DIOPHANTINE APPROXIMATION AND AUTOMORPHIC SPECTRUM

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ABSTRACT. The present paper establishes quantitative estimates on the rate of diophantine approximation in homogeneous varieties of semisimple algebraic groups. The estimates established generalize and improve previous ones, and are sharp in a number of cases. We show that the rate of diophantine approximation is controlled by the spectrum of the automorphic representation, and thus subject to the generalised Ramanujan conjectures.

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1. Introduction

Diophantine approximation can be viewed as an attempt to quantify the density of the set $\mathbb{Q}$ of rational numbers in the reals $\mathbb{R}$ and, more generally, the density of a number field K in its completion K_v. In this paper we will be interested in the problem of Diophantine approximation on more general algebraic varieties. Let X be an algebraic variety defined over a number field K. Given a height function $\mathrm{H} : \mathrm{X}(K) \to \mathbb{R}^+$ and a metric dist_v on $\mathrm{X}(K_v)$, we introduce a function $\omega_v(x,\epsilon)$ which measures the density of $X(K)$ in $X(K_v)$, and thus the Diophantine properties of points x in $\mathrm{X}(K_v)$ with respect to $X(K)$. We define :

$$\omega_v(x,\epsilon) := \min\{\mathrm{H}(z) : \, z \in \mathrm{X}(K),\, \mathrm{dist}_v(x,z) \le \epsilon\} \tag{1.1}$$

(if no such z exists, we set $\omega_v(x,\epsilon) = \infty$). This function is a natural generalization of the uniform irrationality exponent $\hat{\omega}(\xi)$ of a real number ξ (see, for instance, [2]). Note that $\omega_v(x,\epsilon)$ is a non-increasing function which is bounded as $\epsilon \to 0^+$ if and only if $x \in \mathrm{X}(K)$ and is finite if and only if $x \in \overline{\mathrm{X}(K)}$. For $x \in \overline{\mathrm{X}(K)} \backslash \mathrm{X}(K)$, it is natural to consider the growth rate of $\omega_v(x,\epsilon)$ as $\epsilon \to 0^+$, which provides a quantitative measure of irrationality of x with respect to K.

Our paper is motivated by the work [34] of M. Waldschmidt who considered this problem in the case when X is an Abelian variety defined over $\mathbb{Q}$ equipped with the Néron–Tate height. In terms of our notation, he proved upper estimates on the function $\omega_\infty(x,\epsilon)$ and conjectured that for every $\delta > 0$, $\epsilon \in (0, \epsilon_0(\delta))$, and $x \in \overline{\mathrm{X}(K)} \subset \mathrm{X}(\mathbb{R})$,

$$\omega_\infty(x,\epsilon) \le \epsilon^{-\frac{2\dim(\mathrm{X})}{\mathrm{rank}(\mathrm{X}(\mathbb{Q}))} - \delta}.$$

This conjecture is remarkably strong as one can show that the exponent in this estimate is the best possible. We also mention that there is a similar conjecture in the case of algebraic tori (see [35, Conjecture 4.21]).

More generally, let $\mathrm{X} \subset \mathbb{A}^n$ be a quasi-affine variety defined over a number field K. We denote by V_K the set of normalised absolute values $|\cdot|_v$ of K, by K_v the v-completion of K, by k_v the residue field and by q_v its cardinality. We define the height function on $\mathrm{X}(K)$:

$$\mathrm{H}(x) := \prod_{v \in V_K} \max_{1 \le i \le n} (1, |x_i|_v), \tag{1.2}$$

and the metric on $\mathrm{X}(K_v)$:

$$\|x - y\|_v := \max_{1 \le i \le n} |x_i - y_i|_v. \tag{1.3}$$

In this paper we derive upper estimates on the functions $\omega_v(x,\epsilon)$ for quasi-affine varieties which are homogeneous spaces of semisimple algebraic groups. Our upper bounds on the functions $\omega_v(x,\epsilon)$ depend on information about the spectrum of the associated automorphic representations, and the *best possible* upper bounds will be established in several cases. To illustrate the relevance of the Ramanujan-Petersson conjectures to our analysis let us consider first the case of diophantine approximation on hyperboloids. Further examples, including the case of spheres of dimensions 2 and 3 where best possible upper bounds are obtained, will be discussed in §2.

Example 1.1. Let Q be a non-degenerate quadratic form in three variables defined over a number field $K \subset \mathbb{R}$, $a \in K$, and

$$\mathrm{X} = \{Q(x) = a\}.$$

For a finite set of non-Archimedean places of K, we denote by O_S the ring of S-integers. We suppose that Q is isotropic over S and $\mathrm{X}(O_S) \neq \emptyset$. Then assuming the Ramanujan–Petersson conjecture for PGL_2 over K, our main results imply that (w.r.t. the maximum norm $\|\cdot\|_\infty$ on $\mathbb{R}^3$, the completion at $v = \infty$)

(i) for almost every $x \in \mathrm{X}(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(x, \delta))$, there exists $z \in \mathrm{X}(O_S)$ such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-2-\delta},$$

where the exponent 2 is the best possible (cf. (1.4) below).

(ii) for every $x \in \mathrm{X}(\mathbb{R})$ with $\|x\| \leq r$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(r, \delta))$, there exists $z \in X(O_S)$ such that

$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-4-\delta}.$$

Using the best currently known estimates towards the Ramanujan–Petersson conjecture (see [21]), our method gives unconditional solutions to (i) and (ii) with

$$\mathrm{H}(z) \leq \epsilon^{-\frac{18}{7}-\delta} \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-\frac{36}{7}-\delta}$$

respectively. Moreover, when $K = \mathbb{Q}$, (i) and (ii) give unconditional solutions to the problem of diophantine approximation on the hyperboloid $X(\mathbb{R})$ (when Q is isotropic over $\mathbb{R}$), with

$$\mathrm{H}(z) \leq \epsilon^{-\frac{64}{25}-\delta} \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-\frac{128}{25}-\delta}$$

respectively, using [20, Appendix 2].

We also mention that a positive proportion of all places satisfy the bound predicted by the Ramanujan–Petersson conjecture (see [27, 21]). For such S, results (i) and (ii) hold unconditionally.

Finally, we note that for forms unisotropic over $\mathbb{R}$ we will indeed establish the best possible bound unconditionally - see §2.

Before we state our results in full generality, we observe that there is an (obvious) lower bound for the function ω_v. For a subset Y of $\mathrm{X}(K_v)$, we set

$$\omega_v(Y, \epsilon) := \sup_{y \in Y} \omega_v(y, \epsilon).$$

Assuming that Y is not a subset of $\mathrm{X}(K)$, one can give a lower estimate on $\omega_v(Y,\epsilon)$ that depends on the set Y, and more specifically on the Minkowski dimension $d(Y)$ of Y and the size of the set of relevant approximating rational points, namely the exponent $a(Y)$ of Y.

The Minkowski *dimension* of a subset Y of $\mathrm{X}(K_v)$ is defined by

$$d(Y) := \liminf_{\epsilon\to 0^+} \frac{\log D(Y,\epsilon)}{\log(1/\epsilon)},$$

where $D(Y,\epsilon)$ denotes the least number of balls of radius ϵ (w.r.t. the distance dist_v) needed to cover Y. The set of nonsingular points in $\mathrm{X}(K_v)$ has a structure of analytic manifold over K_v. In particular, it is equipped with a canonical measure class. We note that if a subset Y of $\mathrm{X}(K_v)$ has positive measure, then

$$d(Y) = r_v \dim(\mathrm{X})$$

where $r_v = 2$ if $K_v \simeq \mathbb{C}$ and $r_v = 1$ otherwise.

The *exponent* of a subset Y of $\mathrm{X}(K_v)$ is defined by

$$\mathfrak{a}_v(Y) := \inf_{\mathcal{O}\supset Y} \limsup_{h\to\infty} \frac{\log A_v(\mathcal{O},h)}{\log h},$$

where $\mathcal{O}$ runs over open neighborhoods of Y in $\mathrm{X}(K_v)$, and

$$A_v(\mathcal{O},h) := |\{z \in \mathrm{X}(K) : \mathrm{H}(z) \le h,\, z\in\mathcal{O}\}|.$$

Note that since $Y \nsubseteq \mathrm{X}(K)$, we have $\omega_v(Y,\epsilon)\to\infty$ as $\epsilon\to 0^+$. Hence, for a sufficiently small neighbourhood $\mathcal{O}$ of Y, every $\delta_1,\delta_2>0$ and $0<\epsilon<\epsilon_0(\mathcal{O},\delta_1,\delta_2)$, we have

$$\epsilon^{-d(Y)+\delta_1} \le D(Y,\epsilon) \le A_v(\mathcal{O},\omega_v(Y,\epsilon)) \le \omega_v(Y,\epsilon)^{\mathfrak{a}_v(Y)+\delta_2}.$$

This implies that for every $\delta>0$ and sufficiently small $\epsilon>0$ depending on δ,

$$\omega_v(Y,\epsilon) \ge \epsilon^{-\frac{d(Y)}{\mathfrak{a}_v(Y)}+\delta}.$$

In particular, when Y has positive measure, we always have the lower bound

$$\omega_v(Y,\epsilon) \ge \epsilon^{-r_v\frac{\dim(\mathrm{X})}{\mathfrak{a}_v(Y)}+\delta} \tag{1.4}$$

for every $\delta>0$ and $\epsilon\in(0,\epsilon_0(Y,\delta))$.

More generally, we consider the problem of diophantine approximation for points $x=(x_v)_{v\in S}$ with $S\subset V_K$ and $x_v\in\mathrm{X}(K_v)$. Let

$$X_S := \{(x_v)_{v\in S} : x_v\in\mathrm{X}(K_v);\;\; x_v\in\mathrm{X}(O_v) \text{ for almost all } v\}, \tag{1.5}$$

where $O_v=\{x\in K_v : |x|_v\le 1\}$ is the ring of integers in K_v for non-Archimedean v. The set X_S, equipped with the topology of the restricted direct product, is a locally compact second countable space.

One of the fundamental questions in arithmetic geometry is to understand the closure $\overline{\mathrm{X}(K)}$ in X_S where $X(K)$ is embedded in X_S diagonally. We say that the *approximation property* with respect to S holds if $\overline{\mathrm{X}(K)} = X_S$. Alternatively, denote the ring of S-integers of K by

$$O_S = \{x \in K : \ |x|_v \leq 1 \text{ for non-Archimedean } v \notin S\}^1 .$$

Then the approximation property with respect to S can be reformulated as follows: for every $x \in X_S$ and every $\epsilon = (\epsilon_v)_{v \in S'}$, where $\epsilon_v \in (0,1)$ and S' is a finite subset of S, there exists $z \in \mathrm{X}(O_{(V_K \setminus S) \cup S'})$ such that

$$\|x_v - z\|_v \leq \epsilon_v \ \text{ for all } v \in S'.$$

Our aim is to establish a quantitative version of this property. Given x and $(\epsilon_v)_{v \in S'}$ as above, we consider

$$\omega_S(x, (\epsilon_v)_{v \in S'}) := \min \left\{ \mathrm{H}(z) : \ z \in \mathrm{X}(O_{(V_K \setminus S) \cup S'}), \|x_v - z\| \leq \epsilon_v, \ v \in S' \right\} . \tag{1.6}$$

Remark 1.2. To clarify our notation somewhat, consider the group variety $G \subset GL_n$. Fixing the finite set $S' \subset V_K$, we want to establish a rate for simultaneous diophantine approximation of all (or almost all) elements in the group $\prod_{v \in S'} G_v$. The elements in the group $G(K)$ of K-rational points which are allowed in the approximation process are determined by the choice of a set S containing S', which may be finite or infinite. One choice is $S = S'$, in which case the approximations property calls for using elements from $O_{V_K} = G(K)$, namely there are no restrictions at all on the K-rational matrices allowed. Thus for example when $S = \{v\} = S'$, the approximation rate ω_S defined above is given by the function $\omega_v(x, \epsilon)$ as defined in (1.1). Another choice is $S = V_K \setminus v_0$, where v_0 is a valuation not in S'. In this case the approximation property calls for using only K-rational matrices whose elements are in O_v (namely v-integral) for every $v \in V_K$, with the exception of $v \in S'$ and $v = v_0$. We will of course assume that G is isotropic over K_{v_0} in this case. We also admit any other intermediate choice of S containing S', namely we allow imposing *arbitrary* integrality conditions on the set of approximating K-rational matrices. The integrality conditions are that the matrices should be v-integral for $v \in S \setminus S'$, and we assume that G is isotropic over $V_K \setminus S$.

Given a variety X, we define the *exponent* of a subset Y of X_S as

$$\mathfrak{a}_S(Y) := \inf_{\mathcal{O} \supset Y} \limsup_{h \to \infty} \frac{\log A_S(\mathcal{O}, h)}{\log h}, \tag{1.7}$$

where $\mathcal{O}$ runs over open neighborhoods of Y in X_S, and

$$A_S(\mathcal{O}, h) := |\{z \in \mathrm{X}(K) : \ \mathrm{H}(z) \leq h, \ \ z \in \mathcal{O}\}|.$$

[1]Note that $O_{\{v\}} \neq O_v$, the ring of integers defined above !

More appropriately, the notation should be $\mathfrak{a}_S(Y,X)$ but we will suppress the dependence on X in the notation. We also define the *exponent* $\mathfrak{a}_S(\mathrm{X})$ of the variety X as the supremum of $\mathfrak{a}_S(Y)$ as Y runs over bounded subsets of X_S. Since our variety X will be fixed and we will not consider subvarieties of it, this notation should cause no conflict.

As in (1.4), one can show that given $Y \subset X_S$ and finite $S' \subset S$ such that the projection of Y to $X_{S'}$ has positive measure, we have

$$\sup_{y\in Y} \omega_S(y,(\epsilon_v)_{v\in S'}) \geq \prod_{v\in S'} \epsilon_v^{-r_v\frac{\dim(\mathrm{X})}{\mathfrak{a}_S(Y)}+\delta} \tag{1.8}$$

for every $\delta > 0$ and $\epsilon_v \in (0, \epsilon_0(Y, S', \delta))$. It particular, it follows that we have the following universal lower bound

$$\sup_{y\in Y} \omega_S(y,(\epsilon_v)_{v\in S'}) \geq \prod_{v\in S'} \epsilon_v^{-r_v\frac{\dim(\mathrm{X})}{\mathfrak{a}_S(\mathrm{X})}+\delta}, \tag{1.9}$$

provided that the projection of Y to $X_{S'}$ has positive measure.

We shall derive upper estimates on the function ω_S of comparable quality in the case when X is a homogeneous quasi-affine variety of a semisimple group. We shall start by considering the case of the group variety itself.

Let G be a connected almost simple algebraic K-group. Our main results depend on properties of unitary representations π_v of the groups $G_v := \mathrm{G}(K_v)$, which we now introduce We fix a suitable maximal compact subgroup U_v of G_v, whose choice is discussed in Section 3. The group $\mathrm{G}(K)$ embeds in the restricted direct product group G_{V_K} diagonally as a discrete subgroup and $\mathrm{vol}(G_{V_K}/\mathrm{G}(K)) < \infty$ (see [26, §5.3]). We consider the Hilbert space $L^2(G_{V_K}/\mathrm{G}(K))$ consisting of square-integrable functions on $G_{V_K}/\mathrm{G}(K)$. A unitary continuous character χ of G_{V_K} is called automorphic if $\chi(\mathrm{G}(K)) = 1$. Then χ can be considered as an element of $L^2(G_{V_K}/\mathrm{G}(K))$. We denote by $L^2_{00}(G_{V_K}/\mathrm{G}(K))$ the subspace of $L^2(G_{V_K}/\mathrm{G}(K))$ orthogonal to all automorphic characters. The translation action of the group G_v on $G_{V_K}/\mathrm{G}(K)$ defines the unitary representation π_v of G_v on $L^2_{00}(G_{V_K}/\mathrm{G}(K))$. We define the spherical integrability exponent of π_v w.r.t. U_v as follows

$$\mathfrak{q}_v(\mathrm{G}) := \inf\left\{ q>0 : \begin{array}{l} \forall\, U_v\text{-inv. } w \in L^2_{00}(G_{V_K}/\mathrm{G}(K)) \\ \langle \pi_v(g)w, w\rangle \in L^q(G_v) \end{array} \right\}. \tag{1.10}$$

If $\mathfrak{q}_v(\mathrm{G}) = 2$, then we say that the representation π_v is *tempered* [2].

It is one of the fundamental results in the theory of automorphic representations that the integrability exponent $\mathfrak{q}_v(\mathrm{G})$ is finite. (see [6, Theorem 3.1]). Moreover, $\mathfrak{q}_v(\mathrm{G})$ is uniformly bounded over $v \in V_K$ (see [9]). The precise value of $\mathfrak{q}_v(\mathrm{G})$ is related to generalised Ramanujan conjecture and Langlands functoriality conjectures (see [28]). For

[2]This is equivalent to the standard notion of a tempered representation defined in terms of weak containment, by [7, Theorem 1].

instance, the generalised Ramanujan conjecture for SL_2 is equivalent to $\mathfrak{q}_v(\mathrm{SL}_2) = 2$ for all $v \in V_K$, and the best currently known estimate established in [21, 20] gives $\mathfrak{q}_v(\mathrm{SL}_2) \le \frac{18}{7}$ for general number fields K and $\mathfrak{q}_v(\mathrm{SL}_2) \le \frac{64}{25}$ for $K = \mathbb{Q}$.

