Closed manifolds with transcendental L^2-Betti numbers

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Abstract

In this paper, we show how to construct examples of closed manifolds with explicitly computed irrational, even transcendental L^2 Betti numbers, defined via the universal covering.

We show that every non-negative real number shows up as an L^2-Betti number of some covering of a compact manifold, and that many computable real numbers appear as an L^2-Betti number of a universal covering of a compact manifold (with a precise meaning of computable given below).

In algebraic terms, for many given computable real numbers (in particular for many transcendental numbers) we show how to construct a finitely presented group and an element in the integral group ring such that the L^2-dimension of the kernel is the given number.

We follow the method pioneered by Austin [2] but refine it to get very explicit calculations which make the above statements possible.

1 Introduction

In 1974, Atiyah defined L^2-Betti numbers for covering spaces of closed manifolds [1]. A priori these Betti numbers are real and Atiyah asked at the end of his paper to find examples where they are irrational. The question remained open and the fact that these L^2-Betti numbers may always be rational, and even integral for torsion free groups, has become known as the "Atiyah conjecture". Under conditions on the torsion in the group, more refined conjectures have been formulated and popularized as the "strong Atiyah conjectures", [12, Chapter 10], [4, Definition 1.1], which are satisfied for many groups.

Let us observe that the discussion is concerned with two slightly different cases:

- Atiyah from the very beginning studied arbitrary normal coverings of a compact manifold M. The resulting values for the L^2-Betti numbers may be very different depending on which covering of M they are associated with.

- The most important special case, often exclusively considered in later work, uses the universal covering of the manifold M. This way, one defines invariants depending only on M: these are the invariants typically meant by *the* L^2-Betti numbers of M.

The L^2-Betti numbers are homotopy invariants of the underlying manifold M. It follows from this that, when considering only the universal covering, i.e. *the* L^2-Betti numbers, there is in total only a countable set of possible values.

However, a given space can have uncountably many different normal coverings (corresponding to the normal subgroups of the fundamental group) so that the set of possible L^2-Betti numbers of normal coverings of compact manifolds a priori could well be uncountable.

In a recent paper [2], Tim Austin showed that the set of L^2 Betti numbers associated to all possible normal coverings of compact manifolds is uncountable, and in particular contains irrational (and even transcendental) values.

In the present paper, we show how to construct examples of closed manifolds with explicitly given irrational (and transcendental) L^2-Betti numbers for their universal coverings. As explained below, we follow closely the techniques developed by Austin in [2], with refinements which allow us to make explicit dimension calculations. Explicit calculations (and to some extend the basis of all these developments) have been carried out previously in [3, 7, 6], which already lead to unexpected values of L^2-Betti numbers, not however to any which one could prove to be irrational.

The problem at hand has a well known purely algebraic reformulation. The aim is to produce a finitely presented group G and an element Q in the group algebra $\mathbb{Z}[G]$ such that $\dim_G(\ker(Q))$, the von Neumann dimension of the kernel of this operator acting on $l^2(G)$ is irrational. Then there is a standard construction to obtain a closed 7-dimensional manifold M with the fundamental group isomorphic to G and whose third L^2 Betti number (computed using the universal covering) is equal to the $\dim_G(\ker(Q))$ [12, Lemma 10.5] and [6, Proposition 6 and Theorem 7].

If, instead of starting with a finitely presented group one only starts with a finitely generated group G, the standard construction will result in a manifold M with normal covering $\overline{M}$ (which is not necessarily the universal covering) such that the third L^2-Betti number for this covering is equal to $\dim_G(\ker(Q))$.

Actually, we construct a group G which is not finitely presented but admits a recursive presentation and thus embeds into a finitely presented group H by Higman's theorem [8]. For a suitable element $Q \in \mathbb{Q}[G]$ we prove that $\dim_G(\ker(Q))$ is transcendental. Clearing denominators, we can achieve that $Q \in \mathbb{Z}[G]$ without changing its kernel. Finally, it is a standard fact that the dimension of the kernel does not change if we let Q act on $l^2(H)$, compare e.g. [15, Proposition 3.1].

The group G will be of the form

$$(\mathbb{Z}_2^{\oplus \Gamma} / V) \rtimes \Gamma$$

where V is a suitable Γ-invariant subspace of $\mathbb{Z}_2^{\oplus \Gamma}$. For Γ, we will choose either the free group on two generators F_2 (as in [2]) or $\mathbb{Z} \wr \mathbb{Z}$.

The main result of [2] is to construct an uncountable family of groups G_i of the form above and operators $Q_i \in \mathbb{Q}[G_i]$ such that the numbers $\dim_{G_i}(\ker(Q_i))$ are all mutually different. It seems hard to prove that among those groups for which $\dim_{G_i}(\ker(Q_i))$ is irrational are recursively presented groups, as their existence is only inferred from a counting argument.

The new approach of this paper consists of considering different operators Q for which one can explicitly compute $\dim_G(\ker(Q))$. We are able to explicitly produce a recursively presented group for which $\dim_G(\ker(Q))$ is transcendental.

Namely, for any set of natural numbers $I = \{0, n_k\} \subset \mathbb{N}$ (listed in increasing order $0 < n_1 < n_2 < \dots$) we construct a group G_I as above whose presentation is determined by the set I together with $Q_I \in \mathbb{Q}[G_I]$ such that

$$\dim_{G_I}(\ker(Q_I)) = \beta_1 + \beta_2 \sum_{k=1}^{\infty} 2^{-dn_k + k}$$

where β_1 and β_2 are some explicit rational numbers and d is a natural number.

We prove that G_I has a recursive presentation (and therefore embeds into a finitely presented group) if (and only if) I is recursively enumerable. It is now immediate to choose a recursively enumerable set I which leads to an irrational or even transcendental L^2-dimensions, e.g. by asking it to satisfy the Liouville condition. Recall that a real number x is a Liouville number if for any positive integer n, there exist integers p and q with $q > 1$ and such that $0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}$. Liouville [11] showed that such numbers are transcendental.

Let us also stress the fact that we obtain these L^2-Betti numbers for solvable groups (this is the reason why we use the group $\Gamma = \mathbb{Z} \wr \mathbb{Z}$), answering a question of [2]. Note that for torsion-free solvable groups the Atiyah conjecture is known [10, Theorem 1.3].

Using the explicit form of these L^2-dimensions of kernels for the operators we obtain, we can construct out of these for each real number $r \geq 0$ a group G_r (in general not recursively presented) and $A_r \in M_n(\mathbb{Z}[G_r])$ with $\dim(\ker(A_r)) = r$. This relies on explicit knowledge of how the kernel looks like under the operations we employ.

We will discuss possible extensions of the result which can be obtained with the same method. In particular, with suitable modifications and additional effort one could produce many examples of $A \in M_n(\mathbb{Z}[G])$ as above with explicit knowledge of the full spectral measure (as in [3, 7]). This spectral measure would be atomic and many of the L^2-dimensional of the eigenspaces would be transcendental.

We also discuss more about the question which L^2-Betti numbers can (by modifications of the construction) be obtained using finitely presented groups.

Lukasz Grabowski [5] has independently and simultaneously, using an approach which implements Turing machines directly in the integral group ring of a suitable recursively presented group, arrived at results similar to ours. Using yet another strategy, Franz Lehner and Stephan Wagner [9] manage in a clever way to calculate explicitly the L^2-dimension of the kernel of the usual graph Laplacian (as in [3]), but for wreath products of finite groups with non-abelian free groups, and find non-rational algebraic numbers among the values —giving yet another intersting example.

2   Preliminary remarks

We closely follow the notations of [2]. This will hopefully help a reader interested in reading concurrently both papers.

We consider groups of the form $(\mathbb{Z}_2^{\oplus\Gamma}/V) \rtimes_\alpha \Gamma$ where V is some left translation invariant subgroup of $\mathbb{Z}_2^{\oplus\Gamma}$ and the action α of Γ is by translations on the left. Usually, we will omit α in the notation.

For the first few sections, we will only assume that Γ is generated by two elements s_1, s_2 with s_1 of infinite order. Later, concrete computations will be carried out notably in the case of the free group $\Gamma = F_2$ and of the wreath product $\Gamma = \mathbb{Z} \wr \mathbb{Z}$ with natural generating sets.

We denote by $\mathrm{Cay}(\Gamma, S)$ the right Cayley graph of Γ with respect to $S = \{s_1^{\pm 1}, s_2^{\pm 1}\}$. By a path P in $\mathrm{Cay}(\Gamma, S)$ we mean a subset $\{g_1, g_2, g_3, \ldots, g_\ell\} \subset \Gamma$ of mutually distinct consecutive elements of $\mathrm{Cay}(\Gamma, S)$.

We recall that the Pontrjagin dual of $\mathbb{Z}_2^{\oplus\Gamma}$ is isomorphic to the infinite product $\mathbb{Z}_2^\Gamma$. Sometimes we will identify an element $\chi \in \mathbb{Z}_2^\Gamma$ with the subset $\chi^{-1}(1) \subset \Gamma$. We denote by

$$V^\perp = \{\chi \in \mathbb{Z}_2^\Gamma \mid \langle \chi \mid v \rangle = 0 \ \ \forall v \in V\}$$

the dual of $\mathbb{Z}_2^{\oplus\Gamma}/V$ and remind that the Fourier transform

$$\ell^2(\mathbb{Z}_2^{\oplus\Gamma}/V) \simeq L^2(V^\perp, m_{V^\perp})$$

where $m_{V^\perp}$ is the Haar measure on $V^\perp$, induces a spatial isomorphism between the group von Neumann algebra $L((\mathbb{Z}_2^{\oplus\Gamma}/V) \rtimes \Gamma)$ and the cross-product von Neumann algebra $L^\infty(V^\perp) \rtimes_{\hat\alpha} \Gamma$. The dual action $\hat\alpha$ is on $V^\perp$ defined by $\langle \hat\alpha_g \chi \mid$

$u\rangle = \langle \chi \mid \alpha_{g^{-1}} u\rangle$ for $\chi \in V^{\perp}$ and $u \in \mathbb{Z}_2^{\oplus\Gamma}/V$. Note that, if we think of $\chi \in \mathbb{Z}_2^{\Gamma}$ as a subset of Γ, then $\hat\alpha_g \chi$ corresponds to the subset $g\chi^{-1}(1)$, i.e. is obtained by a left translation by g.

