

SURVEY ON ANALYTIC AND TOPOLOGICAL TORSION

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ABSTRACT. The article consists of a survey on analytic and topological torsion. Analytic torsion is defined in terms of the spectrum of the analytic Laplace operator on a Riemannian manifold, whereas topological torsion is defined in terms of a triangulation. The celebrated theorem of Cheeger and Müller identifies these two notions for closed Riemannian manifolds. We also deal with manifolds with boundary and with isometric actions of finite groups. The basic theme is to extract topological invariants from the spectrum of the analytic Laplace operator on a Riemannian manifold.

0. INTRODUCTION

When I was asked to write a contribution to a book in honor of Bernhard Riemann, I was on one side flattered, but on the other side also scared. Although Riemann has done so much foundational and seminal work in many areas, there was no obvious topic, where I may have something to say and on which Riemann has worked. Moreover, I am obviously not an expert on the history of mathematics.

After some thought I decided to choose as topic analytic and topological torsion. This is an interesting example for an interaction between analysis and topology and this seems to be a theme, in which Riemann was interested. The goal is to extract topological invariants from the spectrum of the analytic Laplace operator on a Riemannian manifold.

Finally I had to decide on the structure of the paper and for whom it should be written. A technical paper on latest results did not seem to be appropriate. So I decided to tell the story how one can come from elementary considerations about linear algebra of finite-dimensional Hilbert spaces and elementary invariants such as dimension, trace, and determinant to topological notions, which are in general easy, and then to their analytic counterparts, which are in general much more difficult. Hopefully the first sections are comprehensible even for graduate students and present some important tools and notions, which can be transferred to the analytic setting with some effort. Moreover, this transition explains the basic ideas underlying the analytic notions. For an advanced mathematician, who is not an expert on analytic or topological torsion, it may be interesting to see how this interaction between analysis and topology is developed and what its impact is. We tried to keep the exposition as simple as possible to ensure that the paper is accessible. This also means that for an expert on analytic and topological torsion this article will contain no new information.

Here is a brief summary of the contents of this paper.

In the first section we recall in the framework of linear maps between finite-dimensional Hilbert spaces basic notions such as the trace, the determinant and the spectrum. We will rewrite the classical notion of a determinant in terms of the Zeta-function and the spectral density function. The point will be that in this

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new form they can be extended to the analytic setting, where one has to deal with infinite-dimensional Hilbert spaces. This is not possible if one sticks to the classical definitions.

In the second section we consider finite Hilbert chain complexes, which are chain complexes of finite-dimensional Hilbert spaces for which only finitely many chain modules are not zero. For those we can define Betti numbers and torsion invariants and give an elementary “baby” version of the Hodge de Rham decomposition.

In the third section we pass to analysis. Our first interaction between analysis and topology will be presented by the de Rham Theorem. Then we will explain the Hodge-de Rham Theorem which relates the singular cohomology of a closed Riemannian manifold to the space of harmonic forms.

In the fourth section topological torsion is defined by considering cellular chain complexes of finite CW -complexes or closed Riemannian manifolds. It can be written in terms of the combinatorial Laplace operator in an elementary fashion except that one has to correct the Hilbert space structure on the homology using the isomorphisms of the third section.

The fifth section is devoted to analytic torsion. Its definition is rather complicated, but it should become clear what the idea behind the definition is, in view of the definition of the topological torsion. We will explain that topological and analytical torsion agree for closed Riemannian manifolds. If the compact Riemannian manifold has boundary, then a correction term based on the Euler characteristic of the boundary is needed.

In the sixth section the results of the fifth section are extended to compact Riemannian manifolds with an isometric action of a finite group. Here a new phenomenon occurs, namely a third torsion invariant, the Poincaré torsion, comes into play.

In the seventh section we give a very brief overview over the literature about analytic and topological torsion and its generalization to the L^2 -setting.

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1. OPERATORS OF FINITE-DIMENSIONAL HILBERT SPACES

In this section we review some well-known concepts about a linear map between finite-dimensional (real) Hilbert spaces such as its determinant, its trace, and its spectrum. All of the material presented in this section is accessible to a student in his second year. Often key ideas can easily be seen and illustrated in this elementary context. Moreover, we will sometimes rewrite a well-known notion in a fashion which will later allow us to extend it to more general situations.

1.1. Linear maps between finite-dimensional vector spaces. Let $f: V \rightarrow W$ be a linear map of finite-dimensional (real) vector spaces. Recall that every finite-dimensional vector space V carries a unique topology which is characterized by the property that any linear isomorphism $f: \mathbb{R}^n \xrightarrow{\cong} V$ is a homeomorphism. This definition makes sense since any linear automorphism of \mathbb{R}^n is a homeomorphism. In particular any linear map $f: V \rightarrow W$ of finite-dimensional vector spaces is an *operator*, i.e., a continuous linear map.

We can assign to an endomorphisms $f: V \rightarrow V$ of a finite-dimensional vector space V two basic invariants, its trace and its determinant, as follows. If we write $V^* = \text{hom}_{\mathbb{R}}(V, \mathbb{R})$, then there are canonical linear maps

$$\begin{aligned} \alpha: V^* \otimes V &\rightarrow \text{hom}_{\mathbb{R}}(V, V), & \phi \otimes v &\mapsto (w \mapsto \phi(w) \cdot v); \\ \beta: V^* \otimes V &\xrightarrow{\cong} \text{hom}_{\mathbb{R}}(V, V), & \phi \otimes v &\mapsto \phi(v). \end{aligned}$$

The first one is an isomorphism. Hence we can define the trace map to be the composite

$$\text{tr}: \text{hom}_{\mathbb{R}}(V, V) \xrightarrow{\alpha^{-1}} V^* \otimes V \xrightarrow{\beta} \mathbb{R},$$

and the *trace of f*

$$(1) \quad \text{tr}(f) \in \mathbb{R}$$

to be the image of f under this linear map. The trace has the basic properties that $\text{tr}(g \circ f) = \text{tr}(f \circ g)$ holds for linear maps $f: U \rightarrow V$ and $g: V \rightarrow W$, it is linear, i.e., $\text{tr}(r \cdot f + s \cdot g) = r \cdot \text{tr}(f) + s \cdot \text{tr}(g)$, and $\text{tr}(\text{id}_{\mathbb{R}}: \mathbb{R} \rightarrow \mathbb{R}) = 1$. We leave it to the reader to check that these three properties determine the trace uniquely.

If n is the dimension of V , the vector space $\text{Alt}^n(V)$ of alternating n -forms $V \times V \times \dots \times V \rightarrow \mathbb{R}$ has dimension one. An endomorphism $f: V \rightarrow V$ induces an endomorphism $\text{Alt}^n(f): \text{Alt}^n(V) \rightarrow \text{Alt}^n(V)$. Hence there is precisely one real

number r such that $\text{Alt}^n(f) = r \cdot \text{id}_{\text{Alt}^n(V)}$ and we define the *determinant* of f to be r , or, equivalently, by the equation

$$(2) \quad \det(f) \cdot \text{id}_{\text{Alt}^n(f)} = \text{Alt}^n(f).$$

The determinant has the properties that $\det(g \circ f) = \det(g) \cdot \det(f)$ holds for endomorphisms $f, g: V \rightarrow V$, for any commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{i} & V & \xrightarrow{p} & W & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & U & \xrightarrow{i} & V & \xrightarrow{p} & W & \longrightarrow & 0 \end{array}$$

we get $\det(g) = \det(f) \cdot \det(h)$ and $\det(\text{id}_{\mathbb{R}}) = 1$. We leave it to the reader to check that these three properties determine the determinant uniquely.

Notice that all of our definitions are intrinsic, we do not use bases. Of course if we choose a basis $\{b_1, b_2, \dots, b_n\}$ for V and let A be the (n, n) -matrix describing f with respect to this basis, then we get back the standard definitions in terms of matrices

$$\begin{aligned} \text{tr}(f) &= \sum_{i=1}^n a_{i,i}; \\ \det(f) &= \prod_{\sigma \in S_n} \text{sign}(\sigma) \cdot \prod_{i=1}^n a_{i,\sigma(i)}. \end{aligned}$$

1.2. Linear maps between finite-dimensional Hilbert spaces. Now we consider finite-dimensional Hilbert spaces, i.e., finite-dimensional vector spaces with an inner product. Notice that we do not have to require that V is complete with respect to the induced norm, this is automatically fulfilled. Let $f: U \rightarrow V$ be a linear map. Its *adjoint* is the linear map $f^*: V \rightarrow W$ uniquely determined by the property that $\langle f(v), w \rangle_W = \langle v, f^*(w) \rangle_V$ holds for all $v \in V$ and $w \in W$. If we choose orthonormal basis for V and W and let $A(f)$ and $A(f^*)$ be the matrices describing f and f^* , then $A(f)^*$ is the transpose of $A(f)$. We call an endomorphism $f: V \rightarrow V$ *selfadjoint* if and only if $f^* = f$. This is equivalent to the condition that $A(f)$ is symmetric. We call an endomorphism $f: V \rightarrow V$ *positive* if $\langle f(v), v \rangle \geq 0$ holds for all $v \in V$. This is equivalent to the existence of a linear map $g: V \rightarrow V$ with $f = g^*g$. In particular every positive linear endomorphism is selfadjoint.

The following version of a determinant will be of importance for us. Let $f: V \rightarrow W$ be a linear map of finite-dimensional Hilbert spaces, where V and W may be different. Then $f^*f: V \rightarrow V$ induces an automorphism $(f^*f)^\perp: \ker(f^*f)^\perp \xrightarrow{\cong} \ker(f^*f)^\perp$, where $\ker(f^*f)^\perp$ is the orthogonal complement of $\ker(f^*f)$ in U . Define

$$(3) \quad \det^\perp(f) := \begin{cases} \sqrt{\det((f^*f)^\perp: \ker(f^*f)^\perp \xrightarrow{\cong} \ker(f^*f)^\perp)} & \text{if } f \neq 0 \\ 1 & \text{if } f = 0. \end{cases}$$

The proof of the following elementary lemma is left to the reader, or consult [65, Theorem 3.14 on page 128 and Lemma 3.15 on page 129].

Lemma 1.1.

- (1) If $f: V \rightarrow V$ is a linear automorphism of a finite-dimensional Hilbert space, then $\det^\perp(f) = |\det(f)|$ for $\det(f)$ the classical determinant;
- (2) Let $f: U \rightarrow V$ and $g: V \rightarrow W$ be linear maps of finite-dimensional Hilbert spaces such that f is surjective and g is injective. Then

$$\det^\perp(g \circ f) = \det^\perp(f) \cdot \det^\perp(g);$$

- (3) Let $f_1: U_1 \rightarrow V_1$, $f_2: U_2 \rightarrow V_2$ and $f_3: U_2 \rightarrow V_1$ be linear maps of finite-dimensional Hilbert spaces such that f_1 is surjective and f_2 is injective. Then

$$\det^\perp \begin{pmatrix} f_1 & f_3 \\ 0 & f_2 \end{pmatrix} = \det^\perp(f_1) \cdot \det^\perp(f_2);$$

- (4) Let $f_1: U_1 \rightarrow V_1$ and $f_2: U_2 \rightarrow V_2$ be linear maps of finite-dimensional Hilbert spaces. Then

$$\det^\perp(f_1 \oplus f_2) = \det^\perp(f_1) \cdot \det^\perp(f_2);$$

- (5) Let $f: U \rightarrow V$ be a linear map of finite-dimensional Hilbert spaces. Then

$$\det^\perp(f) = \det^\perp(f^*) = \sqrt{\det^\perp(f^*f)} = \sqrt{\det^\perp(ff^*)}.$$

1.3. The spectrum and the spectral density function. If one is only interested in finite-dimensional Hilbert spaces and in Betti numbers or torsion invariants for finite CW -complexes, one does not need the following material of the remainder of this Section 1. However, we will now lay the foundations to extend these invariants to the analytic setting or to the L^2 -setting, where the Hilbert spaces are not finite-dimensional anymore.

