ASSOUAD-NAGATA DIMENSION OF CONNECTED LIE GROUPS

J.HIGES AND I. PENG

Abstract. We prove that the asymptotic Assouad-Nagata dimension of a connected Lie group G equipped with a left-invariant Riemannian metric coincides with its topological dimension of G/C where C is a maximal compact subgroup. To prove it we will compute the Assouad-Nagata dimension of connected solvable Lie groups and semisimple Lie groups. As a consequence we show that the asymptotic Assouad-Nagata dimension of a polycyclic group equipped with a word metric is equal to its Hirsch length and that some wreath-type finitely generated groups can not be quasi-isometrically embedded into any cocompact lattice on a connected Lie group.

Contents

1. Introduction

The Assouad-Nagata dimension was introduced by Assouad in [1] inspired from the ideas of Nagata. Metric spaces of finite Assouad-Nagata dimension satisfy interesting geometric properties. For example they admit quasisymmetric embeddings into the product of finitely many trees [22] and have nice Lipschitz extension properties (see [22] and [6]). The class of metric

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spaces with finite Assouad-Nagata dimension includes in particular all doubling spaces, metric trees, Euclidean buildings, and homogeneous or pinched negatively curved Hadamard manifolds.

In recent years a small scale version and a large scale version of the Assouad-Nagata dimension have been the focus of interesting research. The small scale version has been studied in the framework of hyperbolic groups under the name of capacity dimension (see [8]). The large scale version of Assouad-Nagata dimension has been referred to by several names, such as asymptotic dimension of linear type, asymptotic dimension with Higson property or asymptotic Assouad-Nagata dimension. This last name would be the one we will use in this paper.

The asymptotic Assouad-Nagata dimension is a linear version of the asymptotic dimension (see [12] for a survey about asymptotic dimension), and it is invariant under quasi-isometries. But while the asymptotic dimension remains invariant under coarse equivalences, the asymptotic Assouad-Nagata dimension does not. Therefore it is reasonable to expect that there would be more relationships between the asymptotic Assouad-Nagata dimension and other quasi-isometric invariants of geometric group theory. For example in [24] and [4] the asymptotic Assouad-Nagata dimension was related with the growth type of amenable groups and wreath products, in [15] it was shown that the asymptotic Assouad-Nagata dimension bounds from above the topological dimension of the asymptotic cones. From the results of [16] it is easy to see that every metric space of finite asymptotic Assoaud-Nagata dimension has Hilbert compression equals one.

On the other hand, estimating the asymptotic Assouad-Nagata dimension of a group is more difficult. Most of the methods developed to calculate the asymptotic dimension of a group behaves poorly for asymptotic Assouad-Nagata dimension. For example, in [2] Bell and Dranishnikov developed a technique to estimate the asymptotic dimension of the fundamental group of a graph of groups. Unfortunately such method can not be applied directly to the asymptotic Assouad-Nagata dimension.

For connected Lie groups, it is natural to study short exact sequences of the form:

$$1 \to H \to G \to G/H \to 1.$$

The idea is to estimate the dimension of G from the dimensions of H and G/H. For example if G is nilpotent then H could be abelian and G/H nilpotent with lower degree of nilpotency.

In [5], Brodskyi, Dydak, Levin and Mitra developed techniques to study the asymptotic Assouad-Nagata dimension of short exact sequences of finitely generated groups with word metrics, and the ideas were applied successfully in [15] to calculate the asymptotic Assouad-Nagata dimension of the Heisenberg group. But the main difficulty in the general case is to understand the distortion of the subgroup H in G.

In this paper we study the Assouad-Nagata dimension (at small and large scale) of connected Lie groups equipped with left invariant Riemannian metrics. First we will analyze the Assouad-Nagata dimension of simply connected solvable Lie groups. The key tool for such goal is a generalization of the results from [5] in the setting of finitely generated groups with word metrics, to general topological groups with left invariant metrics. Using this and some facts from differential geometry, we will show that the Assouad-Nagata dimension of a connected solvable Lie group is equal to its topological dimension. As a consequence of this we will prove that the asymptotic Assouad-Nagata dimension of a polycyclic group is equal to its Hirsch length. This answers an open problem of the asymptotic Assouad-Nagata dimension as Question 4 of [24] and Problem 8.3 of [13](notice that in [13] it is said that Osin solved such problem but no proof has been provided). Also our results increase the catalogue of finitely generated groups with finite asymptotic Assouad-Nagata dimension. So far, the classes of groups known to have finite asymptotic Assouad-Nagata dimension are Coexeter groups, abelian groups, hyperbolic groups, free groups and some types of Baumslag-Solitar groups. Recently it was shown that the asymptotic Assouad-Nagata dimension is preserved under free products (see [7]).

The next step in the proof of the main theorem of this paper will be the study of the Assouad-Nagata dimension of semisimple Lie groups. For such goal we will use the Iwasawa decomposition of a semisimple Lie group. Roughly speaking an Iwasawa decomposition will say that a semisimple Lie group is equivalent to the product of a connected solvable group and a special group K for which we can apply the methods of [5].

After studying the Assouad-Nagata dimension of semisimple Lie groups we will focus in the main result of this paper: the Assouad-Nagata dimension of connected Lie groups. Notice that in [10], Carlsson and Goldfarb proved that the asymptotic dimension of a Lie group G is equal to the topological dimension of G/C where C is its maximal compact subgroup. Therefore it is natural to see if the results of [10] can be extended to the asymptotic Assouad-Nagata dimension. Unfortunately the same techniques of [10] cannot be applied directly to asymptotic Assouad-Nagata dimension. The main tools of [10] use strongly the invariance under coarse equivalences of the asymptotic dimension. For example Proposition 3.4. of [10] uses such property. We will show in Example 4.11 that also the techniques from Theorem 3.5. of [10] can not be applied.

As it was mentioned before, the asymptotic Assouad-Nagata dimension has relationships with the topological dimension of the asymptotic cones. In [15] it was shown that the topological dimension of the asymptotic cone is bounded from above by the asymptotic Assouad-Nagata dimension of the group. De Cornulier computed in [11] the topological dimension of the asymptotic cones of connected Lie groups. He defined the exponential radical of a connected Lie group and used it to compute the dimension of the cones. We will also use the exponential radical to compute the Assouad-Nagata dimension of connected Lie groups. For this final theorem the computations made for connected solvable Lie groups and semisimple Lie groups will play an important role.

We will prove that the asymptotic Assouad-Nagata dimension for a connected Lie group G is equal to the topological dimension of G/C, where C is a maximal compact subgroup. Hence our results can be seen as a bridge among the ones of [10], [11] and [15]. Moreover it is not difficult to show (although tedious) that combining the techniques of [11] and [15] with our main theorem we can improve slightly the results of [11] to compute the Assouad-Nagata dimension of the asymptotic cones of connected Lie groups.

In Section 2 we will study the behaviour of the Assouad-Nagata dimension under some transformations on the metric. Such results will be important when we study the restriction of a metric of a group to one of its subgroups. In Section 3 we will generalize the results of [5] to groups with left invariant metrics. Section 4 is devoted to an intermediate step in our proof: the study of the Assouad-Nagata dimension of nilpotent groups. In Section 5 we will prove that the Assouad-Nagata dimension of simply-connected solvable Lie groups is equal to the topological dimension. As a consequence we will compute the asymptotic Assouad-Nagata dimension of finitely generated polycyclic groups. In section 6 we will study, using Iwasawa decompositions, the Assouad-Nagata dimension of semisimple Lie groups. Finally in section 7 we will prove the main result of the paper.

