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Asymptotic dimension of mapping class groups is finite

Mladen Bestvina, Ken Bromberg and Koji Fujiwara*

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Abstract

We give a general construction of many actions of groups on quasi-trees. We show that mapping class groups act on finite products of δ-hyperbolic spaces so that orbit maps are quasi-isometric embeddings. We prove that mapping class groups have finite asymptotic dimension.

Contents

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1 Introduction

The driving force behind much of the recent development in the geometry of mapping class groups has been the notion of subsurface projections of Masur-Minsky [MM00]. Given a finite binding set of curves α in a surface Σ and an essential subsurface $Y \subset \Sigma$, one may restrict α to Y and obtain a (coarse) point $\pi_Y(\alpha)$ in the curve complex $\mathcal{C}(Y)$ of Y. Thus one has a coarse map defined on the mapping class group

$$\Psi : MCG(\Sigma) \to \prod_Y \mathcal{C}(Y)$$

given by

$$\Psi(g) = (\pi_Y(g(\alpha)))_Y$$

The remarkable Masur-Minsky formula says that the word norm $|g|$ of g is coarsely equal to

$$\sum_Y \{\{d_{\mathcal{C}(Y)}(\pi_Y(\alpha), \pi_Y(g(\alpha)))\}\}_M$$

where the sum goes over all (infinitely many) essential subsurfaces Y, M is sufficiently large, and $\{\{x\}\}_M$ is defined as x if $x > M$ and as 0 if $x \leq M$.

Morally, this formula says that Ψ is a quasi-isometric embedding. However, the product space is not a metric space (the "cut-off" distance is not a metric).

In this paper we show that essential subsurfaces can be grouped in finitely many subcollections $\mathbf{Y}^1, \mathbf{Y}^2, \cdots, \mathbf{Y}^k$ so that subsurfaces in each $\mathbf{Y}^i$ naturally form the vertices of a "projection complex" $\mathcal{P}_K(\mathbf{Y}^i)$ which turns out to be a quasi-tree (graph quasi-isometric to a tree). Moreover, each quasi-tree can be "blown up" to a "quasi-tree of curve complexes" $\mathcal{C}(\mathbf{Y}^i)$ by replacing each vertex of $\mathcal{P}_K(\mathbf{Y}^i)$ by the curve complex of the associated subsurface. Everything can be done equivariantly, so that we have an orbit map

$$MCG(\Sigma) \to \mathcal{C}(\mathbf{Y}^1) \times \mathcal{C}(\mathbf{Y}^2) \times \cdots \times \mathcal{C}(\mathbf{Y}^k)$$

Then the Masur-Minsky formula can be interpreted as saying that this is a quasi-isometric embedding. One consequence of this, our main goal at the beginning of this endeavor, is the main theorem in the paper:

Main Theorem. *Asymptotic dimension of mapping class groups is finite.*

The proof amounts to showing that the quasi-trees of curve complexes have finite asymptotic dimension. The Coarse Baum-Connes conjecture (for torsion free subgroups of finite index) and therefore the Novikov conjecture follows [Yu98]. Various other statements that imply the Novikov conjecture were known earlier (see [Kid08, Ham09, BM]).

We describe the construction of $\mathcal{P}_K(\mathbf{Y}^i)$ and of $\mathcal{C}(\mathbf{Y}^i)$ axiomatically, as it applies to other settings as well, even when the analog of the Masur-Minsky formula is not (yet) known. For example, we construct many actions of various groups, such as non-elementary hyperbolic groups and $Out(F_n)$, on quasi-trees. This has a consequence that second bounded cohomology (even with coefficients in certain representations) is "big". We will investigate this in a future paper. Another consequence is that mapping class groups in even genus can act on quasi-trees with a Dehn twist having unbounded orbits (in the case of odd genus one has to pass to a subgroup of finite index).

By contrast, there are many groups that do not admit nontrivial actions on a quasi-tree. Recall [Man06] that a group G satisfies QFA if every action on a quasi-tree has bounded orbits. Equivalently (see e.g. [Man05]) every quasi-action on a tree has bounded orbits. If G is an irreducible lattice in a higher rank semi-simple Lie group with finite center, it is expected that G has QFA (see [Man06, Oza] for special cases).

1.1 Asymptotic dimension

We give a brief review of asymptotic dimension.

The *asymptotic dimension* $\text{asdim}(\mathcal{X})$ of a metric space $\mathcal{X}$ is said to be $\leq n$ if for every $R > 0$ there is a covering of $\mathcal{X}$ by sets U_i such that every metric R-ball in $\mathcal{X}$ intersects at most $n+1$ of the U_i's, and $\sup \text{diam}\, U_i < \infty$. This definition is due to Gromov [Gro93] and it is invariant under quasi-isometries (or even coarse isometries). In particular, asymptotic dimension of a finitely generated group is well-defined. It is not hard to see that $\text{asdim}(\mathbb{R}^2) \leq 2$ by considering the usual "brick decomposition" of $\mathbb{R}^2$ (with large bricks), and more generally, $\text{asdim}(\mathbb{R}^n) \leq n$. This inequality is also easily seen using the product formula $\text{asdim}(\mathcal{X} \times \mathcal{Y}) \leq \text{asdim}(\mathcal{X}) + \text{asdim}(\mathcal{Y})$.

A generalization of the product formula is Bell-Dranishnikov's Hurewicz theorem [BD06]: Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a Lipschitz map between geodesic metric spaces such that for every $M > 0$ the family $\{f^{-1}(B(y, M))\}$ of preimages of metric balls of radius M has asymptotic dimension $\leq n$ *uniformly* (this means that coverings as in the definition can be found with a diameter bound independent of the center y). Then $\text{asdim}(\mathcal{X}) \leq \text{asdim}(\mathcal{Y}) + n$.

For example, if $1 \to A \to B \to C \to 1$ is a short exact sequence of finitely generated groups then $\text{asdim}(B) \leq \text{asdim}(A) + \text{asdim}(C)$. Likewise, asymptotic dimension of the hyperbolic plane is ≤ 2 by considering the projection to a line whose fibers are horocycles tangent to a fixed point at infinity (e.g. the projection to the y-coordinate in the upper half-space model). More generally one can apply this argument to a semi-simple Lie group and its associated symmetric space.

Gromov proved that δ-hyperbolic groups have finite asymptotic dimension. Here is a proof. Assume that $R \gg \delta$ is an integer. For every vertex v in the Cayley graph of the group at distance $5kR$ from 1, $k = 1, 2, 3, \cdots$, consider the set

$$U_v = \{x \in \Gamma \mid d(1, x) \in [5(k+1)R, 5(k+2)R] \text{ and } v \text{ lies on some geodesic } [1, x]\}$$

An easy thin triangle argument shows that if v, w are two vertices at distance $5kR$ such that both U_v and U_w intersect the same R-ball, then $d(v, w) \leq 2\delta$. This gives a bound on the number of U_v's that can intersect the same R-ball, and this bound is independent of R; thus $\text{asdim}(\Gamma) < \infty$. We can also apply this argument to a tree T to show that $\text{asdim}(T) \leq 1$.

Bell-Fujiwara [BF08] modified this argument to show that curve complexes have finite asymptotic dimension. They are hyperbolic by the celebrated work of Masur-Minsky [MM99], but not locally finite, resulting in an infinite bound. The trick is to use *tight* geodesics in place of arbitrary geodesics. Finiteness properties of tight geodesics proved by Bowditch [Bow08] imply that asymptotic dimension is finite.

1.2 Outline

We now outline a proof of the main theorem:

Main Theorem. *Asymptotic dimension of mapping class groups is finite.*

Let Σ be a surface of finite type and $MCG(\Sigma)$ its mapping class group.

Step 1. Produce an action of $MCG(\Sigma)$ on a finite product $\mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_k$ of metric spaces. An orbit map $MCG(\Sigma) \to \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \times \mathcal{X}_k$

is a quasi-isometric embedding. This reduces us to showing $\text{asdim}(\mathcal{X}_i) < \infty$ for all i.

Step 2. Show each $\mathcal{X}_i$ is hyperbolic and has a Lipschitz map $\mathcal{X}_i \to T_i$ satisfying the Hurewicz theorem with fibers curve complexes of subsurfaces of Σ. This reduces us to showing $\text{asdim}(T_i) < \infty$.

Step 3. Show that each T_i is quasi-isometric to a tree (i.e. it is a *quasi-tree*) and hence $\text{asdim}(T_i) = 1$.

The last step is the most interesting and leads one to wonder which groups admit interesting actions on quasi-trees. An axiomatic construction is as follows:

Let $\mathbf{Y}$ be a set and assume that for every $Y \in \mathbf{Y}$ we have a function $d_Y : (\mathbf{Y} - \{Y\})^2 \to [0, \infty)$ such that

- $d_Y(X, Z) = d_Y(Z, X)$,
- $d_Y(X, W) \leq d_Y(X, Z) + d_Y(Z, W)$,
- there is $\xi > 0$ such that for any $X, Y, Z \in \mathbf{Y}$ at most one of
$$d_X(Y, Z), d_Y(X, Z), d_Z(X, Y)$$
is $> \xi$, and
- there is K_0 such that for any X, Z the set
$$\{Y \in \mathbf{Y} \mid d_Y(X, Z) > K_0\}$$
is finite.

