The dual ultracontractivity and its applications

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1. Introduction

Let $\{T_t\}$ be a symmetric Markov process on a measure space (M, m). The semigroup $\{T_t\}$ is called ultracontractive if T_t are bounded operators from L^1 to L^∞ for all $t > 0$. Some criteria for ultracontractivity are known. For example, for $\mu > 0$, the followings are equivalent to each other:

(i) There exists a constant c_1 so that

$$\|T_t f\|_\infty \le c_1 t^{-\mu/2} \|f\|_1, \quad \forall f \in L^1,\ \forall t > 0. \tag{1.1}$$

(ii) There exists a constant c_2 so that

$$\|f\|_2^{2+4/\mu} \le c_2\, \mathscr{E}(f,f)\, \|f\|_1^{4/\mu}, \quad \forall f \in \mathrm{Dom}(\mathscr{E}) \cap L^1. \tag{1.2}$$

The inequality (1.2) is sometimes called the Nash inequality due to his pioneering work. If $\mu > 2$ the conditions above are equivalent to

(iii) There exists a constant c_3 so that

$$\|f\|_{2\mu/(\mu-2)}^2 \le c_3\, \mathscr{E}(f,f), \quad \forall f \in \mathrm{Dom}(\mathscr{E}). \tag{1.3}$$

Since the ultracontractivity is important in applications, e.g., it deduces the boundedness of transition probability densities, it is well-discussed. On the contrary, exchanging L^1 and L^∞ formally, we discuss the property that the semigroup sends L^∞ to L^1. We call this property as the dual ultracontractivity. As applications, we discuss the one-dimensional diffusion processes. We give some conditions for the dual ultracontractivity and compare them with conditions for the ultracontractivity.

The organization of the paper is as follows. We introduce the notion of dual ultracontractivity and give some conditions for it in Section 2. In Section 3, we consider one-dimensional diffusion processes and give conditions for the dual ultracontractivity.

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2. The dual ultracontractivity

Let (M, m) a measure space and (X_t, P_x) be an m-symmetric Markov process. We denote the life time by ζ. We also denote the measure with the initial distribution m by P_m:

$$P_m = \int_M P_x \, m(dx).$$

We are mainly interested in the case m is a infinite measure and so P_m is not finite in general. The associated semigroup $\{T_t\}$ is defined by

$$T_t f(x) = E_x[f(X_t)].$$

Here E_x stands for the integration with respect to P_x. $\{T_t\}$ is a C_0 contraction semigroup in $L^p(m)$ for $p \geq 1$. We denote the generator by $\mathfrak{A}$:

$$T_t = e^{t\mathfrak{A}}.$$

$\mathfrak{A}$ is a closed operator in $L^p(m)$. It is also a contraction semigroup in $L^\infty(m)$ but not strongly continuous in general. If $p = 2$, $\mathfrak{A}$ is a self-adjoint operator and the associated Dirichlet form $\mathscr{E}$ is defined by

$$\mathscr{E}(f, g) = (\sqrt{-\mathfrak{A}} f, \sqrt{-\mathfrak{A}} g),$$

where $(\ ,\ )$ is the inner product in $L^2(m)$. The domain of $\mathscr{E}$, which we denote by $\mathrm{Dom}(\mathscr{E})$, is $\mathrm{Dom}(\sqrt{-\mathfrak{A}})$.

The semigroup $\{T_t\}$ is called ultracontractive if T_t send $L^1(m)$ into $L^\infty(m)$ for all $t > 0$. Exchanging $L^1(m)$ and $L^\infty(m)$, we introduce a new notion as follows:

Definition 2.1. We say that the semigroup $\{T_t\}$ is *dual ultracontractive* if T_t send $L^\infty(m)$ into $L^1(m)$ for all $t > 0$:

We denote an operator norm of A from L^p into L^q by $\|A\|_{p\to q}$. Then a_t defined

$$a_t = \|T_t\|_{\infty\to 1} \tag{2.1}$$

is decreasing as a function of $t > 0$. Since $T_t 1(x) = P_x(\zeta > t)$, we can easily see that $\|T_t 1\|_1 = a_t$ and hence $a_t = P_m(\zeta > t)$. We have the following.

Proposition 2.1. Set $b_t = \|T_t\|_{2\to 1}$. Then

$$b_{2t}^2 \leq a_{2t} \leq b_t^2. \tag{2.2}$$

Proof. By the duality, we have $\|T_t\|_{\infty\to 2} = \|T_t\|_{2\to 1}$. Hence

$$a_{2t} = \|T_{2t}\|_{\infty\to 1} \leq \|T_t\|_{2\to 1} \|T_t\|_{\infty\to 2} = b_t^2.$$

Conversely, note that $\|T_t\|_{\infty\to 1} = a_t \ \|T_t\|_{\infty\to\infty} \leq 1$, which follows from the Markov property. By using the Riesz-Thorin interpolation theorem, we have

$$b_t = \|T_t\|_{\infty\to 2} \leq a_t^{1/2}$$

as we wanted. □

As was seen in the proposition above, the dual ultracontractivity is closely related to the tail probability of the life time. We will consider this life time problem in the next section.

In the sequel, we give some criteria for the dual ultracontractivity. Criteria for the ultracontractivity and their proofs are well-developed and so, by mimicking them, we can easily have the following theorem.

Theorem 2.2. For a given $\mu > 0$, the followings are equivalent to each other:
(i) There exists a constant c_1 so that

$$\|T_t f\|_1 \le c_1 t^{-\mu/2} \|f\|_\infty, \quad \forall f \in L^\infty, \ \forall t > 0. \tag{2.3}$$

(ii) There exists a constant c_1 so that

$$\|f\|_2^{2+4/\mu} \le c_2 \, \mathscr{E}(f,f) \, \|f\|_\infty^{4/\mu}, \quad \forall f \in \mathrm{Dom}(\mathscr{E}) \cap L^\infty. \tag{2.4}$$

Proof. From (i), we have

$$\|T_{2t} f\|_1 \le C_1 t^{-\mu/2} \|f\|_\infty.$$

Hence, for $0 \le f \in \mathrm{Dom}(\mathscr{E}) \cap L^\infty$

$$\begin{aligned}
C_1 t^{-\mu/2} \|f\|_\infty^2 &\ge \|T_{2t} f\|_1 \|f\|_\infty \\
&\ge (T_{2t} f, f) \\
&= (f,f) + \int_0^t \frac{d}{ds} (T_s f, T_s f) \, ds \\
&= (f,f) - 2 \int_0^t \mathscr{E}(T_s f, T_s f) \, ds \\
&\ge (f,f) - 2t \mathscr{E}(f,f),
\end{aligned}$$

which brings

$$\|f\|_2^2 \le 2t \mathscr{E}(f,f) + C_1 t^{-\mu/2} \|f\|_\infty^2.$$

Now, choosing

$$t = \left(\frac{\|f\|_\infty^2}{\mathscr{E}(f,f)} \right)^{2/(\mu+2)},$$

we have

$$\|f\|_2^2 \le (2 + C_1) \mathscr{E}(f,f)^{\mu/(\mu+2)} \|f\|_\infty^{4/(\mu+2)}.$$

Thus

$$\|f\|_2^{2+4/\mu} \le (2 + C_1)^{(\mu+2)/\mu} \mathscr{E}(f,f) \|f\|_\infty^{4/\mu},$$

which is (ii).

Conversely, suppose (2.4). Take $f \in \mathrm{Dom}(\mathscr{E}) \cap L^\infty$ and set $u(t) = \|T_t f\|_2^2$. Then

$$-\frac{du}{dt} = 2\mathscr{E}(T_t f, T_t f) \geq 2\|T_t f\|_2^{2+4/\mu}/(c_2\|T_t f\|_\infty^{4/\mu}) \geq 2u^{1+2/\mu}/(c_2\|f\|_\infty^{4/\mu}).$$

Hence

$$\frac{d}{dt}(u^{-2/\mu}) \geq \frac{4}{c_2\mu\|f\|_\infty^{4/\mu}}$$

and so

$$u^{-2/\mu}(t) \geq u(t)^{-2/\mu} - u(0)^{-2/\mu} \geq \frac{4t}{c_2\mu\|f\|_\infty^{4/\mu}}.$$

Finally, we have

$$\|T_t f\|_2 = u(t)^{1/2} \leq (c_2\mu/4t)^{\mu/4}\|f\|_\infty \leq (c_2\mu/4)^{\mu/4} t^{-\mu/4}\|f\|_\infty.$$

By approximation, the inequality above holds for all $f \in L^\infty$. Now (2.3) follows from Proposition 2.1. □

From Proposition 2.1, (2.3) is equivalent to the following:

$$\|T_t f\|_1 \leq c_4 t^{-\mu/4}\|f\|_2. \quad \forall f \in L^2,\ \forall t > 0. \tag{2.5}$$

We use this fact without mentioning.