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2. Assouad-Nagata dimension and transformations on the metric

Let s be a positive real number. A *s-scale chain* (or s-path) between two points x and y of a metric space (X, d_X) is defined as a finite sequence of points $\{x = x_0, x_1, ..., x_m = y\}$ such that $d_X(x_i, x_{i+1}) < s$ for every $i = 0, ..., m-1$. A subset S of a metric space (X, d_X) is said to be *s-scale connected* if every two elements of S can be connected by s-scale chain contained in S.

Definition 2.1. A metric space (X, d_X) is said to be of *asymptotic dimension* at most n (notation $\text{asdim}(X, d) \leq n$) if there is an increasing function $D_X : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $s > 0$ there is a cover $\mathcal{U} = \{\mathcal{U}_0, ..., \mathcal{U}_n\}$ so that the s-scale connected components of each $\mathcal{U}_i$ are $D_X(s)$-bounded i.e. the diameter of every component is bounded by $D_X(s)$.

The function D_X is called an *n-dimensional control function* for X. Depending on the type of D_X one can define the following invariants:

Definition 2.2. A metric space (X, d_X) is said to have

- *Assouad-Nagata dimension* at most n (denoted by $\dim_{AN}(X,d) \leq n$) if it has an n-dimensional control function D_X of the form
$$D_X(s) = C \cdot s$$
with $C > 0$ some fixed constant.
- *asymptotic Assouad-Nagata dimension* at most n (denoted by $\mathrm{asdim}_{AN}(X,d) \leq n$) if it has an n-dimensional control function D_X of the form
$$D_X(s) = C \cdot s + k$$
with $C > 0$ and $k \in \mathbb{R}$ two fixed constants.
- *capacity dimension* at most n (notation $\mathrm{cdim}(X,d) \leq n$) if it has an n-dimensional control function D_X such that
$$D_X(s) = C \cdot s$$
in a neighborhood of 0 i.e. with s sufficiently small.

Remark 2.3. For any metric space (X,d) we have $\dim(X,d) \leq dim_{AN}(X,d)$ where dim is the topological dimension (see [1]).

One map $f : (X,d) \to (Y,D)$ between metric spaces is said to be a *quasi-isometric embedding* if there exists two constants $C \geq 1, \lambda \geq 0$ such that:

$$\frac{1}{C} \cdot d(x,y) - \lambda \leq D(f(x), f(y)) \leq C \cdot d(x,y) + \lambda. \tag{1}$$

If in addition there exists a $K > 0$ such that $D(y, f(X)) \leq K$ for every $y \in Y$, f is said to be a *quasi-isometry* and the spaces (X,d) and (Y,D) are said to be quasi-isometric. In (1), if $\lambda = 0$ then f is said to be a *bilipschitz equivalence* and the spaces are said to be *bilipschitz equivalent*. One important fact about Assouad-Nagata dimension is that it is preserved under bi-Lipschitz equivalences. Similarly the asymptotic Assouad-Nagata dimension is preserved under quasi-isometries. The capacity dimension is invariant under maps that are bi-Lipschitz at small scales i.e. bi-Lipschitz when the distances are less than some fix constant $\epsilon > 0$.

It is clear that $\mathrm{asdim}(X,d) \leq \mathrm{asdim}_{AN}(X,d)$. The relation among the Assouad-Nagata dimension, the asymptotic Assouad-Nagata dimension and the capacity dimension in a metric space was studied in [6] via two functors $\max(d,1)$ and $\min(d,1)$, one into the category of bounded metric spaces and the other into the category of discrete metric spaces. We collect from that paper the most important results for our purposes in the following.

Proposition 2.4. [6] *Let (X,d) be a metric space. Then:*

(1) *$cdim(X,d) = dim_{AN}(X, \max(d,\epsilon))$ for every $\epsilon > 0$.*
(2) *$asdim_{AN}(X,d) = dim_{AN}(X, \min(d,\epsilon))$ for every $\epsilon > 0$.*

(3) $dim_{AN}(X,d) = max\{asdim_{AN}(X,d), cdim(X,d)\}$

It was shown in [22] that the Assouad-Nagata dimension is also invariant under quasisymmetric embeddings as for example snow-flake transformations. Recall that a *snow flake transformation* of a metric space (X,d) is of the form (X,d^{α}) where $0<\alpha\leq 1$. We will need the following lemma that is in some sense a generalization of this fact for geodesic spaces:

Lemma 2.5. *Let (X,d) be a (non bounded) geodesic metric space. Suppose $f:\mathbb{R}_+\to\mathbb{R}_+$ is a surjectively non decreasing function such that $(X,f(d))$ is a metric space. Then $dim_{AN}(X,f(d))\leq dim_{AN}(X,d)$.*

Proof. Notice that $f(x+y)\leq f(x)+f(y)$ for every $x,y\in\mathbb{R}_+$. This follows from the fact that (X,d) is geodesic, unbounded, and $(X,f(d))$ is a metric. Let us show it. Pick $a,b\in X$ such that $d(a,b)=x+y$. As (X,d) is geodesic there exists a $c\in X$ such that $d(a,c)=x$ and $d(b,c)=y$. Hence by the triangle inequality in $(X,f(d))$ we get $f(d(a,b))\leq f(d(a,c))+f(d(b,c))$.

Suppose $dim_{AN}(X,d)\leq n$. Let $D(s)=C\cdot s$ be an n-dimensional control function of (X,d). Without loss of generality we can assume $C\in\mathbb{N}$. Let $s>0$. Take some inverse f^{-1} of f. Hence there exists a covering of (X,d) of the form $\mathcal{U}=\bigcup_{i=0}^{n}\mathcal{U}_i$ such that the $f^{-1}(s)$-scale components of each $\mathcal{U}_i$ are $C\cdot f^{-1}(s)$-bounded for every $i\in\{0,\cdots,n\}$. This implies that if $x,y\in X$ are in different $f^{-1}(s)$-scale components of $\mathcal{U}_i$ then $s\leq f(d(x,y))$. Therefore the s-scale components of $\mathcal{U}_i$ in $(X,f(d))$ are contained in the $f^{-1}(s)$-scale components of $\mathcal{U}_i$ in (X,d). Let $x,y\in\mathcal{U}_i$ be two elements that belongs to the same $f^{-1}(s)$-scale component of $\mathcal{U}_i$. We have $d(x,y)\leq C\cdot f^{-1}(s)$. This implies $f(d(x,y))\leq f(C\cdot f^{-1}(s))$. On the other hand by the subadditivity of f and $C\in\mathbb{N}$ we get $f(C\cdot f^{-1}(s))\leq C\cdot f(f^{-1}(s))=C\cdot s$ as desired. □

From the previous lemma we have the following result that can be applied to many remarkable cases as for example unbounded trees or Cayley graphs of finitely generated groups such that the asymptotic dimension coincides with the asymptotic Assouad-Nagata dimension.