We define the *exponent* of a subset S of V_K by

$$\sigma_S := \limsup_{N\to\infty} \frac{1}{\log N} |\{v \in S : q_v \le N\}|, \tag{1.11}$$

where q_v denotes the cardinality of the residue field for non-Archimedean v. Let

$$\mathfrak{q}_S(\mathrm{G}) = (1 + \sigma_S) \sup_{v\in S} \mathfrak{q}_v(\mathrm{G}). \tag{1.12}$$

This parameter will appear below as a bound on the integrability exponent of the automorphic representation restricted to the group G_S

We now turn to state our results, and note that the approximation property for algebraic varieties we defined above, namely the density of the closure of $X(K)$ in X_S has been been studied extensively in the setting of algebraic groups (see [26, Ch. 7]). It is known that when G is connected simply connected almost simple algebraic group defined over K, then the approximation property holds with respect to S provided that G is isotropic over $V_K\backslash S$. Our first result can be viewed as a quantitative version of this fact.

It is convenient to set $I_v = \{q_v^{-n}\}_{n\ge1}$ for non-Archimedean $v \in V_K$ and $I_v = (0, 1)$ for Archimedean $v \in V_K$.

Theorem 1.3. *Let G be a connected simply connected almost simple algebraic K-group and S a (possibly infinite) subset of V_K such that G is isotropic over $V_K\backslash S$. Then*

(i) *There exists a subset Y of full measure in G_S such that for every $\delta > 0$, finite $S' \subset S$, $x \in Y$, and $(\epsilon_v)_{v\in S'}$ with $\epsilon_v \in I_v \cap (0, \epsilon_0(x, S', \delta))$, we have*

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \le \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})/2}.$$

In particular, when the representations π_v, $v \in V_K\backslash S$, are tempered and $\sigma_S = 0$, then the above exponent is the best possible *(cf. (1.8))!*

(ii) *For every $\delta > 0$, bounded $\Omega \subset G_S$, and $(\epsilon_v)_{v\in S'}$ with finite $S' \subset S$ and $\epsilon_v \in I_v \cap (0, \epsilon_v^0(\Omega, \delta))$, where $\epsilon_v^0(\Omega, \delta) \in (0, 1]$ and $\epsilon_v^0(\Omega, \delta) = 1$ for almost all v, we have*

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \le \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})},$$

uniformly over $x \in \Omega$ and finite $S' \subset S$.

Remark 1.4. Comparing the estimate in Theorem 1.3(i) and (1.8), we conclude that the integrability exponents always satisfy $\mathfrak{q}_v(\mathrm{G}) \geq 2$. While this fact was previously known (see, for instance, [3]), it is curious that it follows from diophantine approximation considerations as well. This point is further addressed in Corollary 1.8 below.

We also prove a version of Theorem 1.3(i) which is uniform over finite $S' \subset S$.

Theorem 1.5. *With the notation as in Theorem 1.3, there exists a subset Y of full measure in G_S such that for every $\delta > 0$, $x \in Y$, and $(\epsilon_v)_{v\in S'}$ with finite $S' \subset S$ and $\epsilon_v \in I_v \cap (0, \epsilon_v^0(x,\delta))$, where $\epsilon_v^0(x,\delta) \in (0,1]$ and $\epsilon_v^0(x,\delta) = 1$ for almost all v, we have*

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \leq \left(\prod_{v\in S'} \epsilon_v^{-\frac{r_v \dim(\mathrm{G})+\sigma_S}{\mathfrak{a}_S(\mathrm{G})}-\delta} \right)^{\mathfrak{q}_{V_K\setminus S}(\mathrm{G})/2}.$$

More generally, let $\mathrm{X} \subset \mathbb{A}^n$ be a quasi-affine algebraic variety defined over K equipped with a transitive action of a connected almost simple algebraic K-group $\mathrm{G} \subset \mathrm{GL}_n$. Given a subset S of V_K, we consider the problem of Diophantine approximation in X_S by the rational points in $\mathrm{X}(K)$, or equivalently the problem of Diophantine approximation in $X_{S'}$, where S' is a finite subset of S, by points in $\mathrm{X}(O_{(V_K\setminus S)\cup S'})$. The closure $\overline{\mathrm{X}(O_{(V_K\setminus S)\cup S'})}$ in $\mathrm{X}_{S'}$ can be described explicitly (see Lemma 6.3 below). In particular, it follows that $\overline{\mathrm{X}(O_{(V_K\setminus S)\cup S'})}$ is open $\mathrm{X}_{S'}$ provided that G is isotropic over $V_K\setminus S$. Our next result gives a quantitative version of the density (in the closure) where the estimates are sharp in many cases (see Example 1.1 and Section 2).

Theorem 1.6. *Assume that* G *is isotropic over $V_K\setminus S$. Then*

(i) *For every finite $S' \subset S$, there exists a subset Y of full measure in $\overline{\mathrm{X}(O_{(V_K\setminus S)\cup S'})} \subset X_{S'}$ such that for every $\delta > 0$, $x \in Y$, and $(\epsilon_v)_{v\in S'}$ with $\epsilon_v \in I_v \cap (0, \epsilon_0(x, S', \delta))$, we have*

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \leq \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})}-\delta} \right)^{\mathfrak{q}_{V_K\setminus S}(\mathrm{G})/2}.$$

(ii) *For every finite $S' \subset S$, $\delta > 0$, bounded $Y \subset \overline{\mathrm{X}(O_{(V_K\setminus S)\cup S'})}$, and $(\epsilon_v)_{v\in S'}$ with $\epsilon_v \in I_v \cap (0, \epsilon_0(Y, S', \delta))$, we have*

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \leq \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})}-\delta} \right)^{\mathfrak{q}_{V_K\setminus S}(\mathrm{G})},$$

uniformly over $x \in Y$.

Remark 1.7. We note that the estimates in Theorem 1.6 are stated in terms of $\mathfrak{a}_S(\mathrm{G})$, but not in terms of $\mathfrak{a}_S(\mathrm{X})$. While the latter quantity is difficult to compute in general, in many cases we have $\mathfrak{a}_S(\mathrm{G}) \geq \mathfrak{a}_S(\mathrm{X})$.

For instance, this is so when the rational points in G do not concentrate on a proper subgroup of G, namely when $\mathfrak{a}_S(\mathrm{G}) > \mathfrak{a}_S(\mathrm{Stab}_{\mathrm{G}}(x^0))$ for some $x^0 \in \mathrm{X}(O_{(V_K\setminus S)\cup S'})$. If $\mathfrak{a}_S(\mathrm{G}) \geq \mathfrak{a}_S(\mathrm{X})$ and $\mathfrak{q}_{V_K\setminus S}(\mathrm{G}) = 2$, then Theorem 1.6(i) gives the best possible estimate (cf. (1.9)).

Comparing Theorem 1.6(i) with (1.9), we deduce the lower estimates on the integrability exponents $\mathfrak{q}_v(\mathrm{G})$.

Corollary 1.8. *Let $G < \mathrm{GL}_n$ be a connected almost simple algebraic K-group and $\mathrm{X} \subset \mathbb{A}^n$ a quasi-affine variety on which* G *acts transitively. Assume that* G *is isotropic over $v \in V_K$ and that $\mathrm{X}(O_{V_K\setminus\{v\}})$ is not empty. Then*

$$\mathfrak{q}_v(\mathrm{G}) \geq \frac{2\mathfrak{a}_{V_K\setminus\{v\}}(\mathrm{G})}{\mathfrak{a}_{V_K\setminus\{v\}}(\mathrm{X})}.$$

For example, Corollary 1.8 implies that $\mathfrak{q}_v(\mathrm{SL}_n) \geq 2(n-1)$, which is known to be sharp (see Section 2).

2. Examples

2.1. **Diophantine approximation on spheres.** Let S^d be the unit sphere of dimension d centered at origin, $d \geq 2$, which we view as the level set of the standard quadratic form given by the sum of squares. We fix a prime $p = 1 \mod 4$. Then $\mathrm{S}^d(\mathbb{Z}[1/p])$ is dense in $\mathrm{S}^d(\mathbb{R})$, and here we derive a quantitative density estimates. We treat the cases $d = 2$, $d = 3$, and $d \geq 4$ separately.

For $d = 2$, Theorem 1.6 implies that

- For almost every $x \in \mathrm{S}^2(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(x, \delta))$, there exists $z \in \mathrm{S}^2(\mathbb{Z}[1/p])$ such that
$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-2-\delta}.$$
We note that $\dim(\mathrm{S}^2) = 2$ and $\mathfrak{a}_{V_{\mathbb{Q}}\setminus\{p\}}(\mathrm{S}^2) = 1$, so that this exponent is the best possible (see (1.9)).
- For every $x \in \mathrm{S}^2(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(\delta))$, there exists $z \in \mathrm{S}^2(\mathbb{Z}[1/p])$ such that
$$\|x - z\|_\infty \leq \epsilon \quad \text{and} \quad \mathrm{H}(z) \leq \epsilon^{-4-\delta}.$$

To deduce these estimates, we consider the group $\mathrm{G} \simeq \mathrm{D}^\times/\mathrm{Z}^\times$, where D denotes Hamilton's quaternion algebra and Z the centre of D. This group naturally acts on the variety of pure quaternions of norm one, which can be identified with the sphere S^2. Hence, we are in position to apply Theorem 1.6. Since $p = 1 \mod 4$, the Quaternion algebra split over p and ramifies at ∞. In this case, we have $\mathfrak{q}_{V_{\mathbb{Q}}\setminus\{p\}}(\mathrm{G}) = 2$, which is a consequence of the results of Deligne combined with Jacquet–Langlands correspondence (see [24, Appendix]), $\dim(\mathrm{S}^2) = 2$, and $\mathfrak{a}_{V_{\mathbb{Q}}\setminus\{p\}}(\mathrm{G}) = 1$.

For $d = 3$, Theorem 1.6 implies that

- For almost every $x \in \mathrm{S}^3(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(x,\delta))$, there exists $z \in \mathrm{S}^3(\mathbb{Z}[1/p])$ such that
$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-\frac{3}{2}-\delta}.$$
Since $\dim(\mathrm{S}^3) = 3$ and $\mathfrak{a}_{V_\mathbb{Q}\setminus\{p\}}(\mathrm{S}^3) = 2$, this exponent is the best possible (see (1.9)).
- For every $x \in \mathrm{S}^3(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(\delta))$, there exists $z \in \mathrm{S}^3(\mathbb{Z}[1/p])$ such that
$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-3-\delta}.$$

To deduce these estimates, we now consider the group G of norm one elements of Hamilton's quaternion algebra D, which can be identified with the variety S^3. We have $\mathfrak{q}_p(\mathrm{G}) = 2$, which is a again consequence of the results of Deligne combined with Jacquet–Langlands correspondence (see [24, Appendix]), $\dim(\mathrm{G}) = 3$, and $\mathfrak{a}_{V_\mathbb{Q}\setminus\{p\}}(\mathrm{G}) = 2$. Hence, the claimed estimates follow from Theorem 1.6.

Now let $d \ge 4$. We consider the natural action of the group $\mathrm{G} = \mathrm{SO}_{d+1}$ on S^d. We note that since $p = 1 \mod 4$, the group G splits over $\mathbb{Q}_p$. By [23, 25], we have $\mathfrak{q}_p(\mathrm{G}) \le d$ for even d and $\mathfrak{q}_{V_\mathbb{Q}\setminus\{p\}}(\mathrm{G}) \le d+1$ for odd d. By [11], the parameter $\mathfrak{a}_p(\mathrm{G})$ can be estimated in terms of volumes of the height balls, which in turn can be estimated in terms of the root system data of G. This gives the estimates $\mathfrak{a}_{V_\mathbb{Q}\setminus\{p\}}(\mathrm{G}) = d^2/4$ for even d and $\mathfrak{a}_p(\mathrm{G}) = (d+1)(d+3)/4$ for odd d. Therefore, Theorem 1.6 implies that for even $d \ge 4$,

- For almost every $x \in \mathrm{S}^d(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(x,\delta))$, there exists $z \in \mathrm{S}^d(\mathbb{Z}[1/p])$ such that
$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-2-\delta}.$$
The exponent 2 should be compared with the lower estimate $\frac{d}{d-1}$ given by (1.9). At present we don't know whether this exponent can be improved.
- For every $x \in \mathrm{S}^d(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(\delta))$, there exists $z \in \mathrm{S}^d(\mathbb{Z}[1/p])$ such that
$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-4-\delta}.$$

Similarly, for odd $d \ge 4$,

- For almost every $x \in \mathrm{S}^d(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(x,\delta))$, there exists $z \in \mathrm{S}^d(\mathbb{Z}[1/p])$ such that
$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-\frac{2d}{d+3}-\delta}.$$
- For every $x \in \mathrm{S}^d(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(\delta))$, there exists $z \in \mathrm{S}^d(\mathbb{Z}[1/p])$ such that
$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-\frac{4d}{d+3}-\delta}.$$

It is interesting to compare our results with the results on Diophantine approximation on the spheres obtained in [30] by elementary methods that use rational parametrisations of spheres. It is shown in [30] that for every $x \in \mathrm{S}^d(\mathbb{R})$ and $\epsilon \in (0,1)$, there exists $z \in \mathrm{S}^d(\mathbb{Q})$ such that

$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \text{const}\ \epsilon^{-2\lceil \log_2(d+1)\rceil}.$$

While this result deals with the set of all $\mathbb{Q}$-points on S^d rather than just $\mathbb{Z}[1/p]$-points, the exponent obtained is significantly weaker than ours.

2.2. **Diophantine approximation in the orthogonal group.** Let SO_{d+1}, $d \ge 4$, be the orthogonal group and p be a prime such that $p = 1 \mod 4$. Then $\mathrm{SO}_{d+1}(\mathbb{Z}[1/p])$ is dense in $\mathrm{SO}_{d+1}(\mathbb{R})$. We have $\dim(\mathrm{SO}_{d+1}) = d(d+1)/2$, and $\mathfrak{a}_{V_\mathbb{Q}\setminus\{p\}}(\mathrm{SO}_{d+1})$, $\mathfrak{q}_p(\mathrm{SO}_{d+1})$ are given above. Therefore, Theorem 1.6 gives

- For almost every $x \in \mathrm{SO}_{d+1}(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(x,\delta))$, there exists $z \in \mathrm{SO}_{d+1}(\mathbb{Z}[1/p])$ such that

$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-(d+1)-\delta}, \quad \text{when } d \text{ is even},$$

$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-\frac{d(d+1)}{d+3}-\delta}, \quad \text{when } d \text{ is odd}.$$

  This should be compared with lower estimates on the exponents given by (1.9), which is $2(d+1)/d$ for even d, and $2d/(d+3)$ for odd d.
- For every $x \in \mathrm{SO}_{d+1}(\mathbb{R})$, $\delta > 0$, and $\epsilon \in (0, \epsilon_0(\delta))$, there exists $z \in \mathrm{SO}_{d+1}(\mathbb{Z}[1/p])$ such that

$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-2(d+1)-\delta}, \quad \text{when } d \text{ is even},$$

$$\|x - z\|_\infty \le \epsilon \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-\frac{2d(d+1)}{d+3}-\delta}, \quad \text{when } d \text{ is odd}.$$

2.3. **Diophantine approximation by the Hilbert modular group.** Let K be a totally real number field, O its ring of integers, and S a proper subset of V_K^∞. The Hilbert modular group $\mathrm{SL}_2(O)$ is a dense subgroup of $(\mathrm{SL}_2)_S = \prod_{v\in S} \mathrm{SL}_2(K_v)$. We have $\dim(\mathrm{SL}_2) = 3$ and $\mathfrak{a}_{S\cup V_K^f}(\mathrm{SL}_2) = 2$, and if we assume the Ramanujan conjecture for SL_2 over K, then $\mathfrak{q}_v(\mathrm{SL}_2) = 2$. Hence, Theorem 1.3 implies the following quantitative density results (under the Ramanujan conjecture):

- For almost every $x_v \in \mathrm{SL}_2(K_v)$ with $v \in S$, $\delta > 0$, and $\epsilon_v \in (0, \epsilon_0(x,\delta))$, there exists $z \in \mathrm{SL}_2(O)$ such that

$$\|x_v - z\|_v \le \epsilon_v \text{ for } v \in S \quad \text{and} \quad \mathrm{H}(z) \le \prod_{v\in S} \epsilon_v^{-\frac{3}{2}-\delta}.$$

  The exponent $\frac{3}{2}$ is the best possible by (1.9).

- For every $x_v \in \mathrm{SL}_2(K_v)$ with $v \in S$ such that $\|x_v\|_v \le r$, $\delta > 0$, and $\epsilon_v \in (0, \epsilon_0(r,\delta))$, there exists $z \in \mathrm{SL}_2(O)$ such that
$$\|x_v - z\|_v \le \epsilon_v \text{ for } v \in S \quad \text{and} \quad \mathrm{H}(z) \le \prod_{v\in S} \epsilon_v^{-3-\delta}.$$

Using the best currently known estimates towards the Ramanujan–Petersson conjecture (see [21]), we have $\mathfrak{q}_v(\mathrm{SL}_2) \le \frac{18}{7}$, and Theorem 1.3 gives unconditional solutions to the above inequalities $\|x_v - z\|_v \le \epsilon_v$, $v \in S$, with
$$\mathrm{H}(z) \le \epsilon^{-\frac{27}{14}-\delta} \quad \text{and} \quad \mathrm{H}(z) \le \epsilon^{-\frac{27}{7}-\delta}$$
respectively.

2.4. **Estimates on integrability exponents.** Let us apply Corollary 1.8 to the action of $\mathrm{G} = \mathrm{SL}_n$ on $\mathrm{X} = \mathbb{A}^n \backslash \{0\}$. In this case, we have $\mathfrak{a}_{V_K\backslash\{v\}}(\mathrm{G}) = n^2 - n$ and $\mathfrak{a}_{V_K\backslash\{v\}}(\mathrm{X}) = n$. Hence, we conclude that
$$\mathfrak{q}_v(\mathrm{SL}_n) \ge 2(n-1).$$
This estimate is sharp.

