pi(\text{pr}; \mathbf{E}(p)): H_n^\pi(\tilde{B}; \mathbf{E}(p)) = H_n^\pi(E\pi; \mathbf{E}(p)) \rightarrow H_n^\pi(\text{pt}; \mathbf{E}(p))$$

have the same image, if we identify the targets by the isomorphism ν from (3.3) for the homogeneous space π/π ;

- (iv) Let $f: \tilde{B} \rightarrow E\pi$ be the classifying π -map. Suppose that $\pi_1(X)$ satisfies the K -theoretic FJC, see Definition 2.1. Take $\mathbf{E} = \mathbf{Wh}$, defined in Subsection 3.1.

Then the left vertical arrow in the following commutative diagram is bijective for $n \leq 0$ and surjective for $n = 1$

$$\begin{array}{ccc} H_n^\pi(\tilde{B}; \mathbf{Wh}(p)) & \xrightarrow{H_n^\pi(\text{pr}; \mathbf{Wh}(p))} & H_n^\pi(\text{pt}; \mathbf{Wh}(p)) = \text{Wh}_n(\pi_1(X)) \\ \downarrow H_n^\pi(f; \mathbf{Wh}(p)) & & \downarrow \text{id} \\ H_n^\pi(E\pi; \mathbf{Wh}(p)) & \xrightarrow{H_n^\pi(\text{pr}; \mathbf{Wh}(p))} & H_n^\pi(\text{pt}; \mathbf{Wh}(p)) = \text{Wh}_n(\pi_1(X)) \end{array}$$

In particular the cokernel of the upper horizontal arrow agrees with the cokernel of the lower horizontal arrow for $n \leq 1$.

Proof. (i) Quinn defines a spectrum $\mathbb{E}(p)$ by $\bigvee_\sigma \sigma_+ \wedge \mathbf{E}(p^{-1}(\sigma)) / \sim$ where σ runs through the simplices of B , the pointed space σ_+ is obtained from σ by adding a disjoint base point, and the equivalence relation \sim identifies for an inclusion $\sigma \subset \tau$ the spectrum $\sigma_+ \wedge \mathbf{E}(p^{-1}(\sigma))$ with its image under the map $\sigma_+ \wedge \mathbf{E}(p^{-1}(\sigma)) \rightarrow \tau_+ \wedge \mathbf{E}(p^{-1}(\tau))$ coming from the inclusion $\sigma \subseteq \tau$.

Consider any simplex σ of B . Let $\tilde{\sigma}$ be any lift to \tilde{B} which will be equipped with the obvious simplicial structure coming from B . Then we obtain a map of spectra

$$\tilde{\mathbf{a}}(\tilde{\sigma}): \tilde{\sigma}_+ \wedge \mathbf{E}(\tilde{p}(\pi/1)^{-1}(\tilde{\sigma})) \rightarrow \tilde{B}_+ \wedge \mathbf{E}(X(\pi/1))$$

by the smash product of the inclusion $\tilde{\sigma}_+ \rightarrow \tilde{B}_+$ and the map $\mathbf{E}(\tilde{p}(\pi/1)^{-1}(\tilde{\sigma})) \rightarrow \mathbf{E}(X(\pi/1))$ induced by the inclusion $\tilde{p}(\pi/1)^{-1}(\tilde{\sigma}) \rightarrow X(\pi/1)$. The maps $q: \tilde{B} \rightarrow B$ and $\tilde{p}(\pi/1): X(\pi/1) \rightarrow X$ induce homeomorphisms $\tilde{\sigma} \xrightarrow{\cong} \sigma$ and $\tilde{p}(\pi/1)^{-1}(\tilde{\sigma}) \xrightarrow{\cong} p^{-1}(\sigma)$ and thus an isomorphism of spectra

$$\tilde{\mathbf{b}}(\tilde{\sigma}): \tilde{\sigma}_+ \wedge \mathbf{E}(\tilde{p}(\pi/1)^{-1}(\tilde{\sigma})) \xrightarrow{\cong} \sigma_+ \wedge \mathbf{E}(p^{-1}(\sigma)).$$

For any $w \in \pi$ we obtain a commutative diagram, where the vertical maps are all induced by multiplication with w

$$\begin{array}{ccc}
 & \tilde{\sigma}_+ \wedge \mathbf{E}(\bar{p}(\pi/1)^{-1}(\tilde{\sigma})) & \xrightarrow{\tilde{\mathbf{a}}(\tilde{\sigma})} \tilde{B}_+ \wedge \mathbf{E}(X(\pi/1)) \\
 & \swarrow \mathbb{R} \scriptstyle \mathbf{b}(\tilde{\sigma}) & \downarrow \\
 \sigma_+ \wedge \mathbf{E}(p^{-1}(\sigma)) & & \\
 & \nwarrow \mathbb{R} \scriptstyle \mathbf{b}(\tilde{\sigma} \cdot w) & \\
 & (\tilde{\sigma} \cdot w)_+ \wedge \mathbf{E}(\bar{p}(\pi/1)^{-1}(\tilde{\sigma} \cdot w)) & \xrightarrow{\tilde{\mathbf{a}}(\tilde{\sigma} \cdot w)} \tilde{B}_+ \wedge \mathbf{E}(X(\pi/1))
 \end{array}$$

Thus we obtain a map of spectra

$$\mathbf{a}(\sigma): \sigma_+ \wedge \mathbf{E}(p^{-1}(\sigma)) \rightarrow \tilde{B}_+ \wedge_{\pi} \mathbf{E}(X(\pi/1)).$$

They fit together to a map of spectra

$$\mathbf{a}: \mathbb{E}(p) := \bigvee_{\sigma} \sigma_+ \wedge \mathbf{E}(p^{-1}(\sigma)) / \sim \rightarrow \tilde{B}_+ \wedge_{\pi} \mathbf{E}(X(\pi/1)).$$

After taking homotopy groups it induces the desired map

$$\mu_n: \mathbb{H}_n(B; \mathbf{E}(p)) \xrightarrow{\cong} H_n^{\pi}(\tilde{B}; \mathbf{E}(p)).$$

Notice that Quinn [65, 8. Appendix] replaces $\mathcal{S}(B)$ by the associated Ω -spectrum but this does not matter since it does not change the homotopy groups. In the setting of [19] one does not need Ω -spectra as long as one is only interested in homology, see [19, Lemma 4.4]. The commutativity of the diagram appearing in Lemma 3.19 (i) follows directly from the definitions.

(ii) If Z is a space over B , i.e., a space with a reference map $f: Z \rightarrow B$, then the construction above extends and yields a map

$$\mu_n(Z): \mathbb{H}_n(Z; \mathbf{E}(p)) \rightarrow H_n^{\pi}(\bar{Z}; \mathbf{E}(p))$$

where \bar{Z} is the π -covering given by the pullback

$$\begin{array}{ccc}
 \bar{Z} & \xrightarrow{\bar{f}} & \tilde{B} \\
 \downarrow & & \downarrow \\
 Z & \xrightarrow{f} & B
 \end{array}$$

The map $\mu_n(Z)$ is natural in the space Z over B . Next we show for any space Z over B that $\mu_n(Z)$ is an isomorphism for all $n \in \mathbb{Z}$. Then the claim follows by applying it to the case $Z = B$ with reference map $\text{id}_B: B \rightarrow B$.

In fact both functors are homology theories on the category of spaces over B : For the first one, see [65, Proposition 8.4], for the second one this is true by construction. Hence by the standard Mayer-Vietoris and colimit argument it is enough to check the case where $f: Z \rightarrow B$ is the inclusion of a point $b \in B$. In this case \bar{Z} is isomorphic to $\pi/1$ so $H_n^{\pi}(\bar{Z}; \mathbf{E}(p)) \cong \mathbf{E}(X(\pi/1))$, and the map μ_n is the map $\pi_n \mathbf{E}(p^{-1}(b)) \rightarrow \pi_n(\mathbf{E}(X(\pi/1)))$ induced by the inclusion from the point-preimage into the homotopy fiber. But this map is a homotopy equivalence since p is a fibration and \tilde{B} is contractible. The claim follows as \mathbf{E} is homotopy invariant.

(iii) We can turn p into a fibration \widehat{p} , i.e., there exists a commutative diagram

$$\begin{array}{ccccccc}
 X & \xrightarrow{u} & \widehat{X} & \xrightarrow{i_0} & \widehat{X} \times [0, 1] & \xleftarrow{i_1} & \widehat{X} & \xrightarrow{v} & X \\
 \downarrow p & & \downarrow \widehat{p} & & \downarrow h & & \downarrow h_1 & & \downarrow p \\
 B & \xrightarrow{\text{id}_B} & B & \xrightarrow{i_0} & B \times [0, 1] & \xleftarrow{i_1} & B & \xrightarrow{\text{id}_B} & B
 \end{array}$$

such that $\widehat{p}: \widehat{X} \rightarrow B$ is a fibration, u and v are homotopy inverse homotopy equivalences and the maps i_k denote the obvious inclusions coming from the inclusions $\{k\} \rightarrow [0, 1]$, and h is a homotopy between $p \circ v$ and \widehat{p} . It induces a commutative diagram of abelian groups whose arrows marked with \cong are bijections because of the spectral sequence due to Quinn [65, 8. Appendix] and the assumption that \mathbf{E} sends weak homotopy equivalences to weak homotopy equivalences.

$$\begin{array}{ccc}
 \mathbb{H}_n(B; \mathbf{E}(p)) & \xrightarrow{\text{asmb}_n} & \pi_n(\mathbf{E}(X)) \\
 \downarrow u_* & & \downarrow \pi_n(\mathbf{E}(u)) \cong \\
 \mathbb{H}_n(B; \mathbf{E}(\widehat{p})) & \xrightarrow{\text{asmb}_n} & \pi_n(\mathbf{E}(\widehat{X})) \\
 \downarrow (i_0)_* \cong & & \downarrow \pi_n(\mathbf{E}(i_0)) \cong \\
 \mathbb{H}_n(B \times [0, 1]; \mathbf{E}(h)) & \xrightarrow{\text{asmb}_n} & \pi_n(\mathbf{E}(\widehat{X} \times [0, 1])) \\
 \uparrow (i_1)_* \cong & & \uparrow \pi_n(\mathbf{E}(i_1)) \cong \\
 \mathbb{H}_n(B; \mathbf{E}(\widehat{p})) & \xrightarrow{\text{asmb}_n} & \pi_n(\mathbf{E}(\widehat{X})) \\
 \downarrow v_* & & \downarrow \pi_n(\mathbf{E}(v)) \cong \\
 \mathbb{H}_n(B; \mathbf{E}(p)) & \xrightarrow{\text{asmb}_n} & \pi_n(\mathbf{E}(X))
 \end{array}$$

id

Notice that we are not claiming that the vertical arrows u_* and v_* are isomorphisms. Nevertheless, an easy diagram chase shows that the images of

$$\text{asmb}_n: \mathbb{H}_n(B; \mathbf{E}(p)) \rightarrow \pi_n(\mathbf{E}(X))$$

and

$$\text{asmb}_n: \mathbb{H}_n(B; \mathbf{E}(\widehat{p})) \rightarrow \pi_n(\mathbf{E}(\widehat{X}))$$

agree if we identify their targets with the isomorphism $\pi_n(\mathbf{E}(u))$ whose inverse is $\pi_n(\mathbf{E}(v))$. The following diagram commutes

$$\begin{array}{ccc}
 H_n^\pi(\widetilde{B}; \mathbf{E}(p)) & \xrightarrow{\text{asmb}_n} & H_n^\pi(\text{pt}; \mathbf{E}(p)) \\
 \downarrow H_n^\pi(\widetilde{B}; \mathbf{E}(u)) & & \downarrow H_n^\pi(\text{pt}; \mathbf{E}(u)) \\
 H_n^\pi(\widetilde{B}; \mathbf{E}(\widehat{p})) & \xrightarrow{\text{asmb}_n} & H_n^\pi(\text{pt}; \mathbf{E}(\widehat{p}))
 \end{array}$$

The vertical arrows are bijections because of Lemma 3.9. Hence the images of the upper and the lower horizontal arrows agree if we identify their targets with the right vertical isomorphisms. We conclude that it suffices to prove assertion (iii) for \widehat{p} , or in other words, we can assume without loss of generality that p is a fibration. But then the claim follows directly from assertion (ii).

(iv) By assumption $\pi_1(X)$ satisfies the K -theoretic Farrell-Jones Conjecture; Theorem 2.3 implies that each of its subgroups $\pi_1(X(\pi/H))$ for $H \subseteq \pi$ does. Hence $\pi_q(\mathbf{Wh}(X(\pi/H)))$ vanishes for all $q \leq -2$ and $H \subseteq \pi$, see for instance [54, Subsection 3.1.1]. The map $f: \widetilde{B} \rightarrow E\pi$ is a 2-connected map of free π -CW-complexes.

Now apply the spectral sequence of [19, Theorem 4.7]. This finishes the proof of Lemma 3.19. \square

4. THE TIGHT TORSION AND PROOF OF THEOREM 0.10

4.1. Factorization to an approximate fibration. This subsection is devoted to the following result which will be one of the key ingredients in the definition of tight torsion.

Theorem 4.1 (Factorization to a approximate fibration). *Let $p: M \rightarrow B$ be a map between topological spaces, such that B is a finite simplicial complex and for each $b \in B$, the homotopy fiber of p over B is homotopy finite. Then there exists a homotopy commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{f} & E \\ & \searrow p & \swarrow q \\ & & B \end{array}$$

where f is a homotopy equivalence, q is an approximate fibration, and E is a compact ENR.

We are grateful to Steve Ferry for suggesting the following argument to us, which replaces a more complicated proof in an earlier version.

Proof. We may assume that p is a fibration. We will construct a homotopy equivalence $g: E \rightarrow M$ such that $p \circ g = q$ strictly, and we will moreover make sure that for any subcomplex A of B , the restriction of q over A is an approximate fibration whose total space is a compact ENR and that the restriction of f over A is a homotopy equivalence.

The proof is by induction on the number of simplices of B . The case where B consists of a single 0-simplex is trivial. For the inductive step, assume that B is obtained from B' by attaching a single n -simplex σ along its boundary, and denote by $g': E' \rightarrow M' := M|_{B'}$ and $q': E' \rightarrow B'$ the maps obtained from the inductive assumption.

Let $b \in B$ be the barycenter of σ and choose a homotopy equivalence

$$h: M_b := p^{-1}(b) \rightarrow F$$

where F is a finite CW-complex. As p is a fibration, there is a homotopy equivalence

$$M|_{\sigma} \xrightarrow{\simeq} \text{cyl}(M|_{\partial\sigma} \xrightarrow{t} M_b)$$

relative to $M|_{\partial\sigma}$, where t is given by fiber transport.