In section 2.1 we will describe many natural examples where such functions arise. Of central interest to us is the setting of subsurface projections, where $\mathbf{Y}$ is a family of essential subsurfaces of Σ such that $X, Z \in \mathbf{Y}$ implies $\partial X \cap \partial Z \neq \emptyset$. The distance $d_Y(X, Z)$ is given by restricting ∂X and ∂Z to Y and measuring the distance in the curve complex of Y. The axioms for this case are part of the work of Masur-Minsky [MM99, MM00] and Behrstock [Beh06].

One can then consider the *projection complex* $\mathcal{P}_K(\mathbf{Y})$. Fix a large $K > 0$. The vertices of $\mathcal{P}_K(\mathbf{Y})$ are the elements of $\mathbf{Y}$, and two vertices X, Z are joined by an edge if $d_Y(X, Z) < K$ for all Y. (Technically, we first perturb d by $\leq 2\xi$ before defining $\mathcal{P}_K(\mathbf{Y})$; this is ignored here.) We then argue that $\mathcal{P}_K(\mathbf{Y})$ is a quasi-tree.

To finish the argument, we divide the collection of all essential subsurfaces of Σ into finitely many classes $\mathbf{Y}_1, \cdots, \mathbf{Y}_k$ so that each $\mathbf{Y}_i$ satisfies the

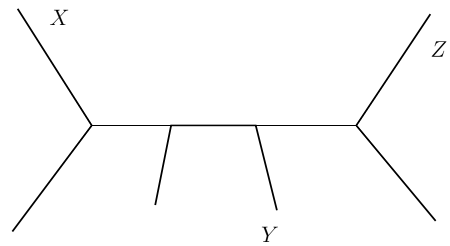

Figure 1: Axiom 3. If $d^\pi_Y(X,Z) \geq \xi$, then $d^\pi_X(Y,Z) < \xi$ and $d^\pi_Z(X,Y) < \xi$.

above "transversality" property. Moreover, this can be done so that each $\mathbf{Y}_i$ is invariant under a certain fixed finite index subgroup $G \subset MCG(\Sigma)$. We obtain quasi-trees $T_1 = \mathcal{P}_K(\mathbf{Y}_1), \cdots, T_k = \mathcal{P}_K(\mathbf{Y}_k)$, and replacing each vertex $Y \in T_i$ by the corresponding curve complex $\mathcal{C}(Y)$ gives rise to $\mathcal{X}_i$. The fact that orbit maps are QI-embeddings follows from a distance formula due to Masur-Minsky [MM00].

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2 The projection complex

2.1 Axioms

Let $\mathbf{Y}$ be a set and assume that for each $Y \in \mathbf{Y}$ we have a function

$$d^\pi_Y : (\mathbf{Y} \setminus \{Y\}) \times (\mathbf{Y} \setminus \{Y\}) \longrightarrow [0, \infty)$$

and a constant $\xi > 0$ such that satisfies the following axioms:

(1) $d^\pi_Y(X,Z) = d^\pi_Y(Z,X)$;

(2) $d^\pi_Y(X,Z) + d^\pi_Y(Z,W) \geq d^\pi_Y(X,W)$;

(3) $\min\{d^\pi_Y(X,Z), d^\pi_Z(X,Y)\} < \xi$;

(4) $\#\{Y | d^\pi_Y(X,Z) \geq \xi\}$ is finite for all $X, Z \in \mathbf{Y}$.

Examples 2.1. (1) Let Γ be a discrete group of isometries of $\mathbb{H}^n$ and $\gamma_1, \cdots, \gamma_k$ a finite collection of loxodromic elements of Γ. Denote by X_i the axis of γ_i and let

$$\mathbf{Y} = \{\gamma X_i | \gamma \in \Gamma, 1 \leq i \leq k\}$$

The reader can check that there is $\nu > 0$ such that the projection $\pi_Y(X)$ (i.e. the image under the nearest point projection map) of any geodesic X in $\mathbf{Y}$ to any other geodesic Y in $\mathbf{Y}$ has diameter bounded by ν. Define $d_Y^\pi(X,Z)$ to be $\mathrm{diam}(\pi_Y(X \cup Z))$. The reader may check that all axioms hold.

(2) Similarly, let Γ be a group of isometries of a connected δ-hyperbolic graph X and fix a finite set $\gamma_1, \cdots, \gamma_k$ of hyperbolic elements of Γ as well as their quasi-axes $X_1, \cdots, X_k$. Let $\mathbf{Y}$ be the set of parallel classes of Γ-translates of the X_i's, where two lines are parallel if each is contained in a Hausdorff neighborhood of the other. Assume in addition that there is $\nu > 0$ such that the projection of any translate γX_i to any nonparallel X_j is bounded by ν (this is equivalent to the Weak Proper Discontinuity condition of [BF02]). As above, define $d_Y^\pi(X,Z)$ to be $\mathrm{diam}(\pi_\gamma(\alpha \cup \beta))$ for any $\alpha \in X$, $\beta \in Z$, $\gamma \in C$. All axioms hold. In particular, this construction applies to the curve complex and the mapping class group of a compact surface.

(3) Let Σ be a closed orientable surface and $\mathbf{Y}$ a set of (isotopy classes of) essential subsurfaces of Σ. Assume that when $X, Z \in \mathbf{Y}$, $X \neq Z$, then ∂X and ∂Z have nonzero intersection number. Masur-Minsky [MM00] define the number $d_Y^\pi(X,Z)$ as the diameter in the curve complex of Y of $\pi_Y(\partial X) \cup \pi_Y(\partial Z)$ (we are really working with the arc-and-curve complex). Axioms other than 3 and 4 are straightforward. Axiom 3 is known as the Behrstock inequality and the original proof [Beh06] uses the Masur-Minsky theory of hierarchies [MM00]. A simple proof due to Leininger is included in [Man10]. Axiom 4 is also proved in [MM00].

(4) Let X be Outer space for $Out(F_n)$ equipped with the Lipschitz metric (which fails to be symmetric). Fully irreducible elements of $Out(F_n)$ have axes in X. Let $\gamma_1, \cdots, \gamma_k$ be a finite collection of fully irreducible automorphisms and let $X_1, \cdots, X_k$ be their axes. Take $\mathbf{Y}$ to be the set of parallel classes of $Out(F_n)$-translates of the X_i's and proceed as in (2). In [AK] Yael Algom-Kfir shows that there is $\nu > 0$ such that the projection of any translate $\gamma(X_i)$ to any nonparallel X_j is bounded by ν. Axiom (3) is explicitly verified in [AK] and Axiom (4) follows quickly from the arguments in [AK].

2.2 Monotonicity

Given distance functions that satisfy the above axioms it is useful to modify their definition slightly. Define $\mathcal{H}(X,Z)$ to be the set of pairs $(X',Z') \in \mathbf{Y}\times\mathbf{Y}$ such that one of the following holds:

- $d^\pi_X(X',Z'), d^\pi_Z(X',Z') > 2\xi$;
- $X = X'$ and $d^\pi_Z(X,Z') > 2\xi$;
- $Z = Z'$ and $d^\pi_X(X',Z) > 2\xi$;
- $(X',Z') = (X,Z)$.

We then define

$$d_Y(X,Z) = \inf_{(X',Z')\in\mathcal{H}(X,Z)} d^\pi_Y(X',Z').$$

Note that it is clear from the definition that $d_Y(X,Z) \leq d^\pi_Y(X,Z)$ and therefore Axiom 3 still holds for d_Y with the same constant. However we need to modify Axiom 2 to a coarse triangle inequality.

Proposition 2.2. *If* $(X',Z') \in \mathcal{H}(X,Z)$ *then*

$$d^\pi_Y(X,Z) - d^\pi_Y(X',Z') < 2\xi.$$

Proof. If $d^\pi_Y(X,Z) < 2\xi$ we are done since the distances are always nonnegative. For the rest of the proof we now assume that $d^\pi_Y(X,Z) \geq 2\xi$.

We first assume that X and Z are distinct form X' and Z'. By the triangle inequality

$$d^\pi_X(X',Y) + d^\pi_X(Y,Z') \geq d^\pi_X(X',Z') > 2\xi$$

and therefore

$$\max\{d^\pi_X(X',Y), d^\pi_X(Y,Z')\} > \xi.$$

Without loss of generality we assume that $d^\pi_X(X',Y) > \xi$.

By Axiom 3 we have $d^\pi_Y(X,X') < \xi$ and again applying the triangle inequality we have

$$d^\pi_Y(X,X') + d^\pi_Y(X',Z) \geq d^\pi_Y(X,Z) \geq 2\xi$$

and therefore

$$d^\pi_Y(X',Z) > 2\xi - \xi = \xi.$$

Another application of Axiom 3 gives us that $d^\pi_Z(X', Y) < \xi$.