For simplicity we sometimes denote $s(\chi) := \hat\alpha_s \chi$. For $F \in L^{\infty}(V^{\perp})$ we denote by $M_F \in L^{\infty}(V^{\perp}) \rtimes_{\hat\alpha} \Gamma$ the twisted pointwise multiplication, defined by

$$M_F f(\chi, g) = F(\hat\alpha_g(\chi)) \cdot f(\chi, g)$$

and by T_s the translation operator given by $T_s f(\chi, g) = f(\chi, s^{-1}g)$. Checking the definitions, we observe the covariant relation

$$T_{s^{-1}} M_F T_s = M_{F \circ \hat\alpha_s}. \tag{2.1}$$

3 The revised operators

We want to construct certain operators in the rational group ring $\mathbb{Q}[\mathbb{Z}_2^{\oplus\Gamma}/V \rtimes \Gamma]$, viewed as acting on $L^2(V^{\perp}, m_{V^{\perp}}) \bar\otimes \ell^2(\Gamma)$. They will be taken to be of the form

$$A = \sum_{s \in S} T_{s^{-1}} (M_{F_s|V^{\perp}} + M_{F_{s^{-1}} \circ \hat\alpha_{s^{-1}}|V^{\perp}}), \tag{3.1}$$

where $F_s : \mathbb{Z}_2^{\Gamma} \to \mathbb{Q}$ will depend only on finitely many coordinates around the origin e. The operator A is self-adjoint as shown in [2, Lemma 3.1].

The essential difference with the operators of [2] is that the function F_s will recognize a very specific family of paths that we call "hooks" and which substitute the paths "with no small horizontal doglegs" of [2, Definition 3.2]. This one ingredient simplifies several computations and is what allows us to calculate the von Neumann dimensions exactly.

3.2 Definition. A path P (finite or not) in $\mathrm{Cay}(\Gamma, S)$ is called a *hook* if it has the form

$$P = \{gs_2^{-n}, \ldots, gs_2^{-1}, g, gs_1, gs_1 s_2^{-1}, \ldots, gs_1 s_2^{-m}\} \tag{3.3}$$

for some $g \in \Gamma$ and $n, m \in \{1, \ldots, \infty\}$. If $n < \infty$ then gs_2^{-n} is called the *left endpoint* of P. We call n the *length of the left leg* and m the *length of the right leg*.

We call a path P (finite or not) a *vertical segment* if

$$P = \{gs_2^{-n}, \ldots, g, \ldots, gs_2^m\} \qquad \text{for some } n, m \in \mathbb{Z}.$$

If $h, hs_2 \in P$, but $hs_2^{-1} \notin P$ we call h a *lower endpoint* of the hook or vertical segment P. If P is a hook and h is additionally the left endpoint, then h is called *left lower endpoint*.

We denote by $B(g, k)$ the ball of radius k around $g \in \Gamma$ in $\mathrm{Cay}(\Gamma, S)$.

3.4 Definition. Let $\chi \in \mathbb{Z}_2^{\Gamma}$. We say that χ is *1-good* if for some hook P in $\mathrm{Cay}(\Gamma, S)$ containing e, its restriction $\chi_{|B(e,1)}$ to $B(e, 1)$ equals 1 on P and 0 outside. We say that χ is *locally good* if χ is 1-good and $s(\chi)$ is 1-good for every $s \in \chi^{-1}(1) \cap B(e, 1)$. We say that χ is *interior good* if χ is locally good and $\left|\chi^{-1}(1) \cap B(e, 1)\right| = 3$. We say that χ is a *good end* if χ is locally good and e is a lower endpoint of the hook P above. This happens exactly if $\left|\chi^{-1}(1) \cap B(e, 1)\right| = 2$.

We now introduce $F_s\colon \mathbb{Z}_2^\Gamma \to \mathbb{Q}$. If $\chi \in \mathbb{Z}_2^\Gamma$ is interior good then

(i) $F_s(\chi) := 1$ if $s^{-1}(\chi)$ is interior good;

(e) $F_s(\chi) := 2$ if $s^{-1}(\chi)$ is a good end;

(b) $F_s(\chi) := \frac{1}{2}$ if $s^{-1}(\chi)$ is 1-good, but not locally good.

Define $F_s(\chi) := 0$ otherwise. Note that this happens if χ is not interior good or if χ is, but $s^{-1}\chi$ is not 1-good.

In the definition of the operator A, both $F_s(\chi)$ and $F_{s^{-1}}\hat{\alpha}_{s^{-1}}(\chi)$ appear. For further reference, we compile a little table showing the values of these two functions for the different possibilities. The columns give the different properties of χ, the rows those of $s^{-1}\chi = \hat{\alpha}_{s^{-1}}\chi$. The first value in each entry is $F_s(\chi)$, the second one $F_{s^{-1}}(s^{-1}\chi)$.

$s^{-1}\chi \setminus \chi$	int. good	good end	1-good, not loc. good	not 1-good
int. good	1; 1	0; 2	0; $\frac{1}{2}$	0; 0
good end	2; 0	0; 0	0; 0	0; 0
1-good, not loc. good	$\frac{1}{2}$; 0	0; 0	0; 0	0; 0
not 1-good	0; 0	0; 0	0; 0	0; 0

This follows by inspecting the definition of F_s. Note that $F_s(\chi)$ depends on χ and $s^{-1}\chi$, whereas $F_{s^{-1}}\hat{\alpha}_{s^{-1}}(\chi) = F_{s^{-1}}(s^{-1}\chi)$ depends on $s^{-1}\chi$ and $(s^{-1})^{-1}s^{-1}\chi = \chi$, which of course also explains why the second matrix we obtain is the transpose of the first.

Finally note that A is a sum of operators of the form $T_{s^{-1}}M_{G_s}$ where $G_s := F_s + F_{s^{-1}}\hat{\alpha}_{s^{-1}}$ itself is a linear combination of characteristic functions, and G_s depends only on the 3-neighborhood of e. For later reference and convenience we list the relevant values of $G_s(\chi)$ next. Note that $G_s(\chi)$ depends on χ and $s^{-1}\chi$; we will give a description now.

3.5 Proposition.   *(1) $G_s(\chi) = 0$ if χ is not a 1-good because then neither χ nor $s^{-1}\chi$ is interior good.*

(2) *(Case where χ is an end): Assume next that χ is 1-good and $\chi^{-1}(1) = \{e, s_2\}$. Then $G_s(\chi) = 0$ if $s \neq s_2$ because then $s^{-1}\chi$ is not 1-good. Moreover, $G_{s_2}(\chi) = 2$ if $s_2^{-1}\chi$ is interior good (i.e. the path extends two more steps) and $G_{s_2}(\chi) = 0$ otherwise.*

(3) *Now assume that χ is 1-good and $\chi^{-1}(1) = \{e, s_2^{-1}, t\}$. For $s \neq s_2^{-1}, t$, $G_s(\chi) = 0$ because then $s^{-1}\chi$ is not 1-good. Moreover,*

   (a) *(case where in one direction the path goes bad): if for $s \in \{s_2^{-1}, t\}$ $s^{-1}\chi$ is not 1-good (i.e. the path doesn't extend in this direction) then $G_s(\chi) = 0$. Write $\{s_2^{-1}, t\} = \{s, s'\}$, then $G_{s'}(\chi) = 0$ if $s'^{-1}\chi$ is not interior good, and $G_{s'}(\chi) = \frac{1}{2}$ otherwise.*

   (b) *(case where in one direction, necessarily s_2^{-1}, the path ends). If $s_2\chi$ is a good end then $G_{s_2^{-1}}(\chi) = 2$ if χ is interior good and $G_{s_2^{-1}}(\chi) = 0$ if χ is not interior good (the latter situation we've just discussed).*

   (c) *Assume now that χ is interior good, $s = \{s_2^{-1}, t\}$ and $s^{-1}\chi$ is interior good (i.e. the hook extends in two directions through e, and in direction s even two steps). Then $G_s(\chi) = 2$.*

3.6 Remark. (1) In other words: we only "move along the path", with weight 2 if one is in an interior situation or arrives at or from a good end point (with some extension of the path in all directions). We use weight $\frac{1}{2}$ if we move to or from a point which is next to a bad point (again the path has to extend a bit in the other direction).

(2) Our definition of F_s involves 1-neighborhoods rather than 10-neighborhoods. This will make calculations later easier, in particular if Γ is not the free group. In the framework of [2] one can economize and can reduce the size of the neighborhoods, albeit not to 1.

(3) We emphasize that our definition of F_s makes the operators A follow the hook itself, rather than its 1-neighborhoods (this convenient simplification will be made precise in a subsequent section).

4 Decomposition of $V^\perp$ into invariant subsets

This section follows pretty much [2, Section 3.2], with slight modifications that we indicate now.

4.1 Definition. Given $\chi \in \mathbb{Z}_2^\Gamma$, a ball $B(g,1)$ is called a *good neighborhood* of χ, if $g^{-1}(\chi) \cap B(e,1) = \{e, s_0\}$. $B(g,1)$ is called a *bad neighborhood* if $g^{-1}(\chi)$ is not 1-good.

Having this definition, and since our notations essentially coincide, we can obtain a partition of $\mathbb{Z}_2^\Gamma$ by simply copying that of [2, Section 3]. Namely, we obtain first a disjoint Borel partition

$$\mathbb{Z}_2^\Gamma = C_0 \cup C_{1,1} \cup C_{1,2} \cup C_{2,2} \cup C_{1,\infty} \cup C_{2,\infty} \cup C_{\infty,\infty}.$$

Here C_0 is the set of χ such that $F_s(\chi)$ and $F_{s^{-1}}(s^{-1}\chi)$ are both zero for all generators s. If $\chi \notin C_0$ then in particular χ is 1-good, i.e. $\chi^{-1}(1) \cap B(e,1)$ contains a piece of a path containing e.