Let $E := E' \cup_{E'|_{\partial\sigma}} \text{cyl}(E'|_{\partial\sigma} \xrightarrow{\text{hot} \circ g'} F)$. Then E is a compact ENR by [12, Corollary E.7] and [36, chapter VI, §1]. The commutative diagram

$$\begin{array}{ccc} E'|_{\partial\sigma} & \xrightarrow{\text{hot} \circ g'} & F \\ \simeq \downarrow g' & & \simeq \uparrow h \\ M|_{\partial\sigma} & \xrightarrow{t} & M_b \end{array}$$

induces a homotopy equivalence $\text{cyl}(h \circ t \circ g') \rightarrow \text{cyl}(t)$ which restricts to g' on $E'|_{\partial\sigma}$, so we obtain a homotopy equivalence $g'': E \rightarrow M$ which extends g' .

We let $q: E \rightarrow B$ extend q' by sending the mapping cylinder canonically to $\sigma = \text{cyl}(\partial\sigma \rightarrow *)$. It follows from the inductive assumption by application of the criterion [42, Theorem 12.15] that q is in fact an approximate fibration (see [75] for more details of this argument). Moreover $p \circ g''$ and q agree over B' so these two

maps are homotopic by the straight-line homotopy in the interior of σ . As p is a fibration, the map g'' is homotopic to a map g such that $p \circ g = q$, via a homotopy which is stationary over B' .

If $A \subset B$ is a subcomplex, then either it does not contain σ , in which case the claim is contained in the inductive assumption. Otherwise A is obtained from some $A' \subset B'$ by attaching the single simplex σ , and the same argument as above applies to show that the restriction of q over A' is an approximate fibration whose total space is a compact ENR, and that the restriction of g over A' is a homotopy equivalence. \square

4.2. Definition of tight torsion. Recall that, given a homotopy equivalence $f: X \rightarrow Y$ between compact ENRs, it is possible to define the Whitehead torsion $\tau(f) \in \text{Wh}(\text{II}Y)$ which has the usual properties. It can be calculated from the classical Whitehead torsion of a homotopy equivalence between compact CW-complexes by the composition rule using the following facts:

- (i) each compact ENR receives a cell-like map $A \rightarrow X$ from a finite CW-complex A ;
- (ii) the Whitehead torsion of a cell-like map between compact ENRs (which is always a homotopy equivalence) is zero.

See [50] for a survey on cell-like maps.

In the sequel $p: M \rightarrow B$ is a π_1 -surjective map between closed topological manifolds such that B is triangulable. In particular M is a compact ENR.

In order to define tight torsion below, we will need the following.

Lemma 4.2. *Given a metric on B , there exists an $\varepsilon > 0$, such that the following holds: For $i = 1, 2$, let $p \simeq q_i \circ f_i$ be two factorizations up to homotopy into a homotopy equivalence, followed by an ε -fibration whose total spaces are compact ENR's. Then the element*

$$(f_1)_*^{-1}\tau(f_1) - (f_2)_*^{-1}\tau(f_2) \in \text{Wh}(\text{II}M)$$

is in the image of the assembly map $H_1^\pi(\text{pr}; \mathbf{Wh}(p)): H_1^\pi(\tilde{B}; \mathbf{Wh}(p)) \rightarrow \text{Wh}(\text{II}M)$.

Recall that we have introduced $\text{NWh}(p)$ in Definition 3.6 as the cokernel of this assembly map. Hence the following definition is meaningful because of Lemma 4.2.

Definition 4.3 (Tight torsion). *Let $p: M \rightarrow B$ be a π_1 -surjective map between closed topological manifolds such that B is triangulable and such that the homotopy fiber of p is homotopy finite. The tight torsion*

$$N\tau(p) \in \text{NWh}(p)$$

is defined whenever there exists a factorization $p \simeq q \circ f$ up to homotopy into a homotopy equivalence followed by an approximate fibration whose total space is a compact ENR; in this case it is defined to be the image of the Whitehead torsion $f_*^{-1}\tau(f)$ of f in the quotient group $\text{NWh}(p)$.

Note that $\text{NWh}(p)$ and the tight torsion $N\tau(p)$ depend only on the homotopy class of p . Our terminology comes from the case where B is the circle S^1 . Then $N\tau(p)$ is the component of the primary torsion obstruction defined in Farrell [23, 25] which lies in the summand

$$\tilde{C}(\mathbb{Z}\Gamma, \alpha) \oplus \tilde{C}(\mathbb{Z}\Gamma, \alpha^{-1})$$

of $\text{Wh}(\pi_1(M))$ where $\Gamma = \ker \pi_1(p)$ and $\pi_1(M) = \Gamma \rtimes_\alpha \pi_1(S^1)$. In this case $N\tau(p)$ vanishes under transfer to the finite sheeted covers $M_n \xrightarrow{p_n} S^1$ for all sufficiently

large integers n where these covers are defined by the following Cartesian diagram:

$$\begin{array}{ccc} M_n & \xrightarrow{p_n} & S^1 \\ \downarrow & & \downarrow z \mapsto z^n \\ M & \xrightarrow{p} & S^1 \end{array}$$

Our terminology comes by thinking that these covers relax the torsion more and more, as $n \rightarrow \infty$, until it becomes zero; cf. [24, 73].

4.3. Proof of Lemma 4.2. In order to guarantee that Definition 4.3 makes sense, we still have to prove Lemma 4.2. This needs some preparations.

Definition 4.4 (Control). *Let*

$$(4.5) \quad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow p & \swarrow q \\ & & B \end{array}$$

be a (not necessarily commutative) diagram of spaces, with B a metric space, and let $\varepsilon > 0$.

- (i) f is called ε -controlled if $d(q \circ f(x), p(x)) < \varepsilon$ holds for all $x \in X$;
- (ii) A homotopy $H: X \times I \rightarrow Y$ is called an ε -homotopy if for all $x \in X$ the path $q \circ H(x, -)$ in B has diameter less than ε ;
- (iii) f is an ε -domination if it is ε -controlled and there exists an ε -controlled map $g: Y \rightarrow X$ and an ε -homotopy $f \circ g \simeq \text{id}_Y$;
- (iv) An ε -domination f is an ε -homotopy equivalence if, in addition, there exists an ε -homotopy $g \circ f \simeq \text{id}_X$.

Remark 4.6. It is easy to see that if $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are ε -homotopy equivalences, then $g \circ f$ is an 4ε -homotopy equivalence. This is the reason to require that not just the homotopies, but also the maps themselves have ε -control. Thus our convention differs from other definitions (for instance, in [15]) that just require the homotopies to be controlled. Note that if a map f is ε -controlled and an ε -homotopy equivalence in this weaker sense, then the triangle inequality implies that it is automatically a 2ε -homotopy equivalence in our sense.

Lemma 4.7. *Suppose that X, Y are separable metric spaces, that B is a compact ANR with a metric and let $\varepsilon > 0$. There is a $\delta > 0$ such that the following holds: If in (4.5), p and q are δ -fibrations, and f is a homotopy equivalence such that the diagram commutes up to homotopy, then f is homotopic to an ε -homotopy equivalence.*

Proof. Factor q into a homotopy equivalence $\lambda: Y \rightarrow \mathcal{E}$ followed by a fibration $r: \mathcal{E} \rightarrow B$. By the fibration property of r , the composite $\lambda \circ f$ is homotopic to a homotopy equivalence f' such that $r \circ f' = p$, i.e., we get a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow[f' \simeq]{} & \mathcal{E} \xleftarrow[\simeq]{\lambda} Y \\ & \searrow p & \downarrow r \swarrow q \\ & & B \end{array}$$

By [15, Proposition 2.3], both f' and λ will be $\varepsilon/4$ -homotopy equivalences, provided p and q are δ -fibrations for a suitably chosen $\delta > 0$. Let λ^{-1} denote a homotopy inverse which is also an $\varepsilon/4$ -homotopy equivalence. Then f is homotopic to the ε -homotopy equivalence $\lambda^{-1} \circ f'$. \square

For later use we also record the following opposite result:

Lemma 4.8. *Let Y be an ANR, let B be a compact ANR with a metric and let $\varepsilon > 0$. There is a $\delta > 0$ such that the following holds: If in (4.5), p is a δ -fibration and f is a δ -domination, then q is an ε -fibration.*

Proof. Let $\delta > 0$ and assume that there exists a δ -section $g: Y \rightarrow X$ of f , i.e., g is δ -controlled and that $f \circ g$ is δ -homotopic to the identity map on Y . We are first going to show that $p \circ g$ is a 5δ -fibration. So suppose we are given a homotopy lifting problem

$$\begin{array}{ccc} Z \times 0 & \xrightarrow{H_0} & Y \\ \downarrow & & \downarrow p \circ g \\ Z \times I & \xrightarrow{h} & B \end{array}$$

for $p \circ g$. By [42, Theorem 12.13], we can assume that Z is a cell. Postcomposing with the map g yields a homotopy lifting problem for p which can be solved up to δ by a map $L: Z \times I \rightarrow X$. Then $f \circ L|_{Z \times 0}$ is δ -homotopic to H_0 , so using the estimated homotopy extension property [14, Proposition 2.1], we can replace $f \circ L$ by a δ -homotopic map H such that $H|_{Z \times 0} = H_0$. By the triangle inequality, H is then a 3δ -lift of h .

Now, since $p \circ g$ is δ -close to q and $p \circ g$ is a 3δ -fibration, it follows that q is an ε -fibration if δ was chosen small enough. This can be seen as follows:

Choose $\delta > 0$ small enough so any two δ -close maps to B are $\varepsilon/4$ -homotopic and $3\delta < \varepsilon/4$ holds, see [36, Theorem 1.1 in Chapter IV on page 111]. In particular $p \circ g$ and q are $\varepsilon/4$ -homotopic. This homotopy may be used to obtain from the homotopy lifting problem displayed on the left a homotopy lifting problem as displayed on the right hand side:

$$\begin{array}{ccc} Z \times \{0\} & \xrightarrow{H_0} & Y \\ \downarrow & & \downarrow q \\ Z \times [0, 1] & \xrightarrow{h} & B \end{array} \qquad \begin{array}{ccc} Z \times \{-1\} & \xrightarrow{H_0} & Y \\ \downarrow & \nearrow L & \downarrow p \circ g \\ Z \times [-1, 1] & \xrightarrow{h} & B \end{array}$$

As $p \circ g$ is a 3δ -fibration, we may solve the problem on the right hand side up to $3\delta < \varepsilon/4$ by a map L . Now by the triangle inequality the restriction of L to $Z \times [0, 1]$ is a $3\varepsilon/4$ -homotopy from the restriction of L to $Z \times \{0\}$ to H_0 . The estimated homotopy extension property (control with respect to the map $p \circ g$) yields a map

$$\widehat{H}: Z \times [-1, 1] \rightarrow Y$$

which is $3\varepsilon/3$ -homotopic (control with respect to the map $p \circ g$) to L and which is H_0 when restricted to $Z \times \{0\}$. It is easily seen that $H := \widehat{H}|_{Z \times [0, 1]}$ is an ε -lift of h . \square

Proof of Lemma 4.2. Let us first consider the special case where E_1 and E_2 happen to be compact CW complexes. Let $g := f_1 \circ f_2^{-1}$. We have a homotopy commutative triangle

$$\begin{array}{ccc} E_2 & \xrightarrow[g \simeq]{} & E_1 \\ q_2 \searrow & & \swarrow q_1 \\ & B & \end{array}$$

By Lemma 4.7, choosing ε small enough, we may assume that g is a δ -homotopy equivalence, for some given $\delta > 0$.

Factor $q_1 = p' \circ \lambda$ into a homotopy equivalence $\lambda: E_1 \rightarrow E'$ followed by a fibration $p': E' \rightarrow B$. As q_1 is an approximate fibration λ is a controlled homotopy equivalence, that is, an ε -homotopy equivalence for every $\varepsilon > 0$.

In this situation, by [65, 1.4] there is a controlled Whitehead torsion in $\mathbb{H}_1(\tilde{B}; \mathbf{Wh}(p'))$; it is mapped under the assembly map of Quinn

$$(4.9) \quad \mathbb{H}_1(B; \mathbf{Wh}(p')) \rightarrow \mathbf{Wh}(\pi_1(E'))$$

to the image of the Whitehead torsion of g under the map induced by λ (see [65, paragraph before 1.7]). We get from the composition rule the following equation in $\mathbf{Wh}(\pi_1(E_1))$:

$$\tau(g) = \tau(f_1) - (f_1)_*(f_2)_*^{-1}\tau(f_2).$$

This implies that $\tau(f_1) - (f_1)_*(f_2)_*^{-1}\tau(f_2)$ lies in the image of the assembly map (4.9). We obtain from Lemma 3.9(i) and Lemma 3.19 (ii) the following commutative diagram

$$\begin{array}{ccc} \mathbb{H}_1(B; \mathbf{Wh}(p')) & \longrightarrow & \mathbf{Wh}(\pi_1(E')) \\ \mu_1 \downarrow \cong & & \cong \uparrow \nu_1 \\ H_1^\pi(\tilde{B}; \mathbf{Wh}(p')) & \longrightarrow & H_1^\pi(\text{pt}; \mathbf{Wh}(p')) \\ (\lambda \circ f_1)_* \uparrow \cong & & \cong \uparrow (\lambda \circ f_1)_* \\ H_1^\pi(\tilde{B}; \mathbf{Wh}(p)) & \longrightarrow & H_1^\pi(\text{pt}; \mathbf{Wh}(p)) = \mathbf{Wh}(\pi_1(M)) \end{array}$$

where the horizontal maps are assembly maps and the vertical maps are bijective. This implies that the image of the assembly map (4.9) agrees with the image of the assembly map $H_1^\pi(\tilde{B}; \mathbf{Wh}(p)) \rightarrow H_1^\pi(\text{pt}; \mathbf{Wh}(p)) = \mathbf{Wh}(\pi_1(M))$. This finishes the proof of Lemma 4.2 in the special case that E_1 and E_2 are compact CW-complexes.