The spectrum $\text{spec}(f)$ of a selfadjoint operator $f: V \rightarrow V$ of finite-dimensional Hilbert spaces consists of the set of eigenvalues λ of f , i.e., real numbers λ for which there exists $v \in V$ with $v \neq 0$ and $f(v) = \lambda \cdot v$. The multiplicity $\mu(f)(\lambda)$ of an eigenvalue λ is the dimension of its eigenspace

$$E_\lambda(f) := \{v \in V \mid f(v) = \lambda \cdot v\}.$$

If $\lambda \in \mathbb{R}$ is not an eigenvalue, we put $\mu(f)(\lambda) = 0$. An elementary but basic result in linear algebra says that for a selfadjoint linear map $f: V \rightarrow V$ there exists an orthonormal basis of eigenvectors of V . A selfadjoint linear endomorphism is positive if $\lambda \geq 0$ holds for each eigenvalue λ .

Next we introduce for a linear map $f: U \rightarrow V$ of finite-dimensional Hilbert spaces its *spectral density function*

$$(4) \quad F(f): [0, \infty) \rightarrow [0, \infty).$$

It is defined as the following right continuous step function. Its value at zero is the dimension of the kernel of f^*f . Notice that $\ker(f^*f) = \ker(f)$ since $v \in \ker(f^*f)$ implies $0 = \langle f^*f(v), v \rangle = \langle f(v), f(v) \rangle$ and hence $f(v) = 0$. The jumps of the step function happen exactly at the square roots of the eigenvalues of f^*f and the height of the jump is the multiplicity $\mu(f^*f)(\lambda)$ of the eigenvalue. There is a number $C \geq 0$ such that $F(f)(\lambda) = \dim(V)$ holds for all $\lambda \geq C$, for instance, take C to be the square root of the largest eigenvalue of f^*f . Obviously f is injective if and only if $F(f)(0) = 0$.

Suppose that f is already a positive operator $f: V \rightarrow V$. Then f^*f is f^2 . Moreover, $F(f)$ has the dimension of $\ker(f)$ as value at zero and the step function jumps exactly at those $\lambda \in \mathbb{R}$ which are eigenvalues of f and the height of the jump is $\mu(f)(\lambda)$.

One can also define the spectral density function of a linear map $f: V \rightarrow W$ of finite-dimensional Hilbert spaces without referring to eigenvalues in a more intrinsic way as follows. Let $\mathcal{L}(f, \lambda)$ be the set of linear subspaces $L \subseteq V$ such that $\|f(v)\| \leq \lambda \cdot \|v\|$ holds for every $v \in L$. Then we from [65, Lemma 2.3 on page 74]

$$(5) \quad F(f)(\lambda) = \sup\{\dim(L) \mid L \in \mathcal{L}(f, \lambda)\}.$$

1.4. Rewriting determinants. The following formula will be of central interest for us. Let $f: V \rightarrow W$ be a linear map of finite-dimensional Hilbert spaces. Notice that $\det^\perp(f) > 0$ so that we can consider the real number $\ln(\det^\perp(f))$. The formula

$$(6) \quad \ln(\det^\perp(f)) = \frac{1}{2} \cdot \sum_{\substack{\lambda \in \text{spec}(f^*f), \\ \lambda > 0}} \mu(f^*f)(\lambda) \cdot \ln(\lambda)$$

is a direct consequence of the fact that we have orthogonal decompositions

$$\begin{aligned} V &= \bigoplus_{\lambda \in \text{spec}(f^*f)} E_\lambda(f^*f); \\ \ker(f^*f)^\perp &= \bigoplus_{\substack{\lambda \in \text{spec}(f^*f), \\ \lambda > 0}} E_\lambda(f^*f). \end{aligned}$$

We can use this orthogonal decomposition to define a new linear automorphism of V by

$$(7) \quad \ln((f^*f)^\perp) := \bigoplus_{\substack{\lambda \in \text{spec}(f^*f), \\ \lambda > 0}} \ln(\lambda) \cdot \text{id}_{E_\lambda(f^*f)}: \ker(f^*f)^\perp \rightarrow \ker(f^*f)^\perp.$$

Then we can rephrase (6) as

$$(8) \quad \ln(\det^\perp(f)) = \frac{1}{2} \cdot \text{tr}(\ln((f^*f)^\perp)).$$

The following observation will be the key to define determinants also for operators between not necessarily finite-dimensional Hilbert spaces, for instance for the analytic Laplace operator acting on smooth p -forms for a closed Riemannian manifold. Namely, we define a holomorphic function $\zeta_f: \mathbb{C} \rightarrow \mathbb{C}$ by

$$(9) \quad \zeta_f(s) = \sum_{\substack{\lambda \in \text{spec}(f^*f), \\ \lambda > 0}} \mu(f^*f)(\lambda) \cdot \lambda^{-s},$$

Then one easily checks using (6)

$$\begin{aligned} (10) \quad -\ln(\det^\perp(f)) &= -\frac{1}{2} \cdot \sum_{\substack{\lambda \in \text{spec}(f^*f), \\ \lambda > 0}} \mu(f^*f)(\lambda) \cdot \ln(\lambda) \\ &= \frac{1}{2} \cdot \sum_{\substack{\lambda \in \text{spec}(f^*f), \\ \lambda > 0}} \mu(f^*f)(\lambda) \cdot \left. \frac{d}{ds} \right|_{s=0} \lambda^{-s} \\ &= \frac{1}{2} \cdot \left. \frac{d}{ds} \right|_{s=0} \zeta_f. \end{aligned}$$

In order to extend the notion of $\det^\perp(f)$ in the L^2 -setting to the Fuglede-Kadison determinant, it is useful to rewrite the quantity $\ln(\det^\perp(f))$ in terms of an integral with respect to measure coming from the spectral density function as follows.

Recall that $F(f)$ is a monotone non-decreasing right-continuous function. Denote by $dF(f)$ the measure on the Borel σ -algebra on \mathbb{R} which is uniquely determined by its values on the half open intervals $(a, b]$ for $a < b$ by $dF(f)((a, b]) = F(f)(b) - F(f)(a)$. The measure of the one point set $\{a\}$ is $\lim_{x \rightarrow 0^+} F(f)(a) - F(f)(a - x)$ and is zero if and only if $F(f)$ is left-continuous in a . We will use here and in the sequel the convention that \int_a^b , $\int_{a^+}^b$, \int_a^∞ and $\int_{a^+}^\infty$ respectively means

integration over the interval $[a, b]$, $(a, b]$, $[a, \infty)$ and (a, ∞) respectively. An easy computation using (6) shows

$$(11) \quad \ln(\det^\perp(f^*f)) = \int_{0+}^{\infty} \ln(\lambda) dF(f).$$

Elementary integration theory shows that we get for $d\lambda$ the standard Lebesgue measure and any $a \geq \dim(U)$

$$(12) \quad \int_{0+}^{\infty} \ln(\lambda) dF = \ln(a) \cdot (F(a) - F(0)) - \int_{0+}^a \frac{F(f)(\lambda) - F(f)(0)}{\lambda} d\lambda.$$

2. FINITE HILBERT CHAIN COMPLEXES

Having in mind the cellular chain complex of a finite CW -complex, we want to consider now finite Hilbert chain complexes. A *finite Hilbert chain complex* $C_* = (C_*, c_*)$ consists of a collection of finite-dimensional Hilbert spaces C_n and linear maps $c_n: C_n \rightarrow C_{n-1}$ for $n \in \mathbb{Z}$ such that $c_n \circ c_{n+1} = 0$ holds for all $n \in \mathbb{Z}$ and there exists a natural number N with $c_n = 0$ for $|n| > N$. A *chain map of finite Hilbert chain complexes* $f_*: C_* \rightarrow D_*$ is a collection of linear maps $f_n: C_n \rightarrow D_n$ for $n \in \mathbb{Z}$ such that $d_n \circ f_n = f_{n-1} \circ c_n$ holds for all $n \in \mathbb{Z}$. (We do not require that the maps f_n are compatible with the Hilbert space structures.) It is obvious what a chain homotopy and a chain homotopy equivalence of finite Hilbert chain complexes means. The homology $H_n(C_*)$ is the Hilbert space $\ker(c_n)/\text{im}(c_{n+1})$, where $\ker(c_n)$ is equipped with the Hilbert space structure coming from C_n and $\ker(c_n)/\text{im}(c_{n+1})$ inherits the quotient Hilbert space structure. Define the *n-th Laplace operator*

$$(13) \quad \Delta_n = c_n^* \circ c_n + c_{n+1} \circ c_{n+1}^*: C_n \rightarrow C_n.$$

The importance of the following notions cannot be underestimated.

Definition 2.1 (Betti numbers and torsion of a finite Hilbert chain complex). Let C_* be a finite Hilbert chain complex.

Define its *n-th Betti number*

$$b_n(C_*) := \dim(H_n(C_*)) \quad \in \mathbb{Z}_{\geq 0}.$$

Define its *torsion*

$$\rho(C_*) := - \sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^\perp(c_n)) \quad \in \mathbb{R},$$

where \det^\perp has been introduced in (3).

2.1. Betti numbers. Next we relate these notions to the Laplace operator. The following result is a “baby”-version of the Hodge-de Rham Theorem, see Subsection 3.2. In the sequel we equip $\ker(\Delta_n) \subseteq C_n$ with the Hilbert space structure induced from the given one on C_n .

Lemma 2.2. *Let C_* be a finite Hilbert chain complex. Then we get for all $n \in \mathbb{Z}$*

$$\ker(\Delta_n) = \ker(c_n) \cap \text{im}(c_{n+1})^\perp,$$

and an orthogonal decomposition

$$C_n = \text{im}(c_n^*) \oplus \text{im}(c_{n+1}) \oplus \ker(\Delta_n).$$

In particular the obvious composite

$$\ker(\Delta_n) \rightarrow \ker(c_n) \rightarrow H_n(C_n)$$

is an isometric isomorphism of Hilbert spaces and we get

$$b_n(C_*) = \dim(\ker(\Delta_n)).$$

Proof. Consider $v \in V$. We compute

$$\begin{aligned} \langle c_n(v), c_n(v) \rangle + \langle c_{n+1}^*(v), c_{n+1}^*(v) \rangle &= \langle c_n^* \circ c_n(v), v \rangle + \langle c_{n+1} \circ c_{n+1}^*(v), v \rangle \\ &= \langle c_n^* \circ c_n(v) + c_{n+1} \circ c_{n+1}^*(v), v \rangle \\ &= \langle \Delta_n(v), v \rangle. \end{aligned}$$

Hence we get for $v \in V$ that $\Delta_n(v) = 0$ is equivalent to $c_n(v) = c_{n+1}^*(v) = 0$. This shows $\ker(\Delta_n) = \ker(c_n) \cap \ker(c_{n+1}^*) = \ker(c_n) \cap \text{im}(c_{n+1})^\perp$. The other claims are now direct consequences. \square

Remark 2.3 (Homotopy invariance of $\dim(\ker(\Delta_n))$). Notice the following fundamental consequence of Lemma 2.2 that $\dim(\ker(\Delta_n))$ depends only on the chain homotopy type of C_* and is in particular independent of the Hilbert space structure on C_* since for a chain homotopy equivalence $f_*: C_* \rightarrow D_*$ we obtain an isomorphism $H_n(f_*): H_n(C_*) \rightarrow H_n(D_*)$ and hence the equality $b_n(C_*) = b_n(D_*)$.

Of course the spectrum of the Laplace operator Δ_n does depend on the Hilbert space structure, but a part of it, namely, the multiplicity of the eigenvalue 0, which is just $\dim(\ker(\Delta_n))$, depends only on the homotopy type of C_* .

Remark 2.4 (Heat operator). One can assign to the Laplace operator $\Delta_n: C_n \rightarrow C_n$ its *heat operator* $e^{-t\Delta_n}$. It is defined analogously to $\ln(f^*f)$, see (7), but now each eigenvalue λ of Δ_n transforms to the eigenvalue $e^{-t\lambda}$.

Then we obviously get

$$b_n(C_*) = \lim_{t \rightarrow \infty} \text{tr}(e^{-t\Delta_n}).$$

2.2. Torsion for finite Hilbert chain complexes. The situation with torsion is more complicated, but in some sense similar, as we explain next. First of all one can rewrite torsion in terms of the Laplace operator.