Another example is given by the orthogonal group $\mathrm{G} = \mathrm{SO}_{d+1}$ acting on the sphere $\mathrm{X} = \mathrm{S}^d$, discussed in Section 2.1. Let p be a prime such that $p = 1 \mod 4$. In this case we have $\mathfrak{a}_{V_\mathbb{Q}\backslash\{p\}}(\mathrm{G}) = d^2/4$ for even $d \ge 4$, $\mathfrak{a}_{V_\mathbb{Q}\backslash\{p\}}(\mathrm{G}) = (d+1)(d+3)/4$ for odd $d \ge 4$, and $\mathfrak{a}_{V_\mathbb{Q}\backslash\{p\}}(\mathrm{X}) = d-1$. Therefore,
$$\mathfrak{q}_v(\mathrm{SO}_{d+1}) \ge \frac{d^2}{2d-2}, \quad \text{when } d \text{ is even},$$
$$\mathfrak{q}_v(\mathrm{SO}_{d+1}) \ge \frac{(d+1)(d+3)}{2d-2}, \quad \text{when } d \text{ is odd}.$$

3. Semisimple groups and spherical functions

Let K be a number field. We denote by V_K the set of all places of K, which consists of the finite set V_K^∞ of Archimedean places and the set V_K^f of non-Archimedean places. For $v \in V_K$, we denote by $|\cdot|_v$ the suitably normalised absolute value and by K_v the corresponding completion. Also for $v \in V_K^f$ we denote by $O_v \subset K_v$ the ring of integers and by q_v the cardinality of the residue field.

3.1. **Structure of semisimple groups.** We recall elements of the structure theory of semisimple algebraic groups over local fields, which is discussed in details in [1, 32]. Let $\mathrm{G} \subset \mathrm{GL}_n$ be a connected semisimple algebraic group defined over a number field K. For all but finitely many places in V_K,

(i) the group $U_v := \mathrm{G}(O_v)$ is a hyperspecial, good maximal compact subgroup of $G_v := \mathrm{G}(K_v)$,
(ii) the group G is unramified over K_v (that is, G is quasi-split over K_v and split over an unramified extension of K_v).

For the other places v, we fix a good, special maximal compact subgroup U_v of G_v. For $S \subset V_K$, we set $U_S = \prod_{v\in S} U_v$.

To simplify notation, for an algebraic group C_v defined over K_v we denote by C_v the set $\mathrm{C}_v(K_v)$ of K_v-points in C_v.

Every subgroup U_v is associated to a minimal parabolic K_v-subgroup P_v of G_v, that has a decomposition $\mathrm{P}_v = \mathrm{N}_v\mathrm{Z}_v$ where Z_v is connected and reductive, and N_v is the unipotent radical of P_v. Let Z_v° be the maximal compact subgroup of Z_v. Then Z_v/Z_v° is a free $\mathbb{Z}$-module of rank equal to the K_v-rank of G. We set $\mathcal{Z}_v = (Z_v/Z_v^\circ)\otimes\mathbb{R}$ and denote by $\nu_v : Z_v \to \mathcal{Z}_v$ the natural map. Let T_v be the maximal K_v-split torus in Z_v. Then $T_v^\circ := T_v \cap Z_v^\circ$ is the maximal compact subgroup of T_v and T_v/T_v° has finite index in Z_v/Z_v°. The group of characters of Z_v is denoted by $X^*(\mathrm{Z}_v)$. For any $\theta \in X^*(\mathrm{Z}_v)$ we associate a linear functional χ_θ on $\mathcal{Z}_v$ defined by

$$|\theta(z)|_v = q_v^{\langle \chi_\theta, \nu_v(z)\rangle}, \quad z \in Z_v, \tag{3.1}$$

where $q_v = e$ for $v \in V_K^\infty$, and q_v is the order of the residue field for $v \in V_K^f$.

The adjoint action of T_v on G defines a root system in the dual space $\mathcal{Z}_v^*$. We fix the ordering on the root system defined by the parabolic subgroup P_v. Let $\mathcal{Z}_v^-$ denote the negative Weyl chamber in $\mathcal{Z}_v$ with respect to this ordering. The relative Weyl group $\mathcal{W}_v := N_{G_v}(T_v)/Z_{G_v}(T_v)$ operates on Z_v and on $\mathcal{Z}_v$. We choose a $\mathcal{W}_v$-invariant scalar product on $\mathcal{Z}_v$. Then we regard the root system as a subset of $\mathcal{Z}_v$, and denote by $\mathcal{Z}_{v,-}$ the cone consisting of negative linear combinations of simple roots. Let $Z_v^- = \nu_v^{-1}(\mathcal{Z}_v^-)$ and $Z_{v,-} = \nu_v^{-1}(\mathcal{Z}_{v,-})$.

The group G_v has the *Cartan decomposition*

$$G_v = U_v Z_v^- U_v,$$

which defines a bijection between the double cosets $U_v\backslash G_v/U_v$ and Z_v^-/Z_v°, and the *Iwasawa decomposition*

$$G_v = N_v Z_v U_v,$$

which defines a bijection between $N_v\backslash G_v/U_v$ and Z_v/Z_v°. For $g \in G_v$, we define $z(g) \in Z_v/Z_v^\circ$ by the property $g \in N_v z(g) U_v$.

For unramified v, we have $T_v/T_v^\circ = Z_v/Z_v^\circ$ and the Cartan decomposition becomes

$$G_v = U_v T_v^- U_v, \tag{3.2}$$

where $T_v^- = \nu_v^{-1}(\mathcal{Z}_v^-)$.

For $S \subset V_K$, we denote by G_S the restricted direct product of the groups G_v for $v \in S$ with respect to the family of subgroups U_v. Then G_S is a locally compact second countable group, and $U_S := \prod_{v\in S} U_v$. is a maximal compact subgroup of G_S.

For every $v \in V_K$, we denote by m_{G_v} the Haar measure on G_v which is normalised so that $m_{G_v}(U_v) = 1$ for non-Archimedean v. Then the restricted product measure $\otimes_{v\in S} m_{G_v}$ defines a Haar measure on the group G_S which we denote by m_S. We also denote by m_{U_v} the probability Haar measure on U_v.

3.2. **Spherical functions.** The theory of spherical functions on semisimple groups over local fields was developed by Harish-Chandra ([13], [16, Chapter IV]) in the Archimedean case, and by Satake ([29], [5]) in the non-Archimedean case.

For $v \in V_K$, we introduce the *Hecke algebra* $\mathcal{H}_v$ consisting of bi-U_v-invariant continuous functions with compact support on G_v and equipped with product given by convolution. The structure of this algebra can be completely described, and in particular, it is commutative.

A continuous bi-U_v-invariant function $\eta : G_v \to \mathbb{C}$ with compact support is called *spherical* if $\eta(e) = 1$ and one of the following equivalent conditions holds:

(a) η is bi-U_v-invariant, and the map

$$\mathcal{H}_v \to \mathbb{C} : \phi \mapsto \int_{G_v} \phi(g)\eta(g^{-1})\, dm_{G_v}(g)$$

is an algebra homomorphism.

(b) for every $\phi \in \mathcal{H}_v$ there exists $\lambda_\phi \in \mathbb{C}$ such that $\phi * \eta = \lambda_\phi \eta$.

(c) For every $g_1, g_2 \in G_v$,

$$\eta(g_1)\eta(g_2) = \int_{U_v} \eta(g_1 u g_2)\, dm_{U_v}(u).$$

We shall use the following parametrisation of the spherical functions as well as some of their basic properties, due to Harish Chandra [13][14][15] and Satake [29].

Theorem 3.1. *Every spherical function on G_v is of the form*

$$\eta_\chi(g) = \int_{U_v} q_v^{\langle \chi, \nu_v(z(ug))\rangle} \Delta_v^{-1/2}(z(ug))\, dm_{U_v}(u) \tag{3.3}$$

where $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$ and Δ_v denotes the modular function of P_v.

Furthermore, $\eta_\chi = \eta_{\chi'}$ if and only if χ and χ' are on the same orbit of the Weyl group $\mathcal{W}_v$.

Let $\rho_v \in \mathcal{Z}_v^*$ denotes the character corresponding to the half-sum of positive roots. Then

$$\Delta_v(z) = q_v^{2\langle \rho_v, \nu_v(z)\rangle}, \quad z \in Z_v.$$

Let $\Pi_v = \{\alpha\} \subset \mathcal{Z}_v^*$ denotes the system of simple roots corresponding to the parabolic subgroup P_v. We denote by $\{\alpha'\}_{\alpha \in \Pi_v}$ the basis dual to Π_v. A character $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$ is called *dominant* if $\mathrm{Re}\,\langle \chi, \alpha'\rangle \geq 0$ for all $\alpha \in \Pi_v$. Note that every χ can be conjugated to a dominant one by $\mathcal{W}_v$. Therefore, in the discussion of spherical functions we may restrict our attention to dominant χ's.

We recall the following well-known properties of spherical functions:

Lemma 3.2. *For dominant $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$,*

- *the spherical function η_χ is bounded if and only if*

$$\operatorname{Re}\langle \chi, \alpha' \rangle \leq \langle \rho, \alpha' \rangle$$

  for all $\alpha \in \Pi_v$.
- *the spherical function $\eta_\chi \in L^p(G_v)$ if and only if*

$$\operatorname{Re}\langle \chi, \alpha' \rangle < (1 - 1/p) \langle \rho, \alpha' \rangle$$

  for all $\alpha \in \Pi_v$.

We shall also need the following estimates.

Lemma 3.3. (i) *For dominant $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$,*

$$|\eta_\chi(z)| \leq q_v^{\operatorname{Re}\langle \chi, \nu_v(g) \rangle} \eta_0(g), \quad g \in Z_v^-.$$

(ii) *For every $\epsilon > 0$,*

$$\eta_0(g) \leq c_\epsilon \, \Delta_v^{-1/2+\epsilon}(g), \quad g \in Z_v,$$

with c_ϵ bounded uniformly in v.

Proof. As to part (i), note that for the Archimedean case, this lemma is proved in [22, Proposition 7.15]. The proof for non-Archimedean case is similar. We have

$$|\eta_\chi(g)| = \int_{U_v} q_v^{\operatorname{Re}\langle \chi, \nu_v(z(ug)) \rangle} \Delta_v^{-1/2}(z(ug)) \, dm_{U_v}(u).$$

Since the double cosets $U_v g U_v$ and $N_v z(ug) U_v = N_v(ug)U_v$ have nontrivial intersection, it follows from [1, Proposition 4.4.4] that for dominant χ,

$$\operatorname{Re}\langle \chi, \nu_v(z(ug)) \rangle \leq \operatorname{Re}\langle \chi, \nu_v(z) \rangle.$$

This implies the first claim.

As to part (ii), the bound stated of the Harish Chandra Ξ-function (denoted η_0 here) was established by Harish Chandra in both the Archimedean and non-Archimedean case (see also the discussion in [17]). The uniformity of the bound, namely the fact that c_ϵ is independent of v, has been observed in [11, §6.2] and follows from the proof in [31, Thm. 4.2.1]. □

3.3. **Unitary representations.** The spherical functions arise naturally as matrix coefficients of unitary representations. We denote by $\hat{G}_v$ the unitary dual of G_v (i.e., the set of equivalence classes of irreducible unitary representations of the group G_v) and by $\hat{G}_v^1$ the spherical unitary dual (i.e., the subset consisting of spherical representations). An irreducible unitary representation is called spherical if it contains a nonzero U_v-invariant vector. Since the Hecke algebra $\mathcal{H}_v$ is commutative, the subspace of U_v-invariant vectors in an irreducible unitary representation is at most one-dimensional. For $\tau_v \in \hat{G}_v^1$, we denote by w_{τ_v} a unit U_v-invariant vector. Then the function

$$\eta_{\tau_v}(g) := \langle \tau_v(g) w_{\tau_v}, w_{\tau_v} \rangle, \quad g \in G_v,$$

is a spherical function on G_v. Moreover, different elements of $\hat{G}_v^1$ give rise to different spherical functions. Therefore, $\hat{G}_v^1$ can be identified with a subset of dominant $\chi \in \mathcal{Z}_v^* \otimes \mathbb{C}$ using Theorem 3.1.

More generally, for $S \subset V_K$, we denote by $\hat{G}_S$ the unitary dual of G_S and by $\hat{G}_S^1$ the spherical unitary dual (with respect to the subgroup U_S). Every $\tau_S \in \hat{G}_S$ is a restricted tensor product of the form $\otimes'_{v\in S}\tau_v$ where $\tau_v \in \hat{G}_v$ and τ_v is spherical for almost all v (see e.g. [8]).

We define the integrability exponent of a U_S-spherical unitary representation $\tau_S : G_S \to \mathcal{U}(\mathcal{H})$ as

$$\mathfrak{q}(\tau_S, U_S) = \inf \{ q > 0 : \forall\ U\text{-inv. } w \in \mathcal{H} : \langle \tau_S(g) w, w \rangle \in L^q(G) \}.$$

Proposition 3.4. *Let τ_v for $v \in S$ be a family consisting of irreducible spherical unitary representations of G_v. Then the integrability exponent of the restricted tensor product representation of G_S satisfies*

$$\mathfrak{q}\left(\otimes'_{v\in S}\tau_v, U_S\right) \le (1+\sigma_S) \sup_{v\in S} \mathfrak{q}(\tau_v, U_v),$$

where σ_S denotes the exponent of the subset S defined in (1.11).

Proof. We have to show that the spherical function $\eta = \prod_{v\in S} \eta_{\tau_v}$ is in $L^q(G_S)$ for $q > (1+\sigma_S)p$ where $p > p(\tau_v, U_v)$ for all $v \in S$. To derive the required estimate we use integration calculated in terms of the Cartan decomposition (3.2). Recall the volume estimate (see [11, §6.2] for a discussion) :

$$m_{G_v}(U_v t U_v) \le c\, \Delta_v(t), \quad t \in T_v^-, \tag{3.4}$$

which is uniform over v. Let the representations τ_v correspond to dominant $\chi_v \in \mathcal{Z}_v \otimes \mathbb{C}$. Since the spherical function η_{χ_v} is in $L^p(G_v)$, it follows from Lemma 3.2 that for $t \in T_v^-$,

$$\left| q_v^{\langle \chi_v, \nu_v(t) \rangle} \right| \le \Delta_v(t)^{\frac{1}{2}-\frac{1}{p}}.$$

Hence, Lemma 3.3 implies the estimate

$$|\eta_{\chi_v}(t)| \le c_\epsilon \Delta_v(t)^{-\frac{1}{p}+\epsilon} \tag{3.5}$$

for every $\epsilon > 0$. Combining (3.4) and (3.5), we deduce that

$$\begin{aligned}\int_{G_v} |\eta_{\chi_v}(g)|^q \, dm_{G_v}(g) &= \sum_{z\in T_v^-/T_v^\circ} |\eta_{\chi_v}(t)|^q m_{G_v}(U_v t U_v) \\ &\le 1 + \sum_{z\in T_v^-/T_v^\circ-\{e\}} (cc_\varepsilon^q)\Delta_v(t)^{-q/p+q\varepsilon+1} \\ &\le 1 + \sum_{z\in T_v^-/T_v^\circ-\{e\}} (cc_\varepsilon^q)\left(\prod_{\chi\in\Pi_v} |\chi(t_v)|_v\right)^{-q/p+q\varepsilon+1} \\ &\le 1 + \sum_{i_1,\ldots,i_r\in\mathbb{Z}_+,(i_1,\ldots,i_r)\ne 0} (cc_\varepsilon^q) q_v^{(-q/p+q\varepsilon+1)\sum_{j=1}^r i_j} \\ &= 1 + O_\varepsilon\left(q_v^{-q/p+q\varepsilon+1}\right)\end{aligned}$$

for every $\epsilon > 0$. It follows from the definition of density σ_S that the partial Euler product $\prod_{v\in S}(1-q_v^{-s})^{-1}$ converges for $s > \sigma_S$. Therefore, we conclude that

$$\prod_{v\in S}\int_{G_v} |\eta_{\tau_v}|^q \, dm_{G_v} < \infty$$

provided that $-q/p + 1 < -\sigma_S$. This completes the proof. □

We denote by π_S the unitary representation of G_S on $L^2_{00}(G_{V_K}/\mathrm{G}(K))$. Proposition 3.4 implies the following estimate on the integrability exponent of π_S.

Corollary 3.5. *For every $S \subset V_K$,*

$$\mathfrak{q}(\pi_S, U_S) \le \mathfrak{q}_S(\mathrm{G}),$$

where $\mathfrak{q}_S(\mathrm{G})$ is defined in (1.12).

Moreover, we also observe that the proof of Proposition 3.4 gives the following uniform estimate, where $\tau \prec \pi_S$ denotes weak containment (see e.g. [7] for a discussion).