Corollary 2.6. *Let (X,d) be a geodesic space and let $f:\mathbb{R}_+\to\mathbb{R}_+$ as in the previous lemma such that $lim_{x\to\infty}f(x)=\infty$. If $asdim(X,d)=dim_{AN}(X,d)$ then $dim_{AN}(X,f(d))=dim_{AN}(X,d)$*

Proof. Just notice that $\text{asdim}(X,d)=\text{asdim}(X,f(d))$ as f induces a coarse equivalence. □

The condition on the equality of dimensions can not be dropped as it is shown in the following example.

Example 2.7. In [4] it was shown that the Cayley graph Γ of the group $\mathbb{Z}_2\wr\mathbb{Z}^2$ has infinite Assouad-Nagata dimension but asymptotic dimension equals two. Moreover from Theorem 5.5 of [4] it follows that the Cayley graph has a 2-dimensional control function of polynomial type. Let d be the metric of the Cayley graph Γ. It is not hard to check (or you can apply directly proposition 2.2 of [17]) that $dim_{AN}(\Gamma, log(d+1)) = 2$.

3. Dimension of exact sequences

In this section we extend a result of [5] about dimension of exact sequences of finitely generated groups equipped with word metrics to dimension of exact sequences of general groups(not necessarily countable) equipped with left invariant metrics. First we will recall some definitions and results from [5].

Definition 3.1. Given a function between metric spaces $f : X \to Y$ and given $m \in \mathbb{N}$. An m-dimensional control function of f is a function $D_f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $r_X > 0$ and $R_Y > 0$, every set $A \subset X$ such that $\text{diam}(f(A)) \leq R_Y$ can be expressed as the union of $m+1$ subsets $\mathcal{U} = \bigcup_{i=0}^{n} \mathcal{U}_i$ such that the r_x-scale components of each $\mathcal{U}_i$ are $D_f(r_X, R_Y)$-bounded.

As in the previous section depending on the type of D_f we could have different notion of dimension of functions (see [5]). We will use just the following:

Definition 3.2. A function between metric spaces $f : X \to Y$ is said to have Assouad-Nagata dimension at most n (notation $\dim_{AN}(f) \leq n$) if there exists an n-dimensional control function D_f of the form $D_f(r_x, R_Y) = a \cdot r_x + b \cdot R_Y$. If n is the minimum number such that f satisfies this property then f is said to have Assouad-Nagata dimension exactly n.

Proposition 3.3. [5, Theorem 7.2] *If $f : X \to Y$ is a Lipschitz function between metric spaces then:*

$$dim_{AN}(X, d_X) \leq dim_{AN}(f) + dim_{AN}(Y, d_Y). \qquad (2)$$

Proposition 3.4. [5, Proposition 3.7] *Suppose A is a subset of a metric space (X, d), $m \geq 0$, $R > 0$. If D_A is an m-dimensional control function of A then $D_B(s) := D_A(s + 2R) + 2R$ is an m-dimensional control function of the R-neighborhood $B(A, R)$.*

Let $1 \to K \to G \to H \to 1$ be an exact sequence of groups. Suppose d_G is a left invariant metric. This metric induces in H a natural metric called the *Hausdorff metric* defined by the norm $\|g \cdot K\|_H = inf\{\|g \cdot k\|_G ; k \in K\}$. Notice that when G is a finitely generated group and d_G is a word metric then the corresponding induced metric d_H is also a word metric.

The induced Hausdorff metric allows us to extend Corollary 8.5 of [5] to general groups as follows.

Proposition 3.5. *Let $1 \to K \to G \to H \to 1$ be an exact sequence of groups. Then*

(1) $dim_{AN}(G, d_G) \leq dim_{AN}(K, d_G|_K) + dim_{AN}(H, d_H)$;
(2) $asdim_{AN}(G, d_G) \leq asdim_{AN}(K, d_G|_K) + asdim_{AN}(H, d_H)$;

(3) $cdim(G, d_G) \leq cdim(K, d_G|_K) + cdim(H, d_H)$,

where d_G is a left invariant metric of G and d_H is the induced (left invariant) Hausdorff metric on H.

Proof. We will prove only the first inequality. The other two follow from the first one and Proposition 2.4 (notice that the induced Hausdorff metrics are preserved by the two functors).

The proof is close to that of Corollary 8.5 in [5]. First, it is clear that if d_H is the Hausdorff metric, then the projection map $f : (G, d_G) \to (H, d_H)$ is 1-Lipschitz. Let $B = B(1_H, R_H)$ and let $A = f^{-1}(B)$. Take $a \in A$, then $\|f(a)\|_H < R_H$. As f is an homomorphism and by the symmetry of the norm we get $\|f(a^{-1})\|_H = \|f(a)^{-1}\|_H = \|f(a)\|_H < R_H$. But on the other hand we have $\|f(a^{-1})\|_H = inf\{\|a^{-1} \cdot k\|_G; k \in K\}$. Therefore $\|f(a^{-1})\|_H = d_G(a, K) < R_H$, and we have shown $A \subset B_G(K, R_H)$. By Proposition 3.4 we have that $D_f(r_G, r_H) := D_K(r_G + 2 \cdot R_H) + 2 \cdot R_H$ is an m-dimensional control function of $B(K, R_H)$, provided D_K is an m-dimensional control function of K. Now if $dim_{AN}(K, d_G|_K) \leq m$ then there exists an $C > 0$ such that $D_K(s) = C \cdot s$. Since all $f^{-1}(B(y, R_H))$ are isometric we can apply Proposition 3.3 to get the inequality:

$$dim_{AN}(G, d_G) \leq dim_{AN}(K, d_G|_K) + dim_{AN}(H, d_H).$$

□

4. Assouad-Nagata dimension of Nilpotent Lie groups

As an intermediate step we study in this section the Assouad-Nagata dimension of nilpotent Lie groups. From Proposition 2.4 we would need to calculate the capacity dimension and the asymptotic Assouad-Nagata dimension. For all connected Lie groups (not necessarily solvable) we will show first that the capacity dimension is less or equal than the topological one. We will use the ideas of Buyalo and Lebedeva. Following definitions and theorem 4.3 come from [9]:

Definition 4.1. A map $f : Z \to Y$ between metric spaces is said to be *quasi-homothetic* with coefficient $R > 0$, if for some $\lambda \geq 1$ and for all z, $z' \in Z$, we have

$$R \cdot d_Z(z, z')/\lambda \leq d_Y(f(z), f(z')) \leq \lambda \cdot R \cdot d_Z(z, z').$$

In this case, it is also said that f is λ-quasi-homothetic with coefficient R.

Definition 4.2. A metric space Z is *locally similar* to a metric space Y, if there is $\lambda \geq 1$ such that for every sufficiently large $R > 1$ and every $A \subset Z$ with $\text{diam}A \leq \Lambda_0/R$, where $\Lambda_0 = \min\{1, \text{diam}Y/\lambda\}$, there is a λ-quasi-homothetic map $f : A \to Y$ with coefficient R.