We can also treat a little relaxed dual ultracontractivity as follows.

Corollary 2.3. For a given $\mu > 0$, the followings are equivalent to each other:
(i) There exists a constant c_1 so that

$$\|T_t f\|_1 \leq c_1 t^{-\mu/2}\|f\|_\infty, \quad \forall f \in L^\infty,\ \forall t \in (0, 1]. \tag{2.6}$$

(ii) There exists a constant c_1 so that

$$\|f\|_2^{2+4/\mu} \leq c_2\left(\mathscr{E}(f,f) + \|f\|_2^2\right)\|f\|_\infty^{4/\mu},\ \forall f \in \mathrm{Dom}(\mathscr{E}) \cap L^\infty. \tag{2.7}$$

Proof. Using Theorem 2.2 for $\mathfrak{A} - 1$, we have that (2.7) is equivalent to

$$\|T_t f\|_1 \leq c_3 t^{-\mu/2} e^t \|f\|_\infty. \tag{2.8}$$

Now (2.6) follows from (2.8).

Conversely suppose (2.6). Note that for $t > 1$

$$\|T_t f\|_1 = \|T_1 T_{t-1} f\|_1 \leq c_1\|T_{t-1} f\|_\infty \leq c_1\|f\|_\infty. \quad (\because \{T_t\} \text{ is contractive})$$

We now easily see that (2.8) holds for all $t > 0$. □

Let us proceed to another kind of criterion. Before that we need to introduce the operator $(-\mathfrak{A})^{-1/2}$ as follows:

$$(-\mathfrak{A})^{-1/2} = \frac{1}{\Gamma(1/2)} \int_0^\infty t^{-1/2} T_t \, dt. \tag{2.9}$$

This can be done if the integral converges.

To consider the continuity of $(-\mathfrak{A})^{-1/2}$, we recall the space $L^{p,\infty}(m)$. For $p \geq 1$, the space $L^{p,\infty}(m)$ is the set of functions satisfying

$$\sup_{\lambda>0} \lambda^p m(\{x;\, |f(x)| > \lambda\}) = \|f\|_{p,\infty}^p < \infty.$$

If an operator T is bounded from L^p into $L^{q,\infty}$, i.e., there exists a constant C such that

$$\|Tf\|_{q,\infty} \leq C\|f\|_p, \quad \forall f \in L^p,$$

T is said to be of weak type (p, q).

Proposition 2.4. Suppose that a constant $\mu > 0$ and indices $1 < q < p < \infty$ satisfy

$$\frac{1}{q} = \frac{1}{p} + \frac{1}{\mu}. \tag{2.10}$$

If (2.3) holds, then $(-\mathfrak{A})^{-1/2}$ is a bounded operator from L^p into L^q.

Proof. (2.10) bears $q > 1$ and hence

$$\frac{1}{\mu} < 1 - \frac{1}{p} = \frac{1}{p'},$$

where p' is the conjugate exponent of p. This means $\mu > p'$.

We will show that $(-\mathfrak{A})^{-1/2}$ is of weak type (p, q). To do this, we assume $\|f\|_p = 1$ and set

$$\Gamma(1/2)(-\mathfrak{A})^{-1/2} f = \int_0^T t^{-1/2} T_t f \, dt + \int_T^\infty t^{-1/2} T_t f \, dt =: I_T + J_T.$$

For any $\lambda > 0$,

$$m(\{x;\, |(-\mathfrak{A})^{-1/2} f(x)| > 2\lambda\}) \leq m(\{x;\, |I_T(x)| > \lambda\}) + m(\{x;\, |J_T(x)| > \lambda\}).$$

As for I_T,

$$\|I_T\|_p = \left\| \int_0^T t^{-1/2} T_t f \, dt \right\|_p \leq 2\|f\|_p T^{1/2}.$$

Hence, by the Chebyshev inequality, we have

$$m(\{x;\, |I_T(x)| > \lambda\}) \leq \frac{\|I_T\|_p^p}{\lambda^p} \leq 2^p \lambda^{-p} T^{p/2}.$$

Now we choose $T = \lambda^{2q/\mu}$. Then

$$m(\{x;\, |I_T(x)| > \lambda\}) \le C_1 \lambda^{-p} \lambda^{2qp/2\mu} = \lambda^{pq\{(1/\mu)-(1/q)\}} = \lambda^{-q}. \quad (\because (2.10)$$

As for J_T, from the assumption, we have

$$\begin{aligned} &\|T_t\|_{\infty\to 1} \le C_2 t^{-\frac{\mu}{2}} \\ &\|T_t\|_{1\to 1}. \le 1 \end{aligned}$$

By using the interpolation theorem, it follows that

$$\|T_t\|_{p\to 1} \le C_3 t^{-\frac{\mu}{2}(1-\frac{1}{p})}.$$

Hence we have

$$\|J_T\|_1 \le \int_T^\infty t^{-1/2} \|T_t f\|_1 \, dt \le C_3 \int_T^\infty t^{-1/2} t^{-\frac{\mu}{2}(1-\frac{1}{p})} \le C_4 T^{\frac{1}{2}-\frac{\mu(p-1)}{2p}}.$$

Since $\mu > p'$, $\frac{1}{2} - \frac{\mu(p-1)}{2p} < 0$ and hence the integral above converges. Again by the Chebyshev inequality, we have

$$m(\{x;\, |I_T(x)| > \lambda\}) \le \frac{\|J_T\|_1}{\lambda} \le C_4 \lambda^{-1} T^{\frac{1}{2}-\frac{\mu(p-1)}{2p}}.$$

Recalling that $T = \lambda^{2q/\mu}$,

$$m(\{x;\, |I_T(x)| > \lambda\}) \le C_4 \lambda^{-1} \lambda^{\frac{2q}{\mu}(\frac{1}{2}-\frac{\mu(p-1)}{2p})} = C_4 \lambda^{-1+\frac{q}{\mu}-\frac{(p-1)q}{p}} = C_4 \lambda^{q(-\frac{1}{q}+\frac{1}{\mu}-1+\frac{1}{p})} = C_4 \lambda^{-q}.$$

Thus we have

$$m(\{x;\, |(-\mathfrak{A})^{-1/2} f(x)| > 2\lambda\}) \le C_5 \lambda^{-q}.$$

This shows that $(-\mathfrak{A})^{-1/2}$ is of weak type (p, q).

Since p, q can be chosen freely under the restriction (2.10), the Marcinkiewicz interpolation theorem yields that $(-\mathfrak{A})^{-1/2}$ is of strong type (p, q), i.e., bounded from L^p into L^q. □

By using this, we can give another necessary and sufficient condition for (2.3) when $\mu > 2$ as follows.

Theorem 2.5. When $\mu > 2$, (2.3) is equivalent to the following: There exists a constant $c_3 > 0$ so that

$$\|f\|^2_{2\mu/(\mu+2)} \le c_3\, \mathscr{E}(f, f) \quad \forall f \in \mathrm{Dom}(\mathscr{E}). \tag{2.11}$$

Proof. When $p = 2$, (2.10) implies $q = 2\mu/(\mu + 2)$. Now Proposition 2.4 shows that $(-\mathfrak{A})^{-1/2}$ is bounded from L^2 into $L^{2\mu/(\mu+2)}$ and hence

$$\|(-\mathfrak{A})^{-1/2} f\|_{2\mu/(\mu+2)} \le C_1 \|f\|_2.$$

Substituting $(-\mathfrak{A})^{1/2}f$ with f, we have

$$\|f\|_{2\mu/(\mu+2)}^2 \le C_1^2\,\|(-\mathfrak{A})^{1/2}f\|_2^2 = C_1^2\,\mathscr{E}(f,f).$$

Conversely, suppose (2.11). Since $\frac{2\mu}{\mu+2} + \frac{4}{\mu+2} = 2$,

$$\begin{aligned}
\int_M f(x)^2\,dm &= \int_M |f(x)|^{2\mu/(\mu+2)}|f(x)|^{4/(\mu+2)}\,dm \\
&\le \|f\|_\infty^{4/(\mu+2)} \int_M |f(x)|^{2\mu/(\mu+2)}\,dm \\
&\le \|f\|_\infty^{4/(\mu+2)}\|f\|_{2\mu/(\mu+2)}^{2\mu/(\mu+2)} \\
&\le C_2\|f\|_\infty^{4/(\mu+2)}\mathscr{E}(f,f)^{\mu/(\mu+2)}.
\end{aligned}$$

Now we can easily have (2.4). □

Note that we do not use the property $\mu > 2$ in the last half of the proof above. This means that for $\mu > 0$, (2.11) deduces (2.4). We will use this fact in the next section.