Lemma 2.5. *If C_* is a finite Hilbert chain complex, then we get*

$$\rho(C_*) = -\frac{1}{2} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \ln(\det^\perp(\Delta_n)).$$

Proof. From Lemma 2.2 we obtain an orthogonal decomposition

$$\begin{aligned} C_n &= \ker(c_n)^\perp \oplus \text{im}(c_{n+1}) \oplus \ker(\Delta_n); \\ \Delta_n &= ((c_n^\perp)^* \circ c_n^\perp) \oplus (c_{n+1}^\perp \circ (c_{n+1}^\perp)^*) \oplus 0, \end{aligned}$$

where $c_n^\perp: \ker(c_n)^\perp \rightarrow \text{im}(c_n)$ is the weak isomorphism induced by c_n . Now we compute using Lemma 1.1

$$\begin{aligned} &-\frac{1}{2} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \ln(\det^\perp(\Delta_n)) \\ &= -\frac{1}{2} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \ln(\det^\perp(((c_n^\perp)^* \circ c_n^\perp) \oplus (c_{n+1}^\perp \circ (c_{n+1}^\perp)^*) \oplus 0)) \\ &= -\frac{1}{2} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \left(\ln(\det^\perp((c_n^\perp)^* \circ c_n^\perp)) \right. \\ &\quad \left. + \ln(\det^\perp(c_{n+1}^\perp \circ (c_{n+1}^\perp)^*)) + \ln(\det^\perp(0)) \right) \\ &= -\frac{1}{2} \cdot \sum_{n \in \mathbb{Z}} (-1)^n \cdot n \cdot \left(2 \cdot \ln(\det^\perp(c_n)) + 2 \cdot \ln(\det^\perp(c_{n+1})) \right) \\ &= -\sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^\perp(c_n)). \end{aligned}$$

\square

A basic property of the torsion is additivity, whose proof can be found in [65, Theorem 3.35 (1) on page 142].

Lemma 2.6. *Consider the short exact sequence of finite Hilbert chain complexes $0 \rightarrow C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \rightarrow 0$. For each $n \in \mathbb{Z}$ we obtain a finite Hilbert chain complex $E[n]_*$ concentrated in dimension 0, 1, and 2 which is given there by $C_n \xrightarrow{i_n} D_n \xrightarrow{p_n} E_n$. The long exact homology sequence associated to $0 \rightarrow C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \rightarrow 0$ can be viewed as finite Hilbert chain complex denoted by LHS_* .*

Then we get

$$\rho(C_*) - \rho(D_*) + \rho(E_*) = \left(\sum_{n \in \mathbb{Z}} (-1)^n \cdot \rho(E[n]_*) \right) - \rho(LHS_*).$$

Let $f: C_* \rightarrow D_*$ be a chain map of finite Hilbert chain complexes. Let $\text{cone}(f_*)$ be its mapping cone whose n -th differential is given by

$$\begin{pmatrix} -c_{n-1} & 0 \\ f_{n-1} & d_n \end{pmatrix} : C_{n-1} \oplus D_n \rightarrow C_{n-1} \oplus D_{n-1}.$$

Define the torsion of f_* by

$$(14) \quad \tau(f_*) := \rho(\text{cone}(f_*)).$$

Lemma 2.7. *Let $f: C_* \rightarrow D_*$ be a chain homotopy equivalence of finite Hilbert chain complexes. Then we get*

$$\tau(f_*) = \rho(D_*) - \rho(C_*) + \sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^\perp(H_n(f_*))).$$

Proof. This follows from Lemma 2.6 applied to the canonical short exact sequence $0 \rightarrow D_* \rightarrow \text{cone}(f_*) \rightarrow \Sigma C_* \rightarrow 0$ using the fact that $H_n(\text{cone}(f_*)) = 0$ holds for $n \in \mathbb{Z}$. \square

Lemma 2.8. *Let $f_*: C_* \rightarrow D_*$ be a chain map of finite contractible Hilbert chain complexes such that f_n is bijective for each $n \in \mathbb{Z}$. Then*

$$\rho(D_*) - \rho(C_*) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^\perp(f_n)).$$

Proof. Because of Lemma 2.7 it suffices to show

$$\rho(\text{cone}(f_*)) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^\perp(f_n)).$$

This is done by induction over the length of C_* which is the supremum $\{m - n \mid C_m \neq 0, C_n \neq 0\}$. The induction step, when the length is less or equal to one, follows directly from the definitions. The induction step is done as follows. Let m be the largest integer with $C_m \neq 0$. Let $C_*|_{m-1}$ obtained from C_* by putting $C_m = 0$ and leaving the rest. Obviously $f_*: C_* \rightarrow D_*$ induces a chain isomorphism $f_*|_{m-1}: C_*|_{m-1} \rightarrow D_*|_{m-1}$. Let $m[C_*]$ be the chain complex concentrated in dimension m whose m -th chain module is C_m . Obviously f_* induces a chain isomorphism $m[f_*]: m[C_*] \rightarrow m[D_*]$. We have the obvious short exact sequence of finite contractible Hilbert chain complexes $0 \rightarrow \text{cone}(m[f_*]) \rightarrow \text{cone}(f_*) \rightarrow \text{cone}(f_*|_{m-1}) \rightarrow 0$. Lemma 2.6 implies

$$\rho(\text{cone}(f_*)) = \rho(\text{cone}(m[f_*])) + \rho(\text{cone}(f_*|_{m-1})).$$

The induction hypothesis applies to $m[C_*]$ and $C_*|_{m-1}$ and thus we have

$$\begin{aligned} \rho(\text{cone}(m[f_*])) &= (-1)^m \cdot \ln(\det^\perp(f_m)); \\ \rho(\text{cone}(f_*|_{m-1})) &= \sum_{n \in \mathbb{Z}, n \neq m} (-1)^n \cdot \ln(\det^\perp(f_n)). \end{aligned}$$

This finishes the proof Lemma 2.8. \square

2.3. Torsion for finite based free \mathbb{Z} -chain complexes. Let C_* be a finite free \mathbb{Z} -chain complex, i.e., a \mathbb{Z} -chain complex whose chain modules are all finitely generated free abelian groups and for which there exists a natural number N such that $c_n = 0$ for $|n| > N$. Given a finitely generated free \mathbb{Z} -module M , we call two \mathbb{Z} -bases $B = \{b_1, b_2, \dots, b_n\}$ and $B' = \{b'_1, b'_2, \dots, b'_n\}$ equivalent if there exists a permutation $\sigma \in S_n$ and elements $\epsilon_i \in \{\pm 1\}$ for $i = 1, 2, \dots, n$ such that $b'_{\sigma(i)} = \epsilon_i \cdot b_i$ holds for $i = 1, 2, \dots, n$. A \mathbb{Z} -basis $B = \{b_1, b_2, \dots, b_n\}$ on M determines on $\mathbb{R} \otimes_{\mathbb{Z}} M$ a Hilbert space structure by requiring that $\{1 \otimes b_1, 1 \otimes b_2, \dots, 1 \otimes b_n\}$ is an orthonormal basis. Obviously this Hilbert space structure depends only on the equivalence class $[B]$ of B .

We call a \mathbb{Z} -chain complex C_* *finite based free* if it is finite free and each C_n comes with an equivalence class $[B_n]$ of \mathbb{Z} -bases. Then $\mathbb{R} \otimes_{\mathbb{Z}} C_*$ inherits the structure of a finite Hilbert chain complex.

Lemma 2.9. *Let C_* be a finite based free contractible \mathbb{Z} -chain complex. Then*

$$\rho(\mathbb{R} \otimes_{\mathbb{Z}} C_*) = 0.$$

Proof. We use induction over the length of C_* which is the supremum $\{m - n \mid C_m \neq 0, C_n \neq 0\}$. The induction step, when the length is smaller than zero, is trivial since then C_* is trivial. The induction step is done as follows. Let n be the smallest integer with $C_n \neq 0$. Then $c_{n+1}: C_{n+1} \rightarrow C_n$ is surjective. We can choose a map of \mathbb{Z} -modules $s_n: C_n \rightarrow C_{n+1}$ with $c_{n+1} \circ s_n = \text{id}_{C_n}$. Then the cokernel $\text{coker}(s_n)$ is a finitely generated free and we can equip it with some equivalence class of \mathbb{Z} -basis. Let $\text{pr}: C_{n+1} \rightarrow \text{coker}(s_n)$ be the projection. We obtain a short exact sequence of finite free \mathbb{Z} -chain complexes by the following diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C_n & \xrightarrow{\text{id}} & C_n \\ & & \downarrow & & \downarrow & & \downarrow s_n & & \downarrow \text{id} \\ \cdots & \xrightarrow{c_{n+4}} & C_{n+3} & \xrightarrow{c_{n+3}} & C_{n+2} & \xrightarrow{c_{n+2}} & C_{n+1} & \xrightarrow{c_{n+1}} & C_n \\ & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{pr} & & \downarrow \\ \cdots & \xrightarrow{c_{n+4}} & C_{n+3} & \xrightarrow{c_{n+3}} & C_{n+2} & \xrightarrow{\text{pr} \circ c_{n+2}} & \text{coker}(s_n) & \longrightarrow & 0 \end{array}$$

If we apply $\mathbb{R} \otimes_{\mathbb{Z}} -$ to the chain complex represented by the upper row, we obtain a finite Hilbert chain complex with trivial torsion. The same is true by the induction hypothesis for the lower row since its length is smaller than the length of C_* . Hence the claim follows from Lemma 2.6 if we can show the same for the 2-dimensional chain complex E_* given in dimensions 0, 1, 2 by $0 \rightarrow C_n \xrightarrow{c_n} C_{n+1} \xrightarrow{\text{pr}} \text{coker}(s_n) \rightarrow 0$. Let E'_* be the 2-dimensional chain complex E'_* given in dimensions 0, 1, 2 by $0 \rightarrow C_n \rightarrow C_n \oplus \text{coker}(s_n) \rightarrow \text{coker}(s_n) \rightarrow 0$ where the differentials are the obvious inclusion and projection and the \mathbb{Z} -bases in dimension 1 is the direct sum of the basis for C_n and $\text{coker}(s_n)$. Obviously we have $\rho(\mathbb{R} \otimes_{\mathbb{Z}} E'_*) = 0$. There is a \mathbb{Z} -chain isomorphism $f_*: E_* \rightarrow E'_*$ such that f_0 and f_2 are the identity. We conclude from Lemma 2.8 that $\rho(\mathbb{R} \otimes_{\mathbb{Z}} E_*) = -\ln(\det^{\perp}(\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f_1))$. Since f_1 is an isomorphism, $\det^{\perp}(\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f_1)$ is the absolute value of the classical determinant of $\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f_1$, which is the classical determinant of f_1 over \mathbb{Z} and hence ± 1 . This finishes the proof of Lemma 2.9. \square

The term $\sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^{\perp}(H_n(f_*)))$ appearing in Lemma 2.7 causes some problems concerning homotopy invariance as the following example shows:

Example 2.10 (Subdivision for $[0, 1]$). Consider $I = [0, 1]$. We specify a CW -structure on I by defining the set of 0-cells by $\{0, 1/n, 2/n, \dots, (n-1)/n, 1\}$ and the set of closed 1-cells by $\{[0, 1/n], [1/n, 2/n], \dots, [(n-1)/n, 1]\}$ for each integer $n \geq 1$. Denote the corresponding CW -complex by $I[n]$. The cellular \mathbb{Z} -chain complex $C_*(I[n])$ is 1-dimensional and its first differential $c[n]_1: \mathbb{Z}^n \rightarrow \mathbb{Z}^{n+1}$ is given by

$$c[n]_1((k_1, k_2, \dots, k_n)) = (-k_1, -k_2 + k_1, -k_3 + k_2, \dots, -k_n + k_{n-1}, k_n).$$

The kernel of $c[n]_1$ is trivial and its image is the kernel of the augmentation homomorphism $\epsilon[n]: \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}$, $(k_1, k_2, \dots, k_{n+1}) \mapsto \sum_{i=1}^{n+1} k_i$. In particular $H_1(C_*(I[n])) = 0$ and we get a \mathbb{Z} -isomorphism

$$\overline{\epsilon[n]}: H_0(C_*(I[n])) \xrightarrow{\cong} \mathbb{Z}$$

induced by $\epsilon[n]$. The Laplace operator $\Delta[n]_1: \mathbb{R}^n \rightarrow \mathbb{R}^n$ in degree 1 is given by the matrix

$$A[n] = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

By developing along the first row we get for its classical determinant for $n \geq 4$

$$\det(A[n]) = 2 \cdot \det(A[n-1]) - \det(A[n-2]).$$

A direct computation shows $\det(A[1]) = 2$, $\det(A[2]) = 3$ and $\det(A[3]) = 4$. This implies $\det(A[n]) = n + 1$ for all $n \geq 1$. Hence we get from Lemma 2.5

$$\begin{aligned} (15) \quad \rho(C_*(I[n])) &= -\frac{1}{2} \cdot (-1)^{-1} \cdot 1 \cdot \ln(\det^\perp(\Delta[n]_1)) \\ &= \frac{\ln(|\det(\Delta[n]_1)|)}{2} \\ &= \frac{\ln(n+1)}{2}. \end{aligned}$$

This shows that $\rho(C_*(I[n]))$ depends on the CW -structure.