Theorem 4.3. [9, Theorem 1.1] *Assume that a metric space Z is locally similar to a compact metric space Y. Then $cdim(Z) < \infty$ and $cdim(Z) \leq dim(Y)$.*

Lemma 4.4. *Let G be a connected Lie group of dimension n with a left-invariant Riemannian metric d_G. Then it is locally similar to a closed ball in $\mathbb{R}^n$.*

Proof. Note that on a smooth manifold M, the exponential map $Exp : T_gM \to M$ is a diffeomorphism between a closed ball of radius ϵ (which might depend on the point p) and a convex neighborhood of p. Now, any diffeomorphism between two compact Riemannian metric spaces is bilipschitz, so there exists a constant λ (which might also depend on the point p) such that the exponential map restricted to the ball of radius ϵ is λ-bilipschitz for some $\lambda \geq 1$. If M is a Lie group, by left translations neither the ϵ nor the λ depends on the point p, so we can assume $p = 1_G$. Let $\log : \bar{B}(1_G, \epsilon) \to \bar{B}(0, K)$ be the inverse of the exponential map where $\bar{B}(0, K)$ is some ball of $\mathbb{R}^n$ that contains the image of $\log(\bar{B}(1_G, \epsilon))$. The metric d of $\mathbb{R}^n$ that we are considering is the usual one. Without loss of generality let us suppose $K > 2 \cdot \lambda$. In such case $\Lambda_0 := min\{1, \frac{2 \cdot K}{\lambda}\} = 1$ and $R_0 = \frac{1}{\epsilon}$. Given $R > R_0$, let A be a closed ball of radius at most $\frac{1}{R}$. By left translation we can assume A is centered in 1_G. Notice $\log(1_G) = 0$. Define the natural dilatation $g : \bar{B}(0, K) \to \bar{B}(0, K \cdot R)$. By the assumption of $K > 2 \cdot \lambda$ the image of the composition of $g \circ \log$ restricted to A lies in $\bar{B}(0, K)$. Moreover it satisfies:

$$\lambda^{-1} \cdot d_G(x, y) \leq 1/R \cdot d(g(\log(x)), g(\log(y))) \leq \lambda \cdot d_G(x, y).$$

□

Combining this lemma with Theorem 4.3 we get:

Corollary 4.5. *Let G be a connected Lie group equipped with a left-invariant Riemannian metric d_G. Then $cdim(G, d_G) \leq dim(G)$.*

Now we study the asymptotic Assouad-Nagata dimension of a nilpotent Lie group G . For such purpose we will apply Proposition 3.5 to the exact sequence $1 \to N^r \to G \to G/N^r \to 1$ where N^r is the last term in the lower central series. Notice that G/N^r is also a nilpotent Lie group. The metrics considered will be a Riemannian metric in G, d_G, and the corresponding induced Hausdorff metric in G/N^r. We will need the following.

Lemma 4.6. *Let G be a connected Lie group and $H \lhd G$ a normal subgroup such that G/H is a connected Lie group. Then there are Riemannian metrics ρ_G and $\rho_{G/H}$ on G and G/H such that the (left-invariant) Hausdorff metric on G/H induced by ρ_G agrees with the path metric induced by $\rho_{G/H}$.*

Proof. Write $\pi : G \to G/H$ for the canonical projection. In G, choose a complement V to T_eH in T_eG. Define ρ_G at T_eG by choosing inner products on T_eH and V, and define $\rho_{G/H}$ at $T_e(G/H)$ to be the inner product chosen on V. By left translating V we obtain a distribution $\triangle_V$ on G, so every differentiable path $\gamma \in G/H$ has a isometric lift $\tilde{\gamma}$ lying entirely in $\triangle_V$, which is unique up to the choice of the starting point.

Fix a coset $pH \in G/H$. Since both metrics are left invariant, it suffices to show that $d_{\rho_{G/H}}(H, pH)$ is the same as $d_{\mathcal{H}}(H, pH)$. Let $\mathcal{K}$ be the set of isometric lifts of differentiable paths connecting H with Hp in G/H that start at the identity, and $\mathcal{B}$ be the set of differentiable paths starting at the identity and end at some point in pH. That is,

$$\mathcal{K} = \{\tilde{\gamma} : \tilde{\gamma}(0) = e, \pi(\tilde{\gamma}) = \gamma, \gamma : [0, L] \to G/H, \gamma(0) = H, \gamma(L) = Hp\}$$

Clearly $\mathcal{K} \subset \mathcal{B}$. However if $\eta \in \mathcal{B}$, $\widetilde{\pi(\eta)}$, the lift of its projection, is also in $\mathcal{B}$, with length no bigger than η, so $\min\{\|\eta\| : \eta \in \mathcal{B}\} = \min\{\|\widetilde{\pi(\eta)}\| : \eta \in \mathcal{B}\}$. But

$$\{\widetilde{\pi(\eta)} : \eta \in \mathcal{B}\} = \mathcal{K}.$$

The claim now follows since

$$d_{\mathcal{H}}(H, pH) = \min\{\|\eta\| : \eta \in \mathcal{B}\} = \min\{\|\zeta\| : \zeta \in \mathcal{K}\} = d_{\rho_{G/H}}(H, Hp)$$

□

Let N be a connected, simply connected nilpotent Lie group, and let N^i be the i-th term in its lower central series. That is, $N^2 = [N, N]$, and $N^{i+1} = [N, N^i]$. By construction, each quotient N^i/N^{i+1} is an abelian Lie group of dimension n_i, so by fixing a subset $\mathcal{K}_i \subset N^i$, a set of N^{i+1} coset representatives in N^i, we have a bijection $\phi_i : \mathbb{R}^{n_i} \to \mathcal{K}_i$, and a map $\phi : N \to \oplus_i \mathbb{R}^{n_i}$ defined as

$$p \xrightarrow{\phi} (p_1, p_2 \cdots), p_i \in \mathbb{R}^{n_i}$$

where $\phi_1(p_1)N^2 = pN^2$, and $\phi_i(p_i)N^{i+1} = (\phi_{i-1}(p_{i-1}))^{-1} \cdots (\phi_1(p_1))^{-1}pN^{i+1}$. One can check that ϕ is a bijection with the inverse given by

$$(p_1, p_2, \cdots) \xrightarrow{\phi^{-1}} \phi_1(p_1)\phi_2(p_2)\phi_3(p_3) \cdots$$

Note that if the degree of nilpotency of N is r, then N^r, the last non-trivial satisfies

$$\phi(N^r) = \{(0, 0, \cdots, p_r) : p_r \in \mathbb{R}^{n_r}\}.$$

With this coordinate system we define $D : N \times N \to \mathbb{R}$ as

$$D(p, q) = D(1, p^{-1}q) = \sum_i^r \|\vec{x}_i\|^{1/i},$$

where $\phi(p^{-1}q) = (\vec{x}_1, \cdots \vec{x}_r)$, and $\|\|$ is the standard Euclidean norm on $\mathbb{R}^n$.

The main result from [20] is the following.

Theorem 4.7. [20, Theorem 4.2] *Let N be a connected, simply connected nilpotent Lie group. Then there is a Riemannian metric d_N on N and constant κ such that for any point p with $d_N(e, p) > 1$,*

$$1/\kappa D(1, p) \le d_N(1, p) \le \kappa D(1, p).$$

Equivalently, $(N, \min(1, D))$ is bilipschitz to $(N, \min(1, d_N))$.

An understanding of distance distortions in nilpotent Lie groups can now be obtained.