In the general case, choose cell-like maps $h_i: E'_i \rightarrow E_i$ for $i = 1, 2$, such that E'_i is a compact CW-complex. As any cell-like map between compact ENRs is an approximate fibration, the composition $q_i \circ h_i$ is a, say, 2ε -fibration. Thus, replacing q_i by $q_i \circ h_i$ and f_i by $h_i^{-1} \circ f_i$, and ε by $\varepsilon/2$, the proof of the special case shows that

$$(f_1)_*^{-1}(h_1)_*\tau(h_1^{-1} \circ f_1) - (f_2)_*^{-1}(h_2)_*\tau(h_2^{-1} \circ f_2)$$

is in the image of the assembly map. Now use the composition rule together with the fact that cell-like maps have zero Whitehead torsion. This finishes the proof of Lemma 4.2. \square

4.4. Relating tight torsion to previously defined torsion invariants. In [31], the authors defined obstructions $\Theta(p)$ and $\tau_{\text{fib}}(p)$ to (actually) fibering a given map $p: M \rightarrow B$ where the homotopy fiber F_p of p is homotopy finite. For simplicity let us assume that M , B , and F_p are path-connected. The element $\tau_{\text{fib}}(p)$ is defined whenever $\Theta(p) = 0$ and lives in the cokernel of

$$\mathbf{Wh}(\pi_1 F_p) \xrightarrow{\chi(B) \cdot i_*} \mathbf{Wh}(\pi_1(M))$$

where $i: F_p \rightarrow M$ is the canonical map and $\chi(B) \in \mathbb{Z}$ denotes the Euler characteristic of B .

Lemma 4.10. *There is a factorization*

$$i_*: \mathbf{Wh}(\pi_1(F_p)) \rightarrow H_1^\pi(\tilde{B}; \mathbf{Wh}(p)) \rightarrow H_1^\pi(\text{pt}; \mathbf{Wh}(p)) = \mathbf{Wh}(\pi_1(M))$$

of the map induced by i .

Proof. In the sequel we will use the notation of Section 3. The canonical map $i: F_p \rightarrow M = M(\pi/\pi)$ may be lifted to a map $i': F_p \rightarrow M(\pi/1)$ (by choosing a lift of the homotopically constant map $F_p \rightarrow X$ to a map $F_p \rightarrow \tilde{X}$). This lift induces a map on Whitehead groups

$$\mathrm{Wh}(\pi_1(F_p)) \rightarrow \mathrm{Wh}(\pi_1(M(\pi/1))) \cong H_1^\pi(\pi; \mathbf{Wh}(p)).$$

As B is path-connected, there is up to homotopy a unique π -map $\pi \rightarrow \tilde{B}$. It induces a map

$$H_1^\pi(\pi; \mathbf{Wh}(p)) \rightarrow H_1^\pi(\tilde{B}; \mathbf{Wh}(p)).$$

The composite of these two maps is by definition the first map in the factorization of i_* . By construction, composing this map with the map induced by the projection $\tilde{B} \rightarrow B$ yields the map i_* . \square

Because of Lemma 4.10 we obtain a well-defined projection

$$\begin{aligned} (4.11) \quad \mathrm{pr}: \mathrm{cok}(\chi(B) \cdot i_*: \mathrm{Wh}(\pi_1(F_p)) \rightarrow \mathrm{Wh}(\pi_1(M))) \\ \rightarrow \mathrm{cok}(i_*: \mathrm{Wh}(\pi_1(F_p)) \rightarrow \mathrm{Wh}(\pi_1(M))) \\ \rightarrow \mathrm{cok}(H_1^\pi(\tilde{B}; \mathbf{Wh}(p)) \rightarrow H_1^\pi(\mathrm{pt}; \mathbf{Wh}(p))) := \mathrm{NWh}(p). \end{aligned}$$

Proposition 4.12. *Suppose that the homotopy fiber F_p of p is homotopy finite, and that $\Theta(f) = 0$. Then $\tau_{\mathrm{fib}}(p)$ maps to $N\tau(p)$ under the projection (4.11).*

Proof. Factorize p into a homotopy equivalence λ followed by a fibration $q: E \rightarrow B$. Recall from [31] that in our situation E carries (after making several choices) a preferred simple structure, i.e., a preferred homotopy equivalence $\varphi: X \rightarrow E$ from some compact ENR. Moreover $\tau_{\mathrm{fib}}(p)$ is represented by the Whitehead torsion of the composite homotopy equivalence from M to X .

Now consider the factorization of q into a homotopy equivalence φ and an approximate fibration from Theorem 4.1. From the inductive construction in the proof of that Theorem, it follows (see [75, Proposition 8.1] for more details) that φ represents the simple structure on E . So both $N\tau(p)$ and $\tau_{\mathrm{fib}}(p)$ are represented by the Whitehead torsion of the same homotopy equivalence. \square

4.5. Proof of Theorem 0.10. We conclude this section by giving the

Proof of Theorem 0.10. Choose a factorization up to homotopy

$$\begin{array}{ccc} M & \xrightarrow{f} & E \\ & \searrow p & \swarrow q \\ & & B \end{array}$$

into a homotopy equivalence and an approximate fibration, where E is a compact ENR, see Theorem 4.1. Since any compact ENR receives a cell-like map (and hence an approximate fibration) from a compact manifold with boundary [50, Corollary 11.2], we may assume that E is a compact manifold with boundary. By assumption, $\tau(f)$ is the image of some element τ' under the assembly map (3.5) and hence by Lemma 3.19 (iii) under the assembly map (3.18). By Quinn's Thin h -Cobordism Theorem [65, 1.2], there is a controlled h -cobordism W from E to some other compact manifold E' , such that the controlled torsion of (W, E) equals $-\tau'$. By Lemma 4.8, the map $W \rightarrow B$ is an approximate fibration, hence the composite $W \times Q \rightarrow W \rightarrow B$ is also an approximate fibration. As the resulting map $M \rightarrow W$ has Whitehead torsion zero, the map $M \times Q \rightarrow W \times Q$ is homotopic to a homeomorphism [13, Main Theorem] and hence $M \times Q$ approximately fibers over B . This finishes the proof of Theorem 0.10. \square

5. PROOF OF THE STABILIZATION THEOREM 1.5 FOR FINITE HOMOTOPY FIBER

Let $p: M \rightarrow B$ be a π_1 -surjective map of closed manifolds such that B is PL and aspherical, and we assume that the L -theoretic FJC holds for $\pi_1(B)$, see Definition 2.2. This section is entirely devoted to the proof of Theorem 1.5 under the stronger assumption that the homotopy fiber of p is finite.

5.1. s -split factorization.

Notation 5.1. Let \overline{M} denote the normal covering space of M corresponding to $\ker(p_*: \pi_1(M) \rightarrow \pi_1(B))$ and \tilde{B} be the universal covering of B . Put $G = \pi_1(B)$ and $E = \tilde{B} \times_G \overline{M}$. Let $\hat{p}: E \rightarrow B$ be the induced fiber bundle with fiber \overline{M} . Finally let $q: E \rightarrow M$ be the induced fiber bundle with fiber \tilde{B} . Since \tilde{B} is contractible, q is a homotopy equivalence.

Finally, let \mathcal{T} be a combinatorial triangulation of B , determining a PL structure on B .

Consider the following diagram:

$$\begin{array}{ccccc} & & \overline{M} & & \\ & & \downarrow & & \\ \tilde{B} & \longrightarrow & E = \tilde{B} \times_G \overline{M} & \xrightarrow{q} & M \\ & & \downarrow \hat{p} & \swarrow p & \\ & & B & & \end{array}$$

An easy exercise shows that the triangle in the diagram commutes after applying π_1 . It follows that it commutes up to homotopy since B is aspherical.

Definition 5.2 (s -split). *Let s be a positive integer. The map q is s -split (with respect to \mathcal{T}) if there exists a homotopy inverse*

$$f: M \times T^s \rightarrow E \times T^s$$

of $q_s = q \times \text{id}_{T^s}$ (called an s -splitting of q relative to \mathcal{T}) and a collection of compact submanifolds M_σ , $\sigma \in \mathcal{T}$ of $M \times T^s$, so that for all simplices $\sigma \in \mathcal{T}$ we have:

- (i) If τ is a proper face of σ , then $M_\tau \subset \partial M_\sigma$,
- (ii) $M_\sigma = f^{-1}(\hat{p}^{-1}(\sigma) \times T^s)$, and
- (iii) f restricts to a homotopy equivalence of pairs

$$f_\sigma: (M_\sigma, \partial M_\sigma) \rightarrow (\hat{p}^{-1}(\sigma) \times T^s, \hat{p}^{-1}(\partial\sigma) \times T^s).$$

The following diagram illustrates the situation, the dotted map g will appear later.

$$\begin{array}{ccc} & & g \\ & & \curvearrowright \\ E \times T^s & \xrightarrow{q_s} & M \times T^s \\ & \xleftarrow{f} & \\ \downarrow \hat{p}_s & & \swarrow p_s \\ B \times T^s & & \end{array}$$

Notation 5.3 (Suppressing orientation homomorphisms). We will suppress the orientation homomorphisms in the notation of the L -groups and for instance write just $L_n(\mathbb{Z}[\pi_1(M)])$ instead of $L_n(\mathbb{Z}[\pi_1(M)], w_1(M))$ for a connected closed manifold M .

Theorem 5.4. *Suppose that the homotopy fiber F_p is finitely dominated. Then there exists a positive integer s such that $q: E \rightarrow M$ is s -split.*

Remark 5.5. By using Wall's 2-sided separating codimension 1 splitting theorem [82, Theorem 12.1], we see that the triangulation \mathcal{T} has a subdivision \mathcal{T}' of arbitrarily small mesh such that q is also s -split (for the same integer s) with respect to \mathcal{T}' .

Proof. We will proceed by an argument similar to one originated by Quinn in his thesis [62], cf. [63]. Let $n = \dim M$ and $\varphi: M \rightarrow E$ be a homotopy inverse to q . We may assume, after a homotopy, that φ is transverse to each submanifold $\widehat{p}^{-1}(\sigma)$ of E , where σ denotes a simplex of \mathcal{T} ; i.e., $\widehat{p} \circ \varphi$ is transverse to each $\sigma \subset B$ [33, 47]. Put $M_\sigma := (\widehat{p} \circ \varphi)^{-1}(\sigma)$ and let

$$\varphi_\sigma: M_\sigma \rightarrow \widehat{p}^{-1}(\sigma)$$

be the restriction of φ .

We first complete the proof of Theorem 1.5 under the extra assumption that the homotopy fiber F_b is a finite complex; we will show afterwards how the general case follows easily from the restricted case.

We conclude from [56, Lemma 11.3], whose proof carries over to the L -theory case word by word, that the assembly map

$$H_*^\pi(\underline{E}\pi; \mathbf{L}^{\langle -\infty \rangle}(\widehat{p})) \rightarrow L_*^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(M)])$$

is an isomorphism for all $* \in \mathbb{Z}$. Since every virtually cyclic subgroup of the torsionfree group π is infinite cyclic, the relative assembly map

$$H_*^\pi(E\pi; \mathbf{L}^{\langle -\infty \rangle}(\widehat{p})) \xrightarrow{\cong} H_*^\pi(\underline{E}\pi; \mathbf{L}^{\langle -\infty \rangle}(\widehat{p}))$$

is bijective for all $* \in \mathbb{Z}$ by [52, Lemma 4.2]. This together with the assumption that B is aspherical implies that the assembly map

$$H_*^\pi(\widetilde{B}; \mathbf{L}^{\langle -\infty \rangle}(\widehat{p})) \rightarrow L_*^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(M)])$$

is bijective for all $* \in \mathbb{Z}$. Since this assembly map is isomorphic to Quinn's assembly map by Lemma 3.19 (ii), we can assume in the sequel that Quinn's assembly map

$$\mathbb{H}_*(B; \mathbf{L}^{\langle -\infty \rangle}(\widehat{p})) \rightarrow L_*^{\langle -\infty \rangle}(\mathbb{Z}[\pi_1(M)])$$

is bijective for all $* \in \mathbb{Z}$.

Although the manifold dimension of E is greater than n , our assumption implies that it is a n -dimensional Poincaré complex. In this situation,

$$\mathcal{P} = \{\varphi_\sigma: M_\sigma \rightarrow \widehat{p}^{-1}(\sigma); \sigma \in \mathcal{T}\}$$

is a conglomerate surgery problem to which we can assign a conglomerate surgery obstruction $\sigma(\mathcal{P}) \in \mathbb{H}_n(B; \mathbf{L}^h(\widehat{p}))$.

Since \mathcal{P} assembles to φ , which is a homotopy equivalence, the image of $\sigma(\mathcal{P})$ under the assembly map

$$\mathbb{H}_n(B; \mathbf{L}^h(\widehat{p})) \rightarrow L_n^h(\pi_1(M))$$

is zero.

There is a sequence of homomorphisms

$$\mathbb{H}_n(B; \mathbf{L}^h(\widehat{p})) = \mathbb{H}_n(B; \mathbf{L}^{\langle 0 \rangle}(\widehat{p})) \rightarrow \mathbb{H}_n(B; \mathbf{L}^{\langle -1 \rangle}(\widehat{p})) \rightarrow \mathbb{H}_n(B; \mathbf{L}^{\langle -2 \rangle}(\widehat{p})) \rightarrow \dots$$

whose colimit is $\mathbb{H}_n(B; \mathbf{L}^{\langle -\infty \rangle}(\widehat{p}))$. The assembly maps fit into a commutative diagram

$$\begin{array}{ccc} \mathbb{H}_n(B; \mathbf{L}^h(\widehat{p})) & \longrightarrow & L_n^h(\pi_1(M)) \\ \downarrow & & \downarrow \\ \mathbb{H}_n(B; \mathbf{L}^{\langle -i \rangle}(\widehat{p})) & \longrightarrow & L_n^{\langle -i \rangle}(\pi_1(M)) \\ \downarrow & & \downarrow \\ \mathbb{H}_n(B; \mathbf{L}^{\langle -\infty \rangle}(\widehat{p})) & \longrightarrow & L_n^{\langle -\infty \rangle}(\pi_1(M)) \end{array}$$

Since the bottom horizontal arrow is injective, there exists $m \geq 0$ such that the image of $\sigma(\mathcal{P})$ under the map

$$\mathbb{H}_n(B; \mathbf{L}^h(\widehat{p})) \rightarrow \mathbb{H}_n(B; \mathbf{L}^{\langle -m \rangle}(\widehat{p}))$$

is trivial. Crossing with T^m yields a map

$$\mathbb{H}_n(B; \mathbf{L}^{\langle -m \rangle}(\widehat{p})) \rightarrow \mathbb{H}_{n+m}(B; \mathbf{L}^h(\widehat{p}_m)).$$

Consider the new conglomerate surgery problem

$$\mathcal{P}_m = \{\varphi_\sigma \times \text{id}_{T^m} : M_\sigma \times T^m \rightarrow \widehat{p}^{-1}(\sigma) \times T^m; \sigma \in \mathcal{T}\}$$

Its conglomerate surgery obstruction $\sigma(\mathcal{P}_m) \in \mathbb{H}_{n+m}(B; \mathbf{L}^h(\widehat{p}_m))$ is the image of $\sigma(\mathcal{P})$ under the composite

$$\mathbb{H}_n(B; \mathbf{L}^h(\widehat{p})) \rightarrow \mathbb{H}_n(B; \mathbf{L}^{\langle -m \rangle}(\widehat{p})) \rightarrow \mathbb{H}_{n+m}(B; \mathbf{L}^h(\widehat{p}_m))$$

and hence trivial. This implies that there exists a conglomerate surgery problem

$$\mathbb{P} = \{\psi_\sigma : W_\sigma \rightarrow \widehat{p}^{-1}(\sigma) \times T^m \times [0, 1]; \sigma \in \mathcal{T}\}$$

such that $\partial^- W_\sigma = M_\sigma \times T^m$ and

$$\psi_\sigma|_{\partial^- W_\sigma} : M_\sigma \times T^m \rightarrow \widehat{p}^{-1}(\sigma) \times T^m \times 0$$

is $\varphi_\sigma \times \text{id}_{T^m}$.