Lastly we will give a remark on $(1-\mathfrak{A})^{-r/2}$. The operator $(1-\mathfrak{A})^{-r/2}$ is defined by

$$(1-\mathfrak{A})^{-r/2} = \frac{1}{\Gamma r/2}\int_0^\infty t^{(r/2)-1}e^{-t}T_t\,dt. \tag{2.12}$$

For $r > 0$, this is a bounded operator in L^p. By using the similar method above, we have the following.

Proposition 2.6. Assume (2.3) for $t \in (0,1]$. If $r > \mu$, then $(1-\mathfrak{A})^{-r/2}$ is a bounded operator from L^∞ into L^1.

Proof. From the assumption, when $t \in (0,1]$, we have

$$\|T_t\|_{\infty\to 1} \le c_1 t^{-\mu/2}.$$

For $t > 1$,

$$\|T_t\|_{\infty\to 1} \le c_1.$$

Then, using $\mu < r$, we have

$$\begin{aligned}
\|(1-\mathfrak{A})^{-r/2}\|_{\infty\to 1} &\le \frac{1}{\Gamma r/2}\int_0^\infty t^{(r/2)-1}e^{-t}\|T_t\|_{\infty\to 1}\,dt \\
&\le \frac{1}{\Gamma r/2}\int_0^1 t^{(r/2)-1}e^{-t}c_1 t^{-\mu/2}\,dt + \frac{1}{\Gamma r/2}\int_1^\infty t^{(r/2)-1}e^{-t}c_1\,dt < \infty.
\end{aligned}$$

This completes the proof. □

3. One dimensional diffusion processes

In this section, we will consider one dimensional diffusion processes and give some conditions for the dual ultracontractivity. To contrast this with the ultracontractivity, we also give some conditions for the ultracontractivity.

To simplify the notation, we give names to necessary notions. As for ultracontractivity

$$R_\mu: \qquad \|T_t f\|_\infty \le C t^{-\mu/2}\|f\|_1, \quad \forall f \in L^1,\ \forall t > 0.$$

Localizing this property, we call

$$R_\mu(0): \qquad \|T_t f\|_\infty \le C t^{-\mu/2}\|f\|_1, \quad \forall f \in L^1,\ \forall t \in (0,1].$$

Asymptotic behavior at $t = \infty$ is also an interesting problem. We will discuss the exponential decay in some cases.

As for dual ultracontractivity,

$$S_\mu: \qquad \|T_t f\|_1 \le C t^{-\mu/2}\|f\|_\infty, \quad \forall f \in L^\infty,\ \forall t > 0,$$
$$S_\mu(0): \qquad \|T_t f\|_1 \le C t^{-\mu/2}\|f\|_\infty, \quad \forall f \in L^\infty,\ \forall t \in (0,1],$$

Next we review fundamental facts of one dimensional diffusion processes. The space is $D = (l_1, l_2)$. We only consider the minimal diffusion on D, i.e., we impose the Dirichlet boundary condition if necessary. A diffusion on D is characterized by speed a measures m and scale function $s(x)$. We assume, without loss of generality, that $s(x) = x$ (i.e., the natural scale). The generator is given by $\frac{d}{dm}\frac{d}{dx}$ and the associated Dirichlet form is given by

$$\mathscr{E}(f,g) = \int_{l_1}^{l_2} \frac{df}{dx}\frac{dg}{dx}dx. \tag{3.1}$$

It is enough to consider the three cases: $D = (0,l)$, $(0,\infty)$ or $(infty,\infty)$.

3.1 The case $D = (0, l)$

Suppose $D = (0,l)$. Take any $\mu > 0$ and introduce the following condition:

$$\sup_{x>0} x^{\mu/(\mu+2)} m([x, l/2)) < \infty, \tag{3.2}$$

$$\sup_{l/2<x<l} (l-x)^{\mu/(\mu+2)} m([l/2, x)) < \infty. \tag{3.3}$$

According to the Feller classification, 0 and l are both exit if and only if

$$\int_0^{l/2} x\, m(dx) < \infty, \tag{3.4}$$

$$\int_{l/2}^{l} (l-x)\, m(dx) < \infty. \tag{3.5}$$

Hence the above condition (3.2) and (3.3) is stronger than this.

Theorem 3.1. S_μ holds if and only if (3.2) and (3.3) hold. Further this condition is equivalent to $S_\mu(0)$ as well.

Proof. We first show that (3.2) and (3.3) are sufficient for S_μ. We divide $(0,l)$ into tow parts $(0,l/2)$, $[l/2,l)$. We only consider the part $(0,l/2)$. The other is similar. Define an operator I by

$$If(x) = \int_0^x f(t)\,dt. \tag{3.6}$$

We regard I as an operator from $L^p((0,l/2),dt)$ into $L^q((0,l/2),dm)$. We investigate the continuity of I.

If $f \in L^1((0,l/2),dt)$, then

$$|If(x)| \le \int_0^x |f(t)|\,dt \le \|f\|_1$$

which yields that $I\colon L^1 \to L^\infty$ is bounded.

On the other hand, for $f \in L^\infty((0,l/2),dt)$,

$$|If(x)| \le \int_0^x |f(t)|\,dt \le x\|f\|_\infty.$$

Now, by the assumption (3.2), setting $M = \sup_{x>0} x^{\mu/(\mu+2)} m([x,l/2))$, we have

$$m(\{x;\, |If(x)| > \lambda\}) \le m(\{x;\, x\|f\|_\infty > \lambda\}) = m((\lambda/\|f\|_\infty, l/2)) \le M\left(\frac{\lambda}{\|f\|_\infty}\right)^{-\mu/(\mu+2)}.$$

This means that I is of weak type $(\infty, \frac{\mu}{\mu+2})$. Thus, by the Marcinkiewicz interpolation theorem, we see that I is a bounded operator from L^2 into $L^{2\mu/(\mu+2)}$.

Note that the boundary condition at 0 is the Dirichlet boundary condition. Hence $g(0) = 0$ for $f \in \mathrm{Dom}(\mathscr{E})$. So, in terms of the Dirichlet form, the above inequality is rewritten as

$$\|g\|_{2\mu/(\mu+2)}^2 \le C_1\,\mathscr{E}(g,g), \quad (g = If).$$

Now, by Theorem 2.5 (to be precise, by the remark after Theorem 2.5), we can see that have that S_μ holds.

Nest we show the sufficiency. Here, we will deduce (3.2), (3.3) from $S_\mu(0)$. Supposing $S_\mu(0)$, (2.7) holds. Take $x \in (0,l/2)$ and define

$$f(u) = \begin{cases} \frac{u}{x}, & 0 < u < x \\ 1, & u \ge x. \end{cases}$$

Applying (2.7) to f, we have

$$\left\{\frac{1}{x^2}\int_{(0,x)} u^2\,dm(u) + m([x,l/2))\right\}^{1+\mu/2} \le c_2\left\{\frac{1}{x} + \frac{1}{x^2}\int_{(0,x)} u^2 dm(u) + m([x,l/2))\right\}.$$

Multiplying $x^{1+\mu/2}$ to the both hands, we have

$$\left\{\frac{1}{x}\int_{(0,x)} u^2\, dm(u) + xm([x,l/2))\right\}^{1+2/\mu} \tag{3.7}$$
$$\leq c_2\left\{x^{2/\mu} + x^{-1+2/\mu}\int_{(0,x)} u^2 dm(u) + x^{1+2/\mu}m([x,l/2))\right\}$$
$$\leq c_2 x^{2/\mu}\left\{1 + \frac{1}{x}\int_{(0,x)} u^2 dm(u) + xm([x,l/2))\right\}.$$

Then it follows that

$$\sup_x\left\{\frac{1}{x}\int_{(0,x)} u^2\, dm(u) + xm([x,l/2))\right\} < \infty.$$

This means that the coefficient of $x^{2/\mu}$ in the right hand side of (3.7) is bounded. Therefore

$$\{xm([x,l/2))\}^{1+2/\mu} \leq C_1 x^{2/\mu},$$

which implies

$$x^{\mu/(\mu+2)}m([x,l/2)) \leq C_2.$$

Thus we have (3.2).
We can show (3.3) in the same manner. □

By Theorem 3.1, S_μ holds under the assumption (3.2), (3.3). Using this fact, we can investigate the asymptotic behavior of the tail probability of life time. That is the motivation to study the dual ultracontractivity. So assume (3.2) and (3.3). Define the Green kernel $G(x,y)$ by

$$G(x,y) = \begin{cases} \frac{x(l-y)}{l}, & x \leq y, \\ \frac{y(l-x)}{l}, & y \leq x. \end{cases} \tag{3.8}$$