Lemma 4.8. *Let N be a connected, simply connected nilpotent Lie group, and H the last non-trivial term in its lower central series. Then there are Riemannian metrics d_N, d_H on N and H such that $(H, \min(1, d_N|_H))$ is bilipschitz to $(H, \min(1, (d_H)^{1/r}))$ where r is the degree of nilpotency of N.*

Proof. Note that $H = N^r$. Equip H with the Riemannian metric induced by $\phi_r : \mathbb{R}^{n_r} \to H$, while putting on G the Riemannian metric given by Theorem 4.7. With these choices we have $D|_H = (d_H)^{1/r}$, and the claim follows since Theorem 4.7 says that $(H, \min(1, D|_H))$ is bilipschitz to $(H, \min(1, d_N|_H))$. □

Lemma 4.9. *Let N be a nilpotent Lie group and let d_N be a left invariant Riemannian metric. Then $asdim_{AN}(N, d_N) \leq dim(N)$*

Proof. Recall the topological dimension of N is the same as the sum of the topological dimensions of factors in its lower central series. We will prove the lemma by induction on the degree of nilpotency. The base case is when N is an abelian Lie group, and in this case we have the equality $asdim_{AN}(N, d_N) = dim(N)$. In general we consider the following short exact sequence

$$1 \to H \to N \to N/H \to 1$$

where H is the last term in the lower central series of N. Let $d_{N/H}$ denotes the induced Hausdorff metric on N/H from d_N and d_H denotes the Riemannian metric on H induced by d_N. By Proposition 3.5 we have:

$$\text{asdim}_{AN}(N, d_N) \leq \text{asdim}_{AN}(H, d_N|_H) + \text{asdim}_{AN}(N/H, d_{N/H}).$$

But $\text{asdim}_{AN}(H, d_N|_H) = \text{asdim}_{AN}(H, d_H)$ because of Lemma 4.8 and the fact that the Assouad-Nagata dimension is invariant under quasi-isometries and snowflake transformations. Moreover $\text{asdim}_{AN}(H, d_H) = dim(H)$ since H is abelian. On the other hand, Lemma 4.6 says that $d_{N/H}$ is a Riemannian metric in N/H, and since N/H is nilpotent with one less degree of nilpotency, induction hypothesis yields $asdim_{AN}(N/h, d_{N/H}) \leq dim(N/H)$, to which the desired claim now follows. □

Theorem 4.10. *Let N be a nilpotent Lie group and let d_G be some Riemannian metric. Then $dim_{AN}(N, d_N) = dim(N)$*

Proof. By Proposition 2.4 the Assouad-Nagata dimension is equal the maximum of the capacity dimension and the asymptotic Assouad-Nagata dimension. Hence applying corollary 4.5 and lemma 4.9 we get $dim_{AN}(N, d_N) \leq dim(N)$. The other inequality follows from the fact that the Assouad-Nagata dimension is always greater or equal to the topological one (see [1]). □

Next example will show that the proof of Theorem 3.5 of [10] can not be applied to compute the Assouad-Nagata dimension of nilpotent groups.

We recommend the reading of such proof in order to understand better the example.

Example 4.11. Consider the 4-dimensional nilpotent Lie group determined by the following Lie algebra.

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4$$

It has a group structure given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \bullet \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + \frac{1}{2}(-x_2 y_1 + x_1 y_2) \\ x_4 + y_4 + \frac{1}{12}(x_1 - y_1)(-x_2 y_1 + x_1 y_2) + \frac{1}{2}(-x_3 y_1 + x_1 y_3) \end{pmatrix}$$

A left-invariant Riemannian metric is given by

$$\left(dx_1 + \frac{1}{2}x_2 dx_3 - \frac{1}{6}x_1^2 x_2 dx_4\right)^2 + \left(dx_2 - \frac{1}{2}x_1 dx_3 + \frac{1}{6}x_1^2 dx_4\right)^2 + \left(\frac{-1}{2}x_1 dx_3\right)^2 + dx_4^2$$

Following the proof of Theorem 3.5 in [10], we express this 4-dimensional nilpotent groups as a semidirect product $T \ltimes N_0$, where N_0 is the subgroup generated by e_2, e_3, e_4, and T is the subgroup generated by e_1. Then the metric on $N_0(x_1 e_1)$ is

$$\left(dx_2 + x_1 dx_3 + x_1^2 dx_4\right)^2 + (dx_3 + x_1 dx_4)^2 + (dx_4)^2$$

Note the role of e_1 coordinate, x_1, plays in the metric. For example if I is an interval of size c in the e_4 direction, then the diameter of $I(x_1 e_1)$, right translate of I by $x_1 e_1 \in T$, is $x_1^2 c$. Similarly if J is an interval of size c in the e_3 direction then the diameter of $J(x_1 e_1)$ is $x_1 c$. In this way we see for a subset $W \subset N_0$ of bounded diameter, the diameter $W(x_1 e_1)$ depends on the value of x_1. Since the metric is left-invariant, the diameter of $W(x_1 e_1)$ is the same as the diameter of $(x_1' e_1)W(x_1 e_1)$ for any $x_1' e_1 \in T$, the same is true for conjugates of W by $x_1 e_1$.

Now to see that the diameter of $S_i^l(U)$ is about D^2 times the diameter of U, we observe that $S_i^l(U)$ consists of right translates of U by elements of T. Express U as $U = t_i W$ for some $W \subset N_0$, $t_i \in T$, we have that

$$S_i^l(U) = \bigcup_{t \in 2D} Ut = \bigcup_{t \in 2D} t_i W t = \bigcup_{t \in 2D} t_i t(t^{-1} W t),$$

which makes the diameter of $S_i^l(U)$ the maximum of $2D$ and the diameter of $t^{-1}Wt$. For $D < 1$, the diameter of $S_i^l(U)$ varies linearly with D (since $D > D^2$ for $D < 1$), but for $D > 1$, the discussion above shows that the diameter depends on D^2.

5. Assouad-Nagata dimension of solvable Lie groups

Definition 5.1. Let G be a connected, simply connected solvable Lie group. The exponential radical, denoted as $Exp(G)$, is a closed normal subgroup such that $G/Exp(G)$ is the biggest quotient with polynomial growth.

Osin has shown in [25] that given a Riemannian metric d_G on G and a Riemannian metric on the exponential radical $d_{Exp(G)}$ then there exists two constants $C > 0$ and $\epsilon \geq 0$ such that fore every $h \in Exp(G)$:

$$\frac{1}{C} log(\|h\|_{Exp(G)} + 1) - \epsilon \leq \|h\|_G \leq C \cdot log(\|h\|_{Exp(G)} + 1) + \epsilon \qquad (3)$$

Also he proved that the exponential radical is contained in the nilradical of G, so it is nilpotent and we have $asdim_{AN}(Exp(G), d_{Exp(G)}) \leq dim(Exp(G)))$ by Lemma 4.9. But if we apply Lemma 2.5 to the particular case $f(s) = log(s+1)$ and the invariance of the asymptotic Assouad-Nagata dimension under quasi-isometries we get $asdim_{AN}(Exp(G), d_G|_{Exp(G)}) \leq dim(Exp(G))$. On the other hand by the definition of exponential radical there is an exact sequence:

$$1 \to Exp(G) \to G \to S \to 1,$$

where S is a solvable Lie group with polynomial growth, and quasi-isometric to a nilpotent group of the same topological dimension [3]. Hence by Lemma 4.9 and Lemma 4.6 we have $\mathrm{asdim}_{AN}(S, d_S) \leq dim(S)$ where d_S is the Riemannian metric induced by d_G. Therefore applying Proposition 3.5 we get:

Proposition 5.2. *Let G be a connected solvable Lie group then:*

$$asdim_{AN}(G, d_G) \leq asdim_{AN}(Exp(G), d_1) + asdim_{AN}(G/Exp(G), d_2) \leq dim(G)$$

where d_1 and d_2 are two Riemannian metrics defined in $Exp(G)$ and $G/Exp(G)$ respectively.