Furthermore, if we put

$$\mathcal{S} = \{\varphi'_\sigma := \psi_\sigma|_{\partial^+ W_\sigma} : \partial^+ W_\sigma \rightarrow \widehat{p}^{-1}(\sigma) \times T^m \times 1; \sigma \in \mathcal{T}\}$$

then each φ'_σ is a homotopy equivalence; i.e., \mathcal{S} is a conglomerate homotopy-topological structure, and we denote the homotopy-topological structure on $E \times T^m$ that \mathcal{S} assembles to by

$$\varphi' : (M')^{n+m} \rightarrow E \times T^m.$$

By topologically assembling \mathbb{P} , we obtain a (single) surgery problem

$$\psi : W^{n+m+1} \rightarrow E \times T^m \times [0, 1]$$

satisfying

- (i) $\psi|_{\partial^- W} : M \times T^m \rightarrow E \times T^m \times 0$ is $\varphi \times \text{id}_{T^m}$;
- (ii) $\psi|_{\partial^+ W} : (M')^{n+m} \rightarrow E \times T^m$ is φ' .

This surgery problem determines an element $\theta \in L_{n+m+1}^h(\pi_1(M) \times \mathbb{Z}^m)$.

Since the assembly map

$$\mathbb{H}_{n+m+1}(B; \mathbf{L}^{\langle -\infty \rangle}(\widehat{p}_m)) \rightarrow L_{n+m+1}^{\langle -\infty \rangle}(\pi_1(M) \times \mathbb{Z}^m)$$

is an epimorphism, we see by an argument similar to the one for T^m that after taking the product with an additional torus T^t of sufficiently large dimension t , there is another conglomerate surgery problem

$$\mathbb{P}' = \{\eta_\sigma : W'_\sigma \rightarrow \widehat{p}^{-1}(\sigma) \times T^m \times T^t \times [1, 2]; \sigma \in \mathcal{T}\}$$

such that $\partial^- W'_\sigma = \partial^+ W_\sigma$ and

$$\eta_\sigma|_{\partial^- W'_\sigma} : \partial^+ W_\sigma \rightarrow \widehat{p}^{-1}(\sigma) \times T^m \times T^t \times 1$$

is $\varphi'_\sigma \times \text{id}_{T^t}$. Furthermore

$$\mathcal{S}' = \{\varphi''_\sigma := \eta_\sigma|_{\partial^+ W'_\sigma} : \partial^+ W'_\sigma \rightarrow \widehat{p}^{-1}(\sigma) \times T^m \times T^t \times 2; \sigma \in \mathcal{T}\}$$

is a conglomerate homotopy-topological structure which assembles to a homotopy-topological structure on $E \times T^m \times T^t$ denoted by

$$\varphi'' : (M'')^{n+m+t} \rightarrow E \times T^m \times T^t.$$

Assembling \mathbb{P}' yields a surgery problem

$$\eta : (W')^{n+m+t+1} \rightarrow E \times T^m \times T^t \times [1, 2]$$

which represents the image of $-\theta$ in $L_{n+m+t+1}^h(\pi_1(M) \times \mathbb{Z}^m \times \mathbb{Z}^t)$ (under the natural map in the Wall-Shaneson formula), and satisfies

- (i) $\eta|_{\partial^- W'} : (M')^{n+m} \times T^t \rightarrow E \times T^m \times T^t$ is $\varphi' \times \text{id}_{T^t}$;
- (ii) $\eta|_{\partial^+ W'} : (M'')^{n+m+t} \rightarrow E \times T^m \times T^t$ is φ'' .

Glueing together the two surgery problems $W \times T^t$ and W' along $\partial^+ W \times T^t = \partial^- W'$ yields a surgery problem representing 0 in $L_{n+m+t+1}^h(\pi_1(M) \times \mathbb{Z}^{m+t})$. Hence $W \times T^t \cup W'$ can be surgered, without touching its boundary, so as to yield an h -cobordism C between $\partial^- C = M \times T^{m+t}$ and $\partial^+ C = \partial^+ W' = (M'')^{n+m+t}$. Since $C \times S^1$ is an s -cobordism, $M \times T^{m+t} \times S^1$ is homeomorphic to $(M'')^{n+m+t} \times S^1$ via a homeomorphism $g : M \times T^{m+t+1} \rightarrow M'' \times S^1$ such that the composition

$$f : M \times T^{m+t+1} \xrightarrow{g} M'' \times S^1 \xrightarrow{\varphi'' \times \text{id}} E \times T^{m+t+1}$$

is homotopic to $\varphi \times \text{id}_{T^{m+t+1}}$. Thus q is s -split for $s = m + t + 1$ by the map f .

This completes the proof of the restricted case of Theorem 1.5, i.e., assuming that F_b is a finite complex. However in the general case we assume that F_b is dominated by a finite complex. But this at least implies that $F_b \times S^1$ has the homotopy type of a finite complex and consequently the homotopy fiber of the composite map

$$M \times S^1 \rightarrow M \xrightarrow{p} B$$

is a finite complex. Therefore the restricted form of Theorem 1.5 applies, showing that $q \times \text{id}_{S^1}$ is s -split. Consequently, q is $(s + 1)$ -split. This finishes the proof of Theorem 5.4. \square

Remark 5.6 (Bounds for s). An analysis of the proof of Theorem 5.4 shows that if $\dim(M) \geq 5$, then s can be taken to be $2(k + 1) + 2$ if k has the property that $H_*^\pi(\widetilde{B}; \mathbf{L}^{\langle -k \rangle}(p)) \xrightarrow{\cong} H_*^\pi(\widetilde{B}; \mathbf{L}^{\langle -\infty \rangle}(p))$ and $L_*^{\langle -k \rangle}(\pi_1(M)) \xrightarrow{\cong} L_*^{\langle -\infty \rangle}(\pi_1(M))$ under the canonical maps. If the *total space* of p satisfies the K -theoretic FJC, then $k = 1$ such that s can be taken to be 6 (and even 5 in the homotopy finite case).

5.2. Gaining control over the torus. Now fix a triangulation \mathcal{T} and an s -splitting f of q relative to \mathcal{T} . Using a cofibration argument, there is a left homotopy inverse

$$g : E \times T^s \rightarrow M \times T^s$$

to f and a homotopy $g \circ f \simeq_H \text{id}_{M \times T^s}$, such that the restriction

$$g_\sigma : (\widehat{p}^{-1}(\sigma) \times T^s, \widehat{p}^{-1}(\partial\sigma) \times T^s) \rightarrow (M_\sigma, \partial M_\sigma)$$

of g is a left homotopy inverse to f_σ , via a homotopy H_σ which is the restriction of H . If the mesh of \mathcal{T} is less than ε , it follows the map g is an ε -domination over B in the sense of Definition 4.4, where the control map from $M \times T^s$ to B is the composite of f with the projection from $E \times T^s$ to B .

If we knew that g was an ε -domination over $B \times T^s$, then we could apply Lemma 4.8 to conclude that $\widehat{p}_s \circ f$ were a δ -fibration. But unfortunately there is no control over the torus yet.

Lemma 5.7. *Suppose that the homotopy fiber F_p is homotopy finite. There is a 4ε -domination g over B in such a way that g is bounded over the torus, i.e., for one (and hence any) lift $\widetilde{g}: E \times \mathbb{R}^s \rightarrow M \times \mathbb{R}^s$ of g along the universal coverings $\mathbb{R}^s \rightarrow \mathbb{R}^s/\mathbb{Z}^s = T^s$ of the torus and for one (and hence all) lifts $\widetilde{f}: M \times \mathbb{R}^s \rightarrow E \times \mathbb{R}^s$ of f there exists $N > 0$ such that*

$$d(\pi \circ \widetilde{f} \circ \widetilde{g}(e, x), x) < N \quad \forall (e, x) \in E \times \mathbb{R}^s.$$

where π is the projection $E \times \mathbb{R}^s \rightarrow \mathbb{R}^s$.

Proof. By Theorem 4.1 and Lemma 4.7, the map \widehat{p}_s is ε -homotopy equivalent to an approximate fibration $E' \rightarrow B \times T^s$ where E' is compact. In particular the identity map on $E \times T^s$ is ε -homotopic (over $B \times T^s$) to a map $z: E \times T^s \rightarrow E \times T^s$ which factors into a compact subset $C \subset E \times T^s$. Then, by the triangle inequality, the map $g' := g \circ z$ is 3ε -controlled over B , and $g' \circ f$ is 4ε -homotopic to the identity map (over B). Thus, g' is a 4ε -domination over B . Choosing a lift \widetilde{z} of z along the universal covering of the torus, we have

$$d(\pi \circ \widetilde{f} \circ \widetilde{g}'(e, x), x) \leq d(\pi \circ \widetilde{f} \circ \widetilde{g} \circ \widetilde{z}(e, x), \pi \circ \widetilde{z}(e, x)) + d(\pi \circ \widetilde{z}(e, x), x).$$

The second summand is less than ε . Choose a compact set $\widetilde{C} \subset E \times \mathbb{R}^s$ which surjects onto C . Then we have

$$\sup_{(e, x) \in E \times \mathbb{R}^s} d(\pi \circ \widetilde{f} \circ \widetilde{g} \circ \widetilde{z}(e, x), \pi \circ \widetilde{z}(e, x)) \leq \sup_{\widetilde{c} \in \widetilde{C}} d(\pi \circ \widetilde{f} \circ \widetilde{g}(\widetilde{c}), \pi(\widetilde{c})) < \infty.$$

□

Note that as $M \times T^s$ is compact, the homotopy H between $g \circ f$ and the identity map is an N -homotopy for some $N > 0$. (Here we measure the diameter of a path in T^s also by first lifting it to the universal cover \mathbb{R}^s .) We might call g from Lemma 5.7 a “bounded domination” over the torus.

Lifting such a map g to a map g_k between coverings over the torus of index k (i.e., to the coverings whose covering projections are determined by the expanding self-maps $x \mapsto x^k$, where $x \in T^s$), we can improve the bound by a factor of k . Choosing k large enough, it follows that g_k is an 5ε -domination over $B \times T^s$ (where $M \times T^s$ is controlled by $\widehat{p}_s \circ f_k$ now). Since $E \times T^s$ is a fiber bundle over $B \times T^s$, we obtain from Lemma 4.8:

Corollary 5.8. *Suppose that the homotopy fiber F_p is homotopy finite. Then for all $\varepsilon > 0$ there is a $\delta > 0$ and an $k > 0$ such that the composite*

$$M \times T^s \xrightarrow{f_k} E \times T^s \xrightarrow{\widehat{p}_s} B \times T^s$$

is an ε -fibration provided that the mesh of \mathcal{T} is less than δ .

Now we can complete the proof of Theorem 1.5 with the help of Corollary 5.8 and Remark 5.5. Since $M \times T^s$ is a closed manifold, an approximation theorem by Chapman [16] shows that $\widehat{p}_s \circ f_k$ is homotopic to an approximate fibration, provided ε was chosen small enough. Now f_k and f induce the same map on fundamental groups, namely $\pi_1(q)^{-1} \times \text{id}_{\mathbb{Z}^s}$. Since B is aspherical, $p_s \simeq \widehat{p}_s \circ f$ is homotopic to $\widehat{p}_s \circ f_k$. This finishes the proof of Theorem 1.5.

6. PROOF OF THE SPLITTING THEOREM 1.6

The content of this section is the proof of Theorems 1.6 and 1.7.

6.1. **Proof of Theorem 1.6 (i).** Let ρ denote the composition

$$\rho: M \times \mathbb{R} \rightarrow M \times S^1 \xrightarrow{\overline{p_1}} B \times S^1 \rightarrow B$$

where $\overline{p_1}$ is the approximate fibration homotopic to p_1 which is assumed to exist. Then ρ is also an approximate fibration. We wish to apply Quinn's End Theorem 1.1 from [65] to complete ρ at “ $+\infty$ ”. The end is tame since $M \times \mathbb{R}$ is an approximate fibration over $B \times \mathbb{R}$. For the same reason, the end has a locally constant fundamental groupoid, which is the fundamental groupoid of the homotopy fiber of ρ , and hence, of p .

Denoting by $p': E \rightarrow B$ a fibration controlled equivalent to p , the end obstruction to obtain a completion is therefore an element

$$q_0(\rho) \in \mathbb{H}_0(\widetilde{B}; \mathbf{Wh}(p'))$$

whose image under Quinn's assembly map

$$\mathbb{H}_0(\widetilde{B}; \mathbf{Wh}(p')) \rightarrow \mathrm{Wh}_0(\pi_1(E)) \cong \widetilde{K}_0(\mathbb{Z}[\pi_1(M)])$$

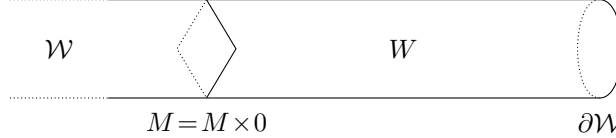
is Siebenmann's obstruction to adding a boundary (see [72] and the paragraph preceding Proposition 1.7 in [65]).