The other sets $C_{i,j}$ are now determined according to the fate of two walkers starting at the origin and moving in opposite directions along this path starting at e. Indeed, for this description we identify χ with the subset $\chi^{-1}(1)$ of Γ. Each walker will have as path a (possibly infinite) hook or vertical segment R', starting at e.

We have three possible disjoint ending scenarios $i,j \in \{1,2,\infty\}$ for each walker:

(∞) the walker never reaches a good or a bad neighborhood, and continues her path forever;

(1) the walker reaches a good neighborhood and ends up at a lower end point of a hook;

(2) the walker reaches a bad neighborhood and stops walking. In this case we let $P' \subset R'$ be the path that the given walker follows up to distance 1 of her stopping point.

Furthermore, in case (1) we set $P' := R'$ to be the path followed by our given walker. Finally, we set R to be the union of the two hooks R' of the two walkers, and similarly we define R. Note that these are hooks or vertical segments and that the 1-neighborhood of each endpoint of P (if it exists) determines the ending scenario at that endpoint.

Next, we refine further and partition $C_{i,j}$ for $i, j < \infty$ according to triples (P, R, ψ), with fate i and j respectively, where we assume that $i \leq j$. Note that R here is finite since $i, j < \infty$) and $\psi \colon B(R, 1) \to \mathbb{Z}_2$ is the characteristic function of the 1-neighborhood of the path R in χ. Note that these have to take the value 1 on R and 0 on all points outside R except for $B(R \setminus P, 1)$. Let $\Omega_{i,j}$ be the set of all such triples and set

$$C_{(P,R,\psi)} = \{\chi \in \mathbb{Z}_2^\Gamma \mid \chi_{|B(R,1)} = \psi\}.$$

We obtain:

4.2 Lemma. *The following is a Borel partition of $\mathbb{Z}_2^\Gamma$:*

$$\mathbb{Z}_2^\Gamma = C_0 \cup \left(\bigcup_{i,j\in 1,2} \bigcup_{i \leq j \, (P,R,\psi)\in\Omega_{i,j}} C_{(P,R,\psi)} \right) \cup C_{1,\infty} \cup C_{2,\infty} \cup C_{\infty,\infty}.$$

By intersection with $V^\perp$ this leads to a Borel partition of $V^\perp$ and therefore to an orthogonal decomposition of $L^2(V^\perp)$. Depending on $V^\perp$, several summands will vanish.

Later, the pile-up of eigenspaces is organised according to the following equivalence relations on triples (P, R, ψ):

4.3 Definition. Two triples (P_1, R_1, ψ_1) and (P_2, R_2, ψ_2) are said to be *translation equivalent* if there exists a $g \in \Gamma$ such that $P_2 = gP_1$, $R_2 = gR_1$ and $\psi_2(gh) = \psi_1(h)$ for all $h \in B(R_1, 1)$.

Equivalence classes in $\Omega_{i,j}$ are finite (since R is finite and contains e) and we denote them by $\mathcal{C} \in \Omega_{i,j}/\sim$. Moreover, note that if $e \in P$ then the set of $g \in \Gamma$ which translates (P, R, ψ) to a translation equivalent pair are exactly the $g \in P$ with $g^{-1} \in P$.

5   Unitary equivalence

We obtain the decomposition

$$L^2(V^\perp)\bar\otimes\ell^2(\Gamma) \simeq \mathcal{H}_0 \oplus \left(\bigoplus_{1 \leq i \leq j \leq 2} \bigoplus_{\mathcal{C}\in\Omega_{i,j}/\sim} \mathcal{H}_\mathcal{C} \right) \oplus \mathcal{H}_{1,\infty} \oplus \mathcal{H}_{2,\infty} \oplus \mathcal{H}_{\infty,\infty} \quad (5.1)$$

where the notation is close to the one in [2, Section 3], namely:

$$\mathcal{H}_\mathcal{C} = \bigoplus_{(P,R,\psi)\in\mathcal{C}} \mathcal{H}_{(P,R,\psi)} \ \ \text{and} \ \ \mathcal{H}_{(P,R,\psi)} = \mathrm{Im}(M_{1_{C(P,R,\psi)}})$$

while $\mathcal{H}_{i,j} = \mathrm{Im}(M_{1_{C_{i,j}}})$, $\mathcal{H}_0 = \mathrm{Im}(M_{1_{C_0}})$.

More precisely, we should have written $C_{i,j} \cap V^\perp$, and we think of the characteristic function $1_{C_{i,j}}$ as a bounded measurable function on $V^\perp$, thus acting by left multiplication on $L^2(V^\perp)$ and also by twisted left multiplication on $L^2(V^\perp)\bar{\otimes}l^2(\Gamma)$. For notational convenience, we have omitted reference to V here.

Note that, because $\mathcal{H}_{(P,R,\psi)}$ is defined by left multiplication of $L^2(V^\perp)\bar{\otimes}\ell^2(\Gamma)$ with a projection in $L^\infty(V^\perp) \rtimes_{\hat{\alpha}} \Gamma$, it is a right Hilbert $\mathbb{Z}_2^{\oplus\Gamma}/V \rtimes \Gamma$-module. Its von Neumann dimension is given by the measure of the subset $C_{P,R,\psi}$:

$$\dim_{\mathbb{Z}_2^{\oplus\Gamma}/V \rtimes \Gamma}(\mathcal{H}_{P,R,\psi}) = m_{V^\perp}(C_{P,R,\psi} \cap V^\perp). \tag{5.2}$$

Corresponding statements apply to the other subspaces.

5.3 Proposition. *For each $\mathcal{C} \in \Omega_{i,j}/\sim$ (with $1 \leq i \leq j \leq 2$) the subspace $\mathcal{H}_{\mathcal{C}}$ is A-invariant.*

Moreover, let $V^{l,i,j}$ be the following weighted graphs: a segment of length $l \geq 2$ where each interior edge has weight two, and if $(i,j) = (1,1)$, both boundary edges have weight 2 as well, whereas if $(i,j) = (1,2)$ then one boundary edge has weight $\frac{1}{2}$ while the other has weight 2, and if $(i,j) = (2,2)$ then both boundary edges have weight $\frac{1}{2}$.

Let $A^{l,i,j}$ be the weighted adjacency matrices, regarded as operators on $l^2(V_v^{l,i,j})$, where $V_v^{l,i,j}$ denotes the vertex set of the graph $V^{l,i,j}$.

Choose one $(P,R,\psi) \in \mathcal{C}$, e.g. the one with e as the (left) lower endpoint of P; "left" if P is a hook (and not a vertical segment).

Then we have a unitary equivalence of Hilbert $\mathbb{Z}_2^{\oplus\Gamma}/V \rtimes \Gamma$-endomorphisms

$$A|_{H_C} \simeq id_{\mathcal{H}_{(P,R,\psi)}} \otimes A^{l,i,j}$$

where l is the length of the path P and (i,j) is the ending scenario of (P,R,ψ).

Proof. The proof is essentially the same as for the corresponding statement [2, Proposition 3.12].

First observe that for $g \in \Gamma$ the operator $T_{g^{-1}}$ (which is a Hilbert-$\mathbb{Z}_2^{\oplus\Gamma}/V \rtimes \Gamma$ isometry) maps $\mathcal{H}_{(P,R,\psi)}$ isometrically to $\mathcal{H}_{(g^{-1}P,g^{-1}R,\hat{\alpha}_{g^{-1}}\psi)}$.

This implies that A, because of its shape, maps a vector in $\mathcal{H}_{(P,R,\psi)}$ indeed to a linear combination of vectors in $\mathcal{H}_{(sP,sR,s\psi)}$ for the generators s. However, inspection of the functions G_s in Proposition 3.5 or Remark 3.6 shows that a non-zero contribution is obtained only if $s \in P$. Consequently, $\mathcal{H}_{\mathcal{C}}$ for $\mathcal{C} \in \Omega_{i,j}/\sim$ is A-invariant. Moreover, inspection of 3.5 further shows that A maps one summand to the other (up to identification with the unitary T_g) exactly with the weights as described by $A^{l,i,j}$; details of the argument follow exactly as in [2, Proposition 3.12], using 3.5 or 3.6.

Moreover, Proposition 3.5 also shows that the operator is zero on $\mathcal{H}_{(P,R,\psi)}$ if $|R| \leq 2$.

Note also that $A_{|\mathcal{H}_0} = 0$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6   The finite dimensional models

We will concentrate now on the particular eigenvalue -2.

6.1 Lemma. *The value -2 is an eigenvalue for $A^{l,1,1}$ acting on $l^2(V^{l,1,1})$ only for $l \equiv 1(3)$ and the eigenspace is one dimensional in this case.*
The value -2 is never an eigenvalue for $A^{l,i,2}$ for $l \in \mathbb{N}$ and $i \in \{1,2\}$.