This name comes from the fact that, defining a operator G by

$$Gf(x) = \int_{(0,l)} G(x,y)f(y)\, m(dy),$$

we have $G = (-\mathfrak{A})^{-1}$.
From the assumption, we can show that G is an operator of trace class. Hence $-\mathfrak{A}$ has a discrete spectrum, which we denote $0 < \lambda_0 < \lambda_1 \leq \cdots$. Let φ_j be an eigenfunction for φ_j and we choose them to consist of c.o.n.s in $L^2(m)$. Then the Green kernel is written as

$$G(x,y) = \sum_{i=1}^{\infty}\frac{1}{\lambda_i}\varphi_i(x)\varphi_i(y).$$

In particular,

$$G(x,x) = \frac{x(l-x)}{l} = \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \varphi_i(x)^2.$$

Moreover the transition probability density is given by

$$p(t,x,y) = \sum_{j=0}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y). \tag{3.9}$$

By Proposition 2.6, $(1-\mathfrak{A})^{-N}$ is a bounded operator from L^2 into L^1 for sufficiently large N. Noting

$$(1-\mathfrak{A})^{-N} \varphi_i = \frac{1}{(1+\lambda_i)^N} \varphi_i,$$

we have

$$\frac{1}{(1+\lambda_i)^N} \|\varphi_i\|_1 = \|(1-\mathfrak{A})^{-N} \varphi_i\|_1 \leq \|(1-\mathfrak{A})^{-N}\|_{2\to 1} \|\varphi_i\|_2 = \|(1-\mathfrak{A})^{-N}\|_{2\to 1}.$$

That is $\|\varphi_i\|_1 \leq C(1+\lambda_i)^N$.

Now we can compute the tail probability of the life time.

$$\begin{aligned}
&P_x[\zeta > t] - e^{-\lambda_0 t} \varphi_0(x) \int_D \varphi_0(y) dm(y) \\
&\qquad = \int_D p(t,x,y) dm(y) - e^{-\lambda_0 t} \varphi_0(x) \int_D \varphi_0(y) dm(y) \\
&\qquad = \int_D \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y) dm(y) - e^{-\lambda_0 t} \varphi_0(x) \int_D \varphi_0(y) dm(y) \\
&\qquad = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \int_D \varphi_i(y) dm(y).
\end{aligned}$$

Hence

$$\begin{aligned}
&|P_x[\zeta > t] - e^{-\lambda_0 t} \varphi_0(x) \int_D \varphi_0(y) dm(y)| \\
&\qquad \leq \sum_{i=1}^{\infty} e^{-\lambda_i t} |\varphi_i(x)| \int_D |\varphi_i(y)| dm(y) \\
&\qquad \leq \sum_{i=1}^{\infty} e^{-\lambda_i t} |\varphi_i(x)| C(1+\lambda_i)^N \\
&\qquad \leq C e^{-\lambda_1 t} \left\{ \sum_{i=1}^{\infty} e^{-2(\lambda_i - \lambda_1)t} (1+\lambda_i)^{2N} \lambda_i \right\}^{1/2} \left\{ \sum_{i=1}^{\infty} \frac{1}{\lambda_i} \varphi_i(x)^2 \right\}^{1/2} \\
&\qquad \leq C e^{-\lambda_1 t} \left\{ \sum_{i=1}^{\infty} e^{-2(\lambda_i - \lambda_1)t} \lambda_i (1+\lambda_i)^{2N} \right\}^{1/2} G(x,x)^{1/2}.
\end{aligned}$$

When $t \geq 1$, noting (3.8) again, we have

$$\sum_{i=1}^{\infty} e^{-2(\lambda_i-\lambda_1)t}\lambda_i(1+\lambda_i)^{2N} \leq \sum_{i=1}^{\infty} e^{-2(\lambda_i-\lambda_1)}\lambda_i^2(1+\lambda_i)^{2N}\frac{1}{\lambda_i}$$
$$\leq \sup_{x\geq l_1}\{e^{-2(x-\lambda_1)}x^2(1+x)^{2N}\}\sum_{i=1}^{\infty}\frac{1}{\lambda_i} < \infty.$$

Thus

$$P_x[\zeta > t] - e^{-\lambda_0 t}\varphi_0(x)\int_D \varphi_0(y)dm(y) = e^{-\lambda_1 t}O(1) \quad \text{as } t \to \infty.$$

So we obtain the exponential decay of the tail probability.

As for the dual ultracontractivity, we have

$$|P_m(\zeta > t) - e^{-\lambda_0 t}\|\varphi_0\|_1^2| = \left|\iint_D p(t,x,y)\,dm(x)dm(y) - e^{-\lambda_0 t}\left\{\int_D \varphi_0(x)\,dm(x)\right\}^2\right|$$
$$\leq \sum_{i=1}^{\infty}\int_D\int_D e^{-\lambda_i t}|\varphi_i(x)|\,|\varphi_i(y)|\,dm(x)dm(y)$$
$$\leq C^2\sum_{i=1}^{\infty} e^{-\lambda_i t}(1+\lambda_i)^{2N}$$
$$\leq C^2 e^{-\lambda_1 t}\sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)t}(1+\lambda_i)^{2N}.$$

When $t \geq 1$, we have

$$\sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)t}(1+\lambda_i)^{2N} \leq \sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)}\lambda_i(1+\lambda_i)^{2N}\frac{1}{\lambda_i}$$
$$\leq \sup_{x\geq l_1}\{e^{-(x-\lambda_1)}x(1+x)^{2N}\}\sum_{i=1}^{\infty}\frac{1}{\lambda_i} < \infty.$$

Thus we obtained the exponential decay of $a_t = P_m(\zeta > t)$ as $t \to \infty$.

When $\mu = 0$, conditions (3.2), (3.3) would be interpreted as $m((0,l)) < \infty$. In this case, $\|T_0 1\|_1 = \|1\|_1 < \infty$ and so $\|T_t\|_{\infty\to 1} \leq \|1\|_1 < \infty$ for $0 < t \leq 1$. This means that $S_\mu(0)$ holds for $\mu = 0$.

If the integrals of (3.2) and (3.3) diverge, i.e.,

$$\int_{(0,l/2)} x\,m(dx) = \infty, \tag{3.10}$$

$$\int_{(l/2,l)} (l-x)\,m(dx) = \infty, \tag{3.11}$$

then the diffusion is conservative and the measure is infinite and so $T_t 1 = 1$ which means the dual ultracontractivity does not hold.

We also discuss the ultracontractivity to compare with the dual ultracontractivity. To show the ultracontractivity, it suffices to prove (1.3). For $\mu > 2$, we introduce the following conditions:

$$\sup_{x>0} x^{\mu/(\mu-2)} m([x,l/2)) < \infty, \tag{3.12}$$

$$\sup_{l/2<x<l} (l-x)^{\mu/(\mu-2)} m([l/2,x)) < \infty. \tag{3.13}$$

The following theorem is essentially proved by Mao [7] but we give here alternative proof based on the interpolation theorem.

Theorem 3.2. Let $\mu > 2$. Then R_μ holds if and only if (3.12) and (3.13) hold.

Proof. We first assume (3.12) and (3.13) and show that R_μ holds.

Define $I\colon L^p((0,l/2),dt) \to L^q((0,l/2),dm)$ by (3.6). As in the proof of Theorem 3.1, I is not only of strong type $(1,\infty)$ but also of weak type and $(\infty, \frac{\mu}{\mu-2})$. By the Marcinkiewicz interpolation theorem, it follows that I is of strong type $(2, \frac{2\mu}{\mu-2})$. That is, we have

$$\|f\|^2_{2\mu/(\mu-2)} \le C\,\mathscr{E}(f,f).$$

Now R_μ follows.

Conversely, assume R_μ for $\mu > 2$. Then (1.3) holds. Under this condition, we will show (3.12) and (3.13).