The latter obstruction is zero since $M \times \mathbb{R}$ compactifies to $M \times [-\infty, +\infty]$. Since we assume that the K -theoretic FJC holds for $\pi_1(B)$ and B is aspherical, the assembly map $H_0^\pi(\widetilde{B}; \mathbf{Wh}(p')) \rightarrow \mathrm{Wh}_0(\pi_1(M))$ is injective. Now we conclude from

Lemma 3.19 (ii) that Quinn's assembly map $\mathbb{H}_0(\widetilde{B}; \mathbf{Wh}(p')) \rightarrow \mathrm{Wh}_0(\pi_1(M))$ is injective and hence $q_0(\rho) = 0$. Let

$$\overline{p}: \mathcal{W} \rightarrow B$$

denote a completion of $\rho: M \times \mathbb{R} \rightarrow B$ given by the End Theorem. It is also an approximate fibration. Now let W denote the compact h -cobordism connecting M to $\partial\mathcal{W}$ inside \mathcal{W} . (Identify M with $M \times 0$.)



So there is an h -cobordism from M to some closed manifold $N := \partial\mathcal{W}$, which is a compact ENR and maps to B by the approximate fibration

$$\overline{p}|_{\partial\mathcal{W}}: \partial\mathcal{W} \rightarrow B.$$

This finishes the proof of Theorem 1.6 (i).

6.2. **Proof of Theorem 1.6 (ii).** (ii) Denote by

$$\bar{\cdot}: \mathrm{Wh}(G) \rightarrow \mathrm{Wh}(G)$$

the involution coming from $w_1(M)$ -twisted involution $a \mapsto \bar{a}$ on $\mathbb{Z}G$. It sends a matrix to its conjugate-transpose.

If we let $x = \tau(W, M)$, then

$$\tau(f) = x - (-1)^n \bar{x}$$

where $f: M \rightarrow \partial\mathcal{W}$ is the homotopy equivalence determined by W and $n = \dim M$. Next note that the class $\tau(f)$ in the cokernel of the assembly map

$$\alpha = H_1^\pi(\mathrm{pr}; \mathbf{Wh}(p)): H_1^\pi(\widetilde{B}; \mathbf{Wh}(p)) \rightarrow \mathrm{Wh}(\pi_1(M))$$

is $N\tau(p)$ and hence vanishes by assumption.

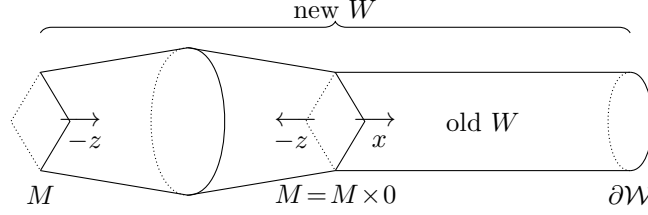
Hence the class of x in $\text{NWh}(p)$ defines an element in the Tate cohomology group $\widehat{H}^n(\mathbb{Z}/2; \text{NWh}(p))$. It follows that we can write

$$x = z + (-1)^n \bar{z} + a$$

where $a \in \text{im}(\alpha)$ because of our assumption on the vanishing of the Tate cohomology group. We can change the embedding of M into $M \times \mathbb{R}$ so that

$$x = a \in \text{im}(\alpha)$$

by inserting an h -cobordism with torsion $-z - (-1)^n \bar{z}$ to the left of $M \times 0$.



Lemma 3.19 (iii) shows that Quinn's assembly map $\mathbb{H}_1(\tilde{B}; \mathbf{Wh}(p)) \rightarrow \text{Wh}(\pi_1(M))$ and $\alpha = H_1^\pi(\text{pr}; \mathbf{Wh}(p)): H_1^\pi(\tilde{B}; \mathbf{Wh}(p)) \rightarrow \text{Wh}(\pi_1(M))$ have the same image. Hence x lies in the image of Quinn's assembly map as well.

We now use Quinn's h -cobordism Theorem 1.2 (b) together with his End Theorem 1.1 (b) from [65] to change the completion of

$$\rho: M \times \mathbb{R} \rightarrow B$$

so that the torsion $\tau(W, M)$ changes by adding to it the element $-a$. Hence we may assume after doing this that $x = 0$ so $W \cong M \times [0, 1]$; in particular that $M \cong \partial W$.

This finishes the proof of Theorem 1.6.

7. PROOF OF THEOREM 1.7 ABOUT THE MAF-RIGIDITY CONJECTURE AND THE SPLITTING OBSTRUCTION

This section is devoted to the proof of Theorem 1.7.

7.1. The MAF Rigidity Conjecture. Let us first recall the statement of the "MAF Rigidity Conjecture" by Hughes-Taylor-Williams [44, page 568]:

Conjecture 7.1 (MAF Rigidity Conjecture). *Let B be a closed aspherical manifold. Then two MAFs $p: M \rightarrow B$ and $q: N \rightarrow B$ are controlled homeomorphic if and only if there is a homeomorphism $h: M \rightarrow N$ such that $q \circ h$ is homotopic to p .*

Here, p and q being controlled homeomorphic means that there is a locally trivial fiber bundle $E \rightarrow [0, 1]$ with $E \times 0 = M$, $E \times 1 = N$, and a fiberwise map $H: E \rightarrow B \times [0, 1]$ which is an approximate fibration, such that $H_0 = p$ and $H_1 = q$. (Compare [42, Proposition 12.17] and [44, page 567].)

The next result is taken from [44, Theorem 1.2].

Theorem 7.2. *The MAF Rigidity Conjecture holds when B is non-positively curved.*

7.2. The splitting obstruction. Suppose now that the MAF Rigidity Conjecture holds for B . Recall the construction of an h -cobordism W from M to ∂W in the proof of Theorem 1.6, (i). Recall also from the beginning of the proof of part (ii) that if we let $x = \tau(W, M)$, we have $[x] - (-1)^n [\bar{x}] = N\tau(p) = 0 \in \text{NWh}(p)$.

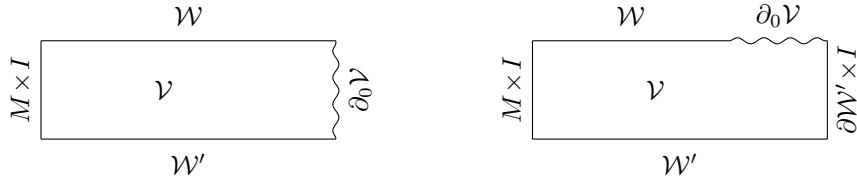
Definition 7.3 (Splitting obstruction). *The splitting obstruction*

$$\kappa_0 \in \widehat{H}^n(\mathbb{Z}/2; \text{NWh}(p))$$

is the class determined by $x = \tau(W, M)$.

We have to show that the element κ_0 is well-defined, i.e., does not depend on the choice of approximate fibration homotopic to p_1 nor on the choice of completion. So suppose that \overline{p}_1 and \overline{p}'_1 are two approximate fibrations homotopic to p_1 . By passing to the infinite cyclic cover, we obtain two maps $\rho, \rho' : M \times \mathbb{R} \rightarrow B$; suppose that $(\mathcal{W}, \overline{p})$ and $(\mathcal{W}', \overline{p}')$ are completions of $(M \times \mathbb{R}, \rho)$ and $(M \times \mathbb{R}, \rho')$ at ∞ .

By the MAF Rigidity Conjecture, we obtain an approximate fibration $H : E \rightarrow B \times S^1 \times [0, 1]$ interpolating between \overline{p}_1 and \overline{p}'_1 . If \overline{E} denotes the infinite cyclic cover corresponding to the kernel of $\pi_1(H)$, we obtain an approximate fibration $F : \overline{E} \rightarrow B$ interpolating between ρ and ρ' . Applying Quinn's relative End Theorem to F , there is a completion $(\mathcal{V}, \overline{F})$ of F that extends the given completions $(\mathcal{W}, \overline{p})$ and $(\mathcal{W}', \overline{p}')$, and the boundary component $\partial_0 \mathcal{V}$ is a thin h -cobordism from $\partial \mathcal{W}$ to $\partial \mathcal{W}'$. (See the left picture.)



We have

$$\tau(\mathcal{W}', M) + \tau(\mathcal{V}, \mathcal{W}') = \tau(\mathcal{W}, M) + \tau(\mathcal{V}, \mathcal{W}),$$

as both are the Whitehead torsions of the inclusion from M to \mathcal{V} . So,

$$x - x' = \tau(\mathcal{V}, \mathcal{W}') - \tau(\mathcal{V}, \mathcal{W}).$$

But the torsions of $(\mathcal{V}, \mathcal{W}')$ and $(\mathcal{V}, \mathcal{W})$ are related by the involution: In fact, bending around the corners transforms the left picture into the right picture, where we see that

$$\tau(\mathcal{V}, \mathcal{W}') = (-1)^{(n+1)} \overline{\tau(\mathcal{V}, \mathcal{W} \cup \partial_0 \mathcal{V})}.$$

Now, since $\partial_0 \mathcal{V}$ is a thin h -cobordism, its Whitehead torsion becomes zero in $\text{NWh}(p)$. Hence

$$[\tau(\mathcal{V}, \mathcal{W}')] = (-1)^{(n+1)} \overline{[\tau(\mathcal{V}, \mathcal{W})]} \in \text{NWh}(p).$$

so that

$$[x] - [x'] = 0 \in \widehat{H}^n(\mathbb{Z}/2; \text{NWh}(p)),$$

establishing that κ_0 is well-defined.

7.3. Finishing the proof of Theorem 1.7. Suppose first that p is homotopic to an approximate fibration q . Then the map $q_1 = q \times \text{id}_{S^1}$ is an approximate fibration homotopic to p_1 , and (again in the notation of the proof of Theorem 1.6) we may take ρ to be the composite

$$M \times \mathbb{R} \rightarrow M \xrightarrow{q} B$$

of the projection and q . This map can obviously be completed by $M \times (-\infty, \infty]$ so that W is just $M \times [0, \infty]$ which is a trivial h -cobordism. So $\tau(W, M) = 0$ and κ_0 vanishes.

On the other hand, if $\kappa_0 = 0$, then we can write $x = z + (-1)^{-1} \bar{z}$ modulo the image of the assembly map and the very same argument as in the proof of part (ii) of Theorem 1.6 shows that p is homotopic to an approximate fibration. Note that for $n > 0$, $N\tau(p_n) = 0$ since it is given by

$$\tau(\varphi \times \text{id}_{S^1}) = \tau(\varphi) \cdot \chi(S^1) = 0$$

by [49].

8. ORIENTABILITY OF CYCLIC SUBGROUPS

If a torsionfree group G contains the fundamental group of the Klein bottle $K := Z \rtimes Z$ as a subgroup, then the cyclic subgroups are certainly not orientable in the sense of Definition 0.7. We conclude from [56, Lemma 8.7 and Lemma 8.8] a converse in a special situation for G .

Theorem 8.1. *Let G be a torsionfree group satisfying one of the following conditions*

- (i) G is hyperbolic;
- (ii) G is a $CAT(0)$ -group and satisfies the Klein bottle condition, i.e., G does not contain the fundamental group $K = Z \rtimes Z$ of the Klein bottle.

Then the cyclic subgroups of G are orientable in the sense of Definition 0.7.

Lemma 8.2. *Let $1 \rightarrow K \xrightarrow{i} G \xrightarrow{p} Q \rightarrow 1$ be an extension of torsionfree groups. Suppose that for any $g \in G$ the conjugation automorphism $K \rightarrow K$ is an inner automorphism of K . If the cyclic subgroups of both K and Q are orientable in the sense of Definition 0.7, then the same is true for G .*

Proof. Fix orientations for the cyclic subgroups of K and of Q . Let $C \subset G$ be a cyclic subgroup of G . If $C = i(C')$ for $C' \subset K$, we let g_C be the image of the chosen generator of C' . Otherwise $C \cap K = \emptyset$ so p defines an isomorphism $C \rightarrow p(C)$, in which case we let g_C be the preimage of the chosen generator of $p(C)$. We leave it to the reader to check that these choices are compatible with the requirements in Definition 0.7. \square

We close this section with a result stated in Proposition 8.4 and mentioned in the introduction. It is not needed elsewhere in the paper. The key ingredient in its proof is [68, Theorem 6.11], which implies the following:

Theorem 8.3. *Let $\Gamma \subset \mathrm{GL}(n, \mathbb{C})$ be a finitely generated subgroup. Then Γ contains a subgroup π of finite index such that for each matrix $A \in \pi$ the eigenvalues of A contain no non-trivial root of unity.*

Proposition 8.4. *Let B be a non-positively curved closed locally symmetric space, then there is a finite sheeted cover $\tilde{B} \rightarrow B$ such that $\pi_1(\tilde{B})$ does not contain a subgroup isomorphic to the fundamental group of the Klein bottle.*

Proof. Put $\Gamma = \pi_1(B)$. There exists a linear centerless semi-simple Lie group G containing Γ as a discrete cocompact subgroup. Consider the adjoint representation of G onto its Lie algebra \mathfrak{g} . Via this representation Γ embeds in $\mathrm{GL}_n(\mathbb{R}) = \mathrm{GL}(\mathfrak{g})$ where $n = \dim_{\mathbb{R}} \mathfrak{g}$. By passing to subgroups of finite index we may assume that G is connected; i.e., is analytic. See Mostow [60, § 2] for terminology. We will apply Theorem 8.3 to this situation, i.e., $\Gamma \subset \mathrm{GL}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{C})$.

Let $\pi \subset \Gamma$ be the finite index subgroup given by Theorem 8.3, and let $\tilde{B} \rightarrow B$ be the covering space corresponding to $\pi \subset \Gamma$. It remains to show that π does not contain a pair of matrices A, B satisfying $BAB^{-1} = A^{-1}$ and $A \neq I$. We do this by showing that the existence of such a pair leads to a contradiction. (Now glance at [60, § 3 on page 12] and also note that π is a discrete subgroup of $\mathrm{GL}_n(\mathbb{R})$.)

By the first paragraph on page 76 of Mostow's book both A and B are semi-simple matrices, i.e., diagonalizable in $\mathrm{GL}_n(\mathbb{C})$. And since A, B^2 commute they are simultaneously diagonalizable. By page 10 of Mostow,

$$A = kp = pk$$

where p has positive real eigenvalues and k has all eigenvalues of length 1. Furthermore $p, k \in G \subset \mathrm{GL}_n(\mathbb{C})$, see p. 12 of Mostow. Also p, k are uniquely defined by A .