Proof. We first study the kernel for the $l \times (l+1)$-matrix $\begin{pmatrix} 2 & \alpha & 0 & \cdots & & & \\ \alpha & 2 & 2 & 0 & \ddots & \cdots & \\ 0 & 2 & 2 & 2 & 0 & \cdots & \\ \cdots & & & & & & \\ & & & \cdots & 0 & 2 & 2 & \beta \end{pmatrix}$
obtained by deleting the last row, and where $\alpha, \beta \in \{1, \frac{1}{2}\}$. A simple linear recursion shows that this kernel is 1-dimensional and spanned by the vector

$$(2\alpha, -4, 4 - \alpha^2, \alpha^2, -4, \ 4 - \alpha^2, \dots, x_l),$$

with

$$x_l = \begin{cases} 2\alpha^2\beta^{-1} & l \equiv 0 \pmod 3 \\ -8\beta^{-1} & l \equiv 1 \pmod 3 \\ 2(4 - \alpha^2)\beta^{-1} & l \equiv 2 \pmod 3. \end{cases}$$

The kernel of $A^{l,i,j}$ is non-trivial if and only if this vector is also mapped to zero by the last row of $A^{l,i,j}$, which is simply the condition

$$\begin{cases} \beta(4 - \alpha^2) + 4\beta^{-1}\alpha^2 = 0 & l \equiv 0 \pmod 3 \\ \beta\alpha^2 - 16\beta^{-1} & l \equiv 1 \pmod 3 \\ -4\beta + 4(4 - \alpha^2)\beta^{-1} = 0 & l \equiv 2 \pmod 3. \end{cases}$$

If $i = j = 1$, i.e. $\alpha = \beta = 2$, this is satisfied if and only if $l \equiv 1 \pmod 3$. However, if $\alpha = \frac{1}{2}$ and either $\beta = 1$ or $\beta = \frac{1}{2}$ we check that the condition is never satisfied. This finishes the proof. $\qquad\square$

7   Dual measures on $\mathbb{Z}_2^\Gamma$

[2, Lemma 5.1] extends readily to our situation:

7.1 Lemma. *Given a subgroup $V \leq \mathbb{Z}_2^{\oplus\Gamma}$, a finite subset $E \subset \Gamma$, and $\psi : E \to \mathbb{Z}_2$, let $C(\psi)$ be the set of characters $\chi \in \mathbb{Z}_2^\Gamma$ such that $\chi_{|E} = \psi$. If $C(\psi) \cap V^\perp \neq \emptyset$, then*

$$m_{V^\perp}(C(\psi)) = \frac{1}{|\{\psi' \in \mathbb{Z}_2^E : \ C(\psi') \cap V^\perp \neq \emptyset\}|}.$$

Proof. Given $\psi_1, \psi_2 : E \to \mathbb{Z}_2$, it is enough to show that

$$m_{V^\perp}(C(\psi_1)) = m_{V^\perp}(C(\psi_2))$$

whenever both $C(\psi_1)$ and $C(\psi_2)$ intersect $V^\perp$. Take $\chi_i \in C(\psi_i) \cap V^\perp$. Then $\chi_2 - \chi_1$ sends $C(\psi_1) \cap V^\perp$ to $C(\psi_2) \cap V^\perp$ and preserves the measure $m_{V^\perp}$. $\quad\square$

We also remark that if $C(\psi) \cap V^\perp = \emptyset$, then certainly $m_{V^\perp}(C(\psi)) = 0$.

Given a finite subset $F \subset \Gamma$ and a subgroup $\Lambda \subset \Gamma$, we define a left invariant subgroup $V_{F,\Lambda}$ of $\mathbb{Z}_2^{\oplus\Gamma}$ in the following way:

$$V_{F,\Lambda} = \text{span}_{\mathbb{Z}_2}\{\mathbf{1}_{gF} - \mathbf{1}_{gtF} \mid \ g \in \Gamma, t \in \Lambda\}, \tag{7.2}$$

where $\mathbf{1}_F$ is the characteristic function of the set F.

Setting $\chi(F) := \sum_{v \in F} \chi(v)$, we have

$$V_{F,\Lambda}^{\perp} = \{\chi \in \mathbb{Z}_2^{\Gamma} \mid \chi(gF) = \chi(gtF), \ \forall g \in \Gamma, t \in \Lambda\}. \tag{7.3}$$

7.4 Definition. Let E, F be subsets of Γ and $\Lambda \leq \Gamma$ be a subgroup. We say that E has the *extension property relative to* (F, Λ) if for any $\psi \colon E \to \mathbb{Z}_2$, then

$$\psi(gF) = \psi(gtF) \ \forall g \in \Gamma, t \in \Lambda \text{ such that } gF \cup gtF \subset E \tag{7.5}$$

implies the existence of $\chi \in V_{F,\Lambda}^{\perp}$ such that $\chi_{|E} = \psi$.

In other words, if the obvious set of conditions on ψ is satisfied on E, then ψ extends to an element of $V_{F,\Lambda}^{\perp}$. Given E, F, Λ, Γ as in Definition 7.4, we let $\Omega_{F,E}$ be the set

$$\Omega_{F,E} := \{gF \subset E, \ g \in \Gamma\}.$$

We denote by $\Omega_{F,E}/\Lambda$ the set of classes of the equivalence relation $\sim_{\Lambda}$ on Ω_E given by right multiplication by Λ on Γ, namely:

$$gF \sim_{\Lambda} g'F \Leftrightarrow g^{-1}g' \in \Lambda.$$

The following is a generalization of [2, Corollary 5.9] with the same proof.

7.6 Lemma. *Let E, F be finite subsets of Γ and $\Lambda \leq \Gamma$ be a subgroup. Assume that E has the extension property relative to (F, Λ). Then*

$$|\{\psi \in \mathbb{Z}_2^E \mid \ C(\psi) \cap V_{F,\Lambda}^{\perp} \neq \emptyset\}| = 2^{|E|-K}$$

where $K = |\Omega_{F,E}| - |\Omega_{F,E}/\Lambda|$.

Proof. Since E has the extension property relative to (F, Λ), the subset

$$\{\psi \in \mathbb{Z}_2^E \mid \ C(\psi) \cap V_{F,\Lambda}^{\perp} \neq \emptyset\}$$

coincide with

$$\{\psi \in \mathbb{Z}_2^E \mid \psi(gF) = \psi(g'F) \text{ whenever } gF, g'F \in \Omega_{F,E}, \ gF \sim_{\Lambda} g'F\}.$$

The latter is the orthogonal of the finite dimensional subspace

$$\operatorname{span}_{\mathbb{Z}_2}\{\mathbf{1}_{gF} - \mathbf{1}_{g'F}, \ gF, g'F \in \Omega_{F,E}, \ gF \sim_{\Lambda} g'F\}$$

and its dimension equals

$$K = \sum_{C \in \Omega_{F,E}/\Lambda} |C| - 1 = |\Omega_{F,E}| - |\Omega_{F,E}/\Lambda|.$$

Therefore,

$$\dim_{\mathbb{Z}_2}\{\psi \in \mathbb{Z}_2^E : \ C(\psi) \cap V_{F,\Lambda}^{\perp} \neq \emptyset\} = |E| - K,$$

hence the result. $\qquad\square$

7.7 Corollary. *Assume, in the situation of Section 5, that $V^{\perp} = V_{F,\Lambda}^{\perp}$ for $\Lambda \subset \Gamma$ a subgroup and $F \subset \Gamma$ finite as above. Moreover, given $\psi \colon B(R,1) \to \mathbb{Z}_2$, assume that it extends to $V^{\perp}$ and that $B(R,1)$ satisfies the extension property for F. Then*

$$\dim_{\mathbb{Z}_2^{\oplus \Gamma}/V \rtimes \Gamma}(\mathcal{H}_{P,R,\psi}) = 2^{-|B(R,1)|+K}, \tag{7.8}$$

where K is the number of equivalence classes of subsets gE of $B(R,1)$ $(g \in \Gamma)$, with $gF \sim gtF$ for $t \in \Gamma$.

Proof. This is a direct consequence of (7.8) and Lemma 7.6. $\qquad\square$

8   Extension lemma

We need a sufficiently general condition for deciding when a set E has the extension property relative to (F, Λ). The following criterion is an analog of [2, Lemma 5.5].

8.1 Lemma. *Suppose that s_1 is of infinite order, $F \subset \{s_1^k \mid k \in \mathbb{Z}\} \subset \Gamma$ is finite and that the subset $B \subset \Gamma$ is* horizontally connected, *i.e. $\forall g \in \Gamma$*

$$\{gs_1^k; k \in \mathbb{Z}\} \cap E$$

is a connected segment.

Then B has the extension property (as in Definition 7.4) relative (F, Λ) for any subgroup Λ of Γ.

Proof. Let B_n be an increasing sequence of subsets of Γ such that $B_0 = B$ and such that B_{n+1} is obtained from B_n by adding an element at distance 1 from B_n, B_{n+1} is horizontally connected and the union of B_n is Γ. We construct a sequence of functions χ_n such that $\chi_0 = \phi$ and $\chi_{n+1}|_{B_{n+1}} = \chi_n$ and the condition (7.5) is true with $E = B_n$. This implies the existence of the extension. Of course it suffices to give a construction of χ_1.

Suppose that we add to B one element h to get $B_1 = B \cup \{h\}$ and we want to construct χ_1. If $gF \subset B_1$ $h \in gF$ then by horizontal connectivity of B and the special shape of F the element h is an end-point of F.

Then, again by horizontal connectivity of B it is not possible that $h \in g'F$ for some other g' with $g'F \subset B_1$.

If there is $e \neq t \in \Lambda$ such that also $gtF \subset B_1$ then necessarily $gtF \subset B$ and by (7.5) $\chi_1(h)$ is imposed by

$$\chi_1(h) = \chi(gF \setminus \{h\}) - \chi(gtF).$$

We only need to show that this is independent of the choice of t. Indeed, if for $t, t' \in \Lambda$ both $gtF, gt'F \subset B$ then, as $t't^{-1} \in \Lambda$ and because B satisfies condition (7.5), $\chi(gtF) = \chi(gt'F)$.

If no pair gF, gtF as considered above exist, we can choose $\chi_1(h)$ at will, e.g. $\chi_1(h) := 0$, as no additional condition has to be satisfied for (7.5) to hold for $E = B_1$. $\square$

9   Subgroups V_I and the effect on the eigenspaces

As before, we consider a group Γ generated by s_1, s_2, but from now on we will mostly concentrate on the case of the free group or of $\mathbb{Z} \wr \mathbb{Z}$.

9.1 Definition. For $n \in \mathbb{N}$, define $t_n := s_2^n s_1 s_2^{-n}$. For $I \subset \mathbb{N}$, define $\Lambda_I := \langle t_i \mid i \in I \rangle \leq \Gamma$.