We now apply the triangle inequality exactly as we did at the start of the proof but replacing X with Z. Again we get that

$$\max\{d^\pi_Z(X', Y), d^\pi_Z(Z', Y)\} > \xi$$

and since we have just seen that $d^\pi_Z(X', Y) < \xi$ we must have $d^\pi_Z(Z', Y) > \xi$. Then by Axiom 3 $d^\pi_Y(Z, Z') < \xi$.

To finish the proof in this case we make one final application of the triangle inequality to see that

$$d^\pi_Y(X, X') + d^\pi_Y(X', Z') + d^\pi_Y(Z', Z) \geq d^\pi_Y(X, Z)$$

and therefore

$$d^\pi_Y(X, Z) - d^\pi_Y(X', Z') < 2\xi.$$

For pairs of the form (X', Z) with $X' \neq X$ the proof is easier. As before we have the inequality

$$d^\pi_X(X', Y) + d^\pi_X(Y, Z) \geq d^\pi_X(X', Z) > 2\xi.$$

Since $d^\pi_Y(X, Z) \geq 2\xi$ we must have $d^\pi_X(Y, Z) < \xi$ and therefore $d^\pi_X(X', Y) > \xi$ and $d^\pi_Y(X, X') < \xi$. We once again apply the triangle inequality to see that

$$d^\pi_Y(X, X') + d^\pi_Y(X', Z) \geq d^\pi_Y(X, Z)$$

and therefore

$$d_Y(X, Z) - d_Y(X', Z) < \xi < 2\xi.$$

The statement is trivial if $(X', Z') = (X, Z)$ so the proof is finished. □

This result has number of important consequences. Before stating them we set notation that helps prevent a proliferation of constants. We say that $x \succ y$ or $y \prec x$ if $x - y$ is bounded above by a constant depending only on ξ. We also define $x \sim y$ if $x \succ y$ and $y \succ x$. For example we can restate Axiom 3 as

$$\min\{d_Y(X, Z), d_Z(X, Y)\} \sim 0.$$

Thus, for the purposes of this notation, we regard ξ as a variable that depends on the particular setting. Note that transitivity holds, i.e. if $x \succ y$ and $y \succ z$ then $x \succ z$, but the constant bounding $z - x$ is worse. Thus it is important to ensure that transitivity is applied only to chains of bounded length.

Next we define $\mathbf{Y}_K(X,Z)$ to be the set of $Y \in \mathbf{Y}$ such that $d_Y(X,Z) > K$.

Here are the properties of the functions d_Y, gathered together in one theorem. One can think of them as axioms.

Theorem 2.3. *There exists a $\boldsymbol{\xi} > 0$, depending only on ξ, such that the following properties hold:*

(A) **Symmetry**

$$d_Y(X,Z) = d_Y(Z,X)$$

(B) **Coarse equality** *For all distinct X, Y and Z*

$$d_Y^\pi(X,Z) \prec d_Y(X,Z) \leq d_Y^\pi(X,Z).$$

(C) **Coarse triangle inequality**

$$d_Y(X,Z) + d_Y(Z,W) \succ d_Y(X,W).$$

(D) **Inequality on triples**

$$\min\{d_Y(X,Z), d_Z(X,Y)\} \sim 0$$

(E) **Finiteness** $\#\{Y | d_Y(X,Z) \geq \boldsymbol{\xi}\}$ *is finite for all $X, Z \in \mathbf{Y}$.*

(F) **Monotonicity** *If $d_Y(X,Z) \geq \boldsymbol{\xi}$ then $d_W(X,Y), d_W(Z,Y) \leq d_W(X,Z)$.*

(G) **Order** *The set $\mathbf{Y}_{\boldsymbol{\xi}}(X,Z) \cup \{X,Z\}$ is totally ordered with least element X and greatest element Z such that if Y_1 is between Y_0 and Y_2 then*

$$d_{Y_1}(X,Z) \prec d_{Y_1}(Y_0,Y_2) \leq d_{Y_1}(X,Z)$$

and if not

$$d_{Y_1}(Y_0,Y_2) \sim 0.$$

(H) **Barrier property** *If $Y \in \mathbf{Y}_{\boldsymbol{\xi}}(X_0,Z)$ and $Y \in \mathbf{Y}_{\boldsymbol{\xi}}(X_1,Z)$ then*

$$d_Z(X_0,X_1) < \boldsymbol{\xi}.$$

Proof. For each property we will see that there is some constant so that the property holds for any larger choice of constant. Throughout the proof one should think of ξ as being fixed but $\boldsymbol{\xi}$ as a variable that won't be fixed until the end of the proof.

The symmetry property follows from the symmetry property for d_Y^π and the definition of d_Y. The coarse inequality property is just a restatement of Proposition 2.2 with our new notation. The coarse triangle inequality, the inequality on triples and the finiteness property all follow from the corresponding properties for d_Y^π plus coarse inequality. Note that the inequality on triples and the finiteness property hold for any $\boldsymbol{\xi} \geq \xi$. This will be important in the proof of the order property.

The monotonicity property requires a bit of work. We show $\mathcal{H}(X,Z) \subseteq \mathcal{H}(X,Y) \cap \mathcal{H}(Z,Y)$ if $\boldsymbol{\xi} \geq 4\xi$. If $(X',Z') \in \mathcal{H}(X,Z)$ then by Proposition 2.2 we have

$$d_Y^\pi(X,Z) - d_Y^\pi(X',Z') < 2\xi$$

and since $d_Y^\pi(X,Z) \geq d_Y(X,Z) \geq \boldsymbol{\xi} \geq 4\xi$ we have $d_Y^\pi(X',Z') > 2\xi$. In particular (X',Z') is in both $\mathcal{H}(X,Y)$ and $\mathcal{H}(Z,Y)$ and the inequalities follow.

The proof of the order property is more involved. We first need to define the order. Applying the coarse triangle inequality we have

$$d_Y(X,W) + d_Y(W,Z) \succ d_Y(X,Z)$$

so if $d_Y(X,Z)$ is sufficiently large then either

$$d_Y(X,W) > \xi \text{ or } d_Y(W,Z) > \xi \qquad (*)$$

but not both (otherwise $d_W(X,Y) < \xi$, $d_W(Y,Z) < \xi$, so by the coarse triangle inequality $d_W(X,Z) < \boldsymbol{\xi}$).

We now define $Y < W$ to mean $d_Y(X,W) > \xi$ (equivalently, $d_Y(W,Z) \leq \xi$). By the inequality on triples, in this case $d_W(X,Y) < \xi$ and therefore $d_W(Y,Z) > \xi$ (by (*) with Y and W interchanged). Thus we could have defined equivalently $Y < W$ to mean $d_W(Y,Z) > \xi$ (equivalently, $d_W(X,Y) \leq \xi$).

If $Y \not< W$ then $d_Y(W,Z) > \xi$ and therefore $d_W(Y,Z) < \xi$, so $W < Y$. Likewise, we cannot have both $Y < W$ and $W < Y$. We also define X to be the least element and Z the greatest element. We have just shown that any two elements can be compared and that if $Y < W$ then $W \not< Y$. It remains to argue the two inequalities and transitivity.

Now assume that $Y_0 < Y_1 < Y_2$ and we'll show the first inequality. If $Y_0 = X$ or $Y_2 = Z$ we're done so assume not. Then by the coarse triangle inequality we have

$$d_{Y_1}(X, Y_0) + d_{Y_1}(Y_0, Y_2) + d_{Y_1}(Y_2, Z) \succ d_{Y_1}(X, Z).$$

Since $Y_0 < Y_1$ and $Y_1 < Y_2$, we have $d_{Y_1}(X, Y_0) \leq \xi$ and $d_{Y_1}(Y_2, Z) \leq \xi$. It follows that

$$d_{Y_1}(Y_0, Y_2) \succ d_{Y_1}(X, Z)$$

and by monotonicity

$$d_{Y_1}(Y_0, Y_2) \leq d_{Y_1}(X, Z).$$

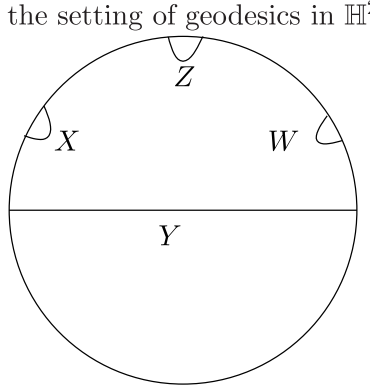
We see that $Y_0 < Y_2$ since

$$d_{Y_0}(X, Y_2) + d_{Y_0}(Y_1, Y_2) \succ d_{Y_0}(X, Y_1) \succ d_{Y_0}(X, Z)$$

and once again if $\boldsymbol{\xi}$ is sufficiently large then $d_{Y_0}(X, Y_2) > \xi$. By the inequality on triples, the second inequality follows from the first.