Corollary 3.6. *For every $S \subset V_K$ and $q > \mathfrak{q}_S(\mathrm{G})$,*

$$\sup\left\{\|\eta_\tau\|_q : \tau \in \hat{G}_S^1,\, \tau \prec \pi_S\right\} < \infty\,.$$

We now use the foregoing spectral considerations to obtain an operator norm estimate. First observe

Proposition 3.7. *Let β be a bi-U_S-invariant finite Borel measure on G_S and $\pi : G_S \to \mathcal{U}(\mathcal{H})$ a strongly continuous unitary representation. Then*

$$\|\pi(\beta)\| \le \sup\left\{\sqrt{\beta^*(\eta_\tau)\beta(\eta_\tau)} : \tau \in \hat{G}_S^1,\, \tau \prec \pi\right\}.$$

Proof. To bound $\|\tau(\beta)\|$ for irreducible representations weakly contained in π, we first consider the case when the measure β is symmetric. Let $\mathcal{H}_\tau^{U_S}$ denote the subspace of $\tau(U_S)$-invariant vectors. It is clear that

$$\tau(\beta)\mathcal{H}_\tau \subset \mathcal{H}_\tau^{U_S}.$$

In particular, $\tau(\beta) = 0$ for $\tau \notin \hat{G}_S^1$. Since β is symmetric, the operator $\tau(\beta)$ is self-adjoint, and hence,

$$\tau(\beta)(\mathcal{H}_\tau^{U_S})^\perp \subset (\mathcal{H}_\tau^{U_S})^\perp.$$

This implies that

$$\|\tau(\beta)\| = \int_{G_S} \langle \tau(\beta)w_\tau, w_\tau\rangle \, d\beta(g) = \beta(\eta_\tau)$$

where $w_\tau \in \mathcal{H}_\tau^{U_S}$ with $\|w_\tau\| = 1$. This completes the proof when β is symmetric. In general, we have

$$\|\pi(\beta)\|^2 = \|\pi(\beta^* * \beta)\|$$

Hence, by the previous argument,

$$\|\pi(\beta)\| \le \sup\left\{\sqrt{(\beta^* * \beta)(\eta_\tau)} : \tau \in \hat{G}_S^1, \tau \prec \pi\right\}.$$

Since β is bi-U_S-invariant, it follows from the properties of spherical functions that

$$(\beta^* * \beta)(\eta_\tau) = \beta^*(\eta_\tau)\beta(\eta_\tau),$$

which implies the claim. □

We can now obtain the following operator norm estimate.

Proposition 3.8. *Let β be a Haar-uniform probability measure supported on a bi-U_S-invariant bounded subset $B \subset G_S$. Then*

$$\|\pi_S(\beta)\| \ll_{S,\delta} m_S(B)^{-\frac{1}{\mathfrak{q}_S(\mathrm{G})}+\delta}.$$

for every $\delta > 0$.

Proof. In view of Proposition 3.7 we need to establish a uniform estimate

$$\beta(\eta_\tau) = \frac{1}{m_S(B)} \int_B \eta_\tau(g)\, dm_S(g),$$

where η_τ is the spherical function of a representation $\tau \in \hat{G}_S^1$ which is weakly contained in π_S. By Hölder's inequality, for $q > \mathfrak{q}_S(\mathrm{G})$, we have

$$\beta(\eta_\tau) \le \frac{1}{m_S(B)} \|\chi_B\|_{(1-1/q)^{-1}} \|\eta_\tau\|_q = m_S(B)^{-1/q}\|\eta_\tau\|_q.$$

Now the claim follows from Proposition 3.7 and Corollary 3.5. □

4. Mean ergodic theorem

We keep the notation from the previous section. In particular, G denotes a connected semisimple group defined over a number field K. Our aim is to prove the mean ergodic theorem for the space $\Upsilon := G_{V_K}/\mathrm{G}(K)$ equipped with the invariant probability measure μ. For $S \subset V_K$, we consider the natural action of $G_{V_K\backslash S}$ on Υ and, given a Haar-uniform probability measure β on $G_{V_K\backslash S}$, the averaging operator

$$\pi_{V_K\backslash S}(\beta)\phi(\varsigma) = \int_{G_{V_K\backslash S}} \phi(g^{-1}\varsigma)\, d\beta(g), \quad \phi \in L^2(\Upsilon). \tag{4.1}$$

We first consider the case when the group G is simply connected. For simply connected groups, the mean ergodic theorem admits a simple version, but the general case requires more delicate considerations because of the presence of nontrivial automorphic characters.

Let $\mathcal{X}_{aut}(G_{V_K})$ be the set of automorphic characters, that is, the set consisting of continuous unitary characters χ of G_{V_K} such that $\chi(\mathrm{G}(K)) = 1$.

Lemma 4.1. *If* G *is simply connected, then* $\mathcal{X}_{aut}(G_{V_K}) = 1$.

Proof. The group G is isotropic over K_v for some $v \in V_K$ (see [26, Theorem 6.7]. Then the group G_v coincides with its commutator (see [26, §7.2]), and hence $\chi(G_v) = 1$ for every character χ of G_{V_K}. Since by the strong approximation property $\mathrm{G}(K)G_v$ is dense in G_{V_K}, the claim follows. □

Lemma 4.1 implies that for simply connected groups, $L^2_{00}(G_{V_K}/\mathrm{G}(K))$ is the space of functions with zero integral. Hence, Proposition 3.8 gives

Theorem 4.2 (mean ergodic theorem)**.** *Assume that* G *is simply connected, and let* β *be the Haar-uniform probability measure supported on bi-*$U_{V_K\backslash S}$*-invariant subset* B *of* $G_{V_K\backslash S}$*. Then for every* $\phi \in L^2(\Upsilon)$,

$$\left\| \pi_{V_K\backslash S}(\beta)\phi - \int_\Upsilon \phi\, d\mu \right\|_2 \ll_{S,\delta} m_{V_K\backslash S}(B)^{-\frac{1}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta}$$

for every $\delta > 0$.

The general case of Theorem 4.2 requires two auxiliary lemmas.

Lemma 4.3. *Let* $p : \tilde{\mathrm{G}} \to \mathrm{G}$ *be a simply connected cover of a connected semisimple group defined over* K*. Then for every* $S \subset V_K$*,* $p(\tilde{G}_S)$ *is a normal co-Abelian subgroup of* G_S*. Moreover, if* S *is finite, then* $p(\tilde{G}_S)$ *has finite index in* G_S*.*

Proof. For every $v \in V_K$, we have the exact sequence in Galois cohomology

$$\tilde{G}_v \xrightarrow{p} G_v \to H^1(K_v, \ker(p)).$$

Since $H^1(K_v, \ker(p))$ is Abelian, $p(\tilde{G}_v)$ is a normal co-Abelian subgroup of G_v. This implies that $p(\tilde{G}_v)$ contains the commutator of G_v. Since $H^1(K_v, \ker(p))$ is finite group (see [26, §6.4]), $p(G_v)$ has finite index in G_v. Also for almost all $v \in V_K$, we have the exact sequence

$$\tilde{\mathrm{G}}(O_v) \xrightarrow{p} \mathrm{G}(O_v) \to H^1\left(K_v^{ur}/K_v, \ker(p)(O_v^{ur})\right)$$

(see [26, Proposition 6.8]), where K_v^{ur} denotes the unramified closure of K_v and O_v^{ur} denotes its ring of integers. Therefore, $p(\tilde{\mathrm{G}}(O_v))$ is a normal co-Abelian subgroup of $\mathrm{G}(O_v)$, and in particular, $p(\tilde{\mathrm{G}}(O_v))$ contains the commutator of $\mathrm{G}(O_v)$.

The group G_S is a union of the subgroups $G_{S,S'}$ defined by

$$G_{S,S'} := \left(\prod_{v\in S'} G_v\right)\left(\prod_{v\in S'} \mathrm{G}(O_v)\right),$$

as S' runs over finite subsets of S. It follows from the first paragraph that $p(\tilde{G}_S)$ contains the commutator of $G_{S,S'}$ for every finite $S' \subset S$. Hence, $p(\tilde{G}_S)$ contains the commutator of G_S, and $p(\tilde{G}_S)$ is a normal co-Abelian subgroup of G_S. □

Let S' be a finite subset of V_K. For an open subgroup U of $G_{V_K^f\backslash S'}$, we denote by $\mathcal{X}_{aut}(G_{V_K})^U$ the subset of $\mathcal{X}_{aut}(G_{V_K})$ consisting of U-invariant characters. Let G^U denote the kernel of $\mathcal{X}_{aut}(G_{V_K})^U$ in G_{V_K}.

Lemma 4.4. *The group $\mathcal{X}_{aut}(G_{V_K})^U$ is finite, and G^U has finite index in G_{V_K}.*

Proof. Let $p : \tilde{\mathrm{G}} \to \mathrm{G}$ be the simply connected cover. For every $\chi \in \mathcal{X}_{aut}(G_{V_K})$, we have $\chi \circ p \in \mathcal{X}_{aut}(\tilde{G}_{V_K})$. Hence, it follows from Lemma 4.1 that $\chi \circ p = 1$. We conclude that every $\chi \in \mathcal{X}_{aut}(G_{V_K})^U$ vanishes on $H := \mathrm{G}(K)p(\tilde{G}_{V_K})U$. By Lemma 4.3, $p(\tilde{G}_{V_K})$ is a normal co-Abelian subgroup of G_{V_K}. Hence it follows that H is a normal co-Abelian subgroup as well. We claim that it has finite index in G_{V_K}.

For every $v \in V_K$, we have the exact sequence

$$\tilde{G}_v \to G_v \to H^1(K_v, \ker(p)),$$

where the last term is finite by [26, §6.4]. This shows that $p(\tilde{G}_{V_K^\infty \cup S'})$ has finite index in $G_{V_K^\infty \cup S'}$. The number of double cosets (i.e., the class number) of the subgroups $\mathrm{G}(K)$ and $G_{V_K^\infty} U_{V_K^f \cap S'} U$ in G_{V_K} is finite (see [26, §8.1]). Then the number of double cosets of $\mathrm{G}(K)$ and $p(\tilde{G}_{V_K^\infty \cup S'})U$ in G_{V_K} is finite as well. From this we conclude that the number of double cosets of $\mathrm{G}(K)$ and $p(\tilde{G}_{V_K})U$ in G_{V_K} is finite, and since $p(\tilde{G}_{V_K})$ is co-Abelian, the factor group G_{V_K}/H is finite.

We have shown above that every $\chi \in \mathcal{X}_{aut}(G_{V_K})^U$ factors through the finite factor group G_{V_K}/H. This implies that $\mathcal{X}_{aut}(G_{V_K})^U$ is finite and G^U has finite index in G_{V_K}, as required. □

Theorem 4.5 (mean ergodic theorem)**.** *Let S be a subset of V_K and S' a finite subset of S. Let U^0 be a finite index subgroup of $U_{V_K^f\cap(S\backslash S')}$ and $U = U_{V_K^f\backslash S}U^0$. Let B be a bounded measurable subset of $G_{V_K\backslash S}\cap G^U$ which is $U_{V_K^f\backslash S}$-biinvariant and β the Haar-uniform probability measure supported on the subset U^0B of $G_{(V_K\backslash S)\cup(V_K^f\backslash S')}$. Then for every $\phi\in L^2(\Upsilon)$ such that* $\mathrm{supp}(\phi)\subset G^U$, *we have*

$$\left\|\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)\phi-\left(\int_\Upsilon\phi\,d\mu\right)\xi_U\right\|_2\ll_{U^0,S,\delta} m_{V_K\backslash S}(B)^{-\frac{1}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta}\|\phi\|_2$$

for every $\delta>0$, where ξ_U is the function on Υ such that $\xi_U=|G_{V_K}:G^U|$ on the open set $G^U/\mathrm{G}(K)\subset\Upsilon$ and $\xi_U=0$ otherwise.

Proof. We have the decomposition

$$L^2(\Upsilon)=\mathcal{H}^0\oplus\mathcal{H}^1\oplus\mathcal{H}^2$$

where $\mathcal{H}^0$ is the orthogonal complement of $\mathcal{X}_{aut}(G_{V_K})$, $\mathcal{H}^1$ is the (finite dimensional) span for $\mathcal{X}_{aut}(G_{V_K})^U$, and $\mathcal{H}^2$ is the orthogonal complement of $\mathcal{H}^0\oplus\mathcal{H}^1$. For $\phi\in L^2(\Upsilon)$, we have the corresponding decomposition

$$\phi=\phi_0+\phi_1+\phi_2.$$

We observe that the set $\mathcal{X}_{aut}(G_{V_K})^U$ can be identified with the group of characters of the finite Abelian group G_{V_K}/G^U. Hence, it follows that

$$\sum_{\chi\in\mathcal{X}_{aut}(G_{V_K})^U}\chi=\xi_U.$$

Since $\mathcal{X}_{aut}(G_{V_K})^U$ forms an orthonormal basis of $\mathcal{H}^1$,

$$\phi_1=\sum_{\chi\in\mathcal{H}^1}\langle\phi,\chi\rangle\,\chi=\left(\int_\Upsilon\phi\,d\mu\right)\sum_{\chi\in\mathcal{X}_{aut}(G_{V_K})^U}\chi=\left(\int_\Upsilon\phi\,d\mu\right)\xi_U.$$

Since the measure β is U-invariant, for every $\chi\in\mathcal{X}_{aut}(G_{V_K})$ and $u\in U$, we have

$$\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)\chi=\chi(u)\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)\chi.$$

Therefore, if χ is not U-invariant, then

$$\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)\chi=0.$$

This implies that

$$\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)|_{\mathcal{H}^2}=0.$$

Also, since $\mathrm{supp}(\beta)\subset G^U$, we have

$$\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)|_{\mathcal{H}^1}=id.$$

We conclude that

$$\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)\phi-\left(\int_\Upsilon\phi\,d\mu\right)\xi_U=\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)\phi_0.$$

By Jensen inequality,

$$\left\|\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)\phi_0\right\|_2 \ll_{U^0} \left\|\pi_{V_K\backslash S}(\beta')\phi_0\right\|_2,$$

where β' is the Haar-uniform probability measure supported on $B \subset G_{V_K\backslash S}$. Therefore, by Proposition 3.8,

$$\left\|\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta)\phi_0\right\|_2 \ll_{U^0,S,\delta} m_{V_K\backslash S}(B)^{-\frac{1}{q_{V_K\backslash S}(\mathrm{G})}+\delta}\|\phi_0\|_2$$

for every $\delta > 0$. This implies the claim. □

5. The duality principle

5.1. **Duality and almost sure approximation on the group variety.** In this section we develop an instance of the duality principle in homogeneous spaces, which is analyzed more generally in [12]. Heuristically speaking, the principle asserts that given a lattice subgroup Γ of G, and another closed subgroup H, the effective estimates on the ergodic behavior of Γ-orbits in G/H can be deduced from the effective estimates on the ergodic behavior of H-orbits in G/Γ. This accounts for the essential role that the automorphic representation and its restriction to subgroups plays in our considerations.

Let $\mathrm{G} \subset \mathrm{GL}_n$ be a linear algebraic group defined over a number field K. In this section we relate the problem of Diophantine approximation on G and on its homogeneous spaces with "shrinking target" properties of suitable dynamical systems. More precisely, we consider the space $\Upsilon = G_{V_K}/\mathrm{G}(K)$ equipped with the natural translation action of the group $G_{V_K\backslash S}$ where $S \subset V_K$. We show that a point in G_S can be approximated by rational points in $\mathrm{G}(K)$ with given accuracy provided that the corresponding orbit of $G_{V_K\backslash S}$ in Υ can be used to approximate the identity coset with a suitable accuracy.

Let $\|\cdot\|_v$ be the maximum norm on K_v^n, that is, $\|x\|_v = \max_i |x_i|_v$ for $x \in K_v^n$. We observe that for $g \in \mathrm{GL}_n(K_v)$,

$$\|g\cdot x\|_v \le c_v(g)\|x\|_v, \tag{5.1}$$

where $c_v(g) = \max_{i,j}|g_{ij}|_v$ for $v \in V_K^f$ and $c_v(g) = n\max_{i,j}|g_{ij}|_v$ for $v \in V_K^\infty$. Note that $c_v(g)$ is bounded on compact subsets of $\mathrm{GL}_n(K_v)$, and $c_v(g) = 1$ for $g \in \mathrm{GL}_n(O_v)$.

The height function H on G_{V_K} is defined by

$$\mathrm{H}(g) = \prod_{v\in V_K}\max_{i,j}(1,|g_{ij}|_v), \quad g \in G_{V_K}. \tag{5.2}$$

This extends the definition of the height function from (1.2).

We note for future reference that given a bounded $\Omega \subset G_{V_K}$,

$$\mathrm{H}(b_1 g b_2) \ll_\Omega \mathrm{H}(g) \tag{5.3}$$

for every $b_1, b_2 \in \Omega$ and $g \in G_{V_K}$.

For $v \in V_K$ and $\epsilon > 0$, we set

$$\mathcal{O}_v(\epsilon) := \{g \in G_v : \|g - e\|_v \le \epsilon\},$$

and for any $S \subset V_K$,

$$\mathcal{O}_S(\epsilon) := \prod_{v \in S} \mathcal{O}_v(\epsilon)^3. \tag{5.4}$$

Let m be the Tamagawa measure on G_{V_K}. We refer to [33, §14] for a detailed construction of m. Given a nonzero left-invariant differential K-form of degree $\dim(\mathrm{G})$ on G, one defines the corresponding local measures m_v on G_v. We assume that the product $\prod_{v \in V_K^f} m_v(\mathrm{G}(O_v))$ converges. Then the Tamagawa measure is given by

$$m = \prod_{v \in V_K} m_v.$$

We note that when G is a semisimple group, then the product of $m_v(\mathrm{G}(O_v))$ is known to converge (see [33, §14]). This property is crucial for us to prove results on Diophantine approximation that are uniform over finite subsets S' of V_K.

Remark 5.1. When the product $\prod_{v \in V_K^f} m_v(\mathrm{G}(O_v))$ diverges, then the Tamagawa measure is defined by $m = \prod_{v \in V_K} \lambda_v m_v$ where $\lambda_v > 0$ are suitably chosen convergence factors. The arguments of this section can be modified to deal with this case as well. However, one loses uniformity over subsets S'.

We use the following estimates for the local measures. Recall that $I_v = \{q_v^n\}_{n>0}$ for $v \in V_K^f$ and $I_v = (0,1)$ for $v \in V_K^\infty$.

Lemma 5.2. *There exist $d_v \in (0,1]$ such that $d_v = 1$ for almost all v and*

$$m_v(\mathcal{O}_v(\epsilon)) \ge d_v \epsilon^{r_v \dim(\mathrm{G})}$$

for all $\epsilon \in I_v$.