Now we can get the main result of this section:

Theorem 5.3. *Let G be a connected solvable Lie group then:*

$$dim_{AN}(G, d_G) = dim(G).$$

Proof. The proof is analogous to the that of Theorem 4.10. On the one hand we have $dim_{AN}(G, d_G) \geq dim(G)$ by [1]. By Proposition 2.4 the Assouad-Nagata dimension is equal the maximum of the capacity dimension and the asymptotic Assouad-Nagata dimension. Hence the other inequality follows from Proposition 5.2 and Corollary 4.5. □

As a consequence of this theorem and Theorem 1.3 of [22] we get the following interesting property of connected solvable Lie groups:

Corollary 5.4. *Let G be a connected solvable Lie group equipped with a Riemannian metric d_G. Then there exists a $0 < p \leq 1$ such that (G, d_G^p) can be bilipschitz embedded into a product of $dim(G) + 1$ many trees.*

Definition 5.5. Let Γ be a finitely generated solvable group. Then the hirsch length is defined as:

$$h(\Gamma) = \sum \mathrm{dim}_{\mathbb{Q}}(\Gamma_i/\Gamma_{i+1} \otimes \mathbb{Q})$$

where $\Gamma_2 = [\Gamma, \Gamma]$ and $\Gamma_{i+1} = [\Gamma_i, \Gamma_i]$

Next Corollary answers Question 4 of [24] and problem 8.3 of [13]:

Corollary 5.6. *Let* (Γ, d_w) *be a polycyclic group equipped with a word metric* d_w. *Then*

$$asdim_{AN}(\Gamma, d_w) = h(\Gamma)$$

Proof. In [14] it was proved that $\text{asdim}(\Gamma, d_w) \geq h(\Gamma)$, so the unique thing we need to show is that $\text{asdim}_{AN}(\Gamma, d_w) \leq h(\Gamma)$. It is known that every polycyclic group is a cocompact lattice in a connected, simply connected solvable Lie group H after modding out a finite torsion subgroup. Moreover $h(\Gamma) = \dim(H)$ and by Theorem 5.3 we have $\dim(H) = \dim_{AN}(H)$. As Γ is a lattice we have $\text{asdim}_{AN}(\Gamma) = \text{asdim}_{AN}(H) \leq \dim_{AN}(H)$, where the last inequality follows from Proposition 2.4. □

Remark 5.7. The condition of word metrics can not be relaxed as it is shown in [18], that a countable nilpotent group G can always be equipped with a proper left invariant metric d_G, such that $\text{asdim}_{AN}(G, d_G)$ is infinite.

6. Assouad-Nagata dimension of semisimple Lie groups

In this section we will compute the asymptotic Assouad-Nagata dimension of semisimple Lie groups. The idea will be to study the Iwasawa decompositions of such groups. Theorem 6.3 from [21] and [19] gives the structural description of a semisimple Lie group that we will need.

Definition 6.1. A Lie algebra is semisimple if it does no have a non-trivial solvable ideal.

Definition 6.2. A Lie group is semisimple if its Lie algebra is semisimple.

Theorem 6.3. *Theorem 6.31, 6.46 from* [21], *Theorem 3.1, Lemma 3.3, Chap XV from* [19]

Let G *be a semisimple Lie group with finitely many component. Then there exist subgroups* K, A *and* N *such that the multiplication map* $A \times N \times K \to G$ *given by* $(a, n, k) \mapsto ank$ *is a diffeomorphism. The groups* A *and* N *are simply connected abelian and simply connected nilpotent respectively, and* A *normalizes* N.

Furthermore, $Z(G)$, *the center of* G, *is contained in* K, *and there is an isometrically embedded connected, simply connected abelian Lie group* $V = Z(K)$, *a compact subgroup* $T < K$ *such that* K *is the semidirect product* $V \rtimes T$, *and that* $Z(G)/V$ *is compact.*

Finally, the group T *is maximal compact in* G *and any compact subgroup can be conjugated into* T.

Remark 6.4. This decomposition of $G = ANK$ is unique up to conjugation. By abuse of terminology we call any one of them the *Iwasawa decomposition.*

Remark 6.5. By definition, a semisimple Lie algebra has no center. Since a semisimple Lie group is one for which its Lie algebra is semisimple, it follows that the center of a semisimple Lie group is necessarily discrete, and that the K in the Iwasawa decomposition is compact if and only if the center is finite.

Corollary 6.6. *Let G be a semisimple Lie group with finite center, and ANK be the Iwasawa decomposition. Then G is bilipschitz diffeomorphic to $AN \times K$.*

Proof. The map $G \to AN \times K$ sends an element $p = ank \mapsto (an, k)$, and we only need to check that the ratio between $d(a_1n_1k_1, a_2n_2k_2)$ and $d((a_1n_1, k_1), (a_2n_2, k_2))$ is bounded. This will follow from the following calculation. Take $k_i \in K$, $g_i \in AN$,

$$d(g_1k_1, g_2k_2) = d(e, k_1^{-1}g_1^{-1}g_2k_2) = d(e, k_1^{-1}k_2\left(k_2^{-1}g_1^{-1}g_2k_2\right))$$

As K is compact,

$$\max\{\frac{d(e,g)}{d(e,kgk^{-1})}, \frac{d(e,kgk^{-1})}{d(e,g)}, g \in G, k \in K\} < \infty.$$

□

Notice that when the center of G is finite then it is easy to compute the Assouad-Nagata dimension applying the previous corollary and the fact that K is compact in such case. The idea of the proof for semisimple Lie groups (not necessarily with finite center) will be to reduce the general case to the finite center case using universal covers. So we have to focus in the fundamental group of a Lie group and its universal cover.

Proposition 6.7. [23] *(Proposition 1.6.4) The fundamental group of any Lie group is a subgroup of its center.*

Proof. Let G be a Lie group and $\tilde{G}$ its universal cover. Then $\pi_1(G)$ is a normal discrete subgroup of $\tilde{G}$ and $G = \tilde{G}/\pi_1(G)$. Since $\pi_1(G)$ is normal, for a fixed $d \in \pi_1(G)$ we can define a map $\alpha : \tilde{G} \to \pi_1(G)$ by $g \mapsto gdg^{-1}$. Since this is a continuous map and $\pi_1(G)$ is discrete, it follows that the image must be a point, namely d. So this shows that π_G lies in the center of $\tilde{G}$. □

Lemma 6.8. *Let G be a semisimple Lie group, and ANK the Iwasawa decomposition. Then $\tilde{G}$, the universal cover of G, is also semisimple and $AN\tilde{K}$ is the Iwasawa decomposition of $\tilde{G}$ where $\tilde{K}$ is the universal cover of K.*

Proof. That $\tilde{G}$ is semisimple follows from the fact that a covering map is a local diffeomorphism and a Lie group is defined to be semisimple if its Lie algebra is semisimple. Since A and N are simply connected, it follows that $\pi_1(G) = \pi_1(K)$, so the second claim follows. □

The next lemma will be used also in the next section. It is just a technical lemma about abelian Lie groups. It will help to compute the Assouad-Nagata dimension of a group of the form G/Γ with G an abelian Lie group and Γ a discrete subgroup.