Finally we prove the barrier property. If the conclusion fails, i.e. if $d_Z(X_0, X_1) \geq \boldsymbol{\xi}$ then $Z \in \mathbf{Y}_{\boldsymbol{\xi}}(X_0, X_1)$ and also, by monotonicity, $Y \in \mathbf{Y}_{\boldsymbol{\xi}}(X_0, X_1)$. If $Y < Z$ in $\mathbf{Y}_{\boldsymbol{\xi}}(X_0, X_1)$ then $d_Y(X_1, Z) \leq \xi$ and if $Z < Y$ then $d_Y(X_0, Z) \leq \xi$. Either way, we have a contradiction. $\square$

Note that the monotonicity property fails for the original distance d^π. Below is an example in the setting of geodesics in $\mathbb{H}^2$ (see Example 2.1(1)).



In the figure, $d^\pi_Y(X, Z)$ can be made arbitrarily large, while $d^\pi_W(Z, Y)$ is slightly larger than $d^\pi_W(X, Z)$.

Also note that one could define in the same way an order on $\mathbf{Y}_K(X, Z) \cup \{X, Z\}$ for any $K \geq \boldsymbol{\xi}$, but this order coincides with the induced order from the larger set $\mathbf{Y}_{\boldsymbol{\xi}}(X, Z) \cup \{X, Z\}$. The order on $\mathbf{Y}_K(Z, X) \cup \{Z, X\}$ is the reverse of the order on (the same set) $\mathbf{Y}_K(X, Z) \cup \{X, Z\}$.

2.3 The projection complex

For $K \geq \boldsymbol{\xi}$ we define a 1-complex $\mathcal{P}_K(\mathbf{Y})$ as follows. The vertex set is $\mathbf{Y}$. We connect two vertices X and Z with an edge if $\mathbf{Y}_K(X,Z)$ is empty. We denote the distance function for this complex by $d(,)$. In particular $d(X,Z) = 1$ if $\mathbf{Y}_K(X,Z) = \emptyset$. Note that for different values of K the spaces $\mathcal{P}_K(\mathbf{Y})$ are not necessarily quasi-isometric to each other (the vertex sets are the same, but for larger K there are more edges). Our goal is to show that $\mathcal{P}_K(\mathbf{Y})$ is quasi-isometric to a tree. We begin by showing that $\mathcal{P}_K(\mathbf{Y})$ is connected and obtained an upper bound on the distance function.

Lemma 2.4. *If X and Z are vertices in $\mathbf{Y}$ then $d(X,Z) \leq |\mathbf{Y}_K(X,Z)| + 1$. In particular, $\mathcal{P}_K(\mathbf{Y})$ is connected.*

Proof. Label the elements of $\mathbf{Y}_K(X,Z) \cup \{X,Z\}$ by $Y_0, Y_1, \ldots, Y_{k+1}$ where the indices respect the order and $k = |\mathbf{Y}_K(X,Z)|$. We claim that $X = Y_0, Y_1, \ldots, Y_{k+1} = Z$ is a path from X to Z. To see this we note that the monotonicity property implies that if $Y \in \mathbf{Y}_K(Y_i, Y_{i+1})$ then $Y \in \mathbf{Y}_K(X,Z)$ and $Y = Y_j$. However, since Y_j cannot be between Y_i and Y_{i+1} we have $d_{Y_j}(Y_i, Y_{i+1}) < \boldsymbol{\xi}$, a contradiction. Therefore $\mathbf{Y}_K(Y_i, Y_{i+1}) = \emptyset$, $d(Y_i, Y_{i+1}) = 1$ and we have our path from X to Z. ☐

On the other hand, the cardinality of $\mathbf{Y}_K(X,Z)$ gives no lower bound on $d(X,Z)$. For example, it is possible that $\mathbf{Y}_K(Y_1, Z) = \emptyset$ and the distance from X to Z is two (even though k is large). This highlights a key difficulty in the paper. From the viewpoint of X, there appear to be many projections larger than the K-threshold between Y_1 and Z. However, from the viewpoint of Y_1 there are no large projections between Y_1 and Z.

A key concept in the paper is the notion of a *guard* and this notion is defined to deal with this problem. Roughly speaking, W is a guard for Y if from every viewpoint there are no large projections between W and Y. The formal condition is that for every vertex $X \in \mathbf{Y}$ with $W \in \mathbf{Y}_{\boldsymbol{\xi}}(X,Y)$ and every $Z \in \mathbf{Y}_K(X,Y) \subset \mathbf{Y}_{\boldsymbol{\xi}}(X,Y)$ then $Z \leq W$. Note that if W is a guard for Y then $d(W,Y) = 1$.

Lemma 2.5. *For K sufficiently large and vertices X, Y, Z and W, if $W \in \mathbf{Y}_{\boldsymbol{\xi}}(X,Y)$, $Z \in \mathbf{Y}_K(X,Y)$ and $W < Z$ in $\mathbf{Y}_{\boldsymbol{\xi}}(X,Y)$, then $Z \in \mathbf{Y}_{K/2}(W,Y)$.*

In particular, if $\mathbf{Y}_{K/2}(W,Y) = \emptyset$ then W is a guard for Y.

Proof. Given X, Y, Z and W as above, by the order property we have

$$d_Z(W,Y) \succ d_Z(X,Y) > K$$

and therefore if K is sufficiently large then

$$d_Z(W, Y) > K/2.$$

$\square$

Note that it follows from this lemma and the order property that the least element of $\mathbf{Y}_{K/2}(X, Z)$ (if nonempty) is a guard for X and the greatest element is a guard for Z.

By definition, the projection of adjacent vertices of $\mathcal{P}_K(\mathbf{Y})$ to another vertex will be bounded above by K. However, if this third vertex is distance two or more from one of the other two vertices we get a stronger bound.

Lemma 2.6. *Let X_0 and X_1 be adjacent vertices in $\mathcal{P}_K(\mathbf{Y})$ and assume W is a vertex in $\mathbf{Y}$ with $d(X_0, W) \geq 2$. Then*

$$d_W(X_0, X_1) \sim 0$$

and

$$d_W(X_0, Z) \sim d_W(X_1, Z)$$

for all $Z \in \mathbf{Y}$.

Proof. Since $d(X_0, W) \geq 2$ there exists $Y \in \mathbf{Y}_K(X_0, W)$. If $d_W(X_0, X_1) > \boldsymbol{\xi}$ then by monotonicity we have

$$d_Y(X_0, X_1) \geq d_Y(X_0, W) > K$$

which contradicts $d(X_0, X_1) = 1$ and therefore $d_W(X_0, X_1) \leq \boldsymbol{\xi}$.

Applying the coarse triangle inequality we have

$$d_W(Z, X_0) + d_W(X_0, X_1) \succ d_W(Z, X_1)$$

which implies half of the second inequality. The other half is proved by swapping X_0 and X_1. $\square$

Lemma 2.7. *If K is sufficiently large the following holds. Let X_0 and X_1 be adjacent vertices with $d(X_i, Z) \geq 3$. Let W be a guard for Z such that $W \in \mathbf{Y}_{K/2}(X_0, Z)$. If $W \notin \mathbf{Y}_{K/2}(X_1, Z)$ then there exists a guard W' for Z such that $W' \in \mathbf{Y}_{K/2}(X_1, Z)$ and $W \in \mathbf{Y}_{\boldsymbol{\xi}}(W', Z)$.*

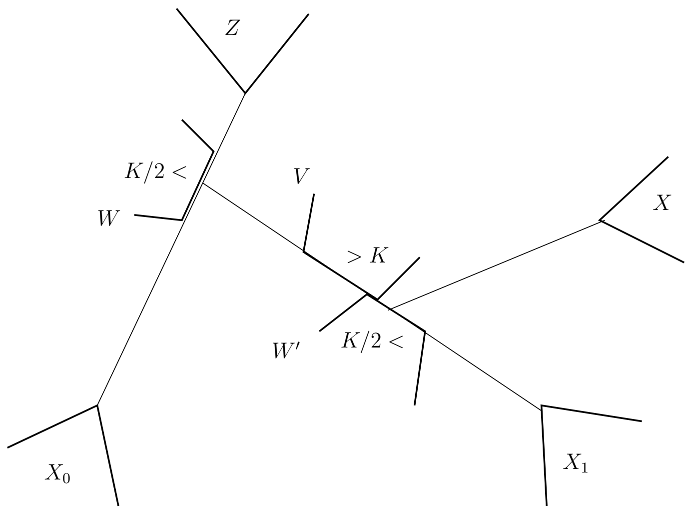


Figure 2: Lemma 2.7

Proof. We assume that $W \notin \mathbf{Y}_{K/2}(X_1, Z)$. Note that $d(W, Z) = 1$ and since $d(X_0, Z) \geq 3$ we have $d(X_0, W) \geq 2$ and we can apply Lemma 2.6. From Lemma 2.6 we see that

$$d_W(X_1, Z) \succ d_W(X_0, Z) > K/2$$

and if K is sufficiently large $W \in \mathbf{Y}_{\boldsymbol{\xi}}(X_1, Z)$.

Since $d(X_1, Z) \geq 3$ we also have $d(X_1, W) \geq 2$ so there must be elements in $\mathbf{Y}_{K/2}(X_1, Z)$ that are less than W in $\mathbf{Y}_{\boldsymbol{\xi}}(X_1, Z)$. We let W' be the greatest such element. By the order property

$$d_W(W', Z) \succ d_W(X_1, Z) \succ K/2$$

and again, if K is sufficient large then $W \in \mathbf{Y}_{\boldsymbol{\xi}}(W', Z)$.