If $F \subset \Gamma$ is finite, define $V_{F,I} := \mathbb{Z}_2[g1_F - gt1_F \mid g \in \Gamma, t \in \Lambda_I] \subset \mathbb{Z}_2^{\oplus\Gamma}$, its Pontryagin dual $V_{F,I}^\perp = \{\chi \in \mathbb{Z}_2^\Gamma \mid \chi(gF) = \chi(gtF), \ \forall g \in \Gamma, t \in \Lambda\}$ as in Definition (7.3). Finally, we specialize to $F_l = \{s_1^{-1}, e, s_1\}$ and set $G_I := \mathbb{Z}_2^{\oplus\Gamma}/V_{F_l,I} \rtimes \Gamma$.

9.2 Lemma. *For*

$$\Gamma = \mathbb{Z} \wr \mathbb{Z} = \langle s_1, s_2 \mid [s_2^k s_1 s_2^{-k}, s_1] = 1 \; \forall k \in \mathbb{Z} \rangle \tag{9.3}$$

the elements t_n are the free generators of a free abelian subgroup, contained in the kernel of the obvious projection to $\mathbb{Z} = \langle s_2 \rangle$.

Proof. This is part of the structure theory of the wreath product: the base $\mathbb{Z}^{\oplus \mathbb{Z}}$ is a free abelian group with generators $s_2^n s_1 s_2^{-n}$ for $n \in \mathbb{Z}$. The group $\langle s_2 \rangle = \mathbb{Z}$ acts on the base by the obvious permutation of the basis elements, the semidirect product is $\mathbb{Z} \wr \mathbb{Z}$. $\square$

9.4 Proposition. *Assume that $P = \left\{ gs_2^{-n}, \ldots, g, gs_1, gs_1 s_2^{-1}, \ldots, gs_1 s_2^{-m} \right\}$ is a hook as in (3.3) with $n, m \in \mathbb{N}_{>0}$ and $I \subset \mathbb{N}$ is given. Assume moreover that Γ is either the free group on free generators s_1, s_2 or $\Gamma = \mathbb{Z} \wr \mathbb{Z}$ as in (9.3).*

Then $xt = y$ for $x \neq y \in P$ and $t \in \Lambda_I$ exactly if $t = t_k$ for some generator t_k with $k \in I$ such that $k \leq n$ and $k \leq m$, $x = gs_2^{-k}$, $y = gs_1 s_2^{-k}$ (or $t = t_k^{-1}$, $x = gs_1 s_2^{-k}$, $y = gs_2^{-k}$).

Proof. Obviously, the t_k, x, y we have given satisfy all the conditions. Since $\mathbb{Z} \wr \mathbb{Z}$ is a quotient of the free group $\langle s_1, s_2 \rangle$, it suffices to show that the conditions are necessary in $\mathbb{Z} \wr \mathbb{Z}$. Now, if $x, y \in P$ satisfy $x^{-1} y \in \Lambda_I$, then in particular $x^{-1} y$ is contained in the base group of the wreath product. First, this means the projection to $\langle s_2 \rangle$ has to be the trivial element, i.e. the s_2-exponent in the words x, y have to be equal. As the base group is free abelian on generators $s_2^v s_1 s_2^{-v}$ ($v \in \mathbb{Z}$) the assertion now follows. $\square$

9.5 Proposition. *As before, assume Γ generated by s_1, s_2 is either $\mathbb{Z} \wr \mathbb{Z}$ or the free group. Set $F_l = \{s_1^{-1}, e, s_1\}$ and let $R \subset \Gamma$ be a hook, $\psi \colon B(R, 1) \to \mathbb{Z}_2$ the characteristic function of R. The ψ extends to $V_{F_l, I}^\perp$. Moreover, $gF_l, hF_l \subset B(R, 1)$ are equivalent if and only if $g, h \in R$ and there is $t \in \Lambda_I$ with $g = ht$.*

Proof. By a normal form argument, we know that whenever $gF_l \subset B(R, 1)$ then $g \in R$. By Lemma 8.1 we only have to check that, whenever xF_l and yF_l for $x, y \in R$ are equivalent, then $\psi(xF_l) = \psi(yF_l)$. By Proposition 9.4 (and normal form in Γ), if xF_l and yF_l are equivalent then $|xF_l \cap B(R, 1)| = 1 = |yF_l \cap B(R, 1)|$: the intersection would have different cardinality only if x (or y) was part of the "bend" of the hook, i.e. xs_1 or $xs_1^{-1} \in R$, but then x is only equivalent to itself. Finally $\psi(xF_l) \equiv |xF_l \cap B(R, 1)| \pmod{2}$, therefore $\psi(xF_l) = \psi(yF_l)$ and the proposition follows. $\square$

9.6 Corollary. *Adopt the situation of Proposition 9.5. Assume that n and m are the length of the left and of the right leg of R, respectively. Then, with $K := |I \cap \{1, \ldots, \min(m, n)\}|$,*

$$\dim_{\mathbb{Z}_2^{\oplus \Gamma} / V_{F_l, I} \rtimes \Gamma} (\mathcal{H}_{R, R, \psi}) = 2^{-3(n+m) - 8 + K}. \tag{9.7}$$

Proof. Because of Proposition 9.5 and Lemma 8.1, we can directly apply Corollary 7.7.

By normal form, we know that inside $B(R, 1)$ there are no relations and that $|R| = n + m + 2$, hence $|B(R, 1)| = 3(n + m + 2) + 2$. Moreover, by Propositions 9.5 and 9.4, the correction term K is exactly as given. $\square$

We conclude this section by showing that in the situation of Proposition 9.5 the sets $C_{1,\infty}$, $C_{2,\infty}$, and $C_{\infty,\infty}$, as defined in Section 4, are negligible with respect to $m_{V_{F_l,I}^\perp}$.

9.8 Lemma. *Given $g \in \Gamma$, let D_g be the measurable subset of $\mathbb{Z}_2^\Gamma$ defined by*

$$D_g = \{\chi \in \mathbb{Z}_2^\Gamma \mid \chi(gs_2^{-k}) = 1 \text{ and } \chi(gs_2^{-k}s_1^a) = 0, \ \forall k \geq 0, \ a = \pm 1\}.$$

Then $m_{V_{F_l,I}^\perp}(D_g) = 0$.

Proof. Given $g \in \Gamma$ and an integer $N \geq 0$ set

$$D_{g,N} = \{\chi \in \mathbb{Z}_2^\Gamma \mid \chi(gs_2^{-k}) = 1 \text{ and } \chi(gs_2^{-k}s_1^a) = 0, \ \forall 0 \leq k \leq N, \ a = \pm 1\}$$

By Lemma 7.6, we have

$$\mu_{V_{F_l,I}^\perp}(D_{g,N}) \leq \frac{1}{2^{3N-K_N}}$$

and it follows from the last assertion of Proposition 9.5 that $K_N \leq N$. Since

$$D_g = \bigcap_{N \geq 1} D_{g,N},$$

we obtain $\mu_{V_{F_l,I}^\perp}(D_g) \leq 2^{-2N}$ for all $N \geq 1$ and the lemma follows. $\square$

Thus, we get the following analog of [2, Prop. 5.8].

9.9 Corollary. *Keeping the notations above, we have*

$$m_{V_{F_l,I}^\perp}(C_{1,\infty}) = m_{V_{F_l,I}^\perp}(C_{2,\infty}) = m_{V_{F_l,I}^\perp}(C_{\infty,\infty}) = 0.$$

In particular,
$$\mathcal{H}_{1,\infty} = \{0\} = \mathcal{H}_{2,\infty} = \mathcal{H}_{\infty,\infty}.$$

Proof. Indeed, $C_{1,\infty} \cup C_{2,\infty} \cup C_{\infty,\infty} \subset \bigcup_{g \in \Gamma} D_g$, so we may apply Lemma 9.8. The second claim follows directly from the corresponding version of (5.2). $\square$

9.10 Remark. Lemma 9.8 and its corollary above can be extended to almost arbitrary subgroups $V_{F,\Lambda}$, where $F \subset \Gamma$ is a finite subset, $\Lambda \leq \Gamma$ is a subgroup as in Section 7, and a large choice of prescriptions on infinite sets $E \subset \Gamma$ having the extension property with respect to (F, Λ). For instance it is enough to have that $|F| \geq 2$ and that the subsets gF ($g \in \Gamma$) which are included in E are pairwise disjoint (as the proof of Lemma 9.8 shows).

10 Explicit calculation of the von Neumann dimension of the eigenspace

We continue with the situation of Section 9. We deal only with the eigenvalue -2 for A and we set $Q = A + 2$.

10.1 Theorem. *Fix $I := \{2, n_1, n_2, \dots\} \subset \mathbb{N}$ with $n_0 := 2 < n_1 < n_2 < \dots$ and such that $n_k \equiv 2 \pmod 3$ $\forall k$. Choose $\Gamma = \mathbb{Z} \wr \mathbb{Z}$ or Γ free on two generators and set $G_I = \mathbb{Z}_2^{\oplus \Gamma} / V_{F_l, I} \rtimes \Gamma$ as in Definition 9.1. Construct $A \in \mathbb{Q}[G_I]$ as above. Then*

$$\dim_{L^2}(Ker(A+2)) = \beta_1 + \beta_2 \sum_{k=1}^{\infty} 2^{-dn_k + k}.$$

Here, $d = 6$ and β_1 and β_2 are explicitly given rational numbers, compare (10.4).

In particular, these numbers show up as L^2-Betti numbers of normal coverings of compact manifolds with covering group G_I.

Proof. We use Proposition 5.3 to decompose A. By Corollary 9.9 and Lemma 6.1 the only contributions to the eigenvalue -2 are obtained on $\mathcal{H}_\mathcal{C}$ if $\mathcal{C} \in \Omega_{1,1} / \sim$ if the length of the associated hook R is congruent 1 modulo 3, and the L^2-dimension of the eigenspace is then

$$\dim_{G_I}(\mathcal{H}_{R,R,\psi}) = 2^{-3(l_1+l_2)-8+|I \cap \{1 \dots, \min\{l_1, l_2\}\}|}, \qquad (10.2)$$

where R is the hook, ψ is the characteristic function of the hook in its 1-neighborhood and l_1, l_2 are the lengths of the left and right leg of the hook, respectively. Note that the length of the hook is $l_1 + l_2 + 1$, so we get a contribution exactly if $l_1 + l_2$ is divisible by 3.