Lemma 6.9. *Let Γ be a discrete subgroup of $\mathbb{Z}^k \times \mathbb{R}^l$. Then $\left(\mathbb{Z}^k \times \mathbb{R}^l\right)/\Gamma$ is quasi-isometric to $\mathbb{Z}^j$ for some j.*

Proof. As $\mathbb{Z}^k \times \mathbb{R}^l$ is q.i. to $\mathbb{Z}^k \times \mathbb{Z}^l$, it is enough to describe $\left(\mathbb{Z}^k \times \mathbb{Z}^l\right)/\Gamma$. Since Γ is a graph of a homomorphism from $\mathbb{Z}^{\min\{k,l\}} \to \mathbb{Z}^{\max\{k,l\}}$, by expressing $\left(\mathbb{Z}^k \times \mathbb{Z}^l\right)$ in terms of a basis that contains a basis representing Γ as a graph, the quotient $\left(\mathbb{Z}^k \times \mathbb{Z}^l\right)/\Gamma = \mathbb{Z}^j$ for some j. □

Now we provide the main theorem of this section.

Theorem 6.10. *Let G ba a semisimple Lie group with a Riemannian metric d. Then $asdim_{AN}(G,d) = dim(G/T)$ where T is a maximal compact subgroup of G*

Proof. By corollary 3.6 of [10] we have $dim(G/T) = asdim(G) \leq asdim_{AN}(G)$. So we only need to show $asdim_{AN}(G) \leq dim(G/T)$.

By Theorem 6.3, $G = ANK$, where $K = V \rtimes T$, and V is connected, simply connected abelian Lie group and T is a maximal compact subgroup. We cannot apply Proposition 3.5 because AN is not normal in G. Instead, we observe that $G/Z(G)$ is a semisimple Lie group with trivial center, and since $Z(G) < K$, it follows that $\hat{G} = AN(K/Z(G))$ is the Iwasawa decomposition for $\hat{G}$, where $K/Z(G)$ is compact. By Corollary 6.6, we have a bilipschitz diffeomorphism $\hat{\phi}$ between $\hat{G}$ and $AN \times K/Z(G)$. Lifting $\hat{\phi}$ up to G we have a bilipschitz diffeomorphism ϕ between G and $AN \times K$ and the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{\phi} & AN \times K \\ \downarrow & & \downarrow \\ \hat{G} & \xrightarrow{\hat{\phi}} & AN \times K/Z(G) \end{array}$$

So now $asdim_{AN}(G) \leq asdim_{AN}(AN) + asdim_{AN}(K)$. But Theorem 6.3 says that K/V is compact and V is isometrically embedded in K, it follows that $asdim_{AN}(K) = asdim_{AN}(V)$, and so $asdim_{AN}(G) \leq asdim_{AN}(AN) + asdim_{AN}(V) = dim(AN) + dim(V)$, since AN and V are both connected, simply connected solvable Lie groups. But $dim(AN)+dim(V) = dim(G/T)$. □

7. Assouad-Nagata dimension of connected Lie groups

In this section we will prove the main result of this paper. We will reduce the general case to the computations made for nilpotent, solvable and semisimple Lie groups in the previous sections. In some sense the proof is similar to the one of solvable Lie group. We will need to study an exact sequence generated by some subgroup that is exponentially distorted in our Lie group. Hence we will need the following concept from [11]:

Definition 7.1. Two Lie groups are said to be locally isomorphic if they have isomorphic Lie algebras.

Remark 7.2. If G_1, G_2 are two Lie groups with isomorphic Lie algebras, then there is a simply connected Lie group $\tilde{G}$ and discrete subgroups $\Gamma_1, \Gamma_2 < Z(\tilde{G})$ such that $G_1 = \tilde{G}/\Gamma_1$, $G_2 = \tilde{G}/\Gamma_2$.

Definition 7.3. If G is a connected Lie group. Its exponential radical $R_{exp}(G)$ is the subgroup of G so that $G/R_{exp}(G)$ is the biggest quotient locally isomorphic to a direct product of a semisimple group and a group with polynomial growth.

Lemma 7.4. *Let G be a connected Lie group and let $R_{exp}(G)$ be its exponential radical. Then $R_{exp}(G)$ is strictly exponentially distorted in G and it is contained in the nilpotent radical of G.*

Proof. See Theorem 6.3. of [11] □

So now we need to study the quotient $G/R_{exp}(G)$. The key will be to understand what means to be locally isomorphic from a large scale point of view. For such purpose we need the following:

Lemma 7.5. *Let G be simply connected semisimple Lie group. Then G is quasi-isometric to $G/Z(G) \times Z(G)$.*

Proof. We already know that G is quasi-isometric to $G/\tilde{K} \times \tilde{K}$ where $\tilde{K}$ is the appropriate factor in the Iwasawa decomposition of G. Since $Z(G) < \tilde{K}$ is a co-compact subgroup, the following sequence

$$1 \to G/Z(G) \to G/\tilde{K} \to \tilde{K}/Z(G)$$

shows that $G/Z(G)$ is quasi-isometric to $G/\tilde{K}$, to which the desired claim follows. □

Theorem 7.6. *(Theorem 2.3 from* [19]*)*
Leg G be a topological group containing a vector group V as a closed normal subgroup. If G/V is compact, then V is a semidirect product of G.

Corollary 7.7. *Let G be a Lie group and $S < G$ a normal solvable Lie group. If G/S is compact, then G is a semidirect product between a solvable group with the same topological dimension as S and a compact group.*

Proof. We induct on the length of the commutator series of S. The base case is when S is abelian and this is given by Theorem 7.6. Now suppose this is true for solvable groups of length $j-1$. Since S is normal in G, it follows that $Z(S)$, is also normal in G. Therefore we can apply the inductive hypothesis to the pair $G/Z(S)$ with subgroup $S/Z(S)$ and conclude that $G/Z(S) = \hat{S} \rtimes \hat{T}$, where $\hat{S}$ is a solvable group and $\hat{T}$ a compact group.

Write $\hat{S} = \tilde{S}/Z(S)$, and $\hat{T} = A/Z(S)$ for subgroups $\tilde{S}, A < G$. Applying Theorem 7.6 to A and $Z(S)$ we conclude that $A = Z(S) \rtimes T$ for some compact subgroup T. Since $Z(S) < \tilde{S}$ acts on $\tilde{S}$ trivially, the semidirect product $G/Z(S) = \hat{S} \rtimes \hat{T}$ implies $G = \tilde{S} \rtimes T$. □

Remark 7.8. Note that the center of a connected, simply connected solvable Lie group is at most exponentially distorted. To justify this claim, we need two observations. First, for such a solvable Lie group S, the exponential map is a diffeomorphism and we can coordinatize S by its Lie algebra. We now describe the second observation which is taken from Section 2 of [3]. Given $\mathfrak{s}$ a solvable Lie algebra, there exist a vector subspace $\mathfrak{v}$ and a nilpotent ideal $\mathfrak{n}$ (the nilpotent radical) such that as a vector space, $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{n}$, and for each $x \in \mathfrak{v}$, $\mathfrak{v}$ lies in the zero generalized eigenspace of $ad(x)$. So if the Lie algebra of a subgroup lies in $\mathfrak{v}$, then the distance of the subgroup is at most polynomially distorted. On the other hand, since $\mathfrak{n}$ is a nilpotent ideal, $\mathfrak{v}$ acts on it, so whenever the Lie algebra of a subgroup lies in $\mathfrak{n}$, the distance of the subgroup is at most exponential-polynomially distorted.