We now show that W' is a guard for Z. Note that for any X with $d_{W'}(X, Z) > \boldsymbol{\xi}$ we also have $d_W(X, Z) > \boldsymbol{\xi}$ by monotonicity. If $V \in \mathbf{Y}_K(X, Z)$ then $V \leq W$ in $\mathbf{Y}_{\boldsymbol{\xi}}(X, Z)$ since W is a guard. If $W' < V$ then $V \in \mathbf{Y}_{K/2}(W', Z) \subseteq \mathbf{Y}_{K/2}(X_1, Z)$ by Lemma 2.5 and monotonicity and therefore $V \neq W$ since $W \notin \mathbf{Y}_{K/2}(X_1, Z)$. However, this contradicts our choice of W' as the greatest element of $\mathbf{Y}_{K/2}(X_1, Z)$ that is less than W. So, $V \leq W'$. ☐

A *barrier* between a path $\{X_0, \ldots, X_k\}$ and a vertex Z is a vertex Y such that $Y \in \mathbf{Y}_{\boldsymbol{\xi}}(X_i, Z)$ for all $i = 0, \ldots, k$. By Theorem 2.3 if there is a barrier between $\{X_0, \ldots, X_k\}$ and Z then $d_Z(X_i, X_j) < \boldsymbol{\xi}$ for all i, j.

Proposition 2.8. *Let $\{X_0, X_1, \cdots, X_k\}$ be a path in $\mathcal{P}_K(\mathbf{Y})$ and Z a vertex of $\mathcal{P}_K(\mathbf{Y})$ such that $d(Z, X_i) \geq 3$ for all i. Then there is a barrier W between the path and Z. In particular, $d_Z(X_0, X_i) \sim 0$ for all i.*

Proof. We will inductively choose a family of guards W_i for Z such that $W_i \in \mathbf{Y}_{K/2}(X_i, Z)$ and if $i > j$ then either $W_i = W_j$ or $W_j \in \mathbf{Y}_{\boldsymbol{\xi}}(W_i, Z)$.

We choose W_0 to be the greatest element of $\mathbf{Y}_{K/2}(X_0, Z)$, so in particular $\mathbf{Y}_{K/2}(W_0, Z) = \emptyset$ by the order and monotonicity properties. By Lemma 2.5, W_0 is a guard for Z. Now assume that W_0 through W_i have been chosen. If $W_i \in \mathbf{Y}_{K/2}(X_{i+1}, Z)$ then we let $W_{i+1} = W_i$. If not, by Lemma 2.7, there exists a guard W_{i+1} in $\mathbf{Y}_{K/2}(X_{i+1}, Z)$ with $W_i \in \mathbf{Y}_{\boldsymbol{\xi}}(W_{i+1}, Z)$. For any $j < i$, by the induction hypothesis, we have that $W_j \in \mathbf{Y}_{\boldsymbol{\xi}}(W_i, Z)$ and by monotonicity therefore $W_j \in \mathbf{Y}_{\boldsymbol{\xi}}(W_{i+1}, Z)$.

Let $W = W_0$. Again applying monotonicity we have that $\mathbf{Y}_{\boldsymbol{\xi}}(X_i, Z) \supseteq \mathbf{Y}_{\boldsymbol{\xi}}(W_i, Z)$, therefore $W \in \mathbf{Y}_{\boldsymbol{\xi}}(X_i, Z)$, so that W is a barrier between the path and Z and that $d_Z(X_0, X_i) < \boldsymbol{\xi}$. ☐

2.4 $\mathcal{P}_K(\mathbf{Y})$ is a quasi-tree

Recall [Man05] that a geodesic metric space $\mathcal{X}$ satisfies the *bottleneck property* if there is a constant $\Delta \geq 0$ such that for any two points $x, z \in \mathcal{X}$ there is a midpoint y (i.e. $d(x, y) = d(y, z) = \frac{1}{2}d(x, z)$) such that any path from x to z intersects the Δ-neighborhood of y. Manning proved in [Man05] that $\mathcal{X}$ is quasi-isometric to a simplicial tree (i.e. it is a *quasi-tree*) if and only if it satisfies the bottleneck property.

There is a slight reformulation of the bottleneck property that is easier to deal with: *$\mathcal{X}$ has the bottleneck property if and only if there is a constant Δ' such that for any two points $x, y \in \mathcal{X}$ there is a path p from x to y such that the Δ'-neighborhood of any other path from x to y contains p.*

We prove this property implies the original property. Let g be a geodesic from x to y, and m be the mid point. We claim that there is a point m' in p which is in the $(2\Delta' + 1)$-neighborhood of m. To see this, let $p(i)$ be points on p from x to y with $d(p(i), p(i + 1)) \leq 1$. By the property, for each i, there is a point $g(j_i)$ on g with $d(p(i), g((j_i)) \leq \Delta'$. By triangle inequality, $d(g(j_i), g(j_{i+1})) \leq 2\Delta' + 1$. Since g is a geodesic, there must be i such that $d(m, g(j_i)) \leq 2\Delta' + 1$. Set $m' = p(i)$. Then, $d(m, m') \leq 3\Delta' + 1$

Now for any path q from x to y, there must be a point m'' in q such that $d(m', m'')$ is at most Δ'. So, $d(m, m'')$ is at most $4\Delta' + 1$.

If the space is a graph, we only need to consider vertices rather than all points in the conditions and arguments.

Theorem 2.9. *For K sufficiently large $\mathcal{P}_K(\mathbf{Y})$ is a quasi-tree.*

Proof. We will verify the modified bottleneck property with $\Delta' = 2$. Let X, Z be two vertices of $\mathcal{P}_K(\mathbf{Y})$. The ordered set $\mathbf{Y}_K(X,Z)$ is a path from X to Z (see the proof of Lemma 2.4). We now check that any path $X = X_0, X_1, \cdots, X_k = Z$ from X to Z passes within 2 of any vertex Y in $\mathbf{Y}_K(X,Z)$. If not, then by Proposition 2.8 we have $d_Y(X,Z) < \boldsymbol{\xi}$ contradicting the fact that $Y \in \mathbf{Y}_K(X,Z)$. □

Question 2.10. *If we use the original distance d^π in the definition of the projection complex $\mathcal{P}_K(\mathbf{Y})$ instead of the modified distance d, would the space still be a quasi-tree?*

Lemma 2.11. *There exists a K' such that if $Y \in \mathbf{Y}_{K'}(X,Z)$ then every geodesic from X to Z in $\mathcal{P}_K(\mathbf{Y})$ contains Y. In particular*

$$d(X,Z) \geq |\mathbf{Y}_{K'}(X,Z)| + 1.$$

Proof. Let $X = X_0, X_1, \dots, X_k = Z$ be a geodesic from X to Z that doesn't contain Y. We will show that $d_Y(X,Z) \prec 5K$.

If $d(X_i, Y) \geq 3$ for all i then by Proposition 2.8 we have $d_Y(X,Z) \sim 0$. Now assume that $d(X_i,Y) < 3$ for some i. Let i^- be the first time that $d(X_{i^-}, Y) < 3$ and i^+ the last time that $d(X_{i^+}, Y) < 3$. Then $i^+ - i^- \leq 4$ since $d(X_{i^-}, X_{i^+}) \leq 4$. For convenience we will assume $i^- > 0$ and $i^+ < k$; an obvious modification of the argument works when this is not the case. Again applying Proposition 2.8 we have that $d_Y(X, X_{i^- -1}) \sim 0$ and $d_Y(X_{i^+ +1}, Z) \sim 0$.

Since the path doesn't contain Y then for all X_i we have $d_Y(X_i, X_{i+1}) \leq K$. Using this estimate and the coarse triangle inequality six times we have

$$d_Y(X_{i^- -1}, X_{i^+ +1}) \prec 5K.$$

Combining with our bounds on $d_Y(X, X_{i^- -1})$ and $d_Y(X, X_{i^+ +1})$ and applying the coarse triangle inequality two more times we have $d_Y(X,Z) \prec 5K$. Therefore there exists a K' with $K' \sim 5K$ such that if $Y \in \mathbf{Y}_{K'}(X,Z)$ then every geodesic from X to Z contains Y. This implies the lemma. □

2.5 Group action on the projection complex

Now assume that G is a group that acts on the set $\mathbf{Y}$ in such a way that projection distances are preserved, i.e. $d^\pi_{g(A)}(g(B), g(C)) = d^\pi_A(B,C)$ for all $A, B, C \in \mathbf{Y}$ and $g \in G$. Then G acts naturally on the projection complex $\mathcal{P}_K(\mathbf{Y})$.

Let K' be the constant from Lemma 2.11.