Proof. The measures m_v in local coordinate are given by

$$|h(x)|_v (dx_1 \wedge \cdots \wedge dx_d)_v,$$

where $h(x)$ is a convergent power series and $d = \dim(\mathrm{G})$. This implies the estimate $m_v(\mathcal{O}_v(\epsilon)) \ge d_v \epsilon^{r_v \dim(\mathrm{G})}$ with some $d_v \in (0,1]$. We note that for $v \in V_K^f$, the neighbourhoods $\mathcal{O}_v(\epsilon)$ with $\epsilon = q_v^{-n}$ are precisely the congruence subgroups

$$\mathrm{G}^{(n)}(O_v) = \{g \in \mathrm{G}(O_v) : g = e \mod \mathfrak{p}_v^n\},$$

where $\mathfrak{p}_v$ denotes the maximal ideal of O_v and q_v denotes the cardinality of the residue field. The computation in [33, §14.1] shows that for almost all v and $n \ge 1$,

$$m_v(\mathrm{G}^{(n)}(O_v)) = q_v^{-nd}.$$

[3] Note that $\mathcal{O}_v(\epsilon) = \mathcal{O}_{\{v\}}(\epsilon)$ and $\mathcal{O}_v(1) = O_v \ne O_{\{v\}}$.

This implies the claim. □

The measure m on G_{V_K} defines the probability Haar measure on the factor space $\Upsilon = G_{V_K}/\mathrm{G}(K)$, which we denote by μ. The measure μ is defined so that for measurable subsets $B \subset G_{V_K}$ that project injectively on Υ, one has

$$\mu(B\mathrm{G}(K)) = \frac{1}{m(G_{V_K}/\mathrm{G}(K))} m(B).$$

Now we establish a series of results that provide a connection between dynamics on the space Υ and Diophantine approximation. The first two results deal with group varieties and the last two results with general homogeneous varieties.

Convention: We shall use the product decompositions of the adele group defined by any subset $Q \subset V_K$

$$G_{V_K} = G_{V_K \backslash Q} \times G_Q \,.$$

In order to simplify notation we identify a subset $B \subset G_{V_K \backslash Q}$ with the subset $B \times \{e\}$ of G_{V_K}.

In Propositions 5.3 and 5.4 below, all implicit constants may depend on the set of places S and the bounded subset Ω, but are independent of other parameters unless stated otherwise.

Proposition 5.3. *Fix $S \subset V_K$ and a bounded subset Ω of G_S. Then there exists a family of measurable subsets Φ_ϵ of Υ indexed by $\epsilon = (\epsilon_v)_{v\in S'}$, where S' is a finite subset of S and $\epsilon_v \in I_v$, that satisfies*

$$\mu(\Phi_\epsilon) \gg \prod_{v\in S'} \epsilon_v^{r_v \dim(\mathrm{G})} \tag{5.5}$$

and the following property holds:

if for a subset $B \subset G_{V_K\backslash S}$, $\epsilon = (\epsilon_v)_{v\in S'}$ as above, $x \in \Omega$ and $\varsigma = (e, x^{-1})\mathrm{G}(K) \in \Upsilon$, we have

$$B^{-1}\varsigma \cap \Phi_\epsilon \neq \emptyset, \tag{5.6}$$

then there exists $z \in \mathrm{G}(K)$ such that

$$\mathrm{H}(z) \ll \max_{b\in B} \mathrm{H}(b) \tag{5.7}$$

and

$$\begin{aligned} \|x_v - z\|_v &\le \epsilon_v \qquad \textit{for all } v \in S', \\ \|x_v - z\|_v &\le 1 \qquad \textit{for all } v \in S\backslash S'. \end{aligned} \tag{5.8}$$

Proof. Since Ω is a bounded subset of G_S, there exists finite $R \subset S$ such that

$$\Omega \subset G_R \times \prod_{v\in S\backslash R} \mathrm{G}(O_v).$$

We fix constants $c_v \ge 1$, $v \in S$, such that $c_v \ge \sup_{g\in\Omega} c_v(g_v^{-1})$, where $c_v(\cdot)$ is the constant given by (5.1). We can take $c_v = 1$ for $v \in S\backslash R$.

We set $\epsilon_v = 1$ for $v \in S\backslash S'$ and $\delta_v = \epsilon_v/c_v$ for $v \in S$, $\delta_v = 1$ for $v \in V_K\backslash S$. Let

$$\tilde{\Phi}_\epsilon = \prod_{v\in V_K} \mathcal{O}_v(\delta_v) \subset G_{V_K} \quad \text{and} \quad \Phi_\epsilon = \tilde{\Phi}_\epsilon \mathrm{G}(K) \subset \Upsilon.$$

Since $\delta_v \le 1$ for all $v \in V_K^f$, we have $\mathcal{O}_v(\delta_v) \subset \mathrm{G}(O_v)$ for every $v \in V_K^f$. Therefore, if $z \in \mathrm{G}(K)$ satisfies

$$\tilde{\Phi}_\epsilon z \cap \tilde{\Phi}_\epsilon \ne \emptyset,$$

then $z \in \mathrm{G}(O)$, where O denotes the ring of integers of K. Since the image of the diagonal embedding of O in $\prod_{v\in V_K^\infty} K_v$ is discrete, there exists $\epsilon_0 > 0$ (depending only on the number field K) such that

$$\mathcal{O}_{V_K^\infty}(\epsilon_0) z \cap \mathcal{O}_{V_K^\infty}(\epsilon_0) = \emptyset \quad \text{for every } z \in \mathrm{G}(O),\ z \ne e.$$

Then when $\epsilon_v \in (0, \epsilon_0)$ for all $v \in V_K^\infty$, we have

$$\tilde{\Phi}_\epsilon z \cap \tilde{\Phi}_\epsilon = \emptyset \quad \text{for every } z \in \mathrm{G}(K),\ z \ne e. \tag{5.9}$$

This implies that for such ϵ,

$$\begin{aligned}\mu(\Phi_\epsilon) \gg m(\tilde{\Phi}_\epsilon) &= \prod_{v\in V_K} m_v(\mathcal{O}_v(\delta_v)) \\ &= \left(\prod_{v\in S'\cup R} m_v(\mathcal{O}_v(\epsilon_v/c_v))\right)\left(\prod_{v\in V_K\backslash S'} m_v(\mathcal{O}_v(1))\right).\end{aligned}$$

Since the product $\prod_{v\in V_K^f} m_v(\mathrm{G}(O_v))$ converges, we deduce using Lemma 5.2 that

$$\mu(\Phi_\epsilon) \gg \prod_{v\in S'\cup R} m_v(\mathcal{O}_v(\epsilon_v/c_v)) \gg \prod_{v\in S'} d_v(\epsilon_v/c_v)^{r_v \dim(\mathrm{G})} \gg \prod_{v\in S'} \epsilon_v^{r_v \dim(\mathrm{G})}.$$

Since $c_v = 1$ for $v \notin R$, the implied constants here are independent of S' (and depend only on Ω). This proves estimate (5.5) provided that $\epsilon_v \in (0, \epsilon_0)$ for all $v \in V_K^\infty$.

To prove this estimate in general, we set $\epsilon'_v = \epsilon_0\epsilon_v \le \epsilon_0$ for $v \in V_K^\infty$ and $\epsilon'_v = \epsilon_v$ for $v \in V_K^f$. Then

$$\begin{aligned}\mu(\Phi_\epsilon) \ge \mu(\Phi_{\epsilon'}) &\gg \prod_{v\in S'} (\epsilon'_v)^{r_v \dim(\mathrm{G})} \\ &\ge \left(\prod_{v\in S'\cap V_K^\infty} \epsilon_0^{r_v \dim(\mathrm{G})}\right) \prod_{v\in S'} \epsilon_v^{r_v \dim(\mathrm{G})}.\end{aligned}$$

This completes the proof of (5.5).

Now we suppose that (5.6) holds. Then for some $b \in B$ and $z \in \mathrm{G}(K)$,

$$(b^{-1}z, x^{-1}z) \in \tilde{\Phi}_\epsilon.$$

Hence, for $v \in S$, we have $x_v^{-1} z \in \mathcal{O}_v(\delta_v)$ and

$$\|x_v - z\|_v \le c_v \, \|x_v^{-1} z - e\|_v \le c_v \delta_v.$$

This proves (5.8).

Finally, we observe that

$$z \in \Omega(b, e)\tilde{\Phi}_\epsilon.$$

Hence, it follows from (5.3) that

$$\mathrm{H}(z) \ll \mathrm{H}(b).$$

This shows (5.7) and completes the proof of the proposition. □

5.2. **Duality and approximation at every points on the group variety.** In order to prove the uniform version of our main theorem (Theorem 1.3(ii)), condition (5.6) in Proposition 5.3 needs to be relaxed. This is achieved by the following proposition:

Proposition 5.4. *Fix $S \subset V_K$ and a bounded subset Ω of G_S. Then there exists a family of measurable subsets $\tilde{\Psi}_\epsilon$ of G_{V_K} indexed by $\epsilon = (\epsilon_v)_{v \in S'}$, where S' is a finite subset of S and $\epsilon_v \in I_v$, that satisfies*

$$\mu(\tilde{\Psi}_\epsilon \mathrm{G}(K)) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(G)}, \tag{5.10}$$

$$\mu(\tilde{\Psi}_\epsilon^{-1}(e, x^{-1})\mathrm{G}(K)) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(G)} \quad \text{for all } x \in \Omega, \tag{5.11}$$

and the following property holds:

if for $B \subset G_{V_K \backslash S}$, $\epsilon = (\epsilon_v)_{v \in S'}$ as above, $x \in \Omega$ and $\varsigma = (e, x^{-1})\mathrm{G}(K) \in \Upsilon$, we have

$$B^{-1}\tilde{\Psi}_\epsilon^{-1}\varsigma \cap \tilde{\Psi}_\epsilon \mathrm{G}(K) \ne \emptyset, \tag{5.12}$$

then there exists $z \in \mathrm{G}(K)$ such that

$$\mathrm{H}(z) \ll \max_{b \in B} \mathrm{H}(b)$$

and

$$\|x_v - z\|_v \le \epsilon_v \quad \text{for all } v \in S', \tag{5.13}$$

$$\|x_v - z\|_v \le 1 \quad \text{for all } v \in S \backslash S'.$$

Proof. Similarly to (5.1), for every $g \in \mathrm{GL}_n(K_v)$ and $y \in \mathrm{M}_n(K_v)$,

$$\|g \cdot y \cdot g^{-1}\|_v \le c_v'(g)\|y\|_v \tag{5.14}$$

where $c_v'(g) \ge 1$, $c_v'(g) = 1$ for $g \in \mathrm{GL}_n(O_v)$, and $c_v'(g)$ is uniformly bounded over bounded subsets. We set $c_v' = \sup_{g \in \Omega} c_v'(g)$.

Let $c_v \ge 1$ be defined as in Proposition 5.3 and $\epsilon_v = 1$ for $v \in S \backslash S'$. We set $\delta_v = 1$ for $v \in V_K \backslash S$, $\delta_v = \epsilon_v/((2n+1)c_v c_v')$ for Archimedean $v \in S$, and $\delta_v = \epsilon_v/(c_v c_v')$ for non-Archimedean $v \in S$. Note $\delta_v = 1$ for almost all v. Let

$$\tilde{\Psi}_\epsilon = \prod_{v \in V_K} \mathcal{O}_v(\delta_v).$$

Since $\delta_v = 1$ for all non-Archimedean $v \notin S' \cup R$, estimate (5.10) can be established by the same argument as in the proof of Proposition 5.3. To prove (5.11), we observe that

$$\mu(\tilde{\Psi}_\epsilon^{-1}(e, x^{-1})\mathrm{G}(K)) = \mu((e, x)\tilde{\Psi}_\epsilon^{-1}(e, x^{-1})\mathrm{G}(K)),$$

and by (5.14),

$$(e, x)\tilde{\Psi}_\epsilon^{-1}(e, x^{-1}) \supset \prod_{v \in V_K} \mathcal{O}_v(\delta_v/c'_v)^{-1}.$$

Hence, estimate (5.11) can be proved similarly to (5.10).

Suppose that (5.12) holds. Then for some $b \in B$, $f' \in \prod_{v \in V_K \setminus S} \mathcal{O}_v(\delta_v)$, $f \in \prod_{v \in S} \mathcal{O}_v(\delta_v)$, and $z \in \mathrm{G}(K)$, we have

$$(b^{-1}(f')^{-1}z, f^{-1}x^{-1}z) \in \tilde{\Psi}_\epsilon.$$

For $v \in S \cap V_K^\infty$, we have $f_v^{-1}x_v^{-1}z \in \mathcal{O}_v(\delta_v)$ and $f_v \in \mathcal{O}_v(\delta_v)$. We observe that $\|f_v\|_v \le 2$, and for every $a \in \mathrm{M}_n(K_v)$,

$$\|f_v a\|_v \le n\|f_v\|_v\|a\|_v \le (2n)\|a\|_v.$$

Hence,

$$\begin{aligned}\|x_v - z\|_v &\le \|x_v - x_v f_v\|_v + \|x_v f_v - z\|_v \\ &\le c_v \left(\|e - f_v\|_v + (2n)\|e - f_v^{-1}x_v^{-1}z\|_v\right) \le (2n+1)c_v\delta_v.\end{aligned}$$

The case $v \in S \cap V_K^f$ is treated similarly. We observe that $\|f_v\|_v \le 1$, and for every $a \in \mathrm{M}_n(K_v)$,

$$\|f_v a\|_v \le \|f_v\|_v\|a\|_v \le \|a\|_v,$$

so that

$$\begin{aligned}\|x_v - z\|_v &\le \max\{\|x_v - x_v f_v\|_v, \|x_v f_v - z\|_v\} \\ &\le c_v \max\{\|e - f_v\|_v, \|e - f_v^{-1}x_v^{-1}z\|_v\} \le c_v\delta_v.\end{aligned}$$

This proves (5.13).

Finally, we observe that

$$z \in \Omega(f', f)(b, e)\tilde{\Psi}_\epsilon.$$

Hence, it follows from (5.3) that

$$\mathrm{H}(z) \ll \mathrm{H}(b).$$

This completes the proof of the proposition. □

5.3. **Duality and approximation on homogeneous varieties.** Consider now the problem of Diophantine approximation on general homogeneous varieties. The results obtained in the present subsection will be used in the proof of Theorem 1.6.

In Propositions 5.5 and 5.6 below, all implicit constants may depend on $S' \subset S \subset V_K$, $x^0 \in X_{S'}$, and $\Omega \subset G_{S'}$, but are independent of other parameters unless stated otherwise.

Proposition 5.5. *Let* $\mathrm{X} \subset \mathbb{A}^n$ *be a homogeneous quasi-affine variety of the group* $\mathrm{G} \subset \mathrm{GL}_n$*. Fix* $S \subset V_K$*, finite* $S' \subset S$*,* $x^0 \in X_{S'}$*, and a bounded subset* Ω *of* $G_{S'}$*. Then there exist* $\epsilon_0 \in (0,1)$ *and a family of measurable subset* Φ_ϵ *of* Υ *indexed by* $\epsilon = (\epsilon_v)_{v\in S'}$ *where* $\epsilon_v \in I_v \cap (0, \epsilon_0)$ *that satisfy*

$$\mu(\Phi_\epsilon) \gg \prod_{v\in S'} \epsilon_v^{r_v \dim(\mathrm{X})} \tag{5.15}$$

and the following property holds:

if for $B \subset G_{V_K\backslash S} \times \prod_{v\in V_K^f\cap(S\backslash S')} \mathrm{G}(O_v)$*,* $\epsilon = (\epsilon_v)_{v\in S'}$ *as above, and* $\varsigma = (e, g^{-1})\mathrm{G}(K) \in \Upsilon$ *with* $g \in \Omega$*, we have*

$$B^{-1}\varsigma \cap \Phi_\epsilon \neq \emptyset, \tag{5.16}$$

then there exists $z \in \mathrm{G}(O_{(V_K\backslash S)\cup S'})$ *such that*

$$\mathrm{H}(z) \ll \max_{b\in B} \mathrm{H}(b) \tag{5.17}$$

and for $x = gx^0 \in X_{S'}$

$$\|x_v - zx_v^0\| \le \epsilon_v \quad \text{for all } v \in S'. \tag{5.18}$$

Proof. For $v \in S'$, let $p_v : G_v \to G_v x_v^0$ be the map $g \mapsto gx_v^0$. There exists a neighbourhood $\mathcal{U}_v$ of x_v^0 in $G_v x_v^0$ and an analytic section $\sigma_v : \mathcal{U}_v \to G_v$ of the map p_v such that $\sigma_v(x_v^0) = e$. (The section σ_v can be constructed using the exponential map in G_v.) Let

$$\mathcal{O}_v(x_v^0, \epsilon) := \{y \in G_v x_v^0 : \|y - x_v^0\|_v \le \epsilon\}.$$

Let L_v be the stabiliser of x_v^0 in G_v, and let L_v^1 be a compact neighbourhood of identity in L_v. Let $c_v = \sup_{g\in\Omega} c_v(g_v^{-1})$, where $c_v(\cdot)$ is the constant given by (5.1). We set

$$\tilde{\Phi}_\epsilon = \left(\prod_{v\in V_K\backslash S'} \mathcal{O}_v\right) \times \left(\prod_{v\in S'} \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c_v))L_v^1\right),$$

where $\epsilon_v > 0$ are sufficiently small, so that $\mathcal{O}_v(x_v^0, \epsilon_v/c_v) \subset \mathcal{U}_v \cap \sigma_v^{-1}(\mathrm{G}(O_v))$, and where $\mathcal{O}_v = \mathrm{G}(O_v)$ for $v \in V_K^f\backslash S'$ and $\mathcal{O}_v$ is a fixed neighbourhood of identity in G_v for $v \in V_K^\infty\backslash S'$. We also set

$$\Phi_\epsilon = \tilde{\Phi}_\epsilon \mathrm{G}(K) \subset \Upsilon.$$

As in the proof of Proposition 5.3, we can choose sufficiently small ϵ_v, L_v^1 and $\mathcal{O}_v$, so that

$$\tilde{\Phi}_\epsilon z \cap \tilde{\Phi}_\epsilon = \emptyset \quad \text{for every } z \in \mathrm{G}(K),\ z \neq e.$$

Then

$$\mu(\Phi_\epsilon) \gg m(\tilde{\Phi}_\epsilon) \gg \prod_{v \in S'} m_v(\sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c_v))L_v^1),$$

and estimating volumes in local coordinates, we obtain

$$\mu(\Phi_\epsilon) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(\mathrm{X})}.$$

This proves (5.15).