Write $I = \{n_1, n_2, \dots\}$ with $n_1 < n_2 < \dots$. We have to add the summand (10.2) for each $1 \le l_1, l_2$ with $l_1 + l_2$ divisible by 3 (each such corresponding to one class of hook passing through e). To facilitate the effect of $|I \cap \{1, \dots, \min\{l_1, l_2\}\}|$, we choose the disjoint decomposition of the (l_1, l_2)-plane into subset $V_k := U_k \setminus U_{k+1}$, where $U_k = \{(l_1, l_2) \mid l_1, l_2 \ge n_k\}$, such that $|I \cap \{1, \dots, \min\{l_1, l_2\}\}| = k$ on V_k.

We obtain (with convention $n_0 = 2$)

$$\dim_{L^2}(Ker(A+2)) = 2^{-8} \sum_{k=0}^{\infty} 2^k \sum_{\substack{(l_1, l_2) \in V_k \\ l_1 + l_2 \equiv 0(3)}} 2^{-3(l_1+l_2)}$$

$$= 2^{-8} \sum_{k=0}^{\infty} 2^k \left(\sum_{\substack{(l_1, l_2) \in U_k \\ l_1 + l_2 \equiv 0(3)}} 2^{-3(l_1+l_2)} - \sum_{\substack{(l_1, l_2) \in U_{k+1} \\ l_1 + l_2 \equiv 0(3)}} 2^{-3(l_1+l_2)} \right). \qquad (10.3)$$

Recall that all n_k are congruent 2 modulo 3. We distinguish the cases $l_1 = 3r_1 + r$ with $r = 0, 1, 2$ (and $l_2 = 3r_2 + 2 - r$ to get $l_1 + l_2 \equiv 0 \pmod 3$) and obtain finally for the sum over U_k

$$\sum_{\substack{(l_1, l_2) \in U_k \\ l_1 + l_2 \equiv 0(3)}} 2^{-3(l_1+l_2)} = \sum_{r=0}^{2} \sum_{r_1=0}^{\infty} 2^{-3(n_k + 3r_1 + r)} \sum_{r_2=0}^{\infty} 2^{-3(n_k + 3r_2 + 2 - r)}$$

$$= 3 \cdot 2^{-6n_k + 2}(1 - 2^{-9})^{-2}.$$

Substituting this in (10.3) we get

$$
\begin{aligned}
\dim_{L^2}(Ker(A+2)) &= \frac{3}{2^6(1-2^{-9})^2} \sum_{k=0}^{\infty} \left(2^k 2^{-6n_k} - \frac{1}{2} 2^{k+1} 2^{-6n_{k+1}} \right) \\
&= \frac{3}{2^6(1-2^{-9})^2} 2^{-12} + \frac{3}{2^7(1-2^{-9})^2} \sum_{k=1}^{\infty} 2^{-6n_k+k}.
\end{aligned}
\tag{10.4}
$$

$\square$

11 Arbitrary real numbers as L^2-Betti numbers for normal coverings

Our main point about the explicit formulas for L^2-Betti numbers is two-fold: on the one hand, we want to show that every positive real number is an L^2-Betti number. This is the goal of the current section.

Secondly, we want to show that we get transcendental L^2-Betti numbers for *universal* coverings, which translates algebraically that we have to use finitely presented groups. This will be done in the last sections.

Now we show how, starting from the L^2-Betti numbers we explicitly obtain in Theorem 10.1, one can construct (again explicitly) more groups and elements in their group rings to finally get the following theorems.

11.1 Theorem. *For every real number $r \geq 0$ their is a finitely generated group Γ_r, an $l \in \mathbb{N}$ and $a_r \in M_l(\mathbb{Z}\Gamma_r)$ such that*

$$
\dim_{\Gamma_r}(\ker(a_r)) = r.
$$

Moreover, from a dyadic expansion $r = \sum \lambda_j 2^j$ with $\lambda_j \in \{0,1\}$ we get (in principle) and "explicit" description of Γ_r and a_r.

Moreover, there is a compact manifold M with a normal covering $\tilde{M}$ (with covering group Γ_r) such that

$$
b_3^{(2)}(\tilde{M}; \Gamma_r) = r.
$$

To prove this from the previous constructions, we review a couple of constructions for which we can control the L^2-Betti numbers in terms of L^2-Betti numbers of the ingredients.

11.2 Lemma. *Let Γ_1, Γ_2 be two groups, $l_1, l_2 \in \mathbb{N}$ and $a_j \in M_{l_j}(\mathbb{Z}[\Gamma_j])$ for $j = 1, 2$. Form the "block sum"*

$$
a := a_1 \oplus a_2 \in M_{l_1+l_2}(\mathbb{Z}[\Gamma_1 \times \Gamma_2]),
$$

where we tacitly identify Γ_j with its image in $\Gamma := \Gamma_1 \times \Gamma_2$ and identify a_j with its image under the induced map. Then

$$
\dim_{\Gamma}(\ker(a)) = \dim_{\Gamma_1}(\ker(a_1)) + \dim_{\Gamma_2}(\ker(a_2)).
$$

Proof. This is well known and essentially clear. First of all, by the induction principle (e.g. [15, Proposition 3.1]), $\dim_{\Gamma}(\ker(a_j)) = \dim_{\Gamma_j}(\ker(a_j))$ for $j = 1, 2$, where we think of a_j either as living over $\mathbb{Z}[\Gamma]$ or over $\mathbb{Z}[\Gamma_j]$.

Secondly, the kernel of a (as block sum) is the direct sum of the kernels of a_1 and of a_2 (in $l^2(\Gamma)^{l_1+l_2}$). As the von Neumann dimension is additive for direct sums, the assertion follows. $\qquad\square$

11.3 Lemma. *Let Γ_1, Γ_2 be two groups, $l_1, l_2 \in \mathbb{N}$ and $a_j \in M_{l_j}(\mathbb{Z}[\Gamma_j])$ for $j = 1, 2$. Assume that a_1 and a_2 are non-negative (if necessary, replace them by $a_j^* a_j$). Form the "tensor sum"*

$$a := a_1 \otimes \operatorname{id} + \operatorname{id} \otimes a_2 \in M_{l_1 \cdot l_2}(\mathbb{Z}[\Gamma_1] \otimes \mathbb{Z}[\Gamma_2]),$$

thinking of $\mathbb{Z}[\Gamma] = \mathbb{Z}[\Gamma_1] \otimes \mathbb{Z}[\Gamma_2]$ acting on $l^2(\Gamma_1 \times \Gamma_2) = l^2(\Gamma_1) \otimes l^2(\Gamma_2)$. Then

$$\dim_\Gamma(\ker(a)) = \dim_{\Gamma_1}(\ker(a_1)) \cdot \dim_{\Gamma_2}(\ker(a_2)).$$

Proof. This lemma is also well known and follows from the fact that in this situation $\ker(a) = \ker(a_1) \otimes \ker(a_2)$. A detailed argument for a special case can be found in the proof of [3, Theorem 4.1].

For the sake of completeness, let us give a more explicit proof here. If p_j is the orthogonal projection onto $\ker(a_j)$ for $j = 1, 2$ (considered as matrices over $\mathcal{N}\Gamma$, induced up from $\mathcal{N}\Gamma_j$), we claim that in this situation $p := p_1 \otimes p_2$ is the projection onto the kernel of a. As $(a_1 \otimes \operatorname{id} + \operatorname{id} \otimes a_2)(p_1 \otimes p_2) = 0$, the image of p is contained in the kernel of a.

Now, $(1 - p_1) \otimes p_2 + (1 - p_1) \otimes (1 - p_2)$ is an orthogonal decomposition of $1 - p$. On the image of $(1 - p_1) \otimes p_2$, which is equal to $\operatorname{im}(1 - p_1) \otimes \operatorname{im}(p_2)$, a coincides with $a_1 \otimes \operatorname{id}$ which is > 0 there.

On the image of $(1 - p_1) \otimes (1 - p_2)$ which coincides with $\operatorname{im}(1 - p_1) \otimes \operatorname{im}(1 - p_2)$, a coincides with $a_1 \otimes \operatorname{id} + \operatorname{id} \otimes a_2$, and both summands are > 0. Altogether, on the complement of $\operatorname{im}(p)$ $a > 0$ and therefore $\ker(a) = \operatorname{im}(p)$.

Finally, we have to compute the Γ-trace of p. Let $e_1, \dots, e_{l_1}$ be the standard basis vectors of $l^2(\Gamma_1)^{l_1}$ and $f_1, \dots, f_{l_2}$ be the standard basis vectors of $l^2(\Gamma_2)^{l_2}$ (the characteristic function of the neutral element in the corresponding component).

Then $\{e_i \otimes f_j\}_{i=1,\dots,l_1;\, j=1,\dots,l_2}$ will be the standard basis for $l^2(\Gamma_1 \times \Gamma_2)^{l_1 \cdot l_2}$. Consequently

$$\operatorname{tr}_\Gamma(p) = \sum_{i=1}^{l_1} \sum_{j=1}^{l_2} \langle p_1 \otimes p_2(e_i \otimes e_j), e_i \otimes e_j \rangle_{l^2(\Gamma_1) \otimes l^2(\Gamma_2)}$$

$$= \sum_{i=1}^{l_1} \langle p_1(e_i), e_i \rangle_{l^2(\Gamma_1)} \cdot \sum_{j=1}^{l_2} \langle p_2(f_j), f_j \rangle_{l^2(\Gamma_2)} = \operatorname{tr}_{\Gamma_1}(p_1) \cdot \operatorname{tr}_{\Gamma_2}(p_2)$$

This proves the claim. $\qquad\square$

11.4 Proposition. *Let $U \subset \mathbb{R}_{\geq 0}$ be a subset of the non-negative real numbers with the following properties*

(1) *U is closed under multiplication and addition of non-negative rational numbers;*

(2) *U is additively closed: if $r, s \in U$ then also $r + s \in U$;*

(3) *there are rational numbers* $a, q \in \mathbb{Q}_{\geq 0}$, $q > 0$ *and* $d \in \mathbb{N}$ *such that for every increasing sequence* $0 \leq n_1 < n_2 < \ldots$ *the number* $a + q \sum_{k=1}^{\infty} 2^k 2^{-dn_k} \in U$.