We have the chain map $f: I[1] \rightarrow I[n]$ given by

$$\begin{aligned} f_1: \mathbb{Z} &\rightarrow \mathbb{Z}^n, \quad k \mapsto (k, k, \dots, k); \\ f_0: \mathbb{Z}^2 &\rightarrow \mathbb{Z}^{n+1} \quad (k_1, k_2) \mapsto (k_1, 0, 0, 0, \dots, k_2). \end{aligned}$$

It induces an isomorphism in homology since $\epsilon[n] \circ C_0(f_*) = \epsilon[1]$ holds. Hence it is a \mathbb{Z} -chain homotopy equivalence. We conclude from Lemma 2.9

$$(16) \quad \rho(\text{cone}(\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} C_*(f))) = 0.$$

The isomorphism $\overline{\epsilon[n]}: H_0(C_*(I[n])) \xrightarrow{\cong} \mathbb{Z}$ induces an explicite isomorphism

$$\alpha[n]: H_0(\mathbb{R} \otimes_{\mathbb{Z}} C_*(I[n])) \xrightarrow{\cong} \mathbb{R} \otimes_{\mathbb{Z}} H_0(C_*(I[n])) \xrightarrow{\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} \overline{\epsilon[n]}} \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z} \xrightarrow{\cong} \mathbb{R}.$$

Recall that $H_0(\mathbb{R} \otimes_{\mathbb{Z}} C_*(I[n]))$ inherits a Hilbert space structure. Then α becomes an isometric isomorphism of Hilbert spaces if we equip \mathbb{R} with the Hilbert space structure for which $1 \in \mathbb{R}$ has norm $(n+1)^{-1/2}$, since the element $(1, 1, \dots, 1) \in \mathbb{R} \otimes_{\mathbb{Z}} C_0(I[n]) = \mathbb{R}^{n+1}$ belongs to $\ker(\text{id} \otimes_{\mathbb{Z}} \epsilon[n])^\perp$, has norm $\sqrt{n+1}$ and its class

in $H_0(\mathbb{R} \otimes_{\mathbb{Z}} C_*(I[n]))$ is sent to $(n+1)$ under $\alpha[n]$. Since $\alpha[n] \circ H_0(f_*) = \alpha[1]$, we conclude

$$(17) \quad \ln((\det^\perp(H_0(f)))) = -\frac{\ln(n+1)}{2}.$$

Notice that (15), (16), and (17) are compatible with Lemma 2.7.

Of course it cannot be desirable that $\rho(C_*(I[n]))$ in the Example 2.10 depends on the CW -structure. This dependency is only due to the dependency of the Hilbert structure on the homology on the CW -structure. Therefore we can get rid of the dependency by fixing a Hilbert space structure on the homology and view this as an extra piece of data.

Definition 2.11 (Torsion for finite based free \mathbb{Z} -chain complex with a given Hilbert structure on homology). Let C_* be a finite based free \mathbb{Z} -chain complex. A *Hilbert space structure* κ on $H_*(\mathbb{R} \otimes_{\mathbb{Z}} C_*)$ is a choice of Hilbert space structure κ_n on each vector space $H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*)$. We define

$$\begin{aligned} \rho(C_*; \kappa) &:= \rho(\mathbb{R} \otimes_{\mathbb{Z}} C_*) + \\ &\quad \sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^\perp(\text{id}: H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*) \rightarrow (H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*), \kappa(C_*)_n))), \end{aligned}$$

where on the source of $\text{id}: H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*) \rightarrow (H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*), \kappa(C_*)_n)$ we use the Hilbert space structure induced by the one on $\mathbb{R} \otimes_{\mathbb{Z}} C_*$.

If we take κ to be the Hilbert space structure induced by the one on $\mathbb{R} \otimes_{\mathbb{Z}} C_*$, then obviously $\rho(C_*; \kappa)$ agrees with $\rho(\mathbb{R} \otimes_{\mathbb{Z}} C_*)$. The desired effect is the following version of homotopy invariance.

Lemma 2.12. *Let $f_*: C_* \rightarrow D_*$ be a \mathbb{Z} -chain homotopy equivalence of finite based free \mathbb{Z} -chain complexes. Let $\kappa(C_*)$ and $\kappa(D_*)$ be Hilbert space structures on $H_*(\mathbb{R} \otimes_{\mathbb{Z}} C_*)$ and $H_*(\mathbb{R} \otimes_{\mathbb{Z}} D_*)$. Then we get*

$$\begin{aligned} &\rho(D_*, \kappa(D_*)) - \rho(C_*, \kappa(C_*)) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^\perp(H_n(\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f_*): (H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*), \kappa(C_*)_n) \\ &\quad \rightarrow (H_n(\mathbb{R} \otimes_{\mathbb{Z}} D_*), \kappa(D_*)_n))). \end{aligned}$$

Proof. We get from Lemma 2.7 and Lemma 2.9

$$\begin{aligned} &\rho(\mathbb{R} \otimes_{\mathbb{Z}} D_*) - \rho(\mathbb{R} \otimes_{\mathbb{Z}} C_*) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(\det^\perp(H_n(\mathbb{R} \otimes_{\mathbb{Z}} f_*): H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*) \rightarrow H_n(\mathbb{R} \otimes_{\mathbb{Z}} D_*))). \end{aligned}$$

Hence it suffices to show for each $n \in \mathbb{Z}$

$$\begin{aligned} &\det^\perp(\text{id}: H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*) \rightarrow (H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*), \kappa(C_*)_n)) \\ &\quad \cdot \det^\perp(H_n(\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f_*): (H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*), \kappa(C_*)_n) \rightarrow (H_n(\mathbb{R} \otimes_{\mathbb{Z}} D_*), \kappa(D_*)_n)) \\ &= \det^\perp(H_n(\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f_*): H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*) \rightarrow H_n(\mathbb{R} \otimes_{\mathbb{Z}} D_*)) \\ &\quad \cdot \det^\perp(\text{id}: H_n(\mathbb{R} \otimes_{\mathbb{Z}} D_*) \rightarrow (H_n(\mathbb{R} \otimes_{\mathbb{Z}} D_*), \kappa(D_*)_n)). \end{aligned}$$

This follows from Lemma 1.1 (2). \square

Example 2.13 (Integral Hilbert structure). Let C_* be a finite based free \mathbb{Z} -chain complex. Choose for each integer n a \mathbb{Z} -basis B_n for $H_n(C_*)/\text{tors}(H_n(C_*))$. Then we get an induced Hilbert structure $\kappa[B_*]$ on $H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*)$ as follows. Obviously

B_n induces an \mathbb{R} -basis on $\mathbb{R} \otimes_{\mathbb{Z}} H_n(C_*) / \text{tors}(H_n(C_*))$. There is a canonical isomorphism

$$\mathbb{R} \otimes_{\mathbb{Z}} H_n(C_*) / \text{tors}(H_n(C_*)) \xrightarrow{\cong} H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*)$$

We equip the target with the Hilbert space structure $\kappa(B_n)$ for which it becomes an isometric isomorphism.

Now consider a chain homotopy equivalence $f_*: C_* \rightarrow D_*$ of finite based free chain complexes. Suppose that we have chosen \mathbb{Z} -basis B_n on $H_*(C_*)$ and B'_n on $H_n(D_*)$. Notice that $H_n(f)$ induces an isomorphism of \mathbb{Z} -modules

$$H_n(C_*) / \text{tors}(H_n(C_*)) \xrightarrow{\cong} H_n(D_*) / \text{tors}(H_n(D_*))$$

The determinant of it with respect to the given integral bases is ± 1 . One easily checks that this implies

$$\det^{\perp}(H_n(\text{id}_{\mathbb{R}} \otimes_{\mathbb{Z}} f_*): (H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*), \kappa(B_n)) \rightarrow (H_n(\mathbb{R} \otimes_{\mathbb{Z}} D_*), \kappa(B'_n))) = 1.$$

Lemma 2.12 implies

$$\rho(C_*; \kappa(B_*)) = \rho(D_*; \kappa(B'_*)).$$

Hence $\rho(C_*; \kappa(B_*))$ is independent of the choice of integral basis on C_* , D_* , $H_n(C_*)$, and $H_n(D_*)$ and is a homotopy invariant of the underlying finite free \mathbb{Z} -chain complexes C_* and D_* . This raises the question what it is?

We leave it to the reader to figure out

$$\rho(C_*; \kappa(B_*)) = \sum_{n \in \mathbb{Z}} (-1)^n \cdot \ln(|\text{tors}(H_n(C_*))|).$$

The proof is straightforward after one has shown using the fact \mathbb{Z} is a principal ideal domain that C_* is homotopy equivalent to a direct sum of \mathbb{Z} -chain complexes each of which is concentrated in two consecutive dimensions and given there by $m \cdot \mathbb{Z} \rightarrow \mathbb{Z}$ for some integer $m \in \mathbb{Z}$.

3. THE HODGE DE RHAM THEOREM

Next we want to give a first classical relation between topology and analysis, the de Rham Theorem and the Hodge-de Rham Theorem.

3.1. The de Rham Theorem. Let M be a (not necessarily compact) manifold (possibly with boundary).

The *de Rham complex* $(\Omega^*(M), d^*)$ is the real cochain complex whose n -th chain module is the real vector space of smooth n -forms on M and whose n -differential is the standard differential for n -forms. The *de Rham cohomology* of M is defined by

$$(18) \quad H_{\text{dR}}^n(M) := \ker(d^n) / \text{im}(d^{n-1}).$$

There is a \mathbb{R} -chain map, natural in M ,

$$A^*(M): \Omega^*(M) \rightarrow C_{\text{sing}, C^\infty}^*(M; \mathbb{R})$$

with the cochain complex of M based on smooth singular simplices with coefficients in \mathbb{R} as target. It sends an n -form $\omega \in \Omega^n(M)$ to the element $A^n(\omega) \in C_{\text{sing}, C^\infty}^n(M; \mathbb{R})$ which assigns to a smooth singular n -simplex $\sigma: \Delta_n \rightarrow M$ the real number $\int_{\Delta_n} \sigma^* \omega$. The Theorem of Stokes implies that this is a chain map. There is a forgetful chain map

$$C_{\text{sing}, C^\infty}^*(M; \mathbb{R}) \rightarrow C_{\text{sing}}^*(M; \mathbb{R})$$

to the standard singular \mathbb{R} -cochain complex, which is based on (continuous) singular simplices with coefficients in \mathbb{R} . Denote by $H_{\text{sing}, C^\infty}^*(M; \mathbb{R})$ the smooth singular cohomology of M with coefficients in \mathbb{R} which is by definition the cohomology of the \mathbb{R} -cochain complex $C_{\text{sing}, C^\infty}^*(M; \mathbb{R})$, and define analogously $H_{\text{sing}}^*(M; \mathbb{R})$. A proof of the next theorem, at least in the case $\partial M = \emptyset$, can be found for instance in [19,

Section V.9.], [41], [43, Theorem 1.5 on page 11 and Theorem 2.4 on page 20], [66, Section 15], [73, Theorem A.31 on page 413].

Theorem 3.1 (De Rham Theorem). *The chain map A^* induces for a smooth manifold M and $n \geq 0$ an isomorphism, natural in M ,*

$$H^n(A^*(M)): H_{\text{dR}}^n(M) \xrightarrow{\cong} H_{\text{sing}, C^\infty}^n(M; \mathbb{R}).$$

The forgetful chain map induces an isomorphism, natural in M ,

$$H_{\text{sing}, C^\infty}^n(M; \mathbb{R}) \xrightarrow{\cong} H_{\text{sing}}^n(M; \mathbb{R}).$$

They are compatible with the multiplicative structures given by the \wedge -product and the \cup -product.