Now suppose that (5.16) holds. Then for some $b \in B$ and $z \in \mathrm{G}(K)$, we have

$$(b^{-1}z, g^{-1}z) \in \tilde{\Phi}_\epsilon.$$

Then for $v \in S'$, we have $g_v^{-1}z \in \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c_v))L_v^1$, and

$$\|g_v^{-1}zx_v^0 - x_v^0\|_v \leq \epsilon_v/c_v.$$

Therefore,

$$\|zx_v^0 - x_v\|_v \leq \epsilon_v \quad \text{for all } v \in S'.$$

Also, for $v \in V_K^f \cap (S\backslash S')$, we have $z \in b_v\mathcal{O}_v$. This implies that $z \in \mathrm{G}(O_{(V_K\backslash S)\cup S'})$. Hence, we have established (5.18).

Finally, we observe that since

$$z \in \Omega(b, e)\tilde{\Phi}_\epsilon,$$

estimate (5.17) follows from (5.3). □

We now turn to proving the estimate crucial in establishing diophantine approximation at every point in a homogeneous variety.

Proposition 5.6. *Let* $\mathrm{X} \subset \mathbb{A}^n$ *be a homogeneous quasi-affine variety of the group* $\mathrm{G} \subset \mathrm{GL}_n$. *Fix* $S \subset V_K$, *finite* $S' \subset S$, $x^0 \in X_{S'}$, *and a bounded subset* Ω *of* $G_{S'}$. *Then there exist* $\epsilon_0 \in (0,1)$ *and a family of measurable subset* $\tilde{\Psi}_\epsilon$ *of* G_{V_K} *indexed by* $\epsilon = (\epsilon_v)_{v \in S'}$ *where* $\epsilon_v \in I_v \cap (0, \epsilon_0)$ *that satisfy*

$$\mu(\tilde{\Psi}_\epsilon \mathrm{G}(K)) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(\mathrm{X})}, \tag{5.19}$$

$$\mu(\tilde{\Psi}_\epsilon^{-1}(e, g^{-1})\mathrm{G}(K)) \gg \prod_{v \in S'} \epsilon_v^{r_v \dim(\mathrm{X})} \quad \textit{for all } g \in \Omega, \tag{5.20}$$

and the following property holds:

if for $B \subset G_{V_K \backslash S} \times \prod_{v \in V_K^f \cap (S\backslash S')} \mathrm{G}(O_v)$, $\epsilon = (\epsilon_v)_{v \in S'}$ *as above, and* $\varsigma = (e, g^{-1})\mathrm{G}(K) \in \Upsilon$ *with* $g \in \Omega$, *we have*

$$B^{-1}\tilde{\Psi}_\epsilon^{-1}\varsigma \cap \tilde{\Psi}_\epsilon \mathrm{G}(K) \neq \emptyset, \tag{5.21}$$

then there exists $z \in \mathrm{G}(O_{(V_K\backslash S)\cup S'})$ *such that*

$$\mathrm{H}(z) \ll \max_{b\in B} \mathrm{H}(b) \tag{5.22}$$

and for $x = gx_0 \in X_{S'}$

$$\|x_v - zx_v^0\| \le \epsilon_v \quad \textit{for all } v \in S'. \tag{5.23}$$

Proof. We use notation introduced in the proof of Proposition 5.5. There exists $c''_v > 0$ such that for every $r_v \in \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_0))L_v^1$, we have

$$\|r_v x\|_v \le c''_v \|x\|_v, \quad x \in K_v^n.$$

Let $c'_v = c_v(1 + c''_v)$ and

$$\tilde{\Psi}_\epsilon = \left(\prod_{v\in V_K\backslash S'} \mathcal{O}_v\right) \times \left(\prod_{v\in S'} \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c'_v))L_v^1\right),$$

Estimate (5.19) is proved as in Proposition 5.5. Taking $\mathcal{O}_v$, ϵ_v, and L_v^1 sufficiently small, we can arrange that

$$(e,g)\tilde{\Psi}_\epsilon^{-1}(e,g^{-1})z \cap (e,g)\tilde{\Psi}_\epsilon^{-1}(e,g^{-1}) = \emptyset \quad \text{for } z \in \mathrm{G}(K),\ z \ne e.$$

Then

$$\mu(\tilde{\Psi}_\epsilon^{-1}(e,g^{-1})\mathrm{G}(K)) \gg m(\tilde{\Psi}_\epsilon^{-1}(e,g^{-1})) = m(\tilde{\Psi}_\epsilon^{-1}).$$

Hence, estimate (5.20) follows from (5.19).

Suppose that (5.21) holds. Then for some $b \in B$, $f' \in \prod_{v\in V_K\backslash S'} \mathcal{O}_v$, $f \in \prod_{v\in S'} \sigma_v(\mathcal{O}_v(x_v^0, \epsilon_v/c'_v))L_v^1$, and $z \in \mathrm{G}(K)$, we have

$$(b^{-1}(f')^{-1}z, f^{-1}g^{-1}z) \in \tilde{\Psi}_\epsilon.$$

For $v \in S'$,

$$\|f_v x_v^0 - x_v^0\|_v \le \epsilon_v/c'_v \quad \text{and} \quad \|f_v^{-1}g_v^{-1}zx_v^0 - x_v^0\|_v \le \epsilon_v/c'_v.$$

Hence,

$$\begin{aligned}\|x_v - zx_v^0\|_v &\le \|g_v x_v^0 - g_v f_v x_v^0\|_v + \|g_v f_v x_v^0 - zx_v^0\|_v \\ &\le c_v\left(\|x_v^0 - f_v x_v^0\|_v + c''_v\|x_v^0 - f_v^{-1}g_v^{-1}zx_v^0\|_v\right) \le \epsilon_v.\end{aligned}$$

Also, for $v \in (S\backslash S') \cap V_K^f$,

$$z \in f'_v b_v \mathrm{G}(O_v) \subset \mathrm{G}(O_v).$$

Hence, $z \in \mathrm{G}(O_{(V_K\backslash S)\cup S'})$. This completes the proof of (5.23).

Finally, since

$$z \in \Omega(f', f)(b, e)\tilde{\Psi}_\epsilon,$$

estimate (5.22) follows from (5.3). □

6. Proof of the main results

6.1. **Almost sure diophantine approximation on the group variety.** In this section we complete the proofs of main results. Our strategy is to combine the duality principle from Section 5 with the mean ergodic theorem from Section 4.

Let $S \subset V_K$ and G be a connected almost simple algebraic K-group. Our first task is to establish an estimate on the exponent $\mathfrak{a}_S(\mathrm{G})$ of the variety G defined after (1.7). It is worth mentioning that $\mathfrak{a}_S(\mathrm{G})$ can, in principle, be computed in terms of the root data of G using methods from [10, 11], but for the purpose of this paper the simple estimate of Lemma 6.1 below will be sufficient. We recall that

$$\mathfrak{a}_S(\mathrm{G}) = \sup_{\Omega \subset G_S} \limsup_{h \to \infty} \frac{\log A_S(\Omega, h)}{\log h},$$

where Ω runs over bounded subset of G_S, and

$$A_S(\Omega, h) = |\{z \in \mathrm{G}(K) : \mathrm{H}(z) \le h, \;\; z \in \Omega\}|.$$

We show that $\mathfrak{a}_S(\mathrm{G})$ can be estimated in terms of volumes of the sets

$$B_h = U_{V_K \backslash S} \{g \in G_{V_K \backslash S} : \mathrm{H}(g) \le h\} U_{V_K \backslash S} \subset G_{V_K \backslash S}. \tag{6.1}$$

Lemma 6.1. *Let Ω be a bounded subset of G_S. Then there exist $c > 1$ and $h_0 > 0$ such that for every $h \ge h_0$*

$$A_S(\Omega, h) \ll_\Omega m_{V_K \backslash S}(B_{ch}).$$

In particular, it follows that for every $\delta > 0$ and $h \ge h_0(S, \delta)$.

$$m_{V_K \backslash S}(B_h) \gg h^{\mathfrak{a}_S(\mathrm{G}) - \delta}.$$

Proof. Let

$$\mathcal{A}_S(\Omega, h) = \{\gamma \in \mathrm{G}(K) \cap G_{V_K \backslash S}\Omega : \mathrm{H}(\gamma) \le h\}.$$

Then $A_S(\Omega, h) = |\mathcal{A}_S(\Omega, h)|$. Since $\mathrm{G}(K)$ is a discrete subgroup of G_{V_K}, there exists a bounded neighbourhood $\mathcal{O}$ of identity in G_{V_K} such that $\mathcal{O}\gamma_1 \cap \mathcal{O}\gamma_2 = \emptyset$ for $\gamma_1 \ne \gamma_2 \in \mathrm{G}(K)$. Then

$$A_S(\Omega, h) = |\mathcal{A}_S(\Omega, h)| \le \frac{m_{V_K}(\mathcal{O}\mathcal{A}_S(\Omega, h))}{m_{V_K}(\mathcal{O})}.$$

We observe that it follows from (5.3) that

$$\mathcal{O}\mathcal{A}_S(\Omega, h) \subset \mathcal{A}_S(\Omega', c_1 h) \subset B_{c_2 h}\Omega'.$$

for a bounded $\Omega' \subset G_S$ and $c_1, c_2 > 0$. This implies the claim. □

We will also need volume estimates for the intersections of B_h with finite index subgroups.

Lemma 6.2. *Let G_0 be a finite index subgroup of G_{V_K}. Then there exists $c \ge 1$ such that*

$$m_{V_K \backslash S}(B_h \cap G_0) \ge \frac{m_{V_K \backslash S}(B_{c^{-1}h})}{|G_{V_K} : G_0|}$$

for every $h > 0$.

Proof. Let $\{g_i\}$ be a finite set of coset representatives of $G_{V_K\backslash S}/(G_{V_K\backslash S}\cap G_0)$. It follows from (5.3) that there exists $c \ge 1$ such that $g_i^{-1}B_h \subset B_{ch}$. Hence,

$$\begin{aligned} m_{V_K\backslash S}(B_h) &= \sum_i m_{V_K\backslash S}(B_h \cap g_i G_0) = \sum_i m_{V_K\backslash S}(g_i^{-1}B_h \cap G_0) \\ &\le |G_{V_K} : G_0| m_{V_K\backslash S}(B_{ch} \cap G_0). \end{aligned}$$

This implies the claim. □

Convention: In the proofs of the Theorems 1.3 and 1.5, implicit constants may depend on the set of places S, but are independent of the subset S' unless stated otherwise. In the proof of Theorem 1.6, implicit constants may depend on S' as well.

Proof of Theorem 1.3(i). Let β_h be the Haar-uniform probability measure supported on the set B_h defined in (6.1). We denote by $\pi_{V_K\backslash S}(\beta_h)$ the corresponding averaging operator acting on $L^2(\Upsilon)$ (see (4.1)). The main idea of the proof is to combine Theorem 4.2 with Proposition 5.3. By Theorem 4.2, for every $\phi \in L^2(\Upsilon)$ and $h, \delta' > 0$,

$$\left\| \pi_{V_K\backslash S}(\beta_h)\phi - \int_\Upsilon \phi\, d\mu \right\|_2 \ll_{\delta'} m_{V_K\backslash S}(B_h)^{-\frac{1}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta'} \|\phi\|_2.$$

Moreover, it follows from Lemma 6.1 that this estimate can be rewritten as

$$\left\| \pi_{V_K\backslash S}(\beta_h)\phi - \int_\Upsilon \phi\, d\mu \right\|_2 \ll_{\delta'} h^{-\frac{\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta'} \|\phi\|_2 \tag{6.2}$$

for every $\delta' > 0$ and $h \ge h_0(\delta')$.

We pick a bounded subset Ω of G_S. Let $\Phi_\epsilon \subset \Upsilon$ be the sets introduced in Proposition 5.3 which are indexed by $\epsilon = (\epsilon_v)_{v\in S'}$ with finite $S' \subset S$ and $\epsilon_v \in I_v$. We denote by ϕ_ϵ the characteristic function of Φ_ϵ. Then (6.2) gives

$$\left\| \pi_{V_K\backslash S}(\beta_h)\phi_\epsilon - \mu(\Phi_\epsilon) \right\|_2 \ll_{\delta'} h^{-\frac{\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta'} \mu(\Phi_\epsilon)^{1/2} \tag{6.3}$$

for every $\delta' > 0$ and $h \ge h_0(\delta')$. We set

$$h_\epsilon = \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})/2} \tag{6.4}$$

with $\delta > 0$. Let

$$\Upsilon_\epsilon = \{\varsigma \in \Upsilon :\ B_{h_\epsilon}^{-1}\varsigma \cap \Phi_\epsilon = \emptyset\}. \tag{6.5}$$

Then for $\varsigma \in \Upsilon_\epsilon$, we have $\pi_{V_K\backslash S}(\beta_{h_\epsilon})\phi_\epsilon(\varsigma) = 0$. Hence, it follows from (6.3) that for every $\delta' > 0$, and ϵ, we have

$$\mu(\Upsilon_\epsilon) \ll_{\delta'} h_\epsilon^{-\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+2\delta'} \mu(\Phi_\epsilon)^{-1} \tag{6.6}$$

provided that $h_\epsilon \geq h_0(\delta')$. Combining this estimate with (6.4) and (5.5), we conclude that

$$\mu(\Upsilon_\epsilon) \ll_{\Omega,\delta'} \prod_{v\in S'} \epsilon_v^{\theta_v} \tag{6.7}$$

for every $\delta' > 0$ and ϵ satisfying $h_\epsilon \geq h_0(\delta')$, where

$$\begin{aligned}\theta_v &= \left(-r_v\frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta\right)\frac{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}{2}\left(-\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})} + 2\delta'\right) - r_v\dim(\mathrm{G})\\ &= \mathfrak{a}_S(\mathrm{G})\delta - \delta'\cdot\mathfrak{q}_{V_K\backslash S}(\mathrm{G})\left(r_v\frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} + \delta\right).\end{aligned}$$

We pick $\delta' = \delta'(\delta) > 0$ so that $\theta_v > 0$.

Let $E_v = I_v$ for $v \in V_K^f$, and let $E_v = \{2^{-n}\}_{n\geq 1}$ for $v \in V_K^\infty$, where we discretize the parameter in the archemedean part in order to take advantage of the Borel-Cantelli lemma. Set $E_{S'} = \prod_{v\in S'} E_v$ for finite $S' \subset S$. We denote by $\Upsilon_{S'}$ the lim sup of the sets Υ_ϵ with $\epsilon \in E_{S'}$. Since $h_\epsilon \geq h_0(\delta')$ holds for all but finitely many $\epsilon \in E_{S'}$, estimate (6.7) holds for all but finitely many $\epsilon \in E_{S'}$ as well, and it follows that

$$\sum_{\epsilon\in E_{S'}} \mu(\Upsilon_\epsilon) < \infty.$$

Hence, by the Borel–Cantelli lemma, $\mu(\Upsilon_{S'}) = 0$. Let $\tilde{\Upsilon}_{S'} \subset G_{V_K}$ be the preimage of $\Upsilon_{S'}$. Then $m(\tilde{\Upsilon}_{S'}) = 0$. We denote by $\tilde{\Upsilon}_0$ the union of $\tilde{\Upsilon}_{S'}$ over finite subsets S' of S. Then $m(\tilde{\Upsilon}_0) = 0$ as well. Moreover, using Fubini's theorem and passing to a superset of $\tilde{\Upsilon}_0$, we can arrange that $\tilde{\Upsilon}_0$ additionally satisfies the property that for every finite $S' \subset V_K$ and $(f', f) \in G_{V_K\backslash S'} \times G_{S'}$ not belonging to $\tilde{\Upsilon}_0$, the set $\{f'' : (f'', f) \notin \tilde{\Upsilon}_0\}$ has full measure in $G_{V_K\backslash S'}$. In particular, it follows that if for $y \in G_{V_K\backslash S}$ and $x \in G_S$, we have $(y, x) \notin \tilde{\Upsilon}_0$, then there exists $(y', x') \notin \tilde{\Upsilon}_0$ such that $x'_v = x_v$ for $v \in S'$, $x'_v \in \mathrm{G}(O_v)$ for $v \in V_K^f \cap (S\backslash S')$, and $y' \in \mathcal{O}_{V_K\backslash S}(1)$. We shall use this property below.

Let

$$\Omega' = \{x \in \Omega : \exists y \in \mathcal{O}_{V_K\backslash S}(1) : (y, x^{-1}) \notin \tilde{\Upsilon}_0\}. \tag{6.8}$$

Since

$$(\mathcal{O}_{V_K\backslash S}(1) \times (\Omega\backslash\Omega')^{-1}) \subset \tilde{\Upsilon}_0,$$

it follows that the set $\Omega\backslash\Omega'$ has measure zero.

Let $G_S = \cup_{j\geq 1}\Omega_j$ be an exhaustion of G_S by compact sets. Then $Y := \cup_{j\geq 1}\Omega'_j$ is a subset of G_S of full measure. Hence, it is suffices to show that given compact $\Omega \subset G_S$, every $x \in \Omega'$ satisfies the claim of the theorem.