Since A has infinite order, $p \neq \text{id}$. (Otherwise $A = k$ which lies in a compact subgroup of $\text{GL}_n(\mathbb{C})$.) Furthermore p^t , $t \in \mathbb{R}$, is a 1-parameter subgroup of G passing through p (see p. 12 of Mostow). Now B^2 commutes with p since it commutes with A (see p. 10 of Mostow). Likewise B^2 commutes with every matrix p^t since p and B^2 are simultaneously diagonalizable. Now

$$p^t = \exp(tv), \quad t \in \mathbb{R}$$

for some $v \in \mathfrak{g}$, $v \neq 0$. And

$$t \mapsto Bp^tB^{-1}$$

is the 1-parameter subgroup $\exp(tu)$ where $u = B(v)$ – through the adjoint action of B on \mathbb{R}^n , i.e., through $B \in \Gamma \subset \text{GL}_n(\mathbb{R})$. And since

$$p^t = B^2p^tB^{-2} = B \exp(tu)B^{-1},$$

we have that $B(u) = v$.

Let V be the subspace of $\mathfrak{g} = \mathbb{R}^n$ spanned by u and v . Then $\dim V$ is 1 or 2, and V is left invariant by B . We now proceed to show that each of the two possibilities leads to a contradiction, thus completing the proof of Proposition 8.4.

Case 1. $\dim V = 2$. Then $\{u, v\}$ form a basis for V and with respect to this basis $B|_V$ is represented by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which has characteristic polynomial $\lambda^2 - 1$. Hence $\lambda = -1$ is an eigenvalue of B , contradicting Theorem 8.3.

Case 2. $\dim V = 1$. Then $u = \lambda v$ for $\lambda \in \mathbb{R}$. Hence

$$v = Bu = \lambda Bv = \lambda u = \lambda^2 v.$$

Since $v \neq 0$, $\lambda^2 = 1$. Therefore $\lambda = \pm 1$. If $\lambda = 1$, then

$$p^t = \exp(tv) = \exp(tu) = Bp^tB^{-1} \quad \forall t \in \mathbb{R}.$$

Setting $t = 1$ yields $p = BpB^{-1}$. Therefore

$$p(BkB^{-1}) = (BpB^{-1})(BkB^{-1}) = BAB^{-1} = A^{-1} = p^{-1}k^{-1}$$

and by the uniqueness of the polar decomposition (Mostow p. 10) we have that

$$p = p^{-1} \quad \text{and} \quad BkB^{-1} = k^{-1}.$$

In particular $p^2 = \text{id}$, so $p = \text{id}$, which contradicts the assumptions on the decomposition $A = kp$.

Therefore $\lambda = -1$, which is an eigenvalue of B , contradicting Theorem 8.3. This finishes the proof of Proposition 8.4 \square

9. THE CASE OF A NON-FINITE HOMOTOPY FIBER

9.1. The homotopy fiber of an approximate fibration.

Lemma 9.1. *Let $q: E \rightarrow B$ be an approximate fibration, where E is a compact ENR and B is a connected topological manifold. Then the homotopy fiber of q is dominated by a finite complex.*

Remark 9.2. This result is probably well-known to the experts but we couldn't find a reference so we include a proof for the reader's convenience. We are grateful to the referee for providing us with the following, alternative argument: By Ferry [32] any CW model for the homotopy fiber is shape equivalent to the actual fiber $q^{-1}(b)$, which is compact, so it follows from [22, Corollary 3.2] that the homotopy fiber is finitely dominated.

Proof of Lemma 9.1. Let $U_1 \subset U_2 \subset B$ two open balls such that $\overline{U_1} \subset U_2$. We form the pull-backs

$$\begin{array}{ccccc} E|_{U_1} & \longrightarrow & E|_{U_2} & \longrightarrow & E \\ \downarrow q & & \downarrow q & & \downarrow q \\ U_1 & \longrightarrow & U_2 & \longrightarrow & B \end{array}$$

It follows from the fact that q is an approximate fibration that the inclusion $E|_{U_1} \rightarrow E|_{U_2}$ is a homotopy equivalence and that both spaces have the homotopy type of the homotopy fiber of q . (See [42, §10] [42, Corollary 2.14 and Theorem 12.15].)

We first complete the argument under the extra assumption that E is a simplicial complex. In this case there is a finite subcomplex K lying in between $E|_{U_1}$ and $E|_{U_2}$ (subdividing E if necessary). So K dominates $E|_{U_1}$ which is the homotopy type of the homotopy fiber of q .

In the general case, E receives a cell-like map from a compact topological manifold [50, 11.2], which in turn admits a disk bundle which is triangulated (see [47, §III.4]). This implies that E receives a cell-like map from a finite simplicial complex X .

Since X is homotopy equivalent to E and the map from X to B is still an approximate fibration, we may just replace E by X and use the argument of the special case above. \square

Remark 9.3. Let $p: X \rightarrow B$ be a π_1 -surjective map of CW -complexes such that B is aspherical and X finite dimensional. Then the homotopy fiber of p is the total space of the covering $q: \overline{X} \rightarrow X$ associated to the kernel of $\pi_1(p)$. In particular it is homotopy equivalent to a finite dimensional CW -complex. This implies that the homotopy fiber is finitely dominated if and only if it has the homotopy type of a CW -complex of finite type, i.e., a CW -complex all whose skeleta are finite.

9.2. Factorization into an ε -fibration. We do not know whether Theorem 4.1 holds when the homotopy fiber is only required to be finitely dominated. (See [32, Theorem 1.1] for a related statement when $B = S^1$.)

Theorem 9.4 (Factorization to an ε -fibration). *Let $p: M \rightarrow B$ be a map of topological spaces, such that B is a finite simplicial complex and the homotopy fiber of p is finitely dominated. Fix a metric on $B \times S^1$.*

Then for each $\varepsilon > 0$ there is a homotopy commutative diagram

$$\begin{array}{ccc} M \times S^1 & \xrightarrow[\simeq]{f} & C \\ & \searrow p_1 := p \times \text{id}_{S^1} & \swarrow q \\ & B \times S^1 & \end{array}$$

where C is a compact ENR, f is a homotopy equivalence, and q is an ε -fibration.

Proof. We can assume that p is a fibration. Given any $\varepsilon > 0$, we will construct a space C which is a compact ENR, a continuous map $q: C \rightarrow B \times S^1$, and an ε -homotopy equivalence $\varphi: C \rightarrow M \times S^1$. (This is good enough by Lemma 4.8.) The map φ will be a composite of several homotopy equivalences which we are going to define now.

Let Z be a finite CW -complex of Euler characteristic 0 with a chosen basepoint. Consider the inclusion and projection maps

$$M \xrightarrow{i} M \times Z \xrightarrow{r} M.$$

Since the composite $r \circ i$ is the identity map, its mapping torus is given by

$$T(r \circ i) = M \times S^1.$$

Denote by $T(r, i)$ the space $M \times [0, \frac{1}{2}] \amalg M \times Z \times [\frac{1}{2}, 1] / \sim$, where we glue $M \times \frac{1}{2}$ to $M \times Z \times \frac{1}{2}$ using the map i and we glue $M \times Z \times 1$ to $M \times 0$ via r . There are well-known homotopy equivalences

$$\varphi_1: T(r \circ i) \rightarrow T(r, i), \quad \varphi_2: T(i \circ r) \rightarrow T(r, i).$$

For instance, φ_1 is the identity on $M \times [0, \frac{1}{2}]$ and $i \times \text{id}$ on $M \times [\frac{1}{2}, 1]$. Its homotopy inverse ψ_1 collapses $M \times Z \times [\frac{1}{2}, 1]$ to M and expands $M \times [0, \frac{1}{2}]$ linearly to $M \times [0, 1]$. As both i and r commute with the projection to B , both homotopies $\psi_1 \circ \varphi_2 \simeq \text{id}$ and $\varphi_1 \circ \psi_2 \simeq \text{id}$ are stationary over B . As $T(r, i) = T(i, r)$, the same is true for the homotopy equivalence φ_2 .

Now, as Z has Euler characteristic zero and the fibers of p are assumed to be finitely dominated, the fibers of the composite fibration $p': M \times Z \rightarrow M \xrightarrow{p} B$ have finiteness obstruction zero and hence are homotopy finite. Choose a triangulation \mathcal{T} of B . By the proof of Theorem 4.1 there exists a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & M \times Z \\ & \searrow \varepsilon & \swarrow p' \\ & & B \end{array}$$

where X is a compact ENR and g is a homotopy equivalence over each simplex. Moreover it follows from the construction of X that for all simplices of B , the inclusions $X|_{\partial\sigma} \rightarrow X|_{\sigma}$ is a cofibration. As p' is a fibration, the inclusion $M|_{\partial\sigma} \rightarrow M|_{\sigma}$ is a cofibration for each simplex σ , too. We may therefore choose a self-map $k: X \rightarrow X$ such that $i \circ r \circ g \simeq g \circ k$ by a homotopy which restricts to a homotopy over each simplex of B . It follows that there is a map $\varphi_3: T(k) \rightarrow T(i \circ r)$ which is a homotopy equivalence over each simplex of B . (Here $T(k)$ is controlled over B via $p' \circ \varphi_3$.)

Hence we obtain a chain of maps

$$E \times S^1 \cong T(r \circ i) \xrightarrow{\varphi_1} T(r, i) \xleftarrow{\varphi_2} T(i \circ r) \xleftarrow{\varphi_3} T(k) =: C$$

where each map is a homotopy equivalence over each simplex of B . Note that C is a compact ENR.

Each of the spaces $T(r \circ i)$, $T(r, i)$, $T(i \circ r)$, and $T(k)$ naturally maps to S^1 and hence also to $B \times S^1$. Now we pull back along self-coverings $\text{id}_B \times n: B \times S^1 \rightarrow B \times S^1$ of index n . Geometrically this corresponds to replacing the mapping tori (such as $T(k)$) by n -fold multiple mapping tori (such as $T(k, \dots, k)$). The pull-backs of the corresponding homotopy equivalences $\varphi_1, \varphi_2, \varphi_3$ are by construction pieced together from homotopy equivalences over each simplex of B and over a single cylinder piece within the multiple mapping torus. Thus, by choosing n and the mesh of the triangulation of B small enough, we can get each of the homotopy equivalences as controlled over $B \times S^1$ as we wish.

This concludes the proof of Theorem 9.4. \square

9.3. Proof of the Stabilization Theorem 1.5 for finitely dominated homotopy fiber. The proof of Theorem 1.5 for the special case of a finite homotopy fiber Section 5 carries over to the finitely dominated case. We just need to replace the use of Theorem 4.1 by Theorem 9.4 in the proof of Lemma 5.7; it follows that Lemma 5.7 and hence Corollary 5.8 are still true under the weaker assumption that the homotopy fiber is finitely dominated, provided $s \geq 1$.

9.4. Tight torsion for finitely dominated homotopy fiber. Next we define the tight torsion of $p_1: M \times S^1 \rightarrow B \times S^1$, provided that the homotopy fiber of p is finitely dominated. Fix a metric on $B \times S^1$. Choose $\epsilon > 0$ such that Lemma 4.2 (for $B \times S^1$ instead of B) applies to it. We define the tight torsion $N\tau(p_1)$ of

$p_1 = p \times \text{id}_{S^1}$ as the class of the Whitehead torsion of f in $\text{NWh}(p_1)$, where f is the map appearing in the factorization $p_1 = q \circ f$ for a ε -fibration q in Theorem 9.4. This is independent of the factorization $p_1 = q \circ f$ by Lemma 4.2. One easily checks that it is also independent of the choice of the metric on $B \times S^1$ since $B \times S^1$ is compact. This definition reduces to the Definition 4.3 of tight torsion $N\tau(p_1)$ in the special case that the homotopy fiber of p is finite since an approximate fibration over $B \times S^1$ is an ε -fibration for any $\varepsilon > 0$ and any metric on B .

9.5. The approximate fiber problem for finitely dominated homotopy fiber. The following theorem extends Theorem 0.9 to the case of a finitely dominated fiber.

Theorem 9.5 (A criterion in the case of a finitely dominated fiber). *Let B be an aspherical closed triangulable manifold. Suppose that $\pi_1(B)$ satisfies FJC and the cyclic subgroups of $\pi_1(B)$ are orientable. Let M be a closed connected manifold of dimension $\neq 4$.*

Then a π_1 -surjective map $p: M \rightarrow B$ is homotopic to a MAF if and only if the following conditions are satisfied:

- (i) *The homotopy fiber of p is finitely dominated;*
- (ii) *If (i) is satisfied, the element $N\tau(p_1) \in \text{NWh}(p_1)$ is defined and we require it to vanish;*
- (iii) *If (i) and (ii) are satisfied, the element $N\tau(p) \in \text{NWh}(p)$ is defined and we require it to vanish.*

Proof. Suppose first that p approximately fibers. Then the homotopy fiber of p is finitely dominated by Lemma 9.1. Note that in this case $N\tau(p)$ is defined and both $N\tau(p)$ and $N\tau(p_1)$ clearly vanish.

Suppose conversely that conditions (i) to (iii) hold. By Theorem 1.5 (see section 9.3) and Theorem 1.6 (i), the map p is h -cobordant to an approximate fibration. Hence there is a homotopy equivalence $f: M \rightarrow M'$ an approximate fibration $q: M' \rightarrow B$ for a closed manifold M' such that $q \circ f \simeq p$. Hence $N\tau(p)$ is defined, namely by the Whitehead torsion of f considered in $\text{NWh}(p)$. (This is well-defined by Lemma 4.2). Condition (iii) together with Theorem 1.6 (ii) implies that p is homotopic to a MAF. This finishes the proof of Theorem 9.5. \square

Remark 9.6. Let M be a closed manifold and $f: M \rightarrow B$ be a manifold approximate fibration whose homotopy fiber is not homotopy equivalent to a finite CW complex. Then f cannot have an (even topological) regular value.

In fact, suppose that $b \in B$ was a regular value, so that $f^{-1}(U) \cong f^{-1}(b) \times U \simeq f^{-1}(b)$ for a small contractible neighborhood U of b in B . Then $f^{-1}(U)$ is an ANR, being an open subset of an ANR, and $f^{-1}(b)$ is a retract of $f^{-1}(U)$, hence also an ANR. By West [87], $f^{-1}(b)$ is homotopy equivalent to a finite CW complex. But f is an approximate fibration, so the homotopy fiber of f is homotopy equivalent to $f^{-1}(U)$ and thus to a finite CW-complex, contradicting the assumption.

9.6. Approximate fibrations with non-finite homotopy fiber. Next we use our results obtained so far to give an example of an approximate fibration $g: M \rightarrow T^2$ whose total space M is a closed smooth manifold, such that the homotopy fiber of g is *not* homotopy equivalent to a finite CW complex. Chapman-Ferry [17] have produced similar examples over arbitrary base manifolds of Euler characteristic zero.