3.2. The Hodge-de Rham Theorem. Now suppose that the smooth manifold M comes with a Riemannian metric and an orientation. Let d be the dimension of M . Denote by

$$(19) \quad *^n: \Omega^n(M) \rightarrow \Omega^{d-n}(M)$$

the *Hodge star-operator* which is defined by the corresponding notion for oriented finite-dimensional Hilbert spaces applied fiberwise. It is uniquely characterized by the property

$$(20) \quad \int_M \omega \wedge *^n \eta = \int_M \langle \omega_x, \eta_x \rangle_{\text{Alt}^n(T_x M)} d\text{vol},$$

where ω and η are n -forms, ω has compact support, and $\langle \omega_x, \eta_x \rangle_{\text{Alt}^n(T_x M)}$ is the inner product on $\text{Alt}^n(T_x M)$ which is induced by the inner product on $T_x M$ given by the Riemannian metric.

Define the *adjoint of the exterior differential*

$$(21) \quad \delta^n = (-1)^{dn+d+1} \cdot *^{d-n+1} \circ d^{d-n} \circ *^n: \Omega^n(M) \rightarrow \Omega^{n-1}(M).$$

Notice that in the definition of δ^n the Hodge star-operator appears twice and the definition is local. Hence we can define δ^n without using an orientation of M , only the Riemannian metric is needed. This is also true for the following definition.

Definition 3.2 (Laplace operator). Define the *n -th Laplace operator* on the Riemannian manifold M

$$\Delta_n = d^{n-1} \circ \delta^n + \delta^{n+1} \circ d^n: \Omega^n(M) \rightarrow \Omega^n(M).$$

Let $\Omega_c^n(M) \subset \Omega^n(M)$ be the *space of smooth p -forms with compact support*. There is the following inner product on it

$$(22) \quad \langle \omega, \eta \rangle_{L^2} := \int_M \omega \wedge *^n \eta = \int_M \langle \omega_x, \eta_x \rangle_{\text{Alt}^n(T_x M)} d\text{vol}.$$

Recall that a Riemannian manifold M is *complete* if each path component of M equipped with the metric induced by the Riemannian metric is a complete metric space. By the Hopf-Rinow Theorem the following statements are equivalent provided that M has no boundary: (1) M is complete, (2) the exponential map is defined for any point $x \in M$ everywhere on $T_x M$, (3) any geodesic of M can be extended to a geodesic defined on \mathbb{R} , see [49, page 94 and 95]. Completeness enters in a crucial way, namely, it will allow us to integrate by parts [48].

Lemma 3.3. *Let M be a complete Riemannian manifold. Let $\omega \in \Omega^n(M)$ and $\eta \in \Omega^{n+1}(M)$ be smooth forms such that ω , $d^n \omega$, η and $\delta^{n+1} \eta$ are square-integrable. Then*

$$\langle d^n \omega, \eta \rangle_{L^2} - \langle \omega, \delta^{n+1} \eta \rangle_{L^2} = \int_{\partial M} (\omega \wedge *^{n+1} \eta)|_{\partial M}.$$

Proof. Completeness ensures the existence of a sequence $f_n : M \rightarrow [0, 1]$ of smooth functions with compact support such that M is the union of the compact sets $\{x \in M \mid f_n(x) = 1\}$ and $\|df_n\|_\infty := \sup\{\|(df_n)_x\|_x \mid x \in M\} < \frac{1}{n}$ holds. With the help of the sequence $(f_n)_{n \geq 1}$ one can reduce the claim to the easy case, where ω and η have compact support. \square

From now on suppose that the boundary of M is empty. Then d^n and δ^n are formally adjoint in the sense that we have for $\omega \in \Omega^n(M)$ and $\eta \in \Omega^{n+1}(M)$ such that ω , $d^n\omega$, η and $\delta^{n+1}\eta$ are square-integrable.

$$(23) \quad \langle d^n(\omega), \eta \rangle_{L^2} = \langle \omega, \delta^{n+1}(\eta) \rangle_{L^2}.$$

Let $L^2\Omega^n(M)$ be the Hilbert space completion of $\Omega_c^n(M)$. Define the *space of L^2 -integrable harmonic smooth n -forms*

$$(24) \quad \mathcal{H}_{(2)}^n(M) := \{\omega \in \Omega^n(M) \mid \Delta_n(\omega) = 0, \int_M \omega \wedge *\omega < \infty\}.$$

The following two results are the analytic versions of Lemma 2.2.

Theorem 3.4 (Hodge-de Rham Decomposition). *Let M be a complete Riemannian manifold without boundary. Then we obtain an orthogonal decomposition, the so called Hodge-de Rham decomposition*

$$L^2\Omega^n(M) = \mathcal{H}_{(2)}^n(M) \oplus \text{clos}(d^{n-1}(\Omega_c^{n-1}(M))) \oplus \text{clos}(\delta^{n+1}(\Omega_c^{n+1}(M))).$$

For us the following result will be of importance. Put

$$(25) \quad \mathcal{H}^n(M) := \{\omega \in \Omega^n(M) \mid \Delta_n(\omega) = 0\}.$$

This is the same as $\mathcal{H}_{(2)}^n(M)$ introduced in (24) if M is compact.

Theorem 3.5 (Hodge-de Rham Theorem). *Let M be a closed smooth manifold. Then the canonical map*

$$\mathcal{H}^n(M) \xrightarrow{\cong} H_{\text{dR}}^n(M)$$

is an isomorphism.

Proof. See for instance [50, Lemma 1.5.3], or [92, (4.2)]. \square

The following remarks are the analytic versions of Remark 2.3 and Remark 2.4

Remark 3.6 (Homotopy invariance of $\dim(\mathcal{H}^n(M))$). Theorem 3.5 implies that $\dim(\ker(\Delta_n))$ depends only on the homotopy type of M and is in particular independent of the Riemannian metric of M . Of course the spectrum of the Laplace operator Δ_n does depend on the Riemannian metric, but a part of it, namely, the multiplicity of the eigenvalue 0, which is just $\dim(\mathcal{H}^n(M))$, depends only on the homotopy type of M .

Remark 3.7 (Heat kernel). To the analytic Laplace operator $\Delta_n : \Omega^n M \rightarrow \Omega^n M$ one can assign its *heat operator* $e^{-t\Delta_n} : \Omega^n M \rightarrow \Omega^n M$ using functional calculus. Roughly speaking, each eigenvalue λ of Δ_n transforms to the eigenvalue $e^{-t\lambda}$. This operator runs out to be given by a kernel, the so called *heat kernel* $e^{-t\Delta_n}(x, y)$. Recall that $e^{-t\Delta_n}(x, y)$ is an element in $\text{hom}_{\mathbb{R}}(\text{Alt}^n(T_x M), \text{Alt}^n(T_y M))$ for x, y in M and we get for $\omega \in \Omega^n(M)$

$$e^{-t\Delta_n}(\omega)_x = \int_M e^{-t\Delta_n}(x, y)(\omega_y) d\text{vol}.$$

For each $x \in M$ we obtain an endomorphism $e^{-t\Delta_n}(x, x)$ of a finite-dimensional real vector space and we have the real number $\text{tr}(e^{-t\Delta_n}(x, x))$. Then we get, see [50, 1.6.52 on page 56]

$$b_n(M) = \lim_{t \rightarrow \infty} \int_M \text{tr}(e^{-t\Delta_n}(x, x)) d\text{vol}.$$

4. TOPOLOGICAL TORSION FOR CLOSED RIEMANNIAN MANIFOLDS

In the section we introduce and investigate the notion of the topological torsion for a closed Riemannian manifold.

4.1. The definition of topological torsion for closed Riemannian manifolds. Let M be a closed Riemannian manifold. The Riemannian metric induces an inner product on $\Omega^n(M)$, see (22), and hence a Hilbert space structure on the finite-dimensional real vector space $\mathcal{H}^n(M)$. Equip $H_{\text{sing}}^n(M; \mathbb{R})$ with the Hilbert space structure $\kappa_{\text{harm}}^n(M)$ for which the composite of the isomorphisms (or their inverses) of Theorem 3.1 and Theorem 3.5

$$H_{\text{sing}}^n(M; \mathbb{R}) \xrightarrow{\cong} H_{\text{sing}; C^\infty}^n(M; \mathbb{R}) \xrightarrow{\cong} H_{\text{dR}}^n(M) \xrightarrow{\cong} \mathcal{H}^n(M)$$

becomes an isometry. There is a preferred isomorphism

$$\text{hom}_{\mathbb{R}}(H_n^{\text{sing}}(M; \mathbb{R}), \mathbb{R}) \xrightarrow{\cong} H_n^{\text{sing}}(M; \mathbb{R}).$$

Equip $H_n^{\text{sing}}(M; \mathbb{R})$ with the Hilbert space structure $\kappa_n^{\text{harm}}(M)$, such that for the induced Hilbert space structure on the dual vector space $\text{hom}_{\mathbb{R}}(H_n^{\text{sing}}(M; \mathbb{R}), \mathbb{R})$ and the Hilbert space structure $\kappa_{\text{harm}}^n(M)$ on $H_n^{\text{sing}}(M; \mathbb{R})$ introduced above this isomorphism becomes an isometry.

Fix a finite CW -complex X and a homotopy equivalence $f: X \rightarrow M$, for instance, a smooth triangulation $t: K \rightarrow M$, i.e., a finite simplicial complex K together with a homeomorphism $t: K \rightarrow M$ such that the restriction of t to a simplex is a smooth immersion, see [83, 99]. Recall that there is a natural isomorphism between singular and cellular homology

$$u_n(X; \mathbb{R}): H_n(X; \mathbb{R}) := H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)) \xrightarrow{\cong} H_n^{\text{sing}}(X; \mathbb{R}).$$

We equip $H_n(X; \mathbb{R}) := H_n(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X))$ with the Hilbert space structure $\kappa_n(f)$ for which the preferred isomorphism

$$H_n(X; \mathbb{R}) \xrightarrow{u_n(X; \mathbb{R})} H_n^{\text{sing}}(X; \mathbb{R}) \xrightarrow{H_n^{\text{sing}}(f; \mathbb{R})} H_n^{\text{sing}}(M; \mathbb{R})$$

is isometric if we equip the target with the Hilbert space structure $\kappa_n^{\text{harm}}(M)$ introduced above.

The cellular \mathbb{Z} -chain complex $C_*(X)$ inherits from the CW -structure a preferred equivalence of \mathbb{Z} -basis. So we can consider

$$\rho(C_*(X); \kappa_*^{\text{harm}}(f)) \in \mathbb{R}$$

as introduced in Definition 2.11. Consider another finite CW -complex X' and a homotopy equivalence $f': X' \rightarrow M$. Choose a cellular homotopy equivalence $g: X \rightarrow X'$ such that $f' \circ g$ is homotopic to f . Then $C_*(g): C_*(X) \rightarrow C_*(X')$ is a \mathbb{Z} -chain homotopy equivalence of finite based free \mathbb{Z} -chain complexes such that $H_n(g; \mathbb{R}): (H_n(X; \mathbb{R}), \kappa_n^{\text{harm}}(f)) \rightarrow (H_n(X'; \mathbb{R}), \kappa_n^{\text{harm}}(f'))$ is an isometric isomorphism for all $n \geq 0$. We conclude from Lemma 2.12

$$\rho(C_*(X); \kappa_*^{\text{harm}}(f)) = \rho(C_*(X'); \kappa_*^{\text{harm}}(f')).$$

Hence the following definition makes sense.

Definition 4.1 (Topological torsion of a closed Riemannian manifold). Let M be a closed Riemannian manifold. Define its *topological torsion*

$$\rho_{\text{top}}(M) := \rho(C_*(X), \kappa_*^{\text{harm}}(f))$$

for any choice of finite CW -complex X and homotopy equivalence $f: X \rightarrow M$.