Then $U = \mathbb{R}_{\geq 0}$.

Proof. Choose $m \in \mathbb{N}$ such that $b := 2^{dm-1}q > a$ is a multiple of q. Adding the rational number $b - a$ and multiplying with the rational number $2q^{-1}$ we see that all real numbers of the form

$$2^0 \cdot 2^{dm} + \sum_{k=1}^{\infty} 2^k 2^{-dn_k} \in U; \qquad 0 \leq n_1 < \ldots. \tag{11.5}$$

Replacing d by $D := dm$ and using only sequences where each n_k is a multiple of m, and multiplying with suitable powers of 2, we see that all real numbers of the form

$$\sum_{k=0}^{\infty} 2^k 2^{-Dn_k}; \qquad 0 \leq n_0 < n_1 < \ldots \tag{11.6}$$

belong to U.

Because U is closed under multiplication with non-negative rational numbers it suffices to show that U contains some non-empty open interval.

Moreover, because U is additively closed and closed under multiplication with powers of 2, it suffices to show that U contains every real number of the form

$$r = \sum_{n \in I} 2^{-Dn}; \qquad I \subset \mathbb{N} \tag{11.7}$$

since an arbitrary real number between 0 and 1 is a sum of at most D multiples (by 2^k with $0 \leq k < d$) of numbers of the form (11.7).

Fix therefore $I \subset \mathbb{N}$. We now describe 2^{D-1} numbers of the form (11.6) with sum equal to r.

Instead of writing down the formulas, we describe the digits of these numbers in dyadic expansion. Note that the relevant feature of any number of the form (11.6) is that the consecutive digits occur at places which are multiples of D (as is true for r), but each new digit shifted one further "to the left".

The first 2^{D-1} dyadic digits of r each give one (the first) digit of the 2^{D-1} numbers to be constructed. The next digit of r (the summand 2^{-Dn_v} with $v = 2^{D-1} + 1$), shifted by 2^{D-1} to the right, gives the second digit of each of the r numbers to be constructed. Note that this is a summand of the form $2^1 \cdot 2^{-D(n_v+1)}$. Note also that the sum of these 2^{D-1} summands is exactly 2^{-Dn_v}, i.e. the corresponding digit of r. The next two digits are used, shifted by 2^{D-2} in the first or last 2^{D-2}, respectively, of our numbers to be constructed. The same reasoning as before shows that these summands have the right form and add up to the right digits of r. The next 4 digits, shifted by 2^{D-3}, are used in one quarter each, i.e. 2^{D-3}, of our numbers to be constructed.

We continue this construction inductively until we arrive at 2^{D-1} digits which are not to be shifted at all. Then we cyclically continue this pattern inductively.

The result are by construction the 2^{D-1} numbers, each of the form (11.6), which therefore belong to U and which add up to r.

As explained above, this implies the assertion.   □

11.8 Corollary. *Every non-negative real number is an L^2-Betti number of some covering of a compact manifold.*

Proof. By a standard reduction, it suffices that for every $r \in \mathbb{R}_{\geq 0}$ there is a finitely generated group Γ, $d \in \mathbb{N}$ and $A \in M_n(\mathbb{Z}\Gamma)$ such that $\dim_\Gamma(\ker(A)) = r$.

However, the main result of this paper asserts that for $I = \{2, n_1, n_2, \dots\}$ with $n_0 = 2 < n_1 < n_2 < \dots$ all congruent 2 modulo 3 and for a certain $\beta_1, \beta_2 \in \mathbb{Q}_{>0}$, $d \in \mathbb{N} > 0$ for

$$r = \beta_1 + \beta_2 \sum_{k=1}^{\infty} 2^k \cdot 2^{-n_k d}$$

there is a finitely generated Γ_r and $a_r \in \mathbb{Z}[\Gamma_r]$ such that $\dim_{\Gamma_r}(\ker(a_r)) = r$.

Using in addition Lemmas 11.2 and 11.3 the set of von Neumann dimensions of kernels satisfies the assumptions of Proposition 11.4. The corollary follows. $\square$

12   Structure of the groups G_I

To show that there are also *universal* coverings with transcendental L^2-Betti numbers —equivalently matrices over the group ring of a finitely presented group with transcendental L^2-dimension of the kernel, we have to analyze the groups used in 10.1 more precisely.

Recall that, starting with $\Gamma = \langle s_1, s_2 \rangle$ either free or $\Gamma = \mathbb{Z} \wr \mathbb{Z}$ and given a subset $I \subset \mathbb{N}$, fixing $F_l = \{s_1^{-1}, e, s_1\}$, in 9.1 we have groups

$$G_I := \left(\mathbb{Z}_2^{\oplus \Gamma} / V_{F_l, I} \right) \rtimes \Gamma.$$

Consider the basis $\{\delta_g \mid g \in \Gamma\}$ of $\mathbb{Z}_2^{\oplus \Gamma} = \mathbb{Z}_2[\Gamma]$ where δ_g is the characteristic function of $\{g\} \subset \Gamma$.

Recall that $V_{F_l, I}$ is the $\mathbb{Z}_2[\Gamma]$-submodule of $\mathbb{Z}_2[\Gamma]$ generated by the elements of the form $\sum_{h \in F_l} (\delta_h - \delta_{th})$, $t \in \Lambda_I$, so as $\mathbb{Z}_2$-vector space it is generated by elements of the form

$$\sum_{h \in F_l} (\delta_{gh} - \delta_{gth}), \qquad g \in \Gamma, \quad t \in \Lambda_I.$$

The following lemma will be useful:

12.1 Lemma. *The subgroup $V_{F_l, I}$ is generated as $\mathbb{Z}_2[\Gamma]$-module by the elements $w_g := \sum_{h \in F_l} (\delta_h - \delta_{gh})$ with $g = t_n$, $n \in I$.*

Assume moreover that s_1 has infinite order in Γ.

For $x \in V_{F_l, I}$ considered as a function on Γ, let $\sigma \subset \Gamma$ be its support. Consider the intersections with the cosets $C_g := g\langle s_1 \rangle$. As s_1 has infinite order, each such coset is a countable ordered set (with $gs_1^k < gs_1^l$ if and only if $k < l$) and we let $[\sigma_g^l, \sigma_g^r]$ be the smallest interval containing $\sigma \cap C_g$ (of course, $\sigma_g^l = gs_1^v$ for some r, and $\sigma_g^r = gs_1^w$ with $w \geq v$).

Then for $x \in V_{F_l, I}$ this interval is either empty or $w - v \geq 2$.

Moreover, for each $b \in [\sigma_g^l s_1, \sigma_g^r s_1^{-1}]$ there is a $1 \neq t \in \Lambda_I$ such that for some $b' \in \{bs_1^{-1}, b, bs_1\}$ the element $b't$ is contained in the interior of the corresponding non-empty interval $[\sigma_h^l, \sigma_h^r]$ containing $\sigma \cap h\langle s_1 \rangle$.

If $g \in \Gamma$ with $w_g \in V_{F_l, I}$ then $g \in \Lambda_I$.

Proof. By definition, $V_{F_l,I}$ is generated by the w_g with $g \in \Lambda_I$. However,

$$w_g + gw_{g'} = \sum_{h \in F_l} (\delta_h - \delta_{gh} + \delta_{gh} - \delta_{gg'h}) = w_{gg'}.$$

As Λ_I by definition is generated by $\{t_n \mid n \in I\}$, the first assertion follows.

The third statement is a direct consequence of the second, as the support of w_g intersects exactly two cosets: $\langle s_1 \rangle$ and $g\langle s_1 \rangle$.

The assertion of the second statement concerning the length of the support intervals (being always ≥ 3) follows by considering the possible cancellations of the left- and right-most element.

The rest of the second statement is proved by induction on the number of summands in

$$x = \sum_{k=1}^{n} g_k w_{t_k} \quad \text{with } g_k \in \Gamma \text{ and } t_k \in \Lambda_I.$$

The statement is trivial if the sum is empty. Assume the statement is correct for x' and form $x = x' + gw_t$ with $t \in \Lambda_I$, $g \in \Gamma$. The statement for x follows directly from the one for x' except if one or several of gs_1^{-1}, gs_1, gts_1^{-1}, gts_1 coincide and therefore cancel an endpoint of one of the intervals $[\sigma_g^l, \sigma_g^r]$ or $[\sigma_{gt}^l, \sigma_{gt}^r]$. But then we can compose the relations (or we have complete cancellation). To illustrate: if gs_1 is the right endpoint, then there is t' such that gt' is in the interior of another (non-empty) interval. If this interval does not cancel with gts_1, then the point gt, which is an interior point of a support interval for x (and potentially wasn't yet for x') satisfies that $gt \cdot t^{-1}t'$ is an interior point of another support interval for x, and $(t^{-1}t') \in \Lambda_I$. $\qquad\square$

12.2 Theorem. *The group G_I does have a recursive presentation if and only if I is recursively enumerable, i.e. there is a Turing machine listing exactly all elements of I.*

Proof. Assume that I is recursively enumerable.

Using Lemma 12.1, a presentation of G_I is given by the generating set $s_1, s_2, \tau =: \delta_e$ with the following relations:

- $\tau^2 = 1$

- $g^{-1}\tau g =: \delta_g$ commutes with $h^{-1}\tau h =: \delta_h$ for each $g, h \in \Gamma$.

- $\prod_{x \in F_l} \delta_{gx}\delta_{gt_n x}$ is trivial for each $n \in I$ and each $g \in \Gamma$.

- If $\Gamma = \mathbb{Z} \wr \mathbb{Z}$, in addition we need the relations of this group: $s_2^n s_1 s_2^{-n}$ commutes with s_1 for each $n \in \mathbb{Z}$.