Proposition 2.12. *(i) Suppose the action of G on $\mathbf{Y}$ has finitely many orbits. Then the action of G on $\mathcal{P}_K(\mathbf{Y})$ is cobounded (i.e. a Hausdorff neighborhood of an orbit is the whole space).*

(ii) Suppose that for every $R > 0$ and $A \in \mathbf{Y}$ there exist $B, C \in \mathbf{Y}$ such that $d^{\pi}_A(B, C) > R$. Then the diameter of $\mathcal{P}_K(\mathbf{Y})$ is infinite.

Proof. (i) is clear. For (ii), choose $A_0, A_1, A_2 \in \mathbf{Y}$ such that $d_{A_1}(A_0, A_2) > K'$. Applying the assumption to A_2, find B, C so that $d_{A_2}(B, C) \succ 3K'$. It follows that for either $A_3 = B$ or $A_3 = C$ we have $d_{A_2}(A_1, A_3) > K'$. Continuing in the same fashion, we can extend the sequence A_i forever with $d_{A_i}(A_{i-1}, A_{i+1}) > K'$. Thus by Lemma 2.11 $d_{\mathcal{P}_K(\mathbf{Y})}(A_0, A_i) \geq i$. □

Lemma 2.13. *Suppose $g \in G$ and $Y \in \mathbf{Y}$ such that*

$$d_Y(g^{-N}(Y), g^N(Y)) > K'$$

for some $N > 0$. Then g acts on $\mathcal{P}_K(\mathbf{Y})$ with positive translation length.

Proof. By equivariance and monotonicity $d_{g^{iN}(Y)}(Y, g^{kN}(Y)) > K'$ for $0 < i < k$ so that Lemma 2.11 gives

$$d_{\mathcal{P}_K(\mathbf{Y})}(Y, g^{kN}(Y)) \geq (k-1)K'$$

which implies that the translation length

$$\tau(g) := \lim_{k \to \infty} \frac{d_{\mathcal{P}_K(\mathbf{Y})}(Y, g^{kN}(Y))}{kN} \geq \frac{K'}{N} > 0$$

□

For $g \in G$ define the *combinatorial axis* as

$$\mathbf{Y}_{K'}(g) = \{Y \in \mathbf{Y} | d_Y(g^{-N}(Y), g^N(Y)) > K' \text{ for some } N > 0\}$$

By Lemma 2.13 this is a quasi-geodesic and g is a hyperbolic isometry.

Proposition 2.14. *(1) There is a total order on $\mathbf{Y}_{K'}(g)$ such that*

- *$Y_1 \in \mathbf{Y}_K(Y_0, Y_2)$ if Y_1 is between Y_0 and Y_2.*
- *If Y_1 is not between Y_0 and Y_2 then $Y_2 \notin \mathbf{Y}_\xi(Y_0, Y_1)$ and $Y_0 \notin \mathbf{Y}_\xi(Y_1, Y_2)$.*

(2) $\mathbf{Y}_{K'}(g)$ is g-invariant and g acts by preserving the order.

(3) The order is unique if we require $g(Y) > Y$ for some (every) $Y \in \mathbf{Y}_{K'}(g)$.

(4) $\mathbf{Y}_{K'}(g)$, if nonempty, is order-isomorphic to $\mathbb{Z}$ and g acts as a translation on $\mathbf{Y}_{K'}(g)$.

Proof. The proof is similar to the proof of Theorem 2.3(G). $\square$

We consider the following axioms on $g \in G$:

(i) g is contained in a unique maximal virtually cyclic subgroup $EC(g)$, the *elementary closure* of g.

(ii) $EC(g)$ is malnormal, i.e. $\psi EC(g)\psi^{-1} = EC(g)$ for $\psi \in G$ implies $\psi \in EC(g)$.

(iii) There is $Y \in \mathbf{Y}_{K'}(g)$ and $m > 0$ such that if $\psi \in G$ fixes $Y, g(Y), \cdots, g^m(Y)$ then $\psi \in EC(g)$.

Note that (iii) implies that g satisfies the assumptions of Lemma 2.13 and so in particular it is hyperbolic.

Assume G is acting on a geodesic metric space X. We say that g is a *WPD element* if $\langle g\rangle$ has an unbouded orbit in X and for every $x \in X$ and $D > 0$ there is $M > 0$ such that

$$\{\phi \in G | d(\phi(g^i(x)), g^i(x)) \leq D, i = \pm M\}$$

is finite.

It suffices to check this property for any fixed x.

Remark 2.15. When X is δ-hyperbolic, $g \in G$ is a WPD element if and only if it is hyperbolic and for some (any) $x \in X$ there is $B > 0$ such that any $\phi \in G - EC(g)$ maps the orbit $< g > x$ to a set whose projection to $< g > x$ has diameter $\leq B$. In this setting the orbit is a quasi-geodesic and the projection is the nearest point projection, coarsely defined. Thus WPD is equivalent to saying that the set of translates of a g-orbit is "discrete" in the sense that any two are either parallel or have bounded "overlap", with parallel orbits coming from translating by elements in $EC(g)$.

Proposition 2.16. *Suppose $g \in G$ satisfies (i)-(iii). Then g is a WPD element with respect to the action of G on $\mathcal{P}_K(\mathbf{Y})$.*

Proof. For a given D we will find M so that

$$\{\phi \in G | d(\phi(g^i(Y)), g^i(Y)) \leq D, i = \pm M\}$$

is finite. Since $\mathbf{Y}_{K'}(g)$ is a quasi-geodesic there exists D' such that for any $M > 0$ if we have $d(\phi(g^i(Y)), g^i(Y)) \leq D, i = \pm M$, then we have $d(\phi(g^i(Y)), g^i(Y)) \leq D'$ for all $-M \leq i \leq M$. For simplicity, we pretend $D = D'$ in the following. Denote by p the maximal number of elements in $\mathbf{Y}_{K'}(g)$ that are within $4D$ of each other (the distance is measured in $\mathcal{P}_K(\mathbf{Y})$ – this number is finite by Proposition 2.14). Let $N > 0$ such that $d_Y(g^{-N}(Y), g^N(Y)) > K'$. Then take $M = p^{m+1} + 1 + N + m + D$.

Now suppose $\phi \in G$ moves each $g^i(Y)$, $-M \leq i \leq M$, distance $\leq D$ (by the discussion in the beginning). Let $h = \phi g \phi^{-1}$. Then $\phi : \mathbf{Y}_{K'}(g) \to \mathbf{Y}_{K'}(h)$ is an order preserving bijection. Note that $g^i(Y) \in \mathbf{Y}_{K'}(h)$ for $0 \leq i \leq p^{m+1} + 1 + m$ since $d_{g^i(Y)}(h^{-N} g^i(Y), h^N g^i(Y)) > K'$ by monotonicity.

Now for each $r = 0, 1, \cdots, m$ the elements $h^{-i} g^i(g^r(Y)) \in \mathbf{Y}_{K'}(h)$, $i = 0, \cdots, p^{m+1} + 1$ are within $2D$ of $g^r(Y)$ and hence within $4D$ of each other. By the choice of p and the pigeon-hole principle it follows that there are $0 \leq i < j \leq p^{m+1} + 1$ such that $h^{-i} g^i(g^r(Y)) = h^{-j} g^j(g^r(Y))$ for each r. Therefore by (iii) we have $g^{-i} h^{i-j} g^j \in EC(g)$, i.e. $h^{i-j} \in EC(g)$. It follows that $EC(h) = EC(g)$ and by (ii) $\phi \in EC(g)$. By (i) $EC(g)$ is virtually cyclic. The subgroup generated by g has finite index, so all but finitely many elements of $EC(g)$ will move Y a distance $> D$. □

Recall [Bow08] that an action of a group G on a graph X is *acylindrical* if for every $D \geq 0$ there exist constants R, C such that whenever x, y are two vertices of X with $d(x, y) \geq M$ then the set

$$\{g \in G \mid d(x, g(x)) \leq D, d(y, g(y)) \leq D\}$$

has cardinality $\leq C$.

Question: Under what conditions is the action of G on $\mathcal{P}_K(\mathbf{Y})$ acylindrical?

In the proof of Proposition 2.16 the size C of the finite set depends on the element g, while for acylindricity it has to be made uniform.

We say that two elements $g_1, g_2 \in G$ satisfying (i)-(iii) are *independent* if $EC(g_1) \cap EC(g_2)$ is finite. To summarize, by a standard Ping-Pong argument we have:

Theorem 2.17. *Suppose a group G acts on a set $\mathbf{Y}$ satisfying our axioms 1-4 such that projection distances are preserved. Therefore, G acts on the quasi-tree $\mathcal{P}_K(\mathbf{Y})$. Further, assume that there exist independent elements*

$g_1, g_2 \in G$ that satisfy (i)-(iii). Then there is a nonabelian free subgroup $F \subset G$ all of whose nontrivial elements act on $\mathcal{P}_K(\mathbf{Y})$ hyperbolically as WPD elements.

For example, if G is a nonelementary word hyperbolic group, the above construction gives many unbounded actions of G on quasi-trees. Recall that many hyperbolic groups have property (T) and don't admit nontrivial actions on trees.