4.2. Topological torsion of rational homology spheres. Let M be a closed oriented Riemannian manifold which is a rational homology sphere, i.e., $H_n(M; \mathbb{Q}) \cong H_n(S^d; \mathbb{Q})$ for $d = \dim(M)$ and $n \geq 0$. We want to show

$$(26) \quad \rho^{\text{top}}(M) = \frac{1 - (-1)^d}{2} \cdot \ln(\text{vol}(M)) + \sum_{n \geq 0} (-1)^n \cdot \ln(|\text{tors}(H_n(M; \mathbb{Z}))|).$$

Choose a finite CW -complex X and a homotopy equivalence $f: X \rightarrow M$. If we equip $H_*(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X))$ with the integral Hilbert space structure $\kappa_*^{\mathbb{Z}}$ as explained in Example 2.13, we get from Example 2.13.

$$\rho(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X); \kappa_*^{\mathbb{Z}}) = \sum_{n \geq 0} (-1)^n \cdot \ln(|\text{tors}(H_n(M; \mathbb{Z}))|).$$

Hence we get

$$\begin{aligned} \rho^{\text{top}}(M) &= \rho(X; \kappa_*^{\text{harm}}) \\ &= \rho(X; \kappa_*^{\text{harm}}) - \rho(X; \kappa_*^{\mathbb{Z}}) + \rho(X; \kappa_*^{\mathbb{Z}}) + \\ &= \rho(X; \kappa_*^{\text{harm}}) - \rho(X; \kappa_*^{\mathbb{Z}}) + \sum_{n \geq 0} (-1)^n \cdot \ln(|\text{tors}(H_n(M; \mathbb{Z}))|). \end{aligned}$$

Lemma 2.12 implies

$$\begin{aligned} &\rho(X; \kappa_*^{\text{harm}}) - \rho(X; \kappa_*^{\mathbb{Z}}) \\ &= \ln(\det^{\perp}(\text{id}: H_0(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)), \kappa_0^{\mathbb{Z}}(X)) \rightarrow H_0(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)), \kappa_0^{\text{harm}}(X))) \\ &\quad + (-1)^d \cdot \ln(\det^{\perp}(\text{id}: H_d(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)), \kappa_d^{\mathbb{Z}}(X)) \rightarrow \\ &\quad \quad \quad H_d(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)), \kappa_d^{\text{harm}}(X))). \end{aligned}$$

Let $1 \in H_0^{\text{sing}}(M; \mathbb{Z})$ and $[M] \in H_d^{\text{sing}}(M; \mathbb{Z})$ be the obvious generators of the infinite cyclic groups $H_0^{\text{sing}}(M; \mathbb{Z})$ and $H_d^{\text{sing}}(M; \mathbb{Z})$. They determine elements in the 1-dimensional vector spaces $H_{\text{sing}}^0(M; \mathbb{R}) = \text{hom}_{\mathbb{Z}}(H_0(M; \mathbb{Z}), \mathbb{R})$ and $H_{\text{sing}}^d(M; \mathbb{R}) = \text{hom}_{\mathbb{Z}}(H_d(M; \mathbb{Z}), \mathbb{R})$. Their image under the composite

$$H_{\text{sing}}^n(M; \mathbb{R}) \xrightarrow{\cong} H_{\text{sing}; C^\infty}^n(M; \mathbb{R}) \xrightarrow{\cong} H_{\text{dR}}^n(M) \xrightarrow{\cong} \mathcal{H}^n(M)$$

is the constant function $c_1: M \rightarrow \mathbb{R}$ with value 1 and $\frac{d\text{vol}}{\text{vol}(M)}$ for $d\text{vol}$ the volume form M for $n = 0, d$. The norm of c_1 and $\frac{d\text{vol}}{\text{vol}(M)}$ with respect to norm coming from (22) is

$$\|c_1\|_{L^2} = \sqrt{\int_M c_1 \wedge *^d(c_1)} = \sqrt{\int_M d\text{vol}} = \sqrt{\text{vol}(M)},$$

and

$$\left\| \frac{d\text{vol}}{\text{vol}(M)} \right\|_{L^2} = \sqrt{\int_M \frac{d\text{vol}}{\text{vol}(M)} \wedge *^d \left(\frac{d\text{vol}}{\text{vol}(M)} \right)} = \sqrt{\int_M \frac{d\text{vol}}{\text{vol}(M)^2}} = \sqrt{\frac{1}{\text{vol}(M)}}.$$

This implies

$$\begin{aligned} &\ln(\det^{\perp}(\text{id}: H_0(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)), \kappa_0^{\mathbb{Z}}(X)) \rightarrow H_0(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)), \kappa_n^{\text{harm}}(X))) \\ &\quad = \frac{\ln(\text{vol}(M))}{2} \end{aligned}$$

and

$$\begin{aligned} &\ln(\det^{\perp}(\text{id}: H_d(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)), \kappa_d^{\mathbb{Z}}(X)) \rightarrow H_d(\mathbb{R} \otimes_{\mathbb{Z}} C_*(X)), \kappa_d^{\text{harm}}(X))) \\ &\quad = \frac{-\ln(\text{vol}(M))}{2} \end{aligned}$$

Now (26) follows.

4.3. Further properties of the topological torsion. Lemma 2.12 implies

Lemma 4.2. *Let $f: M \rightarrow N$ be a homotopy equivalence of closed Riemannian manifolds. Then*

$$\begin{aligned} \rho_{\text{top}}(N) - \rho_{\text{top}}(M) &= \sum_{n \geq 0} (-1)^n \cdot \det^\perp (H_n^{\text{sing}}(f; \mathbb{R}) : H_n^{\text{sing}}(M, \kappa_n^{\text{harm}}(M)) \\ &\quad \rightarrow H_n(N; \mathbb{R}), \kappa_n^{\text{harm}}(N)). \end{aligned}$$

Remark 4.3 (Twisting with finite-dimensional orthogonal representations). In general the topological torsion does depend on the Riemannian metric, see Lemma 4.2. Nevertheless the name topological torsion is justified since this dependency is well understood and depends only on $H_n(M; \mathbb{R})$.

Notice that at least $H_0(M; \mathbb{R})$ cannot be trivial for a smooth manifold. However, there are prominent cases, where one can specify a orthogonal finite-dimensional representation V of $\pi_1(M)$ for a closed Riemannian manifold M such that the V -twisted singular homology $H_n^{\pi_1(M)}(M; V)$ vanishes for all $n \geq 0$. One can also define a V -twisted topological torsion $\rho(M; V)$. If $H_n^{\pi_1(M)}(M; V)$ vanishes for all $n \geq 0$, then $\rho(M; V)$ does not depend on the Riemannian metric at all, and only on the simple homotopy type of M .

Remark 4.4 (Poincaré duality). A direct computation using Poincaré duality and the Universal Coefficient Theorem show that in the situation of Subsection 4.2 the topological torsion vanishes if the dimension of M is even. This is true in general. Namely, if M is a closed Riemannian manifold of even dimension, then $\rho_{\text{top}}(M) = 0$.

Remark 4.5 (Product formula). Let M be closed Riemannian manifolds. Then

$$\rho_{\text{top}}(M \times N) = \chi(M) \cdot \rho_{\text{top}}(N) + \chi(N) \cdot \rho_{\text{top}}(M).$$

One can more generally investigate the behavior of the topological torsion under fiber bundles, see [69].

Remark 4.6 (Compact manifolds with boundary and glueing formula). The topological torsion is also defined for compact Riemannian manifolds with boundary. One has to put the right boundary conditions on the space of harmonic forms so that Theorem 3.5 remains true.

Consider compact Riemannian manifolds M and N together with a diffeomorphism $f: \partial M \xrightarrow{\cong} \partial N$. Equip M , N , ∂N , and $M \cup_f N$ with Riemannian metrics. Then one obtains the glueing formula

$$\rho_{\text{top}}(M \cup_f N) = \rho_{\text{top}}(M) + \rho_{\text{top}}(N) - \rho_{\text{top}}(\partial M) + \rho(LHS_*),$$

where LHS_* is the Hilbert chain complex given by the long exact homology sequence

$$\begin{aligned} \dots \rightarrow H_n^{\text{sing}}(\partial M; \mathbb{R}) \rightarrow H_n^{\text{sing}}(M; \mathbb{R}) \oplus H_n^{\text{sing}}(N; \mathbb{R}) \\ \rightarrow H_n^{\text{sing}}(M \cup_f N; \mathbb{R}) \rightarrow H_{n-1}^{\text{sing}}(\partial M; \mathbb{R}) \rightarrow \dots \end{aligned}$$

for which each homology group is equipped with the harmonic Hilbert space structure κ^{harm} . This follows from Lemma 2.6 and Lemma 2.12.

5. ANALYTIC TORSION FOR CLOSED RIEMANNIAN MANIFOLDS

Recall that we showed that the Betti number of a finite CW -complex X is the dimension of the kernel of the combinatorial Laplace operator $\Delta_n: \mathbb{R} \otimes_{\mathbb{Z}} C_n(X) \rightarrow \mathbb{R} \otimes_{\mathbb{Z}} C_n(X)$. This triggered the question whether the Betti number $b_n(M)$ of a closed Riemannian manifold is the dimension of the kernel of the analytic Laplace operator $\Delta_n: \Omega^n(M) \rightarrow \Omega^n(M)$. We saw that the answer is positive, see Theorem 3.5.

Next we want to apply the same line of thought to torsion. We know how to express the topological torsion in terms of the combinatorial Laplace operator by Lemma 2.5, namely for a finite CW -complex X and a homotopy equivalence $X \rightarrow M$ we get for the combinatorial Laplace operator $\Delta_n: C_n(X) \rightarrow C_n(X)$ the formula

$$\begin{aligned} \rho^{\text{top}}(M) &:= -\frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln(\det^{\perp}(\Delta_n)) \\ &\quad + \sum_{n \geq 0} (-1)^n \cdot \ln(\det^{\perp}(\text{id}: H_n(\mathbb{R} \otimes_{\mathbb{C}} C_*) \rightarrow (H_n(\mathbb{R} \otimes_{\mathbb{C}} C_*), \kappa_n^{\text{harm}}))). \end{aligned}$$

One can hope that the rather complicated correction term given by the sum of terms involving κ_*^{harm} is not necessary in the analytic setting, since the analytic Laplace operator $\Delta_n: \Omega^n(M) \rightarrow \Omega^n(M)$ is closely related to harmonic forms. This suggests to try to make sense of the following expression involving the analytic Laplace operator

$$\rho_{\text{an}}(M) := -\frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \ln(\det^{\perp}(\Delta_n)).$$

The problem is that the analytic Laplace operator Δ_n acts on infinite-dimensional vector spaces and therefore the expression $\det^{\perp}(\Delta_n)$ is a priori not defined. To give it nevertheless a meaning, one has to take a closer look on the spectrum of the analytic Laplace operator Δ_n for a closed Riemannian manifold.

5.1. The spectrum of the Laplace operator on closed Riemannian manifolds. Let M be a closed Riemannian manifold. Next we record some basic facts about the spectrum of the analytic Laplace operator $\Delta_n: \Omega^n(M) \rightarrow \Omega^n(M)$. Denote by $E_{\lambda}(\Delta_n) = \{\omega \in \Omega^n(M) \mid \Delta_n(\omega) = \lambda \cdot \omega\}$ the eigenspace of Δ_n for $\lambda \in \mathbb{C}$. We call λ an eigenvalue of Δ_n if $E_{\lambda}(\Delta_n) \neq 0$. It turns out that each eigenvalue λ of Δ_n is a real number satisfying $\lambda \geq 0$. Notice that $E_{\lambda}(\Delta_n)$ and $E_{\mu}(\Delta_n)$ are orthogonal in $L^2\Omega^n(M)$ for $\lambda \neq \mu$ since we get from (23) for $\nu_0, \nu_1 \in \Omega^n(M)$

$$(27) \quad \langle \Delta_n(\nu_0), \nu_1 \rangle_{L^2} = \langle \nu_0, \Delta_n(\nu_1) \rangle_{L^2},$$

and hence we get for $\omega \in E_{\lambda}(\Delta_n)$ and $\eta \in E_{\mu}(\Delta_n)$

$$\lambda \cdot \langle \omega, \eta \rangle_{L^2} = \langle \lambda \cdot \omega, \eta \rangle_{L^2} = \langle \Delta_n(\omega), \eta \rangle_{L^2} = \langle \omega, \Delta_n(\eta) \rangle_{L^2} = \langle \omega, \mu \cdot \eta \rangle_{L^2} = \mu \cdot \langle \omega, \eta \rangle_{L^2}.$$

Moreover, we have the orthogonal decomposition

$$\bigoplus_{\lambda \geq 0} E_{\lambda}(\Delta_n) = L^2\Omega^n(M).$$

We define the n -th-Zeta-function for $s \in \mathbb{C}$

$$(28) \quad \zeta_n(s) = \sum_{\lambda > 0} \dim_{\mathbb{R}}(E_{\lambda}(\Delta_n)) \cdot \lambda^{-s},$$

where λ runs through all eigenvalues of Δ_n with $\lambda > 0$. Of course it is a priori not clear whether this sums converges. However, the following result holds, see for instance [50, Section 1.12].