As it is easy to list all elements of Γ, starting with the Turing machine for I we can produce a Turing machine listing all these relations, i.e. this presentation is recursive.

Assume, on the other hand, that there is a Turing machine producing all the relations in G_I. In particular, it will list all the words $w_{t_n} = \prod_{x \in F_l} \delta_x \delta_{t_n x}$ for the $n \in I$. Because the word problem in Γ and in $\mathbb{Z}_2[\Gamma]$ is solvable, we can recognize these words and determine the $n \in I$ from them. On the other hand, by Lemma 12.1 if $b \notin I$ (i.e. $t_b \notin \Lambda_I$) then $w_{t_b} \notin V_{F_l,I}$ i.e. w_{t_b} and therefore b

is not listed. In other words, this algorithm produces exactly the elements of I, and hence I is recursive. $\square$

12.3 Theorem. *The group G_I does have solvable word problem if and only if I is recursive, i.e. there is a Turing machine listing the elements of I and another one listing those of the complement of I.*

Proof. Assume that G_I has a solvable word problem. This means that, if we write down $w_{t_n} = \prod_{x \in F_l} \delta_x \delta_{t_n g x}$ we can decide whether $w_{t_n} = 1$ or not, i.e. $w_{t_n} \in V_{F_l, I}$ or not. By Lemma 12.1 this means that we can decide whether $t_n \in \Lambda_I$ or not, i.e. whether $n \in I$.

There is a (computable) normal form for each element of $\mathbb{Z}_2^{\oplus \Gamma} \rtimes \Gamma$, written it as the product of an element of $\mathbb{Z}_2^{\oplus \Gamma}$ and of an element of Γ. It follows that, since Γ has solvable world problem, the word problem in G_I is solvable if and only if it is solvable in the normal subgroup $\mathbb{Z}_2^{\oplus \Gamma}/V_{F_l, I}$.

This is equivalent to solving whether an element $x \in \mathbb{Z}_2^{\oplus \Gamma}$ belongs to $V_{F_l, I}$. This can be done as follows (provided I is recursive):

The function x is finitely supported on Γ with values in $\mathbb{Z}_2$. Consider, as in the proof of Lemma 12.1, its restriction to the coset $C := g\langle s_1 \rangle$ and form the support interval $[gs_1^{-1}, \sigma_g^r = gs_1^d]$ as in Lemma 12.1 (we choose g in its coset appropriately).

Assume this support interval is not empty. By Lemma 12.1 if $d \leq 1$ and not $g, gs_1 \in \sigma$ then $x \notin V_{F_l, I}$.

Otherwise we now check, using that I is recursive together with Lemma 9.2 of [2, Lemma 5.2], whether there is $t \in \Lambda_I$ such that gt is in the interior of *another* support interval.

If this is not the case, then by Lemma 12.1 $x \notin V_{F_l, I}$. Otherwise, subtract $\delta_{gs_1^{-1}} + \delta_g + \delta_{gs_1} + \delta_{gts_1^{-1}} + \delta_{gt} + \delta_{gts_1}$ (which is an element of $V_{F_l, I}$) from x and continue as above.

After finitely many steps, either we observe that $x \notin V_{F_l, I}$ or the intersection of the support with the coset C is empty. As there are only finitely many cosets, we observe in finitely many steps that $x \notin V_{F_l, I}$ or we arrive at the empty word, which implies that $x \in V_{F_l, I}$. $\square$

13 Finitely presented groups

13.1 Theorem. *There is an explicitly given finitely presented group G and element $A \in \mathbb{Z}[G]$ such that $\dim_G \ker(A)$ is transcendental.*

Consequently, there is a compact manifold M such that an L^2-Betti number of the universal covering is transcendental.

Proof. In Theorem 10.1 we give an explicit construction of G and A such that $\dim_G \ker(A) = \beta_1 + \beta_2 \sum_{k=1}^{\infty} 2^{-dn_k + k}$ for every subset $I = \{n_1 < n_2 < \dots\} \subset \mathbb{N}$.

Moreover, if I is recursively enumerable, e.g. $I = \{k! \mid k \in \mathbb{N}\}$ then the corresponding group G has a recursive presentation by Theorem 12.2. If we use this set I, then the resulting $\sum_{k=1}^{\infty} 2^{-dk! + k}$ is transcendental as it is a Liouville number [11].

Finally, as explained in the introduction, we use e.g. [15, Proposition 3.1] and replace G by a finitely presented supergroup. To produce such a finitely

presented group which contains the recursively presented groups G, we use Higman's theorem [8]. How to explicitly construct the supergroup and its presentation is shown nicely in [14, Chapter 12, p. 450 ff].   □

13.2 Remark. The finitely presented groups in Theorem 13.1 are obtained via application of Higman's embedding theorem. Unfortunately, although the recursively presented groups used as input for this theorem can be arranged to be solvable, this can not be expected for the resulting finitely presented group (indeed, the method of proof will produce groups which contain non-abelian free subgroups. Moreover, the construction in principle is explicit, but in practice the finite presentation obtained will be extremely cumbersome.

Some of the examples of Grabowski [5] are much more explicit and give solvable (hence amenable) groups.

13.3 Remark. Using the method of proof of Proposition 11.4 one can use "many" transcendental numbers which occur as L^2-Betti numbers of universal coverings of manifolds, or equivalently as kernel-dimensions for elements in the group ring of finitely presented groups. In particular, one can obtain all numbers of the form $\sum_{n \in I} 2^{-n}$ for a subset $I \subset \mathbb{N}$ which is recursive. In this case, moreover, we can arrange that the groups in question have a solvable word problem by Theorem 12.3.

13.4 Remark. Grabowski obtains all numbers $\sum_{n \in I} 2^{-n}$ where I is recursively enumerable. We obtain all $\sum_{k=1}^{\infty} 2^{-dn_k + k}$ for $I = \{n_1 < n_2 < \dots\}$ recursively enumerable. Itai and Dror Bar-Natan explained to us that these two classes of groups do not coincide[1]. Variations of the constructions will yield yet other values.

It is clear that there are all together only countably many possible L^2-Betti numbers using the integral group ring of finitely presented groups (as the set of isomorphism classes of these groups is countable).

It is an open question how this set exactly looks like. In [13] it is shown that for any L^2-Betti number r of the universal covering of a compact manifold there is a Turing machine which produces a sequence of rational numbers whose limes superior is r. But there seem to be many groups which satisfy this property and which are not obtained by the methods of this paper or of [5].

In [13], it is also shown that an L^2-Betti number obtained from a finitely presented group with solvable word problem is of the form $\sum_{n \in I}^{\infty} 2^{-n}$ for a recursive subset $I \subset \mathbb{N}$. Consequently, these are precisely the L^2-Betti numbers obtained with groups which have a solvable word problem.

13.5 Remark. The construction we have described here allows for many modifications. Essentially, we can make an operator A which accepts *local* patterns in the Cayley graph of Γ. One interesting modification would be to only accept 1-neighborhoods of hooks with a thickened neighborhood of the ends.

Then one could replace in the definition of the quotient groups the set F_l by a slightly large set $F = \{e, s_1, s_1^2, s_1^3\}$. Translates of this do only fit into the

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[1]Let $J = \{n_1 < n_2 < \dots\}$ be an infinite recursively enumerable, but not recursive. Set $I := \{2^{n_k}\}$ which is then also recursively enumerable. But $TI := \{2^{n_k} + k\}$ is not recursively enumerable: otherwise, as $k \le n_k << 2^{n_k}$ one could recover from the $2^{n_k} + k$ also k (and n_k). But the information that n_k is the k-th smallest element of J allows us, by waiting until $k-1$ smaller elements of J are listed, to determine exactly the elements of J which are smaller than n_k and eventually to decide which numbers are in J and which are not —contradicting that J is not recursive.

relevant set (the 1-neighborhood of the hood with thickened ends) at the ends. This way one could arrange to have identifications in such subsets only if the two legs of the hood have equal length, and to have exactly one identification in this case. Nothing else changes, but the final sum corresponding to the calculation of Theorem 10.1 does give

$$\beta_1' + \beta_2' \sum_{k=1}^{\infty} 2^{-dn_k}.$$

It is then easy to see that, using recursively presented groups, we can get all numbers $\sum_{k=1}^{\infty} 2^{-n_k}$ with $I = \{n_1 < n_2 < \dots\}$ recursively enumerable. Consequently, these numbers are also obtained as L^2-Betti numbers of universal coverings of compact manifolds.

13.6 Remark. We can go even one step further with our modifications and instead of hooks with two vertical legs work with hooks with left leg vertical as before, but right leg horizontal $\{g, gs_1, \dots, gs_1^d\}$. Again, one looks at the 1-neighborhood of such hooks, but with thickened ends.

Finally, one modifies F to be a cross: $F = \{s_1^{-2}, s_1^{-1}, e, s_1, s_2, s_2^2, s_2^{-1}\}$.

Translates of F fit only into our neighborhoods of the hook if they are placed in the end.

Instead of the subgroups Λ_I one works with subgroups Λ_I' generated by $s_1^n s_2^n$ for $n \in I$. At least if Γ is free, this subgroup is free one these generators and has appropriate properties corresponding to those of Λ_I we used above. At least if Γ is free, the extension lemma works for the cross F (use the proof of [2, Lemma 5.5]).

We can then arrange the local patterns at the two ends (which are locally different: one is horizontal, the other vertical) to differ for those which contribute to the eigenvalue -2 in such a way that extension is *not* possible. It follows that, instead of an identification which increases the weight of the contribution to the spectrum, those paths where both legs have length $n_k \in I$ do not contribute at all.

Carrying out the calculations, we obtain for I recursively enumerable an L^2-Betti number for a recursively presented group) of the form

$$\beta_1'' - \beta_2'' \sum_{k \in I}^{\infty} 2^{-dk}; \quad \text{with } \beta_1'', \beta_2'' \in \mathbb{Q}, d \in \mathbb{N}.$$

We haven't checked, but expect, that the same works with $\Gamma = \mathbb{Z} \wr \mathbb{Z}$.

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