Let p be an odd prime such that $h_1(p)$ has an odd prime factor. (Here $h_1(p)$ denotes the first factor of the class number of the ring $\mathbb{Z}[e^{2\pi i/p}]$. For instance, the primes $p = 23, 31, 37, 41, 43,$ and 47 will do [11, Table 8]. Any irregular prime p satisfies this condition on $h_1(p)$.) Then Wall has shown [81, Corollary 5.4.2]

that there exists a Poincaré 4-complex Y with $\pi_1 Y = \mathbb{Z}/p$ whose Wall finiteness obstruction $\sigma(Y) \in \widetilde{K}_0(\mathbb{Z}[\mathbb{Z}/p])$ is non-zero and in fact not of order two.

We claim that $Y \times S^1 \times S^2$ is homotopically equivalent to a closed smooth 7-manifold M' . In fact, the only obstruction to lifting the Spivak fibration of Y to a stable vector bundle is in

$$H^3(Y; \mathbb{Z}/2) = H_1(Y; \mathbb{Z}/2) = \mathbb{Z}/p \otimes \mathbb{Z}/2 = 0.$$

The Wall obstruction is $\sigma(Y \times S^1) = \chi(S^1)\sigma(Y) = 0$. Therefore there exists a closed smooth 5-manifold N and a normal map $N \rightarrow Y \times S^1$. Crossing f with $\text{id}: (D^3, S^2) \rightarrow (D^3, S^2)$ and using the π - π -theorem yields that

$$f \times \text{id}: N \times (D^3, S^2) \rightarrow Y \times S^1 \times (D^3, S^2)$$

can be surgered to a homotopy equivalence. Restricting this to the boundary yields a closed smooth 7-manifold M' homotopy equivalent to $Y \times S^1 \times S^2$.

Notice that M' comes with a canonical map $g': M' \rightarrow S^1$, corresponding to $\pi_1(M') = \mathbb{Z}/p \times \mathbb{Z} \rightarrow \mathbb{Z}$. Its homotopy fiber $F_{g'}$ is dominated by a finite complex but *not* homotopy equivalent to a finite complex, as can be seen by the Wall obstruction:

$$\sigma(F_{g'}) = \sigma(Y \times S^2) = \chi(S^2) \cdot \sigma(Y) = 2 \cdot \sigma(Y) \neq 0.$$

Following the proof of Theorem 9.5, we see that the map

$$g' \times \text{id}_{S^1}: M' \times S^1 \rightarrow T^2$$

is h -cobordant to an approximate fibration $g: M \rightarrow T^2$. Of course the homotopy fiber F_g of g is homotopy equivalent to $F_{g'}$.

10. THREE-DIMENSIONAL MANIFOLDS

Theorem 10.1 (3-manifolds as source). *Let M be a closed 3-manifold, B be an aspherical closed manifold, and $p: M \rightarrow B$. Then the following assertions are equivalent:*

- (i) *The kernel of the $p_*: \pi_1(M) \rightarrow \pi_1(B)$ is finitely generated and its image has finite index;*
- (ii) *The homotopy fiber of p is finitely dominated;*
- (iii) *The homotopy fiber of p is homotopy finite;*
- (iv) *p is homotopic to MAF;*
- (v) *p is homotopic to the projection of a locally trivial fiber bundle of closed manifolds.*

Proof. Obviously it suffices to show that the first condition implies the last one. Following Remark 1.8, it is enough to consider the case where p is π_1 -surjective.

We begin with the case, where M is homotopy equivalent to $\mathbb{R}\mathbb{P}^2 \times S^1$. Then $\pi_1(B)$ is a quotient of $\mathbb{Z}/2 \times \mathbb{Z}$ and hence $\pi_1(B) \cong \mathbb{Z}$ and $B = S^1$. Since Thurston's Geometrization Conjecture is true by the work of Perelman (see [59]), M is actually homeomorphic to $\mathbb{R}\mathbb{P}^2 \times S^1$ by [71, page 457]. Since any map $\mathbb{R}\mathbb{P}^2 \rightarrow S^1$ factors up to homotopy through the projection $\text{pr}: \mathbb{R}\mathbb{P}^2 \times S^1 \rightarrow S^1$ and p is π_1 -surjective, we conclude that p is homotopic to either pr or $c \circ \text{pr}$ where $c: S^1 \rightarrow S^1$ is the reflection. Both maps are fiber bundle projections. Hence we can assume in the sequel that M is not homotopy equivalent to $\mathbb{R}\mathbb{P}^2 \times S^1$.

Next we treat the case, where $H = \ker p_*$ is not isomorphic to \mathbb{Z} . By [34, Theorem 11.1 (2) on page 100] the group $\pi_1(B)$ is virtually cyclic. Since B is by assumption a closed aspherical manifold, $\pi_1(B) \cong \mathbb{Z}$ and $B = S^1$. We conclude from [34, Theorem 11.6 (i) on page 104] and Perelman's proof of the Poincaré Conjecture (see [48] or [58]) that there exists a fiber bundle $F \xrightarrow{i} M \xrightarrow{q} S^1$ with a connected 2-manifold as fiber such that the image of $\pi_1(i)$ is H . Since q or

$c \circ q$ induce the same map on the fundamental group as p , the map p is homotopy equivalent to the one of the bundle projections q or $c \circ q$.

It remains to treat the case, where $H \cong \mathbb{Z}$. If $\pi_1(B) \cong S^1$, then $\pi_1(M)$ is \mathbb{Z}^2 or $\mathbb{Z} \times \mathbb{Z}$ which is not possible for a closed 3-manifold M since otherwise by [34, Lemma 3.13 on page 28 and Theorem 7.1 on page 66] the manifold M is irreducible and hence by [34, Theorem 4.3] aspherical and therefore the cohomological dimension of $\pi_1(M)$ is precisely 3. Hence B is an aspherical closed 2-manifold. We conclude from [34, Theorem 11.10 on page 112] that p is homotopy equivalent to a bundle projection since any automorphism of $\pi_1(B)$ is induced by a homeomorphism. \square

Remark 10.2. The equivalence between (iv) and (v) in this theorem holds even if B is not aspherical: For B of dimension 1 or 2, this follows from [45, Theorem A] together with Perelman's solution of the Poincaré conjecture. For B of dimension 3 this is due to [1] plus the Poincaré conjecture. We are grateful to the referee for pointing this out to us.

11. A COUNTEREXAMPLE TO THE NAIVE CONJECTURE 0.4

Let B denote the Klein bottle and K its fundamental group.

Theorem 11.1 (Counterexample to the Naive Conjecture 0.4). *For $n \geq 6$ there exists a continuous map $p: M \rightarrow B \times S^1$, where M is a closed smooth $(n+1)$ -manifold whose homotopy fiber F_p is a finite complex and whose tight torsion $N\tau(p) = 0$, but where p is not homotopic to an approximate fibration.*

Addendum 11.2. In fact M is of the form $\mathcal{M} \times S^1$ for a smooth n -manifold \mathcal{M} , and p factors as the composite $(q \circ \phi) \times \text{id}_{S^1}$ where $q: B \times \mathcal{N} \rightarrow B$ is the projection onto the first factor, \mathcal{N} is a closed smooth manifold, and $\phi: \mathcal{M} \rightarrow B \times \mathcal{N}$ is a homotopy equivalence.

Here is the strategy of proof of Theorem 11.1. We will construct the homotopy equivalence $\phi: \mathcal{M} \rightarrow B \times \mathcal{N}$ such that $\tau(\phi)$ is not in the image of the assembly map

$$(11.3) \quad \text{asmb} = H_1^\pi(\text{pr}, \mathbf{Wh}(q)): H_1^\pi(\tilde{B}; \mathbf{Wh}(q)) \rightarrow \text{Wh}(\pi_1(B \times \mathcal{N}))$$

and hence the tight torsion

$$y = N\tau(q \circ \phi) \neq 0 \in \text{NWh}(\pi_1(B \times \mathcal{N})).$$

As $\bar{y} = (-1)^{n+1}y$, where $n = \dim \mathcal{M}$, y defines a Tate cohomology class

$$[y] \in \hat{H}^{n+1}(\mathbb{Z}/2; \text{NWh}(q))$$

which, in our construction, will be non-zero. We then set $M = \mathcal{M} \times S^1$, $N = \mathcal{N} \times S^1$, and $p = (q \circ \phi) \times \text{id}_{S^1}$. Note that $N\tau(p) = 0$ since it can be calculated using $f = \varphi \times \text{id}_{S^1}$ in (0.3) and since

$$\tau(\varphi \times \text{id}_{S^1}) = \tau(\varphi) \cdot \chi(S^1) = 0$$

by [49].

Provided we can implement our construction, our proof is by contradiction, i.e., we assume that $p: \mathcal{M} \times S^1 \rightarrow B \times S^1$ is homotopic to an approximate fibration \hat{p} . Then by Theorem 1.6 (i), we obtain an h -cobordism W between $q \circ \phi: \mathcal{M} \rightarrow B$ and an approximate fibration. Here $y = x - (-1)^n \bar{x}$ where x is the image of $\tau(W, \mathcal{M})$ in $\text{NWh}(q)$ and therefore

$$[y] = 0 \in \hat{H}^{n+1}(\mathbb{Z}/2; \text{NWh}(q)).$$

This contradiction proves Theorem 11.1, provided we can implement our construction.

The following lemma is a consequence of work of Bass and Murthy [10].

Lemma 11.4. *The group $S = \mathbb{Z}/4$ has the properties that*

$$NK_0(\mathbb{Z}[S]) \cong \bigoplus_{i=1}^{\infty} \mathbb{Z}/2$$

and

$$\hat{}: NK_0(\mathbb{Z}[S]) \rightarrow NK_0(\mathbb{Z}[S])$$

is the identity where $\hat{}$ is the involution induced by the involution of the polynomial ring $\mathbb{Z}[S][x]$ determined by $x \mapsto x$ and $s \mapsto s^{-1}$ for $s \in S$.

Proof. There is a Cartesian square

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}/4] & \longrightarrow & \mathbb{Z}[i] \\ \downarrow & & \downarrow \\ \mathbb{Z}[\mathbb{Z}/2] & \longrightarrow & R := \mathbb{Z}_2[x]/(x^2) \end{array}$$

Applying the functor NK_* to it gives a Mayer-Vietoris sequence

$$NK_1(\mathbb{Z}[\mathbb{Z}/2]) \oplus NK_1(\mathbb{Z}[i]) \rightarrow NK_1(R) \rightarrow NK_0(\mathbb{Z}[\mathbb{Z}/4]) \rightarrow NK_0(\mathbb{Z}[\mathbb{Z}/2]) \oplus NK_0(\mathbb{Z}[i])$$

Now it is known that $NK_i(\mathbb{Z}[\mathbb{Z}/2]) = 0$ for $i = 0, 1$ and $NK_i(\mathbb{Z}[i]) = 0$ for $i = 0, 1$ since the Gaussian integers are a regular ring.

Hence the map

$$NK_1(R) \rightarrow NK_0(\mathbb{Z}[\mathbb{Z}/4])$$

is an isomorphism, which respects the involution since the maps in the Cartesian square do. On the other hand the only involution on R is id. And since R is quasi-regular, i.e., $R/(x)$ is regular and x is nilpotent, $NK_1(R)$ is easy to compute; namely it is $(R[t])^\times / \{1, 1+x\}$. Thinking of these as $(1, 1)$ -matrices, we see that the involution on $NK_1(R)$ is id. \square

Choose an oriented closed $(n-2)$ -manifold \mathcal{N} so that $\pi_1(\mathcal{N}) = S = \mathbb{Z}/4$. To construct $\phi: \mathcal{M} \rightarrow B \times \mathcal{N}$ we examine the Rothenberg exact sequence and in particular the map

$$L_{n+1}^h(\mathbb{Z}[S \times K]) \rightarrow \widehat{H}^{n+1}(\mathbb{Z}/2; \text{Wh}(S \times K)).$$

Remark 11.5. All the L -groups occurring in the following discussion are with respect to $w: S \times K \rightarrow \mathbb{Z}/2$ given by the composition of projection $S \times K \rightarrow K$ with the first Stiefel-Whitney class of the Klein bottle.

The quotient map

$$\text{Wh}(S \times K) \rightarrow \text{NWh}(q)$$

preserves the involutions and hence induces a homomorphism

$$\widehat{H}^{n+1}(\mathbb{Z}/2; \text{Wh}(S \times K)) \rightarrow \widehat{H}^{n+1}(\mathbb{Z}/2; \text{NWh}(q)).$$

Denote its composition with the Rothenberg homomorphism by

$$\eta: L_{n+1}^h(\mathbb{Z}[S \times K]) \rightarrow \widehat{H}^{n+1}(\mathbb{Z}/2; \text{NWh}(q)).$$

If η is not identically zero, just select x such that $\eta(x) \neq 0$ and it will determine $\phi: \mathcal{M} \rightarrow B \times \mathcal{N}$ such that $[N\tau(\bar{q} \circ \phi)] \neq 0$ showing that our construction can be implemented.

We proceed to show that such an element x exists. Let \mathcal{C} be the class of all finitely generated abelian groups. The claim clearly follows from the two assertions:

Assertion (i): η is a mod- \mathcal{C} monomorphism, i.e., $\ker \eta$ is in \mathcal{C} ;

Assertion (ii): $L_{n+1}^h(\mathbb{Z}[S \times K])$ is not in \mathcal{C} .

To prove the assertions, we use a variant of the Rothenberg exact sequence where we replace L^s by L^X and $X \subset \text{Wh}(S \times K)$ is the image of the homomorphism

$$\text{Wh}(S \times \mathbb{Z}) \rightarrow \text{Wh}(S \times K)$$

induced by the inclusion $S \times \mathbb{Z} \rightarrow (S \times \mathbb{Z}) \rtimes \mathbb{Z} = S \times K$ (see [80, page 4]):

$$(11.6) \quad \cdots \rightarrow L_{n+1}^X(\mathbb{Z}[S \times K]) \rightarrow L_{n+1}^h(\mathbb{Z}[S \times K]) \xrightarrow{\psi} \widehat{H}^{n+1}(\mathbb{Z}/2; \text{Wh}(S \times K)/X) \\ \rightarrow L_n^X(\mathbb{Z}[S \times K]) \rightarrow \cdots$$

We start by showing

Assertion (iii): ψ is a mod- \mathcal{C} isomorphism, i.e., both the kernel and cokernel of ψ are in \mathcal{C} .