3 A quasi-tree of metric spaces

3.1 Axiom and construction

In our list of examples the set $\mathbf{Y}$ and the functions d_Y^π all arose from geometric settings. We now formalize this. For each $Y \in \mathbf{Y}$ let $\mathcal{C}(Y)$ be a geodesic metric space, and let π_Y be a function, called *projection*, from $\mathbf{Y}\backslash\{Y\}$ to subsets of $\mathcal{C}(Y)$. We then define π_Y on $x \in X \neq Y$ by $\pi_Y(x) = \pi_Y(X)$. On Y itself we define π_Y to be the identity map. (Strictly speaking π_Y takes points in Y to singleton subsets of Y.) We now add another axiom:

(0) $\operatorname{diam}(\pi_Y(X)) < \xi$;

We then define

$$d_Y^\pi(X, Z) = \operatorname{diam}\{\pi_Y(X) \cup \pi_Y(Z)\}.$$

Axioms (1) and (2) are clear and we assume that axioms (3) and (4) hold.

Note that the examples that were discussed at the start of the paper all arise in this way. We also define $d_Y^\pi(x, z) = \operatorname{diam}\{\pi_Y(x) \cup \pi_Y(b)\}$, and similarly for $d_Y^\pi(x, Z)$. Note that $d_Y^\pi(x, z)$ still makes sense if $x \in Y$ and/or $z \in Y$ as does $d_Y^\pi(x, Z)$ if $x \in Y$.

We define $d_Y(X, Z)$ exactly as before and if neither $x \in Y$ nor $z \in Y$ then we set $d_Y(x, z) = d_Y(X, Z)$. If either $x \in Y$ or $z \in Y$ then $d_Y(x, z) = d_Y^\pi(x, z)$ and $x \in Y$ then $d_Y(x, Z) = d_Y^\pi(x, Z)$. In these last two cases we don't have the monotonicity lemma and in fact the lemma doesn't even make sense. Finally we define $\mathbf{Y}_K(x, z)$ to be the set of Y such that $d_Y(x, z) > K$. These sets are almost the same as $\mathbf{Y}_K(X, Z)$ although they may possibly contain X or Z. We similarly define $\mathbf{Y}_K(x, Z)$.

We construct a path metric space $\mathcal{C}(\mathbf{Y})$ by taking the disjoint union of the metric spaces $\mathcal{C}(Y)$ for $Y \in \mathbf{Y}$ and if $d(X, Z) = 1$ in $\mathcal{P}_K(\mathbf{Y})$ we attach an edge of length L from every point in $\pi_X(Z)$ to every point in $\pi_Z(X)$. For any two choices of L the corresponding complexes will be quasi-isometric; however, by choosing L to be a function of K we can assure that L is sufficiently large so that the metrics spaces $\mathcal{C}(Y)$ will be totally geodesically embedded in $\mathcal{C}(\mathbf{Y})$ but that L will still be comparable to K. This will streamline some of our proofs.

We will prove this in the following lemma. Note that in this lemma we use the unmodified projection functions, d^{π}_{Y} as we will need to apply the triangle inequality indeterminate number of times. To simplify notation we will restrict the discussion to the case when each $\mathcal{C}(Y)$ is a connected graph endowed with length metric with each edge of length 1. The general case is an easy modification, or indeed, one may replace $\mathcal{C}(Y)$ by the Vietoris-Rips complex whose vertices are the points of $\mathcal{C}(Y)$, and edges correspond to pairs of points at distance ≤ 1.

Lemma 3.1. *There exists an $L = L(K)$ with $L \sim K$ such that*

$$d_{\mathcal{C}(\mathbf{Y})}(x, z) \geq d^{\pi}_{Y}(x, z)$$

for all $Y \in \mathcal{C}(\mathbf{Y})$ with equality if and only if both x and z are in Y. In particular each $\mathcal{C}(Y)$ is totally geodesically embedded in $\mathcal{C}(\mathbf{Y})$.

Proof. Let $\mathcal{C}'(\mathbf{Y})$ be the space obtained by collapsing $\mathcal{C}(Z)$ for every $Z \in \mathbf{Y}\backslash\{Y\}$. Let $x_0, x_1, \ldots, x_k$ be a shortest path of adjacent vertices between the images of x and z in $\mathcal{C}'(\mathbf{Y})$. Thus each x_i is either a vertex in $\mathcal{C}(Y)$ or it is some $Z \in \mathbf{Y} \setminus \{Y\}$.

We'll show that $d^{\pi}_{Y}(x_i, x_{i+1}) \leq d_{\mathcal{C}'(\mathbf{Y})}(x_i, x_{i+1})$ with equality if and only if both x_i and x_{i+1} are in $\mathcal{C}(Y)$. There are three cases. If neither x_i or x_{i+1} are in $\mathcal{C}(Y)$ then by the coarse equality

$$d^{\pi}_{Y}(x_i, x_{i+1}) \prec d_Y(x_i, x_{i+1}) < K$$

and

$$d_{\mathcal{C}'(\mathbf{Y})}(x_i, x_{i+1}) = L$$

so

$$d^{\pi}_{Y}(x_i, x_{i+1}) < d_{\mathcal{C}'(\mathbf{Y})}(x_i, x_{i+1})$$

if L is sufficiently large but $L \sim K$. If x_i and x_{i+1} are both in $\mathcal{C}(Y)$ then $d_{\mathcal{C}'(\mathbf{Y})}(x_i, x_{i+1}) = d^{\pi}_{Y}(x_i, x_{i+1}) = 1$. If exactly one of the two is in $\mathcal{C}(Y)$ we have $d^{\pi}_{Y}(x_i, x_{i+1}) \sim 0$ and $d_{\mathcal{C}'(\mathbf{Y})}(x_i, x_{i+1}) = L$ so $d^{\pi}_{Y}(x_i, x_{i+1}) < d_{\mathcal{C}'(\mathbf{Y})}(x_i, x_{i+1})$ for sufficiently large L. Again L can be chosen such that $L \sim K$.

The triangle inequality then shows that

$$d_{\mathcal{C}'(\mathbf{Y})}(x_0, x_k) \geq d^{\pi}_{Y}(x_0, x_k) = d^{\pi}_{Y}(x, z)$$

with equality if and only if all of the x_i are in $\mathcal{C}(Y)$. Since the projection to $\mathcal{C}'(\mathbf{Y})$ is 1-Lipschitz we have

$$d_{\mathcal{C}(\mathbf{Y})}(x,z) \geq d^{\pi}_Y(x,z)$$

with equality if and only if x and z are in $\mathcal{C}(Y)$.

To see that $\mathcal{C}(Y)$ is totally geodesically embedded in $\mathcal{C}(\mathbf{Y})$ we observe that d^{π}_Y is the metric on $\mathcal{C}(Y)$ and we have just shown that if x and z are in $\mathcal{C}(Y)$, any path in $\mathcal{C}(\mathbf{Y})$ that leaves $\mathcal{C}(Y)$ has length strictly longer than $d^{\pi}_Y(x,z)$. Therefore every geodesic from x to z is contained in $\mathcal{C}(Y)$. □

3.2 Distance estimate in $\mathcal{C}(\mathbf{Y})$

This section will not be needed for our main result that asdim$(MCG(\Sigma)) < \infty$, but rather for Theorems 3.12 and 3.10 that have independent applications. We start by writing down a straightforward estimate for an upper bound for the distance in $\mathcal{C}(\mathbf{Y})$. This is obtained by constructing a "standard path" joining two points and computing its length.

Definition 3.2. A *standard path* from $x \in \mathcal{C}(X)$ to $z \in \mathcal{C}(Z)$ is any path that passes through $\mathcal{C}(W)$ if and only if $W \in \mathbf{Y}_K(X,Z) \cup \{X,Z\}$, it passes through them in the natural order, and within each $\mathcal{C}(W)$ the path is a geodesic.

Lemma 3.3. *For K sufficiently large*

$$d_{\mathcal{C}(\mathbf{Y})}(x,z) \leq 6K + 4 \sum_{Y \in \mathbf{Y}_K(x,z)} d_Y(x,z)$$

for all $x, z \in \mathcal{C}(\mathbf{Y})$, and moreover the length of any standard path from x to z is bounded above by the same expression.

Proof. Let X and Z be the vertices in $\mathbf{Y}$ with $x \in \mathcal{C}(X)$ and $z \in \mathcal{C}(Z)$. Let $\mathbf{Y}_K(X,Z) \cup \{X,Z\} = \{X = Y_0, Y_1, \ldots, Y_k = Z\}$ with labeling respecting the order. Let x_i^+ be a point in $\pi_{Y_i}(Y_{i+1})$ and x_i^- a point in $\pi_{Y_i}(Y_{i-1})$, where defined. At the endpoints let $x_0^- = x$ and $x_k^+ = z$. Since the distance between x_i^+ and x_{i+1}^- is L we have

$$d_{\mathcal{C}(\mathbf{Y})}(x,z) \leq kL + \sum d_{\mathcal{C}(Y)}(x_i^-, x_i^+).$$

Now we estimate $d_{\mathcal{C}(\mathbf{Y})}(x_i^-, x_i^+)$. For $i \in \{1, \ldots, k-1\}$ we have

$$\begin{aligned} d_{\mathcal{C}(\mathbf{Y})}(x_i^-, x_i^+) &\leq d^{\pi}_{Y_i}(Y_{i-1}, Y_{i+1}) \\ &\prec d_{Y_i}(Y_{i-1}, Y_{i+1}) \\ &\prec d_{Y_i}(x,z) \end{aligned}$$

where the second line follows from the coarse equality property and the third follows from the order property. Since $d_{Y_i}(x,z) > K$ this implies that

$$d_{\mathcal{C}(\mathbf{Y})}(x_i^-, x_i^+) < 2d_{Y_i}(x,z)$$

for K sufficiently large.