Lemma 5.1. *The Zeta-function ζ_n converges absolutely for $s \in S = \{s \in \mathbb{C} \mid \text{Real}(s) > \dim(M)/2\}$ and defines a holomorphic function on S . It has a meromorphic extension to \mathbb{C} which is analytic in zero and whose derivative at zero $\frac{d}{ds}\big|_{s=0} \zeta_n(s)$ lies in \mathbb{R} .*

5.2. The definition of analytic torsion for closed Riemannian manifolds.

In view of Lemma 5.1 the following definition make sense. It is due to Ray-Singer [89] and motivated by (10) and Lemma 2.5.

Definition 5.2 (Analytic torsion). Let M be a closed Riemannian manifold. Define its *analytic torsion*

$$\rho_{\text{an}}(M) := \frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \frac{d}{ds} \Big|_{s=0} \zeta_n(s).$$

Remark 5.3 (Analytic torsion in terms of the heat kernel). One can rewrite the analytic torsion also in terms of the heat kernel by

$$\rho_{\text{an}}(M; V) := \frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \frac{d}{ds} \frac{1}{\Gamma(s)} \cdot \int_0^\infty t^{s-1} \cdot \theta_n(M)^\perp dt \Big|_{s=0},$$

where

$$\begin{aligned} \theta_n(M)(t) &:= \int_M \text{tr}(e^{-t\Delta_n}(x, x)) \, d\text{vol}; \\ \theta_n(M)^\perp &= \theta_n(M)(t) - \dim_{\mathbb{R}}(H_n(M; \mathbb{R})); \\ \Gamma(s) &= \int_0^\infty t^{s-1} e^{-t} dt \quad \text{for } \text{Real}(s) > 0, \end{aligned}$$

the Gamma-function $\Gamma(s)$ is defined for $s \in \mathbb{C}$ by meromorphic extension with poles of order 1 in $\{n \in \mathbb{Z} \mid n \leq 0\}$ and satisfies $\Gamma(s+1) = s \cdot \Gamma(s)$ and $\Gamma(n+1) = n!$ for $n \in \mathbb{Z}, n \geq 0$, see for instance [65, Section 3.5.1].

5.3. Analytic torsion of S^1 and the Riemann Zeta-function. Fix a positive real number μ . Equip \mathbb{R} with the standard metric and the unit circle S^1 with the Riemannian metric for which $\mathbb{R} \rightarrow S^1, t \mapsto \exp(2\pi i \mu^{-1} t)$ is isometric. Then S^1 has volume μ . The Laplace operator $\Delta^1: \Omega^1(\mathbb{R}) \rightarrow \Omega^1(\mathbb{R})$ sends $f(t)dt$ to $-f''(t)dt$. By checking the μ -periodic solutions of $f''(t) = -\lambda f(t)$, one shows that $\Delta^1: \Omega^1(S^1) \rightarrow \Omega^1(S^1)$ has eigenspaces

$$E_\lambda(\Delta_1) = \begin{cases} \text{span}_{\mathbb{R}}\{f_n dt, g_n dt\} & \text{for } \lambda = (2\pi\mu^{-1}n)^2, n \geq 1; \\ \text{span}_{\mathbb{R}}\{dt\} & \text{for } \lambda = 0; \\ \{0\} & \text{otherwise,} \end{cases}$$

where $f_n(\exp(2\pi i \mu^{-1} t)) = \cos(2\pi\mu^{-1} nt)$ and $g_n(\exp(2\pi i \mu^{-1} t)) = \sin(2\pi\mu^{-1} nt)$. Denote by

$$(29) \quad \zeta_{\text{Riem}}(s) = \sum_{n \geq 1} n^{-s}$$

the *Riemannian Zeta-function*. We have

$$\zeta_1(s) = \sum_{n \geq 1} 2 \cdot (2\pi\mu^{-1}n)^{-s}.$$

As $\zeta_{\text{Riem}}(0) = -\frac{1}{2}$ and $\zeta'_{\text{Riem}}(0) = -\frac{\ln(2\pi)}{2}$ hold (see Titchmarsh [96]), we obtain

$$\begin{aligned}
\rho_{\text{an}}(S^1) &= \frac{1}{2} \cdot \sum_{n \geq 0} (-1)^n \cdot n \cdot \left. \frac{d}{ds} \right|_{s=0} \zeta_n(s) \\
&= -\frac{1}{2} \cdot \left. \frac{d}{ds} \right|_{s=0} \zeta_1(s) \\
&= -\left. \frac{d}{ds} \right|_{s=0} \left(\sum_{n \geq 1} ((2\pi\mu^{-1}n)^2)^{-s} \right) \\
&= -\left. \frac{d}{ds} \right|_{s=0} (\exp(-2 \cdot \ln(2\pi\mu^{-1}) \cdot s) \cdot \zeta_{\text{Riem}}(2s)) \\
&= -\left(\left. \frac{d}{ds} \right|_{s=0} \exp(-2 \cdot \ln(2\pi\mu^{-1}) \cdot s) \right) \cdot \zeta_{\text{Riem}}(0) \\
&\quad - \exp(-2 \cdot \ln(2\pi\mu^{-1}) \cdot 0) \cdot \left. \frac{d}{ds} \right|_{s=0} \zeta_{\text{Riem}}(2s) \\
&= 2 \cdot \ln(2\pi\mu^{-1}) \cdot \zeta_{\text{Riem}}(0) - 2 \cdot \left. \frac{d}{ds} \right|_{s=0} \zeta_{\text{Riem}}(s) \\
&= 2 \cdot \ln(2\pi\mu^{-1}) \cdot \frac{-1}{2} - 2 \cdot \frac{-\ln(2\pi)}{2} \\
&= -\ln(2\pi) + \ln(\mu) + \ln(2\pi) \\
&= \ln(\mu).
\end{aligned}$$

Notice that this agrees with $\rho^{\text{top}}(S^1)$ by (26).

5.4. The equality of analytic and topological torsion for closed Riemannian manifolds: The Cheeger-Müller Theorem. The following celebrated result was proved independently by Cheeger [37] and Müller [79].

Theorem 5.4 (Equality of analytic and Reidemeister torsion). *Let M be a closed Riemannian manifold. Then*

$$\rho_{\text{an}}(M) = \rho_{\text{top}}(M).$$

It was already known before the final proof of Theorem 5.4 that the difference $\rho_{\text{an}}(M) - \rho_{\text{top}}(M)$ is independent of the Riemannian metric and $\rho_{\text{an}}(M)$ and $\rho_{\text{top}}(M)$ satisfies analogous product formulas so that the desired equality holds for a product $M \times N$ if it holds for both M and N . Müller's strategy was to show that the difference $\rho_{\text{an}}(M) - \rho_{\text{top}}(M)$ depends only on the bordism class of M and then verify the equality on generators of the oriented bordism ring. He also uses an interesting result of Dodziuk and Patodi [42, Theorem 3.7] that the eigenvalues of the combinatorial Laplace operator $\Delta_n(K)$ of a smooth triangulation K of M converge to the eigenvalues of the analytic Laplace operator $\Delta_n(M)$ if the mesh, which is the supremum over the distances with respect to the metric coming from the Riemannian metric of any two vertices spanning a 1-simplex, of the triangulation K goes to zero.

5.5. The relation between analytic and topological torsion for compact Riemannian manifolds. Let M be a compact Riemannian manifold. Suppose that its boundary ∂M is written as disjoint union $\partial_0 M \coprod \partial_1 M$, where $\partial_i M$ itself is a disjoint union of path components of ∂M . In particular $\partial_i M$ itself is a closed manifold. We will assume that the Riemannian metric on M is a product near the boundary and we will equip ∂M with the induced Riemannian metric. By

introducing appropriate boundary condition for the Laplace operator one can define $\rho_{\text{an}}(M, \partial_0 M)$ and $\rho_{\text{top}}(M, \partial_0 M)$. The next result is proved in [63, Corollary 5.1].

Theorem 5.5 (The relation between analytic and topological torsion for compact Riemannian manifolds). *We get under the conditions above*

$$\rho_{\text{an}}(M, \partial_0 M) = \rho_{\text{top}}(M, \partial_0 M) + \frac{\ln(2)}{2} \cdot \chi(\partial M).$$

Example 5.6 (Unit interval). Equip $I = [0, 1]$ with the standard metric scaled by $\mu > 0$. The volume form is then μdt . The analytic Laplace operator $\Delta_1: \Omega^1 I \rightarrow \Omega^1 I$ maps $f(t)dt$ to $-\mu^{-2}f''(t)dt$. Denote by $E_\lambda(\Delta_1(I))$ and $E_\lambda(\Delta_1(I, \partial I))$ the eigenspace of Δ_1 for $\lambda \geq 0$, where for $\Delta_1(I)$ and $\Delta_1(I, \partial I)$ respectively we require for a 1-form $f(t)dt$ the boundary condition $f(0) = f(1) = 0$ and $f'(0) = f'(1) = 0$ respectively. If $\lambda = (\pi\mu^{-1}n)^2$ for $n \in \mathbb{Z}, n \geq 1$, then

$$\begin{aligned} E_\lambda(\Delta_1(I)) &= \text{span}_{\mathbb{R}}\{\sin(\pi nt)dt\}; \\ E_\lambda(\Delta_1(I, \partial I)) &= \text{span}_{\mathbb{R}}\{\cos(\pi nt)dt\}, \end{aligned}$$

and $E_\lambda(\Delta_1(I)) = E_\lambda(\Delta_1(I, \partial I)) = 0$ if λ is not of this form. As $\zeta_{\text{Riem}}(0) = -\frac{1}{2}$ and $\zeta'_{\text{Riem}}(0) = -\frac{\ln(2\pi)}{2}$ hold (see Titchmarsh[96]), we get

$$\zeta_1(I) = \zeta_1(I, \partial I) = \left(\frac{\pi}{\mu}\right)^{-2s} \cdot \zeta_{\text{Rie}}(2s).$$

This implies

$$\rho_{\text{an}}(I) = \rho_{\text{an}}(I, \partial I) = \ln(2\mu).$$

A calculation similar to the one of Subsection 4.2 shows

$$\rho_{\text{top}}(I) = \rho_{\text{top}}(I, \partial I) = \ln(\mu).$$

This is compatible with Theorem 5.5 since $\chi(\partial I) = 2$.

Remark 5.7 (Twisting with finite-dimensional orthogonal representations). For an orthogonal finite-dimensional representation V of $\pi_1(M)$ for a compact Riemannian manifold M one can also define the V -twisted analytic torsion $\rho_{\text{an}}(M, \partial_0 M; V)$ and V -twisted topological torsion $\rho_{\text{top}}(M, \partial_0 M; V)$. Theorem 5.5 generalizes to

$$\rho_{\text{an}}(M, \partial_0 M; V) = \rho_{\text{top}}(M, \partial_0 M; V) + \frac{\ln(2)}{2} \cdot \dim_{\mathbb{R}}(V) \cdot \chi(\partial M).$$

Remark 5.8 (Elliptic operators and indices). The Euler characteristic term in Theorem 5.5 can be interpreted as the index of the de Rham complex. This leads to the following question.

Let P^* be an elliptic complex of partial differential operators. Denote by $\Delta(P^*)$ the associated Laplace operator. It is an elliptic positive self-adjoint partial differential operator in each dimension. Hence its analytic torsion $\rho_{\text{an}}(P^*)$ can be defined as done before for the ordinary Laplace operator. Suppose that the complex P^* restricts on the boundary of M to an elliptic complex ∂P^* in an appropriate sense. Can one find a more or less topological invariant $\rho_{\text{top}}(P^*)$ such that the following equation holds

$$\rho_{\text{an}}(P^*) = \rho_{\text{top}}(P^*) + \frac{\ln(2)}{2} \cdot \text{index}(\partial P^*).$$

If we take P^* to be the de Rham complex and put $\rho_{\text{top}}(P^*)$ to be the topological torsion $\rho_{\text{top}}(M)$, then the equation above just reduces to Theorem 5.5.

6. EQUIVARIANT TORSION FOR ACTIONS OF FINITE GROUPS

Throughout this section G is a finite group. Let M be a compact Riemannian manifold. Suppose that its boundary ∂M is written as the disjoint union $\partial_0 M \amalg \partial_1 M$, where $\partial_i M$ itself is a disjoint union of path components of ∂M . Let G be a finite group acting by isometries on M .