We consider on a interval $(0,l/2)$. For fixed $x \in (0,l/2)$, define f by

$$f(u) = \begin{cases} \frac{u}{x}, & 0 < u < x \\ 1, & u \ge x. \end{cases}$$

Since f satisfies (1.3), we have

$$\left\{ \int_{(0,x)} \left(\frac{u}{x}\right)^{2\mu/(2\mu-2)} dm(u) + m([x,l/2)) \right\}^{2(\mu-2)/2\mu} \le C \int_0^x \frac{1}{x^2} du.$$

Hence

$$m([x,l/2))^{(\mu-2)/\mu} \le C/x$$
$$x^{\mu/(\mu-2)} m([x,l/2)) \le C^{\mu/(\mu-2)}.$$

Thus we have obtained (3.12). (3.13) is proved similarly. □

When $\mu = 2$, R_2 always holds. In fact, since D is a finite interval, we have

$$\|f\|^2_\infty \le C_4 \mathscr{E}(f,f)$$

and so R_2 holds. When

$$\int_{(0,l/2)} x\, m(dx) = \infty, \quad \int_{(l/2,l)} (l-x)\, m(dx) = \infty,$$

the diffusion become conservative but still R_2 holds i.e., the transition probability density converges to 0 uniformly.

Lastly we give a sufficient condition to ensure the exponential decay of the transition probability densities as $t \to \infty$. Let us assume that the Green operator is of Hilbert-Schmidt class. That is, $G(x, y)$ defined by (3.8) satisfies

$$\int_{(0,l)\times(0,l)} G(x,y)^2 \, dm(x)dm(y) < \infty. \tag{3.14}$$

This condition if equivalent to the following condition:

$$\int_{(0,l/2)} x^2 m([x, l/2)) \, dm(x) < \infty, \tag{3.15}$$

$$\int_{[l/2,l)} x^2 m([l/2, x]) \, dm(x) < \infty. \tag{3.16}$$

To see this, we first derive (3.16) from (3.15) and (3.16). We consider the part of $x \le y$ of the integral region of (3.14) and divide it into the following three parts:

$$\begin{array}{ll} \text{(i)} & 0 < x < l/2, \quad x \le y < l/2, \\ \text{(ii)} & 0 < x \le l/2, \quad l/2 \le y < l, \\ \text{(iii)} & l/2 \le y < l, \quad l/2 < x \le y. \end{array}$$

As for (i), we have

$$\begin{aligned} \int_{(0,l/2)} x^2 \, dm(x) \int_{[x,l/2)} (l-y)^2 \, dm(y) &\le l^2 \int_{(0,l/2)} x^2 \, dm(x) \int_{[x,l/2)} dm(y) \\ &\le l^2 \int_{(0,l/2)} x^2 m([x, l/2)) \, dm(x). \end{aligned}$$

As for (ii),

$$\int_{(0,l/2)} x^2 \, dm(x) \int_{[l/2,l)} (l-y)^2 \, dm(y) < \infty,$$

and as for (iii),

$$\begin{aligned} \int_{(l/2,l)} (l-y)^2 \, dm(y) \int_{(l/2,y]} x^2 \, dm(x) &\le l^2 \int_{(l/2,l)} (l-y)^2 \, dm(y) \int_{(l/2,y]} dm(x) \\ &\le l^2 \int_{(l/2,l)} (l-y)^2 m((l/2, y]) \, dm(y). \end{aligned}$$

Thus (3.14) was shown.

As for (i), (iii), reversed inequality holds by changing constants. So we can easily see that (3.15) and (3.16) is derived from (3.14).

(3.15) and (3.16) are also equivalent to

$$\int_{(0,l/2)} x m([x, l/2))^2 \, dx < \infty, \tag{3.17}$$

$$(3.18) \qquad \int_{[l/2,l)} (l-x)^2 m([l/2,x])dm(x) < \infty.$$

To see this, note that

$$\begin{aligned} m([x,l/2))^2 &= \int_{[x,l/2)\times[x,l/2)} dm(u)dm(v) \\ &= \int_{[x,l/2)} dm(u) \int_{(u,l/2)} dm(v) + \int_{[x,l/2)} dm(v) \int_{[v,l/2)} dm(u) \\ &= \int_{[x,l/2)} \{m((u,l/2) + m([u,l/2)\}\, dm(u). \end{aligned}$$

Then

$$\begin{aligned} \int_{(0,l/2)} xm([x,l/2))^2\, dx &= \int_{(0,l/2)} x\, dx \int_{[x,l/2)} \{m((u,l/2)) + m([u,l/2))\}\, dm(u) \\ &\leq 2\int_{(0,l/2)} x\, dx \int_{[x,l/2)} m([u,l/2))\, dm(u) \\ &\leq 2\int_{(0,l/2)} m([u,l/2))\, dm(u) \int_{(0,u]} x\, dx \\ &\leq 2\int_{(0,l/2)} u^2 m([u,l/2))\, dm(u). \end{aligned}$$

Thus (3.17) is obtained form (3.15). Conversely if we assume (3.17), then

$$\begin{aligned} \int_{(0,l/2)} u^2 m([u,l/2))\, dm(u) &= 2\int_{(0,l/2)} m([u,l/2))\, dm(u) \int_{(0,u]} x\, dx \\ &= 2\int_{(0,l/2)} x\, dx \int_{[x,l/2)} m([u,l/2))\, dm(u) \\ &= 2\int_{(0,l/2)} x\, dx \int_{[x,l/2)} \{m([u,l/2)) + m((u,l/2))\}\, dm(u) \\ &= 2\int_{(0,l/2)} xm([x,l/2)^2\, dx. \end{aligned}$$

So we have (3.15).

Under the assumption that G is of Hilbert-Schmidt class, the transition probability density is expressed as (3.9). By taking N to be large enough, $(1-\mathfrak{A})^{-N}\colon L^2(m) \to L^\infty(m)$ becomes a bounded operator. Since $(1-\mathfrak{A})^{-N}\varphi_i = \varphi_i/(1+\lambda)^N$, it holds that

$$\frac{1}{(1+\lambda_i)^N}\|\varphi_i\|_\infty = \|G^N\varphi_i\|_\infty = \|G^N\|_{2\to\infty}\|\varphi_i\|_2.$$

Therefore

$$\|\varphi_i\|_\infty \leq C(1+\lambda_i)^N.$$

Now we have

$$
\begin{aligned}
|p(t,x,y) - e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y)| &\le \sum_{i=1}^{\infty} e^{-\lambda_i t}|\varphi_i(x)\varphi_i(y)| \\
&\le C^2 \sum_{i=1}^{\infty} e^{-\lambda_i t}(1+\lambda_i)^{2N} \\
&\le C^2 e^{-\lambda_1 t} \sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)t}(1+\lambda_i)^{2N}.
\end{aligned}
$$

If $t \ge 1$, then

$$
\begin{aligned}
|p(t,x,y) - e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y)| &\le Ce^{-\lambda_1 t} \sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)}(1+\lambda_i)^{2N} \\
&\le Ce^{-\lambda_1 t} \sum_{i=1}^{\infty} e^{-(\lambda_i-\lambda_1)}\lambda_i^2(1+\lambda_i)^{2N}\frac{1}{\lambda_i^2} \\
&\le Ce^{-\lambda_1 t} \sup_{x\ge\lambda_1}\{e^{-(x-\lambda_1)}x^2(1+x)^{2N}\} \sum_{i=1}^{\infty}\frac{1}{\lambda_i^2}.
\end{aligned}
$$

Hence we have that as

$$|p(t,x,y) - e^{-\lambda_0 t}\varphi_0(x)\varphi_0(y)| = e^{-\lambda_1 t}O(1) \quad \text{as } t \to \infty,$$

i.e., $p(t,x,y)$ decays exponentially.

3.2 The case $D = (0,\infty)$

We assume $D = (0,\infty)$. For $\mu > 0$, we consider the following condition:

$$\sup_{0<x<1} x^{\mu/(\mu+2)} m([x,1)) < \infty, \tag{3.19}$$

$$\sup_{x\ge 1} x^{\mu/(\mu+2)} m([x,\infty)) < \infty. \tag{3.20}$$

The condition that 0 is exit and ∞ is non-exit & entrance is

$$\int_{(0,1)} x\, m(dx) < \infty, \tag{3.21}$$

$$\int_{[1,\infty)} x\, m(dx) < \infty. \tag{3.22}$$

(3.19) is stronger than (3.21) and (3.20) is weaker than (3.22). We have the following.

Theorem 3.3. (3.19) and (3.20) are necessary and sufficient for S_μ.

Proof. We first show the sufficiency.

As in the proof of Theorem 3.1, we define I by

$$If(x) = \int_0^x f(t)\,dt. \tag{3.23}$$

We regard I as an operator from $L^p((0,\infty),dt)$ into $L^q((0,\infty),dm)$. We investigate the continuity of I.