For this it suffices, because of (11.6), to show that $L_i^X(\mathbb{Z}[S \times K])$ is finitely generated for both $i = n$ and $i = n + 1$. And these groups can be analyzed by using a variant of the Wall-Shaneson exact sequence; in particular by using Ranicki [69, Theorem 5.2] where $A = \mathbb{Z}[S \times \mathbb{Z}]$, $\alpha: A \xrightarrow{\cong} A$ is the ring automorphism induced by the group automorphism by $\text{id}_S \times -\text{id}_{\mathbb{Z}}: S \times \mathbb{Z} \xrightarrow{\cong} S \times \mathbb{Z}$, and $R = K_1(\mathbb{Z}[S \times \mathbb{Z}])$. Then $\widetilde{V}_n^{\varepsilon R}(A_\alpha) = L_n^X(\mathbb{Z}[S \times K])$ and it suffices to show that both $V_n^R(A) = L_n^h(\mathbb{Z}[S \times \mathbb{Z}])$ and $V_{n-1}^{(1-\alpha)^{-1}R}(A) = L_{n-1}^h(\mathbb{Z}[S \times \mathbb{Z}])$ are finitely generated. Now Wall [80] showed that both $L_i^h(\mathbb{Z}[F])$ and $L_i^s(\mathbb{Z}[F])$ are finitely generated for all i and every finite group F . Hence the Wall-Shaneson Theorem shows that $L_i^s(\mathbb{Z}[S \times \mathbb{Z}])$ is in \mathcal{C} for all i . And using the Rothenberg exact sequence together with Bass's result that $\text{Wh}(F)$ is in \mathcal{C} for every finite group F , we see that $L_i^h(\mathbb{Z}[S \times \mathbb{Z}])$ is in \mathcal{C} , completing the verification of Assertion (iii).

We use this result to show Assertion (i). Consider the following commutative diagram of $\mathbb{Z}/2$ -modules:

$$\begin{array}{ccc} \text{Wh}(S \times K) & \xrightarrow{\quad\quad\quad} & \text{Wh}(S \times K)/X \\ \downarrow & & \downarrow \gamma \\ \text{Wh}(S \times K)/\text{im}(\text{asmb}) & \xrightarrow{\quad\beta\quad} & \text{Wh}(S \times K)/(X + \text{im}(\text{asmb})) \end{array}$$

where asmb is the assembly map of (11.3). Because of it, it clearly suffices to show that γ induces a mod- \mathcal{C} isomorphism

$$\gamma_*: \widehat{H}^{n+1}(\mathbb{Z}/2; \text{Wh}(S \times K)/X) \rightarrow \widehat{H}^{n+1}(\mathbb{Z}/2; \text{Wh}(S \times K)/(X + \text{im}(\text{asmb}))).$$

Using the exact sequence in Tate cohomology induced from a short exact sequence of $\mathbb{Z}/2$ -modules, we see that this is true provided we can verify the following Assertion:

Assertion (iv): For all $n \in \mathbb{Z}$, $\widehat{H}^{n+1}(\mathbb{Z}/2; \text{im}(\text{asmb})/(X \cap \text{im}(\text{asmb})))$ is in \mathcal{C} .

In fact, the domain of asmb is finitely generated because of the Atiyah-Hirzebruch spectral sequence and the fact that $K_i(\mathbb{Z}[F])$ is finitely generated when F is a finite group and $i \leq 1$. Therefore $\text{im}(\text{asmb})/(X \cap \text{im}(\text{asmb}))$ and any of its sub-quotients are also finitely generated, which establishes Assertion (iv).

So Assertion (i) is proven; let us now show Assertion (ii), which, by Assertion (iii), is equivalent to showing that $\widehat{H}^{n+1}(\mathbb{Z}/2; \text{Wh}(S \times K)/X)$ is not in \mathcal{C} .

By the main result of [27], $\text{Wh}(S \times K)/X$ is the direct sum of two $\mathbb{Z}/2$ -modules W and Z where $\widehat{H}^{n+1}(\mathbb{Z}/2; Z) = 0$ and W is $\mathbb{Z}/2$ -isomorphic to $NK_0(\mathbb{Z}[S])$ with involution $\widehat{} = \text{id}$ because of Lemma 11.4. Therefore

$$\widehat{H}^{n+1}(\mathbb{Z}/2; \text{Wh}(S \times K)/X) = NK_0(\mathbb{Z}[S])/2 \cdot NK_0(\mathbb{Z}[S]).$$

By Lemma 11.4, $NK_0(S)/2NK_0(S)$ is not in \mathcal{C} .

This concludes the proof of Theorem 11.1.

12. AN EXAMPLE ON BLOCK FIBERING

The following example shows that the vanishing of our obstructions does, in general, *not* imply that a map is homotopic to a block bundle.

Theorem 12.1 (MAF versus block bundle). *Let $m \geq 6$ be even. There exist a pair of smooth closed (connected) manifolds M and N of dimension $m + 1$, a simple homotopy equivalence $f: M \rightarrow N$, and a smooth fibre bundle map $p: N \rightarrow S^1 \times S^1$ such that the composite map $p \circ f: M \rightarrow S^1 \times S^1$ is not homotopic to a block bundle projection but is homotopic to an approximate fibration.*

Remark 12.2. In fact $N = L \times (S^1 \times S^1)$ where L is closed connected smooth manifold, and p is the projection onto the second factor.

We will need the following result to prove Theorem 12.1.

Lemma 12.3. *There exist a pair of closed connected smooth manifolds L and M with $m = \dim(M)$ being any given even integer ≥ 6 , and a homotopy equivalence $\varphi: M \rightarrow L \times S^1$ such that $q(\tau(\varphi))$ cannot be expressed as $x + \bar{x}$ for some element $x \in \tilde{K}_0(\mathbb{Z}[\pi_1(L)])$. Here $q: \text{Wh}(\pi_1(L \times S^1)) \rightarrow \tilde{K}_0(\mathbb{Z}[\pi_1(L)])$ denotes the projection map in the Bass-Heller-Swan formula.*

Proof. We begin by recalling some needed calculations. These calculations can be found in [2] and [57, page 30]. Let p be an odd prime.

- (i) $L_i^s(\mathbb{Z}[\mathbb{Z}/p]) = L_i^h(\mathbb{Z}[\mathbb{Z}/p]) = 0$ if i is odd;
- (ii) $\hat{H}^i(\mathbb{Z}/2; \text{Wh}(\mathbb{Z}/p)) = 0$ if i is odd;
- (iii) $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29]) = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

The following result is a consequence of (iii) and the fact that $\mathbb{Z}/2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/2$ equipped with some $\mathbb{Z}/2[\mathbb{Z}/2]$ -module structure has either the trivial $\mathbb{Z}/2$ -action or is $\mathbb{Z}/2[\mathbb{Z}/2]$ -isomorphic the direct

sum of $\mathbb{Z}/2[\mathbb{Z}/2]$ and the trivial $\mathbb{Z}/2[\mathbb{Z}/2]$ -module $\mathbb{Z}/2$:

- (iv) $\hat{H}^i(\mathbb{Z}/2; \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29])) \neq 0$ for all i .

Next consider the Rothenberg exact sequence for $\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/29]$:

$$(12.4) \quad \cdots \rightarrow L_{m+1}^h(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/29]) \xrightarrow{\beta} \hat{H}^{m+1}(\mathbb{Z}/2; \text{Wh}(\mathbb{Z} \times \mathbb{Z}/29)) \\ \rightarrow L_m^s(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/29]) \xrightarrow{\alpha_m} L_m^h(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/29]) \cdots \rightarrow .$$

Let us examine the map α_m of (12.4) in terms of the Wall-Shaneson formula. Since

$$L_m^s(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/29]) = L_m^s(\mathbb{Z}[\mathbb{Z}/29]) \oplus L_{m-1}^h(\mathbb{Z}[\mathbb{Z}/29]) = L_m^s(\mathbb{Z}[\mathbb{Z}/29])$$

because of calculation (i) and our assumption that m is even, α_m factors as the composition of the Rothenberg map

$$\alpha: L_m^s(\mathbb{Z}[\mathbb{Z}/29]) \rightarrow L_m^h(\mathbb{Z}[\mathbb{Z}/29])$$

and the functorial inclusion

$$L_m^h(\mathbb{Z}[\mathbb{Z}/29]) \rightarrow L_m^h(\mathbb{Z}[\mathbb{Z} \times \mathbb{Z}/29]).$$

But α is monic by the Rothenberg exact sequence for $\mathbb{Z}[\mathbb{Z}/29]$ and calculation (ii); namely, that $\hat{H}^{m+1}(\mathbb{Z}/2; \text{Wh}(\mathbb{Z}/29)) = 0$.

Consequently, α_m is monic and hence the map β in (12.4) is an epimorphism.

Let L be any closed smooth (connected) $(m - 1)$ -dimensional manifold with $\pi_1(L) = \mathbb{Z}/29$ and fix an element $y \in \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29])$ satisfying $y = \bar{y}$ but not of the form $x + \bar{x}$ where $x \in \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29])$. This is possible because of statement (iv). Let

$$\sigma: \tilde{K}_0(\mathbb{Z}[\mathbb{Z}/29]) \rightarrow \text{Wh}(\mathbb{Z} \times \mathbb{Z}/29)$$

be the usual splitting of q and let $z = \sigma(y)$. Then

$$\bar{z} = -z$$

since $\overline{\sigma(y)} = -\sigma(\bar{y})$. Since β is onto, there exists a closed smooth manifold \mathcal{M} together with a homotopy equivalence

$$\varphi: \mathcal{M} \rightarrow L \times S^1$$

such that $\tau(\varphi) = z$. But

$$q(\tau(\varphi)) = q(\sigma(y)) = y$$

so we have completed the construction. This finishes the proof of Lemma 12.3. \square

Proof of Theorem 12.1. Let $N = L \times S^1 \times S^1$, $M = \mathcal{M} \times S^1$, $f = \varphi \times \text{id}$, and p be the projection described in Remark 12.2. Since

$$\tau(f) = \chi(S^1) \tau(\varphi) = 0$$

by the Kwun-Szczarba formula [49]. Hence $p \circ f: \mathcal{M} \times S^1 \rightarrow S^1 \times S^1$ is homotopic to an approximate fibration by Theorem 0.9, since the homotopy fiber of $f \circ p$ is L and $\pi_1(S^1 \times S^1)$ satisfies FJC by Theorem 2.3.

Hence it remains to show that $p \circ f$ is *not* homotopic to a block bundle projection. We do this by assuming the opposite and showing this assumption leads to a contradiction. Our assumption clearly forces M to contain a closed connected (locally flat) codimension one submanifold \mathcal{M}_0 where \mathcal{M}_0 also contains a closed connected (locally flat) codimension one submanifold \mathcal{L}_0 with the following properties:

- (i) \mathcal{M}_0 lifts to the covering space $\mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M} \times S^1$ and the inclusion $\mathcal{M}_0 \subset \mathcal{M} \times \mathbb{R}$ is a homotopy equivalence;
- (ii) The composite map

$$\mathcal{L}_0 \subset \mathcal{M}_0 \subset \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M} \xrightarrow{\varphi} L \times S^1$$

lifts to the covering space $L \times \mathbb{R} \rightarrow L \times S^1$ and this lift induces a homotopy equivalence.

Remark 12.5. We can assume that \mathcal{M}_0 and \mathcal{L}_0 are smooth submanifolds satisfying properties (i) and (ii) by applying Hirsch's codimension one smoothing theorem in [35].

The following diagram (12.6) is useful in “visualizing” properties (i) and (ii): the dotted arrows in this diagram are the lifts posited in (i) and (ii). Note that all the horizontal arrows are homotopy equivalences.

$$(12.6) \quad \begin{array}{ccccccc} & & \mathcal{M} \times S^1 & & & & \\ & & \uparrow & \swarrow & & & \\ & & \mathcal{M}_0 & \cdots \cdots \cdots & \mathcal{M} \times \mathbb{R} & \longrightarrow & \mathcal{M} \xrightarrow{\varphi} L \times S^1 \\ & & \uparrow & & & & \uparrow \\ & & \mathcal{L}_0 & \cdots \cdots \cdots & L \times \mathbb{R} & \longrightarrow & L \end{array}$$

Denote the composite of the horizontal arrows in line 2 of (12.6) by

$$\psi: \mathcal{M}_0 \rightarrow L \times S^1$$

and identify L with the codimension one submanifold $L \times 1$ of $L \times S^1$. It is easily seen using property (ii) that ψ is homotopic to a smooth homotopy equivalence η which is split along L ; i.e.

- (i) η is transverse to L , and
- (ii) $\eta|_{\eta^{-1}(L)}: \eta^{-1}(L) \rightarrow L$ is a homotopy equivalence.

It was shown in [28] (see also [26]) that η split forces its Whitehead torsion $\tau(\eta)$ to lie in the image of $\text{Wh}(\pi_1(L))$ in $\text{Wh}(\pi_1(L \times S^1))$; consequently $q(\tau(\eta)) = 0$. And therefore

$$(12.7) \quad q(\tau(\psi)) = 0$$

since $\tau(\psi) = \tau(\eta)$.

Now observe that, for r a sufficiently large real number, $\mathcal{M} \times r$ is disjoint from \mathcal{M}_0 and the region W between them is a compact smooth h -cobordism with $\partial_- W = \mathcal{M}_0$ and $\partial_+ W = \mathcal{M} \times r$. Let y denote $\tau(W, \partial_- W)$ and let $g: \partial_- W \rightarrow \partial_+ W$ be the composite of the inclusion $\partial_- W \subset W$ with a retraction $W \rightarrow \partial_+ W$, then

$$\tau(g) = y - \bar{y}$$

because m is assumed to be even. After identifying $\mathcal{M} \times r$ with \mathcal{M} in the natural (diffeomorphic) way, the composite $\varphi \circ g$ is homotopic to ψ since g is easily seen to be homotopic to the composition

$$\mathcal{M}_0 \subset \mathcal{M} \times \mathbb{R} \rightarrow \mathcal{M}$$

of the first two arrows on line 2 of 12.6. Therefore

$$\tau(\psi) = \tau(\varphi) + \varphi_*(y - \bar{y});$$

applying q to this equation yields

$$q(\tau(\psi)) = q(\tau(\varphi)) - (x + \bar{x})$$

where $x = q(\varphi_*(\bar{y}))$. (Note that $q(\bar{z}) = -\overline{q(z)}$.)

Substituting identity (12.7) into this equation yields

$$q(\tau(\varphi)) = x + \bar{x}$$

which is the contradiction proving the Theorem 12.1. \square

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