Let $\text{Rep}_{\mathbb{R}}(G)$ be the real representation ring of G . Denote by $K_1(\mathbb{R}G)^{\mathbb{Z}/2}$ the $\mathbb{Z}/2$ -fixed point set of the $\mathbb{Z}/2$ -action on $K_1(\mathbb{R}G)$ which comes from the involution of rings $\mathbb{R}G \rightarrow \mathbb{R}G$, $\sum_{g \in G} r_g \cdot g \mapsto \sum_{g \in G} r_g \cdot g^{-1}$. Then one can define *the equivariant analytic torsion*

$$\rho_{\text{an}}^G(M, \partial_1 M) \in \mathbb{R} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{R}}(G),$$

the *equivariant topological torsion*

$$\rho_{\text{top}}^G(M, \partial_1 M) \in K_1(\mathbb{R}G)^{\mathbb{Z}/2},$$

the *Poincaré torsion*

$$\rho_{\text{pd}}^G(M, \partial_1 M) \in K_1(\mathbb{R}G)^{\mathbb{Z}/2},$$

and the *equivariant Euler characteristic*

$$\chi^G(\partial M) \in \mathbb{R} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{R}}(G).$$

The analytic torsion $\rho_{\text{an}}^G(M, \partial_0 M)$ is defined analogously to the analytic torsion in the non-equivariant case, one just takes into account that the eigenspaces $E_{\lambda}(\Delta_n)$ determine elements in $\text{Rep}_{\mathbb{R}}(G)$ and counts it as an element in $\text{Rep}_{\mathbb{R}}(G)$ instead of only counting its dimension. The topological torsion is defined in terms of the cellular chain complex of an equivariant triangulation. The equivariant Euler characteristic $\chi^G(\partial_0 M)$ is given by $\sum_{n \geq 0} (-1)^n \cdot [H_n(\partial M; \mathbb{R})]$ taking again into account that $H_n(\partial M; \mathbb{R})$ is a finite-dimensional G -representation. A new phenomenon is represented by the Poincaré torsion $\rho_{\text{pd}}^G(M, \partial_0 M)$ which measures the deviation from equivariant Poincaré duality being simple. It is defined in terms of the $\mathbb{Z}G$ -chain map $-\cap [M]: C^{\dim(M)-*}(M, \partial_1 M) \rightarrow C_*(M, \partial_0 M)$ which is a \mathbb{Z} -chain homotopy equivalence but not necessarily a $\mathbb{Z}G$ -chain homotopy equivalence. If M has no boundary and has odd dimension, or if G acts freely, then $\rho_{\text{pd}}^G(M, \partial_0 M)$ vanishes.

Denote by $\widehat{\text{Rep}}_{\mathbb{R}}(G)$ the subgroup of $\text{Rep}_{\mathbb{R}}(G)$ generated by the irreducible representations of real or complex type. The following result is proved in [63, Theorem 4.5].

Theorem 6.1 (Equivariant torsion). *Suppose that the Riemannian metric on M is a product near the boundary. Then there is an isomorphism*

$$\Gamma_1 \oplus \Gamma_2 : K_1(\mathbb{R}G)^{\mathbb{Z}/2} \xrightarrow{\cong} \mathbb{R} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{R}}(G) \oplus (\mathbb{Z}/2 \otimes_{\mathbb{Z}} \widehat{\text{Rep}}_{\mathbb{R}}(G)),$$

and we have

$$\rho_{\text{an}}^G(M, \partial_1 M) = \Gamma_1(\rho_{\text{top}}^G(M, \partial_1 M)) - \frac{1}{2} \cdot \Gamma_1(\rho_{\text{pd}}^G(M, \partial_1 M)) + \frac{\ln(2)}{2} \cdot \chi^G(\partial M),$$

and

$$\Gamma_2(\rho_{\text{top}}^G(M, \partial_1 M)) = \Gamma_2(\rho_{\text{pd}}^G(M, \partial_1 M)) = 0.$$

Remark 6.2 (The strategy of proof). Let M be a compact G -manifold with G invariant Riemannian metric which is a product near the boundary. Then its double $M \cup_{\partial M} M$ inherits a $G \times \mathbb{Z}/2$ -action and a $G \times \mathbb{Z}/2$ -invariant metric. It turns out that the equivariant torsion $\rho_{\text{an}}^{G \times \mathbb{Z}/2}(M \cup_{\partial M} M)$ carries the same information as $\rho_{\text{an}}^G(M)$ and $\rho_{\text{an}}^G(M, \partial M)$ together. This is also true for $\rho_{\text{top}}^{G \times \mathbb{Z}/2}(M \cup_{\partial M} M)$ but the concrete formulas are different for the topological and analytical setting, the difference term is essentially $\frac{\ln(2)}{2} \cdot \chi^G(\partial M)$. Thus one can reduce the case of a compact G -manifold to the case of a closed $G \times \mathbb{Z}/2$ -manifold.

Lott-Rothenberg [61] handled the odd-dimensional case without boundary using ideas of Cheeger [37] and Müller [79]. They noticed that in the even-dimensional case the analytic and topological torsion do not agree without computing the correction term, which turns out to be the Poincaré torsion.

Remark 6.3 (Unit spheres in representations). The Poincaré duality torsion can be used to prove the celebrated result of de Rham [40] that two orthogonal G -representations V and W are isometrically $\mathbb{R}G$ -isomorphic if and only if their unit spheres are G -diffeomorphic, see [63, Section 5]. Similar proofs can be found in Rothenberg [93] and Lott-Rothenberg [61]. The result is an extension of the classification of lens spaces which is carried out for example in Cohen [38] and Milnor [77].

The result of de Rham does not hold in the topological category. Namely, there are non-linearly isomorphic G -representations V and W whose unit spheres are G -homeomorphic, see Cappell-Shaneson [32]), and also [33, 34, 51].

However, if G has odd order, G -homeomorphic implies G -diffeomorphic for unit spheres in G -representations as shown by Hsiang-Pardon [52] and Madsen-Rothenberg [72].

Example 6.4 (S^1 with complex conjugation). Fix a positive real number μ . Equip \mathbb{R} with the standard metric and the unit circle S^1 with the Riemannian metric for which $\mathbb{R} \rightarrow S^1, t \mapsto \exp(2\pi i \mu^{-1} t)$ is isometric. Then S^1 has volume μ . Let $\mathbb{Z}/2$ act on S^1 by complex conjugation. We get by a direct computation, see [63, Example 1.15]

$$\rho_{\text{an}}^{\mathbb{Z}/2}(S^1) = \ln(\mu) \cdot ([\mathbb{R}] + [\mathbb{R}^-]) \in \mathbb{R} \otimes_{\mathbb{Z}} \text{Rep}_{\mathbb{R}}(\mathbb{Z}/2),$$

where \mathbb{R} is the trivial 1-dimensional real $\mathbb{Z}/2$ -representation and \mathbb{R}^- is the 1-dimensional $\mathbb{Z}/2$ -representation for which the generator of $\mathbb{Z}/2$ acts by $-\text{id}_{\mathbb{R}}$. We obtain in $K_1(\mathbb{R}[\mathbb{Z}/2])^{\mathbb{Z}/2}$ by a direct computation, see [63, Example 3.25],

$$\begin{aligned} \rho_{\text{top}}^{\mathbb{Z}/2}(S^1) &= [\mu/2 \cdot \text{id}: \mathbb{R} \rightarrow \mathbb{R}] + [2\mu \cdot \text{id}: \mathbb{R}^- \rightarrow \mathbb{R}^-]; \\ \rho_{\text{pd}}^{\mathbb{Z}/2}(S^1) &= [4 \cdot \text{id}: \mathbb{R} \rightarrow \mathbb{R}] + [1/4 \cdot \text{id}: \mathbb{R}^- \rightarrow \mathbb{R}^-]. \end{aligned}$$

This is compatible with Theorem 6.1.

Example 6.5 (S^1 with antipodal action). Fix a positive real number μ . Equip \mathbb{R} with the standard Riemannian metric and the unit circle S^1 with the Riemannian metric for which $\mathbb{R} \rightarrow S^1, t \mapsto \exp(2\pi i \mu^{-1} t)$ is isometric. Then S^1 has volume μ . Let $\mathbb{Z}/2$ act on S^1 by the antipodal map which sends z to $-z$. This is a free orientation preserving action. Then

$$\begin{aligned} \rho_{\text{an}}^{\mathbb{Z}/2}(S^1) &= \ln(\mu) \cdot [\mathbb{R}[\mathbb{Z}/2]]; \\ \rho_{\text{top}}^{\mathbb{Z}/2}(S^1) &= [\mu \cdot \text{id}: \mathbb{R}[\mathbb{Z}/2] \rightarrow \mathbb{R}[\mathbb{Z}/2]]; \\ \rho_{\text{pd}}^{\mathbb{Z}/2}(S^1) &= 0. \end{aligned}$$

As an illustration we state the following corollary of Theorem 6.1 and basic considerations about Poincaré duality, which explains the role of the Poincaré torsion that does not appear in the non-equivariant setting, see [63, Corollary 5.6].

Corollary 6.6. *Let M be a Riemannian G -manifold with invariant Riemannian metric. Suppose that M is closed and orientable and G acts orientation preserving.*

(1) *If $\dim(M)$ is odd, we have*

$$\begin{aligned} \rho_{\text{an}}^G(M) &= \Gamma_1(\rho_{\text{top}}(M)); \\ \rho_{\text{pd}}^G(M) &= 0; \end{aligned}$$

(2) If $\dim(M)$ is even, we get

$$\begin{aligned}\rho_{\text{an}}(M) &= 0; \\ \rho_{\text{top}}^G(M) &= \frac{\rho_{\text{pd}}^G(M)}{2}.\end{aligned}$$

Remark 6.7 (Twisting with equivariant coefficient systems). There are also versions of the notions and results of this section for appropriate equivariant coefficient system as explained in [63].

7. OUTLOOK

7.1. Analytic torsion. There are many important papers about analytic torsion and variations of it in the literature. We have to leave it to the reader to figure out the relevant authors and papers since an appropriate discussion would go far beyond the scope of this article. At least we give a list of references which is far from being complete. They concern for instance determinant lines, holomorphic versions, higher versions, equivariant versions, singular spaces, algebraic varieties, and hyperbolic manifolds, see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 39, 44, 45, 46, 47, 57, 58, 59, 60, 74, 78, 80, 81, 82, 90, 98, 103, 104, 105].

7.2. Topological torsion. Also for topological torsion there are many important papers in the literature. In particular Whitehead torsion and Reidemeister torsion have been intensively studied. Again we have to leave it to the reader to figure out the relevant authors and papers since an appropriate discussion would go far beyond the scope of this article. At least we give a list of references which is far from being complete, they concern for instance s -cobordisms, knot theory, classification of manifolds, equivariant versions, and higher versions, see [2, 3, 35, 36, 38, 53, 54, 55, 56, 62, 64, 70, 71, 75, 76, 77, 84, 85, 86, 87, 88, 91, 94, 95, 97, 100, 101, 102].

7.3. L^2 -versions. The next level is to pass to non-compact spaces, essentially to a G -covering $\overline{M} \rightarrow M$ for a closed Riemannian manifold M and the induced G -invariant Riemannian metric on \overline{M} or to a G -covering $\overline{X} \rightarrow X$ for a finite CW -complex X , where G is a (not necessarily finite) discrete group. This requires to extend our basic invariants of finite-dimensional Hilbert spaces of Sections 1 and 2 to an appropriate setting of infinite dimensional Hilbert spaces taking the cocompact free proper group action on \overline{M} or \overline{X} into account. Here group von Neumann algebras play a key role. The first instance where this has been carried out is the paper by Atiyah [1]. Generalizations of the ideas about the spectral density function presented in Subsection 1.3 and 1.4. come into play and lead for instance to the notion of the Fuglede-Kadison determinant, which generalizes the classical determinant to this setting. The material presented in Subsection 1.3 and 1.4 is helpful if one wants to understand the L^2 -versions.

All this leads to the notions of L^2 -Betti numbers and of L^2 -torsion, which have been intensively studied in the literature and have many applications to problems arising in topology, geometry, group theory and von Neumann algebras. A discussion of these L^2 -invariants would go far beyond the scope of this article. For more information about the circle of these ideas and invariants we refer for instance to [65, 67, 68].

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