For $f \in L^1((0,\infty),dt)$, it holds that

$$|If(x)| \le \|f\|_1$$

and hence $I\colon L^1((0,\infty),dt) \to L^\infty((0,\infty),dm)$ is bounded. If $f \in L^\infty((0,\infty),dt)$, then

$$|If(x)| \le x\|f\|_\infty.$$

From the assumptions (3.19) and (3.20), $M = \sup_{x>0} x^{\mu/(\mu+2)} m([x,\infty)) < \infty$. Therefore

$$m(\{x;\, |If(x)| > \lambda\}) \le m(\{x;\, x\|f\|_\infty > \lambda\}) = m((\lambda/\|f\|_\infty,\infty)) \le M\left(\frac{\lambda}{\|f\|_\infty}\right)^{-\mu/(\mu+2)}.$$

This means that I is of weak type $(\infty, \mu/(\mu+2))$. By the Marcinkiewicz interpolation theorem, we see that I is an bounded operator from $L^2((0,\infty),dt)$ into $L^{2\mu/(\mu+2)}((0,\infty),dm)$.

Since the boundary condition at 0 is the Dirichlet condition, $f(0) = 0$ for $f \in \mathrm{Dom}(\mathscr{E})$. So the we can rewrite the inequality above as

$$\|f\|_{2\mu/(\mu+2)}^2 \le C_1\,\mathscr{E}(f,f).$$

Now, by Theorem 2.5, S_μ follows.

Conversely we assume S_μ. Then, by Theorem 2.2, we have

$$\|f\|_2^{2+4/\mu} \le C_1 \mathscr{E}(f,f)\|f\|_\infty^{2+4/\mu}.$$

For $x > 0$, we define

$$f(u) := u \wedge x = \min\{u, x\}, \quad u > 0.$$

Applying the above inequality to this function f, we have

$$\left\{\int_{(0,x)} u^2 dm(u) + x^2 \int_{[x,\infty)} dm(u)\right\}^{1+2/\mu} \le C\int_0^x du \times x^{4/\mu}.$$

Hence

$$\begin{gathered} x^{2+4/\mu} m([x,\infty))^{(\mu+2)/\mu} \le C x^{1+4/\mu} \\ x m([x,\infty))^{(\mu+2)/\mu} \le C \\ x^{\mu/(\mu+2)} m([x,\infty)) \le C^{\mu/(\mu+2)}, \end{gathered}$$

which shows (3.19), (3.20). □

Relaxing the condition, we can get the following.

Theorem 3.4. $S_\mu(0)$ holds if and only if (3.12) and $m(([1,\infty)) < \infty$ holds.

Proof. First we assume (3.12) and $m(([1,\infty)) < \infty$. As in the proof of Theorem 3.3, it follows, from the assumption (3.12), that

$$\left\{\int_{(0,1]} |f(x)|^{2\mu/(\mu+2)} m(dx)\right\}^{(\mu+2)/\mu} \leq C_1 \mathscr{E}(f,f).$$

On the other hand, since $\frac{\mu}{\mu+2} + \frac{2}{\mu+2} = 1$, we can apply the Hölder inequality on $[1,\infty)$ and get

$$\begin{aligned}\int_{[1,\infty)} |f(x)|^{2\mu/(\mu+2)}\, m(dx) &\leq \left\{\int_{[1,\infty)} |f(x)|^{\frac{2\mu}{\mu+2}\cdot\frac{\mu+2}{\mu}}\, m(dx)\right\}^{\mu/(\mu+2)} \left\{\int_{[1,\infty)} 1^{\frac{\mu+2}{2}}\, m(dx)\right\}^{2/(\mu+2)} \\ &\leq \left\{\int_{[1,\infty)} |f(x)|^2\, m(dx)\right\}^{\mu/(\mu+2)} m([1,\infty))^{2/(\mu+2)}.\end{aligned}$$

Therefore

$$\left\{\int_{[1,\infty)} |f(x)|^{2\mu/(\mu+2)}\, m(dx)\right\}^{(\mu+2)/\mu} \leq \left\{\int_{[1,\infty)} |f(x)|^2\, m(dx)\right\} m([1,\infty))^{2/\mu}.$$

By combining both of them, we have

$$\|f\|^2_{2\mu/(\mu+2} \leq C_2(\|f\|_2^2 + \mathscr{E}(f,f)),$$

which leads $S_\mu(0)$.

Conversely, we assume $S_\mu(0)$. By Corollary 2.3, we have

$$\|f\|_2^{2+4/\mu} \leq C_3(\|f\|_2^2 + \mathscr{E}(f,f))\|f\|_\infty^{4/\mu}.$$

As for f, we take the following function f_n $(n \in \mathbb{N})$:

$$f_n(x) = \begin{cases} x, & 0 < x \leq 1, \\ 1, & 1 < x \leq n, \\ n+1-x, & n < x \leq n+1, \\ 0, & x > n+1. \end{cases}$$

Then, we have

$$\left\{\int_{(0,\infty)} f_n(x)^2\, dm(x)\right\}^{1+2/\mu} \leq \left\{2 + \int_{(0,\infty)} f_n(x)^2\, dm(x)\right\}.$$

Dividing the both hand by $\int_{(0,\infty)} f_n(x)^2\, dm(x)$, we have

$$\left\{\int_{(0,\infty)} f_n(x)^2\, dm(x)\right\}^{1+2/\mu} \leq \left\{\frac{2}{\int_{(0,\infty)} f_n(x)^2\, dm(x)} + 1\right\}.$$

From this, it follows that $\lim_{n\to\infty} \int_{(0,\infty)} f_n(x)^2 \, dm(x) < \infty$, which shows $m([1,\infty)) < \infty$.

On the interval $(0,1)$, making a similar argument as in Theorem 3.3, we can get (3.19). □

If $\int_{(0,1)} x \, dm(x) = \infty$, then the boundary 0 is non-exit and ∞ is non-exit as well. Now the diffusion is conservative and the measure is infinite and so the dual ultracontractivity never holds.

Lastly we investigate the asymptotic behavior of the tail probability of the life time. We assume (3.19) and the following:

$$\int_{(1,\infty)} x^2 m([x,\infty)) \, dm(x) < \infty. \tag{3.24}$$

Recall that the Green kernel is given by

$$G(x,y) = x \wedge y.$$

Under this condition, we show that the Green operator is of Hilbert-Schmidt class. Since the Green kernel is symmetric, we consider on the region $x \le y$. For $0 < x \le 1$, we have

$$\int_{(0,1]} dm(x) \int_{[x,1]} (x \wedge y)^2 \, dm(y) = \int_{(0,1]} x^2 m([x,\infty)) \, dm(x) < \infty.$$

For $x > 1$,

$$\int_{(1,\infty)} dm(x) \int_{[x,\infty)} (x \wedge y)^2 \, dm(y) = \int_{(1,\infty)} x^2 m([x,\infty)) \, dm(x) < \infty.$$

Hence the Green operator is of Hilbert-Schmidt class. We can also prove that (3.24) holds if G is of Hilbert-Schmidt class. Moreover we can easily show that (3.24) is also equivalent to

$$\int_{(1,\infty)} x m([x,\infty))^2 \, dm(x) < \infty. \tag{3.25}$$

Under this condition, the dual ultracontractivity holds. Since G has discrete spectrum, the transition probability density $p(t,x,y)$ has the following eigen-function expansion:

$$p(t,x,y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y).$$

Moreover the tail probability of the life time ζ is expressed as

$$P_x(\zeta > T) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(x) \int_D \varphi_i(y) \, dm(y).$$

So, as in the case of $D = (0,l)$, we can show that

$$\left| P_x(\zeta > T) - e^{-\lambda_0 t} \varphi_0(x) \int_D \varphi_0(y) \, dm(y) \right| = e^{-\lambda_1 t} O(1).$$

We also have

$$\|T_t\|_{\infty\to 1} = e^{-\lambda_0 t}O(1) \quad \text{as } t\to\infty$$

in the same manner.

Now we proceed to the ultracontractivity.
For $\mu > 2$, we introduce the following condition:

$$\sup_{0<x<1} x^{\mu/(\mu-2)} m([x,1)) < \infty, \tag{3.26}$$

$$\sup_{x\geq 1} x^{\mu/(\mu-2)} m([x,\infty)) < \infty. \tag{3.27}$$

Then we have the following.