For $x \in \Omega'$, there exists $\tilde{\varsigma} := (y', (x')^{-1}) \notin \tilde{\Upsilon}_0$ such that $x'_v = x_v$ for $v \in S'$, $x'_v \in \mathrm{G}(O_v)$ for $v \in V_K^f \cap (S\backslash S')$, and $y' \in \mathcal{O}_{V_K\backslash S}(1)$.

Since $\tilde{\varsigma}\mathrm{G}(K) \notin \Upsilon_0$, it follows that $\tilde{\varsigma}\mathrm{G}(K) \notin \Upsilon_\epsilon$ for all but finitely many $\epsilon \in E_{S'}$. That is, there exists $\epsilon_0(x, S', \delta) > 0$ such that for $\epsilon \in E_{S'} \cap (0, \epsilon_0(x, S', \delta))^{S'}$, we have

$$B_{h_\epsilon}^{-1}\tilde{\varsigma}\mathrm{G}(K) \cap \Phi_\epsilon \neq \emptyset,$$

and

$$(\mathcal{O}_{V_K \setminus S}(1)^{-1} B_{h_\epsilon})^{-1}(e, (x')^{-1})\mathrm{G}(K) \cap \Phi_\epsilon \neq \emptyset.$$

Now we are in position to apply Proposition 5.3. It follows that there exists $z \in \mathrm{G}(K)$ such that

$$\mathrm{H}(z) \ll_\Omega \max_{b \in \mathcal{O}_{V_K \setminus S}(1)^{-1} B_{h_\epsilon}} \mathrm{H}(b) \ll h_\epsilon$$

and

$$\begin{aligned} \|x'_v - z\|_v &\leq \epsilon_v \quad \text{for all } v \in S', \\ \|x'_v - z\|_v &\leq 1 \quad \text{for all } v \in S \setminus S'. \end{aligned}$$

Since $x'_v \in \mathrm{G}(O_v)$ for $v \in V_K^f \cap (S \setminus S')$, it follows that $z \in \mathrm{G}(O_{(V_K \setminus S) \cup S'})$ Hence, for every $\delta > 0$ and $\epsilon \in E_{S'} \cap (0, \epsilon_0(x, S', \delta))^{S'}$, we have

$$\omega_S(x, \epsilon) \ll_\Omega h_\epsilon = \left(\prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2}.$$

This implies that for all $\delta > 0$ and sufficiently small ϵ, depending on δ and Ω, we also have (absorbing the constant by passing from δ to 2δ)

$$\omega_S(x, \epsilon) \leq \left(\prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - 2\delta} \right)^{\mathfrak{q}_{V_K \setminus S}(G)/2}.$$

Now in order to finish the proof of the theorem, we need to extend this estimate to the case when $\epsilon_v \in (0, \epsilon_0(x, S', \delta))$ for $v \in V_K^\infty \cap S'$. For such ϵ_v, there exists $\epsilon'_v \in E_v \cap (0, \epsilon_0(x, S', \delta))$ such that $\epsilon'_v \leq \epsilon_v \leq 2\epsilon'_v$. For $v \in V_K^f \cap S'$, we set $\epsilon'_v = \epsilon_v$. Then

$$\begin{aligned} \omega_S(x, \epsilon) \leq \omega_S(x, \epsilon') &\leq \left(\prod_{v \in S'} (\epsilon'_v)^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(\mathrm{G})/2} \\ &\ll_{V_K^\infty \cap S'} \left(\prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K \setminus S}(\mathrm{G})/2} \end{aligned} \tag{6.9}$$

for every $\delta > 0$ and $\epsilon = (\epsilon_v)_{v \in S'}$ with $\epsilon_v \in I_v \cap (0, \epsilon_0(x, S', \delta))$. This also implies that

$$\omega_S(x, \epsilon) \leq \left(\prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - 2\delta} \right)^{\mathfrak{q}_{V_K \setminus S}(\mathrm{G})/2}$$

for all $\delta > 0$ and sufficiently small ϵ (depending on δ and $V_K^\infty \cap S'$). This completes the proof. $\square$

Proof of Theorem 1.5. The proof of the theorem follows the same strategy as the proof of Theorem 1.3, but we choose the parameter h_ϵ to be

$$h_\epsilon = \left(\prod_{v\in S'} \epsilon_v^{-\frac{r_v \dim(\mathrm{G})+\sigma_S}{\mathfrak{a}_S(\mathrm{G})}-\delta}\right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})/2}$$

with $\delta > 0$.

We fix a bounded subset Ω of G_S and consider the sets

$$\Upsilon_\epsilon = \{\varsigma \in \Upsilon :\ B_{h_\epsilon}^{-1}\varsigma \cap \Phi_\epsilon = \emptyset\},$$

indexed by $\epsilon = (\epsilon_v)_{v\in S'}$ with finite $S' \subset S$ and $\epsilon_v \in I_v$, where the sets Φ_ϵ is defined in Proposition 5.3. Arguing as in the proof of Theorem 1.3, we obtain the estimate

$$\mu(\Upsilon_\epsilon) \ll_{\Omega,\delta'} \prod_{v\in S'} \epsilon_v^{\theta_v}, \tag{6.10}$$

where

$$\begin{aligned}\theta_v &= \left(-\frac{r_v \dim(\mathrm{G})+\sigma_S}{\mathfrak{a}_S(\mathrm{G})}-\delta\right)\frac{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}{2}\left(-\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+2\delta'\right) - r_v\dim(\mathrm{G})\\ &= \sigma_S + \mathfrak{a}_S(\mathrm{G})\delta - \delta'\cdot \mathfrak{q}_{V_K\backslash S}(\mathrm{G})\left(\frac{r_v \dim(\mathrm{G})+\sigma_S}{\mathfrak{a}_S(\mathrm{G})}+\delta\right).\end{aligned}$$

This estimate is valid for every $\delta, \delta' > 0$ and ϵ as above provided that $h_\epsilon \ge h_0(\delta')$. Moreover, we emphasize that this estimate is uniform over finite $S' \subset S$. We choose $\delta' = \delta'(\delta) > 0$ such that $\theta_v \ge \theta > \sigma_S$.

Let

$$E_S = \{(\epsilon_v)_{v\in S} : \epsilon_v = q_v^{-n_v} \text{ with } n_v \ge 0,\ n_v = 0 \text{ for a.e. } v\},$$

where we set $q_v = 2$ for $v \in V_K^\infty$. We note that $h_\epsilon \ge h_0(\delta')$ for all but finitely many $\epsilon \in E_S$. Therefore, estimate (6.10) holds for all but finitely many $\epsilon \in E_S$, and the sum $\sum_{\epsilon\in E_S}\mu(\Upsilon_\epsilon)$ can be estimated (except possibly finitely many terms) by

$$\sum_{\epsilon\in E_S}\left(\prod_{v\in S}\epsilon_v^{\theta_v}\right) \le \prod_{v\in S}(1-q_v^{-\theta})^{-1},$$

where the last product converges because $\theta > \sigma_S$.

Let Υ_0 be the lim sup of the sets Υ_ϵ as $\epsilon \in E_S$. Then by the Borel-Cantelli lemma, $\mu(\Upsilon_0) = 0$. Let $\tilde{\Upsilon}_0 \subset G_{V_K}$ be the preimage of Υ_0. Then $m(\tilde{\Upsilon}_0) = 0$. Moreover, as in the proof of Theorem 1.3(i), we can pass to a superset of $\tilde{\Upsilon}_0$ to arrange that if for $y \in G_{V_K\backslash S}$ and $x \in G_S$, we have $(y,x) \notin \tilde{\Upsilon}_0$, then there exists $(y',x') \notin \tilde{\Upsilon}_0$ such that $x'_v = x_v$ for $v \in S'$, $x'_v \in \mathrm{G}(O_v)$ for $v \in V_K^f \cap (S\backslash S')$, and $y' \in \mathcal{O}_{V_K\backslash S}(1)$.

Given bounded $\Omega \subset G_S$, we define Ω' as in (6.8). Then $\Omega\backslash\Omega'$ has measure zero, and it suffices to show that every $x \in \Omega'$ satisfies the claim of the theorem.

For $x \in \Omega'$, there exists $\tilde{\varsigma} := (y', (x')^{-1}) \notin \tilde{\Upsilon}_0$ such that $x'_v = x_v$ for $v \in S'$, $x'_v \in \mathrm{G}(O_v)$ for $v \in V_K^f \cap (S \backslash S')$, and $y' \in \mathcal{O}_{V_K \backslash S}(1)$. Since $\tilde{\varsigma}\mathrm{G}(K) \notin \Upsilon_0$, we conclude that $\tilde{\varsigma}\mathrm{G}(K) \notin \Upsilon_\epsilon$ for all but finitely many $\epsilon \in E_S$. In particular, there exists $\epsilon_v^0(x, \delta) \in (0, 1]$, $v \in S$, such that $\epsilon_v^0(x, \delta) = 1$ for almost all v and for all $\epsilon = (\epsilon_v)_{v \in S'}$ with finite $S' \subset S$ and $\epsilon_v \in E_v \cap (0, \epsilon_v^0(x, \delta))$, we have

$$B_{h_\epsilon}^{-1} \tilde{\varsigma}\mathrm{G}(K) \cap \Phi_\epsilon \neq \emptyset.$$

Applying Proposition 5.3, we deduce that there exists $z \in \mathrm{G}(K)$ such that

$$\mathrm{H}(z) \ll_\Omega h_\epsilon$$

and

$$\begin{aligned} \|x'_v - z\|_v &\leq \epsilon_v \quad \text{for all } v \in S', \\ \|x'_v - z\|_v &\leq 1 \quad \text{for all } v \in S \backslash S', \end{aligned}$$

provided that $\epsilon_v \in \{q_v^{-n}\}_{n \geq 0} \cap (0, \epsilon_v^0(x, \delta))$. Since $x'_v \in \mathrm{G}(O_v)$ for $v \in V_K^f \cap (S \backslash S')$, it follows that $z \in \mathrm{G}(O_{(V_K \backslash S) \cup S'})$.

We conclude that

$$\omega_S(x, \epsilon) \ll_\Omega h_\epsilon = \left(\prod_{v \in S'} \epsilon_v^{-\frac{r_v \dim(G) + \sigma_S}{\mathfrak{a}_S(G)} - \delta} \right)^{\mathfrak{q}_{V_K \backslash S}(G)/2} \tag{6.11}$$

for every finite $S' \subset S$ and $\epsilon \in E_{S'}$ such that $\epsilon_v \in \{q_v^{-n}\}_{n \geq 0} \cap (0, \epsilon_v^0(x, \delta))$. Moreover, this implies that for all $\delta > 0$ and sufficiently small ϵ (depending on δ and Ω), we have

$$\omega_S(x, \epsilon) \leq \left(\prod_{v \in S'} \epsilon_v^{-\frac{r_v \dim(G) + \sigma_S}{\mathfrak{a}_S(G)} - 2\delta} \right)^{\mathfrak{q}_{V_K \backslash S}(G)/2}$$

Finally, it remains to extend this estimate to $\epsilon_v \in (0, \epsilon_v^0(x, \delta))$ when $v \in V_K^\infty \cap S'$. This can be done as in (6.9). □

6.2. **Diophantine approximation at every point on the group variety.**

Proof of Theorem 1.3(ii). Without loss of generality, we may assume that $\Omega = \prod_{v \in S} \Omega_v$ where Ω_v is a bounded subset G_v and $\Omega_v \supset \mathrm{G}(O_v)$ for all $v \in V_K^f$.

Let the sets $\tilde{\Psi}_\epsilon \subset G_{V_K}$ be as defined in Proposition 5.4 indexed by $\epsilon = (\epsilon_v)_{v \in S'}$ with finite $S' \subset S$ and $\epsilon_v \in I_v$. We set

$$h_\epsilon = \left(\prod_{v \in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K \backslash S}(\mathrm{G})}$$

with $\delta > 0$ and

$$\Upsilon_\epsilon = \{\varsigma \in \Upsilon : \, B_{h_\epsilon}^{-1} \varsigma \cap \tilde{\Psi}_\epsilon \mathrm{G}(K) = \emptyset\}.$$

Let ψ_ϵ be the characteristic function of the set $\Psi_\epsilon := \tilde{\Psi}_\epsilon \mathrm{G}(K) \subset \Upsilon$. As in the proof of Theorem 1.6, we get

$$\left\|\pi_{V_K\backslash S}(\beta_h)\psi_\epsilon - \mu(\Psi_\epsilon)\right\|_2 \ll_{\delta'} h^{-\frac{\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta'} \mu(\Psi_\epsilon)^{1/2}$$

for every $\delta' > 0$ and $h \geq h_0(\delta')$. Since $\pi_{V_K\backslash S}(\beta_{h_\epsilon})\psi_\epsilon = 0$ on Υ_ϵ, it follows that

$$\mu(\Upsilon_\epsilon) \ll_{\delta'} h_\epsilon^{-\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+2\delta'} \mu(\Psi_\epsilon)^{-1},$$

and using (5.10), we conclude that

$$\mu(\Upsilon_\epsilon) \ll_{\Omega,\delta'} \prod_{v\in S'} \epsilon_v^{\theta_v}, \tag{6.12}$$

where

$$\begin{aligned}\theta_v &= \left(r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} + \delta\right) \mathfrak{q}_{V_K\backslash S}(\mathrm{G}) \left(\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})} - 2\delta'\right) - r_v \dim(\mathrm{G}) \\ &= r_v \dim(\mathrm{G}) + \delta \cdot \mathfrak{q}_{V_K\backslash S}(\mathrm{G}) \left(\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})} - 2\delta'\right) - \delta' \cdot 2r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})}.\end{aligned}$$

This estimate holds provided that $h_\epsilon \geq h_0(\delta')$, so that it holds for all but finitely many $\epsilon \in E_S$.

We choose $\delta' = \delta'(\delta)$ such that $\theta_v > r_v \dim(\mathrm{G})$.

For $x \in \Omega$, there exists $x' \in \Omega$ such that $x'_v = x_v$ for $v \in S'$ and $x'_v \in \mathrm{G}(O_v)$ for $v \in V_K^f \cap (S\backslash S')$. For $\varsigma := (e, (x')^{-1})\mathrm{G}(K)$, we have by (5.11)

$$\mu(\tilde{\Psi}_\epsilon^{-1}\varsigma) \gg_\Omega \prod_{v\in S'} \epsilon_v^{r_v \dim(\mathrm{G})}. \tag{6.13}$$

Comparing (6.12) and (6.13), we conclude that the inequality

$$\mu(\tilde{\Psi}_\epsilon^{-1}\varsigma) > \mu(\Upsilon_\epsilon) \tag{6.14}$$

holds for all but finitely many $\epsilon \in E_S$. If (6.14) holds, then $\tilde{\Psi}_\epsilon^{-1}\varsigma \not\subseteq \Upsilon_\epsilon$, and

$$B_{h_\epsilon}^{-1}\tilde{\Psi}_\epsilon^{-1}\varsigma \cap \tilde{\Psi}_\epsilon \mathrm{G}(K) \neq \emptyset.$$

Therefore, by Proposition 5.4, for all but finitely many $\epsilon \in E_S$, there exists $z \in \mathrm{G}(K)$ such that

$$\mathrm{H}(z) \ll_\Omega h_\epsilon$$

and

$$\begin{aligned}\|x'_v - z\|_v &\leq \epsilon_v \quad \text{for all } v \in S', \\ \|x'_v - z\|_v &\leq 1 \quad \text{for all } v \in S\backslash S'.\end{aligned}$$

Since $x'_v \in \mathrm{G}(O_v)$ for $v \in V_K^f \cap (S\backslash S')$, it follows that $z \in \mathrm{G}(O_{(V_K\backslash S)\cup S'})$.

This shows that there exists $\epsilon_v^0(\Omega,\delta)\in(0,1]$ satisfying $\epsilon_v^0(\Omega,\delta)=1$ for almost all v such that for every $x\in\Omega$ and $\epsilon\in E_S$ with $\epsilon_v\in(0,\epsilon_v^0(\Omega,\delta))$, we have the estimate

$$\omega_S\left(x,(\epsilon_v)_{v\in S'}\right)\ll_\Omega h_\epsilon=\left(\prod_{v\in S'}\epsilon_v^{-r_v\frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})}-\delta}\right)^{\mathfrak{q}_{V_K\setminus S}(\mathrm{G})}.$$

Therefore, for all $\delta>0$ and sufficiently small ϵ (depending on δ and Ω), we also have

$$\omega_S\left(x,(\epsilon_v)_{v\in S'}\right)\le\left(\prod_{v\in S'}\epsilon_v^{-r_v\frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})}-2\delta}\right)^{\mathfrak{q}_{V_K\setminus S}(\mathrm{G})}.$$

Now it remains to extend this estimate to the case when $\epsilon_v\in(0,\epsilon_v^0(\Omega,\delta))$ for $v\in V_K^\infty\cap S'$, which can be achieved as in (6.9). This completes the proof of the theorem. □

6.3. **Diophantine approximation on homogeneous varieties.** Let X be an quasi-affine algebraic variety defined over K equipped with a transitive action of a connected almost simple algebraic K-group G. Let S be a subset of V_K and S' a finite subset of S. Before we start the proof of Theorem 1.6, we need to describe the structure of the topological closure $\overline{\mathrm{X}(O_{(V_K\setminus S)\cup S'})}$ in $X_{S'}$.

Lemma 6.3. *Assume that* G *is isotropic over* $V_K\backslash S$, *and let* $p:\tilde{\mathrm{G}}\to\mathrm{G}$ *denote the simply connected cover of* G. *Then the closure* $\overline{\mathrm{X}(O_{(V_K\setminus S)\cup S'})}$ *in* $X_{S'}$ *is open and*

$$\overline{\mathrm{X}(O_{(V_K\setminus S)\cup S'})}=p(\tilde{G}_{S'})\mathrm{X}(O_{(V_K\setminus S)\cup S'}).$$

Moreover, it is a union of finitely many orbits of $p(\tilde{G}_{S'})$.