The claim now follows since we can always split the center of S into a direct sum of subgroups, whose Lie algebras lie in $\mathfrak{v}$ and $\mathfrak{n}$ respectively.

Theorem 7.9. *Let G be a connected Lie group. Then $asdim_{AN}(G) = dim(G/C')$, where C' is a maximal compact subgroup of G.*

Proof. By Corollary 3.6 of [10] we have $dim(G/C) = asdim(G) \leq asdim_{AN}(G)$, so it is enough to show that $asdim_{AN}(G) \leq dim(G/T)$.

Let $R_{exp}(G)$ be the exponential radical of G. Then we have the following short exact sequence.

$$1 \to R_{exp}(G) \to G \to G/R_{exp}(G) \to 1, \tag{4}$$

where quotient $U = G/R_{exp}(G)$ is locally isomorphic to a direct product of a semisimple Lie group and a solvable Lie group. By Remark 7.2 there are a simply connected semisimple Lie group L and a simply connected solvable Lie group S such that U is isomorphic to $(L\times S)/\Gamma$ for some discrete subgroup $\Gamma < Z(L \times S) = Z(L) \times Z(S)$. By Lemma 7.5 we have

$$U \overset{q.i.}{\sim} L/Z(L) \times (Z(L) \times S)/\Gamma. \tag{5}$$

In addition, we also have

$$1 \to (Z(L) \times Z(S))/\Gamma \to (Z(L) \times S)/\Gamma \to S/Z(S) \to 1, \tag{6}$$

where we note that $Z(S)$ is at most exponentially distorted in S by Remark 7.8, and the same is true for $(Z(L) \times Z(S))/\Gamma$ in $(Z(L) \times S)/\Gamma$.

Putting (4), (5) and (6) together we see that

$$\begin{aligned}
asdim_{AN}(G) &\leq asdim_{AN}(R_{exp}(G)) + asdim_{AN}(G/R_{exp}(G)) \\
&= dim(R_{exp}(G)) + asdim_{AN}(U) \\
&\leq dim(R_{exp}(G)) + asdim_{AN}(L/Z(L)) + asdim_{AN}((Z(L) \times S)/\Gamma) \\
&= dim(R_{exp}(G)) + dim(L/Z(L)) + asdim_{AN}((Z(L) \times S)/\Gamma) \\
&\leq dim(R_{exp}(G)) + dim(L/Z(L)) \\
&\quad + asdim_{AN}((Z(L) \times Z(S))/\Gamma) + asdim_{AN}(S/Z(S)) \\
&= dim(R_{exp}(G)) + dim(L/Z(L)) + dim((Z(L) \times Z(S))/\Gamma) + dim(S/Z(S))
\end{aligned}$$

Now let $L = ANK$ be the Iwasawa decomposition, and $K = V \rtimes T$ as given by Theorem 6.3. Then $dim(AN) = dim(L/Z(L))$, and

$$U = AN\left((V \times S)/\Gamma \rtimes T\right).$$

But $(V \times S)/\Gamma$ is diffeomorphic to a product $E \times T_1$ where T_1 is a maximal subgroup in $(V \times S)/\Gamma$, and E is a manifold diffeomorphic to $\mathbb{R}^n$, where $n = dim(E) = dim(S/Z(S)) + dim((Z(L) \times Z(S))/\Gamma)$. In other words, $U = G/R_{exp}(G)$ is diffeomorphic to $AN \times E \times C$ where C is the compact subgroup $T_1 \rtimes T$. By Corollary 7.7 we see that the subgroup $\tilde{T} < G$ such that $\tilde{T}/R_{exp}(G) = C$ is a semidirect product between a solvable group S_1 of the same dimension as $R_{exp}(G)$ and a maximal subgroup $C' < G$. Therefore

$$\begin{aligned}
dim(G/C') &= dim(AN) + dim(E) + dim(R_{exp}(G)) \\
&= dim(L/Z(L)) + dim(S/Z(S)) + dim((Z(L) \times Z(S))/\Gamma) + dim(R_{exp}(G))
\end{aligned}$$

□

We now have an analogous result to Corollary 3.6 of [10].

Corollary 7.10. *Let Γ be a cocompact lattice in a connected Lie group G. Then*

$$asdim_{AN}(\Gamma) = dim(G/K),$$

where K is a maximal compact subgroup.

Proof. Since Γ is quasi-isometric to G, it follows from Theorem 6 that

$$asdim_{AN}(\Gamma) = asdim_{AN}(G) = dim(G/K)$$

□

For the global Assouad-Nagata dimension the result is the following:

Corollary 7.11. *Let G be a connected Lie group equipped with a Riemannian metric. Then $dim_{AN}(G) = dim(G)$.*

Proof. On one hand we have $dim_{AN}(G) = \max\{asdim_{AN}(G), cdim(G)\}$, by corollary 4.5 we get $cdim(G) \leq dim(G)$. Combining this inequality with Theorem 7.9 we get $dim_{AN}(G) \leq dim(G)$.

On the other hand $dim(G) \leq dim_{AN}(G)$ (see the original paper of Assouad [1]). □

The asymptotic Assouad-Nagata dimension gives us an obstruction for a finitely generated group to be quasi-isometrically embeddable in a connected Lie group in particular quasi-isometric to a lattice. For example for certain classes of wreath products we have the following:

Corollary 7.12. *Let H be a finite group ($H \neq 1$) and let G be a finitely generated group such that the growth is not bounded by a linear function. Then $H \wr G$ equipped with a word metric can not be embedded quasi-isometrically in any connected Lie group.*

Proof. In corollary 5.2 of [4] it was proved that $asdim_{AN}(H \wr G) = \infty$. By Theorem 7.9 the asymptotic Assouad-Nagata dimension of any connected Lie group is finite. Therefore $H \wr G$ can not be embedded quasi-isometrically in any connected Lie group. □

Remark 7.13. Notice that the previous corollary implies that the solvable groups $\mathbb{Z}_2 \wr \mathbb{Z}^n$ with $n > 1$ can not be a quasi-isometric to a cocompact lattice in a connected Lie group.

For general Lie groups we also have the following embedding result that is a consequence of the results of [22]. For more direct consequences we recommend to check such paper.

Corollary 7.14. *Let G be a connected Lie group equipped with a Riemannian metric d_G. Then there exists a $0 < p \leq 1$ such that (G, d_G^p) can be bilipschitz embedded into a product of $dim(G) + 1$ many trees.*

Proof. Combine 7.11 with Theorem 1.3 of [22]. □

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E-mail address: `josemhiges@yahoo.es`

E-mail address: `ipeng@indiana.edu`