Theorem 3.5. *R_μ holds if and only if (3.26) and (3.27) hold.*

Proof. Define $If(x) = \int_0^x f(t)\,dt$. As we have seen, $I\colon L^1((0,\infty),dt) \to L^\infty((0,\infty),dm)$ is bounded. Further, for $f \in L^\infty((0,\infty),dt)$, we have $|If(x)| \leq x\|f\|_\infty$. Hence, by (3.26) and (3.27), we have

$$m(\{x;\,|If(x)|>\lambda\}) \leq m(\{x;\,x\|f\|_\infty > \lambda\}) = m((\lambda/\|f\|_\infty,\infty)) \leq C\left(\frac{\lambda}{\|f\|_\infty}\right)^{-\mu/(\mu-2)}.$$

This means that I is of weak type $(\infty,\mu/(\mu-2))$. Now the Marcinkiewicz interpolation theorem yields that I is a bounded operator from $L^2((0,\infty),dt)$ into $L^{2\mu/(\mu-2)((0,\infty),dm)}$. This leads to

$$\|f\|^2_{2\mu/(\mu-2)} \leq C_1\,\mathscr{E}(f,f), \tag{3.28}$$

which implies R_μ.

Next we show the converse. So we assume . As f in (3.28), we take the following f_n ($n\in\mathbb{N}$):

$$f_n(x) = \begin{cases} x, & 0<x\leq 1,\\ 1, & 1<x\leq n,\\ n+1-x, & n<x\leq n+1,\\ 0, & x>n+1.\end{cases}$$

Then we have

$$\left\{\int_{(0,\infty)} f_n(x)^{2\mu/(\mu-2)}\,dm(x)\right\} \leq 2C_2.$$

which brings that $m([1,\infty)) < \infty$.

For $x > 0$, define $f(u) = u \wedge x$. From EqEOD.84, we have

$$\|f\|_{2\mu/(\mu-2)}^2 = \left\{ \int_{(0,x)} u^{2\mu/(\mu-2)} dm(u) + x^{2\mu/(\mu-2)} \int_{[x,\infty)} dm(u) \right\}^{(\mu-2)/\mu} \leq C_3 \int_0^x du.$$

Hence

$$x^2 m([x,\infty))^{\mu-2)/\mu} \leq C_3 x.$$
$$x^{\mu/(\mu-2)}\, m([x,\infty)) \leq C_3^{\mu/(\mu-2)}.$$

Since x is arbitrary, we can see that (3.26) and (3.27) hold. □

The necessary and sufficient condition for $R_\mu(0)$ is given in the following theorem. We take $\mu > 2$.

Theorem 3.6. Assume $m([1,\infty)) < \infty$. Under this condition, $R_\mu(0)$ holds if and only if (3.27) holds.

Proof. We divide the interval into two parts: $(0,1)$ and $[1,\infty)$. On $(0,1)$,

$$|f(x)| = \left| \int_0^x \frac{df}{dt}(t)\, dt \right| \leq \sqrt{x} \left\{ \int_0^x \left(\frac{df}{dt} \right)^2 dt \right\}^{1/2}.$$

Hence

$$\sup_{0<x<1} |f(x)| \leq \mathscr{E}(f,f)^{1/2}.$$

Noticing this, we have

$$\begin{aligned} \int_{(0,1)} |f(x)|^{2\mu/(\mu-2)}\, dm(x) &= \int_{(0,1)} |f(x)|^2 |f(x)|^{4/(\mu-2)}\, dm(x) \\ &\leq \left(\sup_{0<x<1} |f(x)| \right)^{4/(\mu-2)} \int_{(0,1)} |f(x)|^2\, dm(x) \\ &\leq \|f\|_2^2 \mathscr{E}(f,f)^{2/(\mu-2)}. \end{aligned}$$

Therefore

$$\left\{ \int_{(0,1)} |f(x)|^{2\mu/(\mu-2)}\, dm(x) \right\}^{(\mu-2)/\mu} \leq (\|f\|_2^2)^{(\mu-2)/\mu} \mathscr{E}(f,f)^{2/\mu}.$$

Now we recall the following inequality: for $0 < \alpha < 1$, $x^\alpha y^{1-\alpha} \leq \alpha x + (1-\alpha) y$. Using this inequality with $\alpha = \frac{\mu-2}{\mu}$, we have

$$\left\{ \int_{(0,1)} |f(x)|^{2\mu/(\mu-2)}\, dm(x) \right\}^{(\mu-2)/\mu} \leq \frac{\mu-2}{\mu} \|f\|_2^2 + \frac{2}{\mu} \mathscr{E}(f,f).$$

So the estimate on $(0,1)$ is done. On the interval $[1,\infty)$, consider the operator

$$If(x) = \int_1^x f(u)du, \quad x \geq 1.$$

We investigate the continuity of I as an operator from $L^p([1,\infty), dt)$ into $L^q([1,\infty), dm)$. Under our condition, we can show that it is of strong type $(2, \frac{2\mu}{\mu-2})$. From this, we have

$$\left\{\int_{[1,\infty)} |f(x) - f(1)|^{2\mu/(\mu-2)}\, dm(x)\right\}^{(\mu-2)/2\mu} \leq C\left\{\int_1^\infty \left(\frac{df}{dt}\right)^2 dt\right\}^{1/2} \leq C\mathscr{E}(f,f)^{1/2}.$$

We now recall that $|f(1)| \leq \mathscr{E}(f,f)^{1/2}$. Then

$$\begin{aligned}\left\{\int_{[1,\infty)} |f(x)|^{2\mu/(\mu-2)}\, dm(x)\right\}^{(\mu-2)/2\mu} &\leq \left\{\int_{[1,\infty)} |f(x) - f(1)|^{2\mu/(\mu-2)}\, dm(x)\right\}^{(\mu-2)/\mu} \\ &\quad + \left\{\int_{[1,\infty)} |f(1)|^{2\mu/(\mu-2)}\, dm(x)\right\}^{(\mu-2)/\mu} \\ &\leq \mathscr{E}(f,f)^{1/2} + |f(1)| m([1,\infty))^{(\mu-2)/2\mu} \\ &\leq (1 + m([1,\infty))^{(\mu-2)/2\mu})\mathscr{E}(f,f)^{1/2}.\end{aligned}$$

Combining both of them, we have

$$\|f\|^2_{2\mu/(\mu-2)} \leq C(\|f\|_2^2 + \mathscr{E}(f,f)). \tag{3.29}$$

So $R_\mu(0)$ follows.

Conversely, assuming (3.29), we will show (3.27). To avoid the complexity of notation, we take D to be $(-1,\infty)$. For $x > 0$, define a function f by $f(u) = u \wedge x$. For $u < 0$, we set $f(u) = 0$. Using (3.29), we have

$$\begin{aligned}\|f\|^2_{2\mu/(\mu-2)} &= \left\{\int_{(0,x)} u^{2\mu/(\mu-2)}\, dm(u) + x^{2\mu/(\mu-2)} m([x,\infty))\right\}^{(\mu-2)/\mu} \\ &\leq C\left\{\int_0^x 1\, du + \int_{(0,x)} u^2\, dm(u) + x^2 m([x,\infty))\right\}.\end{aligned}$$

Hence

$$x^2 m([x,\infty))^{(\mu-2)/\mu} \leq C\left\{x + \int_{(0,x)} u^2\, dm(u) + x^2 m([x,\infty))\right\}.$$

Now, noting that

$$\begin{aligned}\int_{(0,x)} u^2\, dm(u) &= \int_{(0,x)} dm(u) \int_{(0,u]} 2t\, dt = \int_{(0,x)} 2t\, dt \int_{[t,x)} dm(u) \\ &= \int_{(0,x)} 2t m([t,x))\, dt \leq \int_{(0,x)} 2t m([t,\infty))\, dt,\end{aligned}$$

we have

$$x^2 m([x,\infty))^{(\mu-2)/\mu} \le C\Big\{ x + \int_{(0,x)} 2tm([t,\infty))\,dt + x^2 m([x,\infty)) \Big\}.$$

Dividing the both hands by x, we have

$$xm([x,\infty))^{(\mu-2)/\mu} \le C\Big\{ 1 + \frac{1}{x}\int_{(0,x)} 2tm([t,\infty))\,dt + xm([x,\infty)) \Big\}.$$

For the constant C above, choose x_0 so that

$$m([x_0,\infty))^{1-(\mu-2)/\mu} \le \frac{1}{4C}$$

and define

$$K(x) = \sup_{x_0 \le t \le x} t\, m([t,\infty)^{(\mu-2)/\mu}.$$

For $x > x_0$,

$$\begin{aligned} &xm([x,\infty))^{(\mu-2)/\mu} \\ &\quad \le C\Big\{ 1 + \frac{1}{x}\int_{(0,x_0)} 2tm([t,\infty))\,dt + \frac{1}{x}\int_{[x_0,x)} 2tm([t,\infty))^{(\mu-2)/\mu} m([t,\infty))^{1-(\mu-2)/\mu}\,dt \\ &\qquad\quad + xm([x,\infty))^{(\mu-2)/\mu} m([x,\infty))^{1-(\mu-2)/\mu} \Big\} \\ &\quad \le C\Big\{ 1 + \frac{1}{x_0}\int_{(0,x_0)} 2tm([t,\infty))\,dt + \frac{1}{x}\int_{[x_0,x)} 2K(x)\frac{1}{4C}\,dt \\ &\qquad\quad + xm([x,\infty))^{(\mu-2)/\mu}\frac{1}{4C} \Big\}. \end{aligned}$$