Proof. Given $x^0\in\mathrm{X}(O_{(V_K\setminus S)\cup S'})$, we consider the map

$$P:\tilde{\mathrm{G}}\to\mathrm{X}:g\mapsto p(g)x^0.$$

Let

$$\Gamma=\tilde{\mathrm{G}}(K)\cap\left(\bigcap_{v\in(S\setminus S')\cap V_K^f}p^{-1}(\mathrm{G}(O_v))\right).$$

Then $P(\Gamma)\subset\mathrm{X}(O_{(V_K\setminus S)\cup S'})$. Since $\tilde{G}$ is isotropic over $V_K\backslash S$, the subgroup Γ is dense in $\tilde{G}_{S'}$, and

$$P(\tilde{G}_{S'})\subset\overline{P(\Gamma)}\subset\overline{\mathrm{X}(O_{(V_K\setminus S)\cup S'})}.$$

Since the variety X is a homogeneous space of $\tilde{\mathrm{G}}$ with the action $g\cdot x=p(g)x$, $x\in X$, it follows from [26, §3.1] that every orbit of $\tilde{G}_{S'}$ is open and closed in $X_{S'}$. This implies the first claim.

The last claim follows from finiteness of Galois cohomology over local fields (see [26, §6.4]). □

Proof of Theorem 1.6(i). According to Lemma 6.3, the set $\overline{\mathrm{X}(O_{(V_K\backslash S)\cup S'})}$ is a finite union of open (and closed) orbits of $p(\tilde{G}_{S'})$. We intend to show that given $x^0 \in \mathrm{X}(O_{(V_K\backslash S)\cup S'})$ there exists a set Y of full measure in $p(\tilde{G}_{S'})x^0$ whose points satisfies the claim of the theorem. In fact, we prove that for every bounded $Y \subset p(\tilde{G}_{S'})x^0$ there exists a conull subset $Y' \subset Y$ whose points satisfy the claim of the theorem. This will complete the proof.

Let Ω be a bounded subset of $p(\tilde{G}_{S'})$ such that $\Omega x^0 = Y$.

We set

$$U^0 = \prod_{v\in V_K^f\cap(S\backslash S')} (U_v \cap \mathrm{G}(O_v)) \quad \text{and} \quad U = U_{V_K^f\backslash S} U^0.$$

Note that that $U_v = \mathrm{G}(O_v)$ for almost all v. Since both U_v and $\mathrm{G}(O_v)$ are open and compact, it follows that the subgroup $U_v \cap \mathrm{G}(O_v)$ has finite index in U_v. Hence, U^0 has finite index in $U_{V_K^f\cap(S\backslash S')}$.

Recall that G^U denotes the kernel of U-invariant automorphic characters of G_{V_K} (see Section 4). We note that $\mathrm{G}(K) \subset G^U$, and G^U is a finite index subgroup of G_{V_K} by Lemma 4.4. Let β'_h be the Haar-uniform probability measure supported on $B'_h := U^0(B_h \cap G^U)$. By Theorem 4.5, for every $\phi \in L^2(\Upsilon)$ such that $\mathrm{supp}(\phi) \subset G^U/\mathrm{G}(K)$ and $h, \delta' > 0$,

$$\left\| \pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta'_h)\phi - \left(\int_\Upsilon \phi\, d\mu\right)\xi_U \right\|_2 \ll_{\delta'} m_{V_K\backslash S}(B_h\cap G^U)^{-\frac{1}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta'} \|\phi\|_2.$$

By Lemmas 6.1 and 6.2,

$$m_{V_K\backslash S}(B_h \cap G^U) \gg m_{V_K\backslash S}(B_h) \gg_\Omega h^{\mathfrak{a}_S(\mathrm{G})-\delta'}$$

for every $\delta' > 0$ and $h \geq h_0(\delta')$. Hence, combining these estimates, we conclude that for every $\delta' > 0$ and $h \geq h_0(\delta')$, we have

$$\left\| \pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta'_h)\phi - \left(\int_\Upsilon \phi\, d\mu\right)\xi_U \right\|_2 \ll_{\delta'} h^{-\frac{\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta'} \|\phi\|_2. \tag{6.15}$$

We apply (6.15) in the case when ϕ is the characteristic function ϕ_ϵ of the set Φ_ϵ introduced in Proposition 5.5, which is indexed by $\epsilon = (\epsilon_v)_{v\in S'}$ with finite $S' \subset S$ and $\epsilon_v \in I_v \cap (0, \epsilon_0)$. Note that it follows from the construction of the sets Φ_ϵ that we can arrange that for sufficiently small ϵ, we have $\Phi_\epsilon \subset G^U/\mathrm{G}(K)$, and hence (6.15) is applicable to $\phi = \phi_\epsilon$. We obtain

$$\left\| \pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta'_h)\phi_\epsilon - \mu(\Phi_\epsilon)\xi_U \right\|_2 \ll_{\delta'} h^{-\frac{\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta'} \mu(\Phi_\epsilon)^{1/2} \tag{6.16}$$

for every $\delta' > 0$ and $h \geq h_0(\delta')$. Let

$$h_\epsilon = \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})/2} \tag{6.17}$$

with $\delta > 0$, and

$$\Upsilon_\epsilon = \{\varsigma \in G^U/\mathrm{G}(K) :\ (B'_{h_\epsilon})^{-1}\varsigma \cap \Phi_\epsilon = \emptyset\}.$$

For $\varsigma \in \Upsilon_\epsilon$, we have

$$\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta'_{h_\epsilon})\phi_\epsilon(\varsigma) = 0.$$

Hence, (6.16) implies the estimate

$$\mu(\Upsilon_\epsilon) \ll_{\delta'} |G_{V_K} : G^U|^{-1} h_\epsilon^{-\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+2\delta'} \mu(\Phi_\epsilon)^{-1},$$

and using (6.17) and (5.15), we conclude that

$$\mu(\Upsilon_\epsilon) \ll_{\Omega,\delta'} \prod_{v\in S'} \epsilon_v^{\theta_v}, \tag{6.18}$$

where

$$\theta_v = \mathfrak{a}_S(\mathrm{G})\delta - \delta' \cdot \mathfrak{q}_{V_K\backslash S}(\mathrm{G}) \left(r_v \frac{\dim(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})} + \delta \right).$$

This estimate holds provided that $h_\epsilon \ge h_0(\delta')$. Hence, it holds for all but finitely many $\epsilon \in E_{S'} \cap (0, \epsilon_0)^{S'}$.

We choose $\delta' = \delta'(\delta) > 0$ sufficiently small, so that $\theta_v > 0$.

Now we may argue as in the proof of Theorem 1.3(i). We denote by Υ_0 the lim sup of the sets Υ_ϵ with $\epsilon \in E_{S'} \cap (0, \epsilon_0)^{S'}$. Since (6.18) holds for all but finitely many $\epsilon \in E_{S'}$, it follows that

$$\sum_{\epsilon\in E_{S'}\cap(0,\epsilon_0)^{S'}} \mu(\Upsilon_\epsilon) < \infty,$$

and by the Borel–Cantelli lemma, $\mu(\Upsilon_0) = 0$. We fix a bounded neighbourhood V of identity in $G_{V_K^\infty\backslash S'}$ contained in G^U. Then UV is a neighbourhood of identity in $G_{V_K\backslash S'}$. If we set

$$\Omega' = \{g \in \Omega :\ \exists y \in UV :\ (y, g^{-1})\mathrm{G}(K) \notin \Upsilon_0\},$$

then

$$(UV \times (\Omega\backslash\Omega')^{-1})\mathrm{G}(K) \subset \Upsilon_0,$$

and, hence, the set $\Omega\backslash\Omega'$ has measure zero. This implies that the subset $Y' := \Omega' x^0$ has full measure in $Y = \Omega x^0$.

For $g \in \Omega'$, there exists $y \in UV$ such that $\varsigma := (y, g^{-1})\mathrm{G}(K) \notin \Upsilon_0$. This implies that there exists $\epsilon_0(g, \delta) > 0$ such that for $\epsilon \in E_{S'} \cap (0, \epsilon_0(g, \delta))^{S'}$, we have

$$(B'_{h_\epsilon})^{-1}\varsigma \cap \Phi_\epsilon \neq \emptyset,$$

and hence,

$$((UV)^{-1}B'_{h_\epsilon})^{-1}(e, x^{-1}) \cap \Phi_\epsilon \neq \emptyset.$$

Therefore, it follows from Proposition 5.5 that there exists $z \in \mathrm{G}(O_{(V_K\backslash S)\cup S'})$ such that

$$\mathrm{H}(z) \ll_\Omega \max_{b\in(UV)^{-1}B'_{h_\epsilon}} \mathrm{H}(b) \ll h_\epsilon,$$

and for $x = gx^0 \in Y'$,

$$\|x_v - zx^0\|_v \le \epsilon_v \quad \text{for all } v \in S'.$$

We have $zx^0 \in \mathrm{X}(O_{(V_K\backslash S)\cup S'})$ and $\mathrm{H}(zx^0) \ll \mathrm{H}(z)$.

Since these estimates are valid for arbitrary bounded Y, we conclude that for every $\delta > 0$, almost every $x \in \overline{\mathrm{X}(O_{(V_K\backslash S)\cup S'})}$, and $(\epsilon_v)_{v\in S'}$ with finite $S' \subset S$ and $\epsilon_v \in E_v \cap (0, \epsilon_0(x, S', \delta))$, we have

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \ll_\Omega h_\epsilon = \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})} - \delta}\right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})/2}.$$

Moreover, this implies that

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \le \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{X})}{\mathfrak{a}_S(\mathrm{G})} - 2\delta}\right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})/2}$$

for every $\delta > 0$ and sufficiently small ϵ (depending on δ and Ω). Finally, it remains to extend this estimate to $\epsilon_v \in (0, \epsilon_0(x, S', \delta))$ for $v \in V_K^\infty \cap S'$, which can be done as in (6.9). □

Proof of Theorem 1.6(ii). By Lemma 6.3, the set $\overline{\mathrm{X}(O_{(V_K\backslash S)\cup S'})}$ is a finite union of open (and closed) orbits of $p(\tilde{G}_{S'})$. Hence, it suffices to prove the theorem for every compact subset Y contained in $p(\tilde{G}_{S'})x^0$ with $x^0 \in \mathrm{X}(O_{(V_K\backslash S)\cup S'})$.

Let Ω be a bounded subset of $p(\tilde{G}_{S'})$ such that $\Omega x^0 = Y$. Take $g \in \Omega$ and set $\varsigma := (e, g^{-1})\mathrm{G}(K) \in \Upsilon$. We note that $p(\tilde{G}_{S'}) \subset G^U$. In particular, $g \in G^U$.

Let B'_h and U be defined as in the proof of Theorem 1.6(i), and let $\tilde{\Psi}_\epsilon \subset G_{V_K}$ be the sets defined as in Proposition 5.6 which are indexed by $\epsilon = (\epsilon_v)_{v\in S'}$ with finite $S' \subset S$ and $\epsilon_v \in I_v \cap (0, \epsilon_0)$. As in the proof of Theorem 1.6(i), we will work on the space $G^U/\mathrm{G}(K)$. We note that taking the components of ϵ, L_v^0, and $\mathcal{O}_v$ sufficiently small, we can arrange that $\tilde{\Psi}_\epsilon \subset G^U$. Then Theorem 4.5 is applicable to the function ψ_ϵ which is the characteristic function of the set $\Psi_\epsilon := \tilde{\Psi}_\epsilon \mathrm{G}(K) \subset G^U/\mathrm{G}(K)$. Hence, as in the proof of Theorem 1.6(i), we have

$$\left\|\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta'_h)\psi_\epsilon - \mu(\Psi_\epsilon)\xi_U\right\|_2 \ll_{\delta'} h^{-\frac{\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+\delta'} \mu(\Psi_\epsilon)^{1/2} \quad (6.19)$$

for every $\delta' > 0$ and $h \ge h_0(\delta')$. We set

$$h_\epsilon = \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta}\right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}$$

with $\delta > 0$, and

$$\Upsilon_\epsilon = \{\varsigma \in G^U/\mathrm{G}(K) :\ (B'_{h_\epsilon})^{-1}\varsigma \cap \tilde{\Psi}_\epsilon \mathrm{G}(K) = \emptyset\}.$$

Since $\pi_{(V_K\backslash S)\cup(V_K^f\backslash S')}(\beta'_{h_\epsilon})\psi_\epsilon = 0$ on Υ_ϵ, inequality (6.19) implies that

$$\mu(\Upsilon_\epsilon) \ll_{\delta'} |G_{V_K} : G^U|^{-1} h_\epsilon^{-\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}+2\delta'} \mu(\Psi_\epsilon)^{-1}$$

when $h_\epsilon \geq h_0(\delta')$. Combining this estimate with (5.19), we conclude that

$$\mu(\Upsilon_\epsilon) \ll_{\Omega,\delta'} \prod_{v\in S'} \epsilon_v^{\theta_v}, \tag{6.20}$$

where

$$\theta_v = r_v \dim(\mathrm{X}) + \delta \cdot \mathfrak{q}_{V_K\backslash S}(\mathrm{G}) \left(\frac{2\mathfrak{a}_S(\mathrm{G})}{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})} - 2\delta' \right) - \delta' \cdot 2r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{X})}.$$

This estimate holds for all ϵ satisfying $h_\epsilon \geq h_0(\delta')$, and hence for all ϵ with sufficiently small components (depending on δ').

We pick $\delta' = \delta'(\delta) > 0$ such that $\theta_v > r_v \dim(\mathrm{G})$.

Let $g \in \Omega$ and $\varsigma := (e, g^{-1})\mathrm{G}(K)$. Then by (5.20),

$$\mu(\tilde{\Psi}_\epsilon^{-1}\varsigma) \gg_\Omega \prod_{v\in S'} \epsilon_v^{r_v \dim(\mathrm{X})},$$

Comparing this estimate with (6.20), we conclude that

$$\mu(\tilde{\Psi}_\epsilon^{-1}\varsigma) > \mu(\Upsilon_\epsilon)$$

provided that $\epsilon_v \leq \epsilon_0(Y, S', \delta)$, $v \in S'$. Then $\tilde{\Psi}_\epsilon^{-1}\varsigma \not\subseteq \Upsilon_\epsilon$, and

$$(B'_h)^{-1}\tilde{\Psi}_\epsilon^{-1}\varsigma \cap \Psi_\epsilon \neq \emptyset.$$

Hence, Proposition 5.6 implies that there exists $z \in \mathrm{G}(O_{(V_K\backslash S)\cup S'})$ such that

$$\mathrm{H}(z) \ll_\Omega \max_{b\in B'_{h_\epsilon}} \mathrm{H}(b) \ll h_\epsilon = \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})},$$

and for $x = gx_0$, we have

$$\|x_v - zx^0\| \leq \epsilon_v \quad \text{for all } v \in S'.$$

Since $\mathrm{H}(zx^0) \ll \mathrm{H}(z)$, we deduce that for every $\delta > 0$ and $(\epsilon_v)_{v\in S'}$ with $\epsilon_v \in I_v \cap (0, \epsilon_0(Y, \delta))$, we have

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \ll_\Omega \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - \delta} \right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}.$$

Moreover, this implies that for all sufficiently small ϵ (depending on δ and Ω), we have

$$\omega_S(x, (\epsilon_v)_{v\in S'}) \ll_\Omega \left(\prod_{v\in S'} \epsilon_v^{-r_v \frac{\dim(\mathrm{G})}{\mathfrak{a}_S(\mathrm{G})} - 2\delta} \right)^{\mathfrak{q}_{V_K\backslash S}(\mathrm{G})}.$$

This completes the proof of the theorem. □

Notation

K	a number field,
V_K	the set of places of K,
V_K^f	the subset of non-Archimedean places,
V_K^∞	the subset of Archimedean places,
O	the ring of integers of K,
O_S	the ring of S-integers of K,
K_v	the completion of K at $v \in V_K$,
O_v	the ring of integers of K_v,
q_v	the cardinality of the residue field of O_v,
r_v	2 if v is a complex place and 1 otherwise,
I_v	$(0,1)$ for $v \in V_K^\infty$ and $\{q_v^{-n}\}_{n\geq 1}$ for $v \in V_K^f$,
G_v	the set of K_v-points of G,
U_v	the maximal compact subgroup of G_v chosen in Section 3,
G_S	the restricted direct product of (G_v, U_v) with $v \in S$,
m_{G_v}	the Haar measure on G_v normalised,
	so that $m_{G_v}(U_v) = 1$ for non-Archimedean v,
m_S	the product of m_{G_v} over $v \in S$,
m_v	the Haar measure on G_v defined by G-invariant differential form,
m	the Tamagawa measure on G_{V_K} which is the product of m_v's,
H	the height function, see (1.2), (5.2),
ω_S	the Diophantine approximation function, see (1.6),
$\mathfrak{a}_S(\mathrm{X})$	the exponent of a variety X, see (1.7),
$\mathfrak{q}_v(\mathrm{G})$	the integrability exponent of automorphic representations, see (1.10),
σ_S	the exponent of a subset S of V_K, see (1.11),
$\mathfrak{q}_S(\mathrm{G})$	see (1.12),
Υ	$G_{V_K}/\mathrm{G}(K)$,
μ	the probability Haar measure on Υ,
$\mathcal{X}_{aut}(G_{V_K})$	the set of automorphic characters of G_{V_K},
$\mathcal{X}_{aut}(G_{V_K})^U$	the set of automorphic characters invariant under U,
G^U	the kernel of $\mathcal{X}_{aut}(G_{V_K})^U$ in G_{V_K},
$\mathcal{O}_S(\epsilon)$	the neighborhood of identity in G_S, see (5.4),
B_h	see (6.1).

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