Hence

$$\frac{3}{4}xm([x,\infty))^{(\mu-2)/\mu} \le C\Big\{ 1 + \frac{1}{x_0}\int_{(0,x_0)} 2tm([t,\infty))\,dt \Big\} + \frac{K(x)}{2}$$

In the left hand side, running x through $x_0 \le x \le y$ and taking the supremum, we have

$$\frac{3}{4}K(y) \le C\Big\{ 1 + \frac{1}{x_0}\int_{(0,x_0)} 2tm([t,\infty))\,dt \Big\} + \frac{K(y)}{2}.$$

Thus we have

$$\frac{1}{4}K(y) \le C\Big\{ 1 + \frac{1}{x_0}\int_{(0,x_0)} 2tm([t,\infty))\,dt \Big\}.$$

Since y is arbitrary, we see that K is bounded and $\sup_{x>0} x\, m([x,\infty)^{(\mu-2)/\mu} < \infty$ follows. Now (3.27) is shown and the proof is completed. □

Lastly we discuss the exponential decay of the transition probability. We assume the following.

$$\int_0^1 x\, m([x,1))^2 dx < \infty \tag{3.30}$$

$$\sup_{x\ge 1} x^{\mu/(\mu-2)} m([x,\infty)) < \infty. \tag{3.31}$$

(3.31) is same as (3.27). Under this condition, we will show that the transition probability density decays exponentially. Recall that the Green kernel is given by

$$G(x,y) = x \wedge y$$

and the Green operator $G = (-\mathfrak{A})^{-1}$ is given by

$$Gf(x) = \int_{(0,\infty)} G(x,y) f(y)\, dm(x).$$

From the assumption, we can show that G is of Hilbert-Schmidt class. We show this as follows. We only consider on the region $y \ge x$ by the symmetry. Further , we divide the region into two parts according to $0 < x \le 1$ and $x > 1$. For the first region,

$$\begin{aligned}\int_{0<x\le 1} dm(x) \int_{y\ge x} G(x,y)^2\, dm(y) &= \int_{0<x\le 1} dm(x) \int_{y\ge x} x^2\, dm(y)\\ &= \int_{0<x\le 1} x^2\, m([x,\infty))\, dm(x) < \infty.\end{aligned}$$

And for the second region,

$$\int_{x>1} dm(x) \int_{y\ge x} G(x,y)^2\, dm(y) = \int_{x>1} x^2\, m([x,\infty))\, dm(x) < \infty.$$

Thus G is of Hilbert-Schmidt class and so it has the discrete spectrum and the ultracontractivity holds as before. So we can show the exponential decay of the probability density function $p(t,x,y)$ in a similar way as in the case $D = (0,l)$.

3.3 The case $D = (-\infty, \infty)$

In this case, the diffusion is conservative and

$$T_t 1 = 1.$$

Hence, if m is finite, then the semigroup is always dual ultracontractive. But, since $\|T_t\|_{1,\infty} = m(\mathbb{R})$, the operator norm $\|T_t\|\|_{\infty\to 1}$ does not decay as in (2.3). If m is infinite, the dual ultracontractivity never holds since $1 \notin L^1$.

The dual ultracontractivity is of no interest in this case. We are much interested in the ultracontractivity. As for this problem, we only consider the case $m((-\infty,\infty)) < \infty$. For $\mu > 2$, we introduce the following conditions.

$$\sup_{x\geq 1} x^{\mu/(\mu-2)} m([x,\infty)) < \infty, \tag{3.32}$$

$$\sup_{x\geq 1} x^{\mu/(\mu-2)} m((-\infty,-x]) < \infty. \tag{3.33}$$

We have the following theorem.

Theorem 3.7. Assume $m((-\infty,\infty)) < \infty$ and take any $\mu > 2$. Then $R_\mu(0)$ holds if and only if (3.32) and (3.33) hold.

Proof. The necessity of (3.32) and (3.33) can be proved in the same way as Theorem 3.6. We show the sufficiency. As in the same way as Theorem 3.6, we have

$$\left\{\int_{[x,\infty)} |f(y)-f(x)|^{2\mu/(\mu-2)}\, dm(y)\right\} \leq C\mathscr{E}(f,f)^{1/2}.$$

From this, for $-1 \leq x \leq 0$, we get

$$\begin{aligned}\left\{\int_{[0,\infty)} |f(y)|^{2\mu/(\mu-2)}\, dm(y)\right\}^{(\mu-2)/2\mu} &= \left\{\int_{[0,\infty)} |f(y)-f(x)+f(x)|^{2\mu/(\mu-2)}\, dm(y)\right\}^{(\mu-2)/2\mu} \\ &\leq \left\{\int_{[0,\infty)} |f(y)-f(x)|^{2\mu/(\mu-2)}\, dm(y)\right\}^{(\mu-2)/2\mu} \\ &\quad + \left\{\int_{[0,\infty)} |f(x)|^{2\mu/(\mu-2)}\, dm(y)\right\}^{(\mu-2)/2\mu} \\ &\leq C\mathscr{E}(f,f)^{1/2} + |f(x)| m([0,\infty))^{(\mu-2)/2\mu}.\end{aligned}$$

Integrating the both hands side on $[-1,0]$ with respect to x, we have

$$\begin{aligned}&\left\{\int_{[0,\infty)} |f(y)|^{2\mu/(\mu-2)}\, dm(y)\right\}^{(\mu-2)/2\mu} m([-1,0]) \\ &\quad\leq C\mathscr{E}(f,f)^{1/2} m([-1,0]) + \int_{[0,1]} |f(x)|\, dm(x) m([0,\infty))^{(\mu-2)/2\mu} \\ &\quad\leq C\mathscr{E}(f,f)^{1/2} m([-1,0]) + \left\{\int_{[0,1]} |f(x)|^2\, dm(x)\right\}^{1/2} m([-1,0])^{1/2} m([0,\infty))^{(\mu-2)/2\mu}.\end{aligned}$$

Hence

$$\begin{aligned}&\left\{\int_{[0,\infty)} |f(y)|^{2\mu/(\mu-2)}\, dm(y)\right\}^{(\mu-2)/2\mu} \\ &\quad\leq C\mathscr{E}(f,f)^{1/2} + \left\{\int_{[0,1]} |f(x)|^2\, dm(x)\right\}^{1/2} m([-1,0])^{-1/2} m([0,\infty))^{(\mu-2)/2\mu}.\end{aligned}$$

We can get similar estimate on $(-\infty,0]$ and so we have

$$\|f\|^2_{2\mu/(\mu-2)} \leq C(\mathscr{E}(f,f) + \|f\|_2^2),$$

which is the desired result. □

Under the assumption of (3.32) and (3.33), the resolvent $G_\alpha = (\alpha - \mathfrak{A})^{-1}$ becomes a trace class operator and the transition probability density $p(t,x,y)$ has an eigen-function expansion. Since the lowest eigenvalue of $-\mathfrak{A}$ is 0 and the eigen-function is a constant function 1, we can show that $|p(t,x,y) - m((-\infty,\infty))^{-1}|$ converges to 0 exponentially. Moreover this convergence is uniform in x and y.

Lastly, we consider the Ornstein-Uhlenbeck process. Its generator is $\mathfrak{A} = \frac{d^2}{dx^2} - x\frac{d}{dx}$. It is well-known that the process does not satisfy the ultracontractivity. The speed measure is

$$m(dx) = e^{-x^2/2}dx$$

and the scale function is

$$s(x) = \int_0^x e^{y^2/2}dy \sim \frac{e^{x^2/2}}{x} \quad \text{as } x \to \infty.$$

Further, we have

$$m([x,\infty)) \sim \frac{e^{-x^2/2}}{x} \quad \text{as } x \to \infty$$

and hence

$$s(x)m([x,\infty)) \sim \frac{1}{x^2} \to 0 \quad \text{as } x \to \infty.$$

Therefore we have

$$\sup_{x\geq 0} s(x)m([x,\infty)) < \infty.$$

This shows that, in the natural scale, $\sup_{x\geq 0} x^{\mu/(\mu-2)}m([x,\infty)) < \infty$ is not sufficient for the ultracontractivity.

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