Since $L = L(K) \sim K$ we also have that $L < 2K$ if K is sufficiently large and since $d_{Y_i}(x,z) > K$ we have $L < 2d_{Y_i}(x,z)$ and

$$L + d_{\mathcal{C}(\mathbf{Y})}(x_i^-, x_i^+) \leq 4d_{Y_i}(x,z).$$

We similarly have that $d_{C(\mathbf{Y})}(x_i^-, x_i^+) \prec d_{Y_i}(x,z)$ when $i = 0, k$.

If $d_{Y_i}(x,z) > K$ we then have $d_{\mathcal{C}(\mathbf{Y})}(x_i^-, x_i^+) < 2d_{Y_i}(x,z)$ while if $d_{Y_i}(x,z) \leq K$ then $d_{\mathcal{C}(\mathbf{Y})}(x_i^-, x_i^+) < 2K$. We can write this as a single inequality

$$d_{\mathcal{C}(\mathbf{Y})}(x_i^-, x_i^+) < 2\max\{K, d_{Y_i}(x,z)\}$$

that applies to both cases. Now

$$\begin{aligned}
d_{\mathcal{C}(\mathbf{Y})}(x,z) &\leq kL + \sum d_{\mathcal{C}(Y)}(x_i^-, x_i^+) \\
&\leq L + 4\sum_{i=1}^{k-1} d_{Y_i}(x,z) + 2\sum_{i=0,k} \max\{K, d_{Y_i}(x,z)\} \\
&\leq 6K + 4\sum_{Y\in\mathbf{Y}_K(x,z)} d_Y(x,z)
\end{aligned}$$

□

We aim to find a lower bound in the spirit of Lemma 2.11 for the projection complex $\mathcal{P}_K(\mathbf{Y})$. See Theorem 3.12. We will need a version of Proposition 2.8 for $\mathcal{C}(\mathbf{Y})$. The proof will be a word for word repeat of Proposition 2.8 but first we need a new version of Lemma 2.6.

Lemma 3.4. *Let X_0 and X_1 be vertices in $\mathcal{P}_K(\mathbf{Y})$ with $d(X_0, X_1) = 1$ and let x_0 and x_1 be points in $\mathcal{C}(X_0)$ and $\mathcal{C}(X_1)$ such that $x_0 \in \pi_{X_0}(X_1)$ and $x_1 \in \pi_{X_1}(X_0)$. Let W be a vertex in $\mathbf{Y}$ and w a point in $\mathcal{C}(W)$ with $d_{\mathcal{C}(\mathbf{Y})}(x_i, w) \geq 2L$. Then either*

$$d_W(x_0, x_1) \sim 0$$

or

$$d_W(x_i, w) \succ L \text{ for } i = 0, 1.$$

Proof. First assume $X_0 = W$. Since $x_0 \in \pi_W(x_1) = \pi_{X_0}(x_1)$ we have $d^\pi_W(x_0, x_1) \le \operatorname{diam}(\pi_W(X_1)) \sim 0$. Of course, we get the same bound if $X_1 = W$.

If either $d(X_0, W) \ge 2$ or $d(X_1, W) \ge 2$ then $d^\pi_W(x_0, x_1) = d^\pi_W(X_0, X_1) \sim 0$ by Lemma 2.6.

This leaves us with the case where $d(X_0, W) = d(X_1, W) = 1$. We first observe that if $d_{X_0}(X_1, W) > \boldsymbol{\xi}$ then $d_W(x_0, x_1) = d_W(X_0, X_1) \sim 0$. The same estimate holds if $d_{X_1}(X_0, W) > \boldsymbol{\xi}$.

The final sub-case is when both $d_{X_0}(X_1, W) \le \boldsymbol{\xi}$ and $d^\pi_{X_1}(X_0, W) \le \boldsymbol{\xi}$. It is here that we use the lower bound $d_{\mathcal{C}(\mathbf{Y})}(x_i, w) \ge 2L$. To do so we need the upper bound

$$d_{\mathcal{C}(\mathbf{Y})}(x_0, w) \le d_{X_0}(x_0, w) + L + d_W(x_0, w)$$

which is obtained by taking the path made up of a path in $\mathcal{C}(X_0)$ connecting x_0 to $\pi_{X_0}(w)$, an edge from $\pi_{X_0}(W)$ to $\pi_W(X_0)$ and a path in $\mathcal{C}(W)$ from $\pi_W(X_0)$ to w. Since $x_0 \in \pi_{X_0}(X_1)$ we have $d_{X_0}(x_0, w) \prec d_{X_0}(X_1, W)$. Combining the bounds gives $d_W(x_0, w) \succ L$ and the same bound holds for $d_W(x_1, w)$. □

Lemma 3.5. *For K sufficiently large the following holds. Let x_0 and x_1 be adjacent vertices in $\mathcal{C}(\mathbf{Y})$ and let Y be a vertex in $\mathcal{P}_K(\mathbf{Y})$ with $d_{\mathcal{C}(\mathbf{Y})}(x_i, \mathcal{C}(Y)) \ge 3L$. If W is a guard for Y with $W \in \mathbf{Y}_{K/2}(x_0, Y)$ and $W \notin \mathbf{Y}_{K/2}(x_1, Y)$ then there exists a guard W' for Y with $W' \in \mathbf{Y}_{K/2}(x_1, Y)$ and $W \in \mathbf{Y}_{\boldsymbol{\xi}}(W', Y)$.*

Proof. Let X_0 and X_1 be the vertices of $\mathcal{P}_K(\mathbf{Y})$ such that $x_i \in \mathcal{C}(X_i)$. If $X_0 = X_1 \ne W$ then $W \in \mathbf{Y}_{K/2}(x_1, Y)$ and the lemma is vacuous. If $X_0 = X_1 = W$ then

$$\begin{aligned} 3L &\le d_{\mathcal{C}(\mathbf{Y})}(x_i, \mathcal{C}(Y)) \\ &\le d_{\mathcal{C}(\mathbf{Y})}(x_i, \pi_Y(W)) \\ &\le d_W(x_i, Y) + L \end{aligned}$$

and therefore $d^\pi_W(x_i, \pi_W(Y)) \ge 2L$. Since $L \sim K$ if K is sufficiently large then $2L \ge K$ and $W \in \mathbf{Y}_{K/2}(x_1, Y)$, therefore the lemma is vacuous as well.

We now assume that $X_0 \ne X_1$. We can now apply Lemma 3.4 with w a point in $\pi_W(Y)$. Note that $d_{\mathcal{C}(\mathbf{Y})}(w, \mathcal{C}(Y)) = L$ so $d_{\mathcal{C}(\mathbf{Y})}(x_i, w) \ge 2L$.

Lemma 3.4 gives us two possibilities. First we may have $d_W(x_1, w) \succ L \succ K$ in which case $W \in \mathbf{Y}_{K/2}(x_1, Y)$ for K sufficiently large.

Therefore if $W \notin \mathbf{Y}_{K/2}(x_1, Y)$ then Lemma 3.4 gives $d_W(x_0, x_1) \sim 0$. For K sufficiently large the coarse triangle inequality then implies that $W \in$

$\mathbf{Y}_{\boldsymbol{\xi}}(x_1, Y)$ as $W \in \mathbf{Y}_{K/2}(x_0, Y)$. Since W is a guard for Y every vertex in $\mathbf{Y}_K(x_1, Y)$ must be less than W in $\mathbf{Y}_{\boldsymbol{\xi}}(x_1, Y)$. Furthermore $\mathbf{Y}_K(x_1, Y)$ can't be empty for if it was then, as above, $d(x_1, \mathcal{C}(Y)) \leq d_{X_1}(x_1, Y) + L \leq K + L < 3L$ if K is sufficiently large. Therefore there must be elements ($\neq W$, could be $= X_1$) of $\mathbf{Y}_K(x_1, Y)$ that are less than W in $\mathbf{Y}_{\boldsymbol{\xi}}(x_1, Y)$. The rest of the proof now is a repeat of the proof of Lemma 2.7. Namely, we take W' to be the greatest element of $\mathbf{Y}_{K/2}(x_1, Y)$ that is less than W in $\mathbf{Y}_{\boldsymbol{\xi}}(x_1, Y)$. The proof that $W \in \mathbf{Y}_{\boldsymbol{\xi}}(W', Y)$ and that W' is a guard is exactly as in the proof of Lemma 2.7. □

We define the notion of a *barrier* for a path in $\mathcal{C}(\mathbf{Y})$ just as we did for paths in $\mathcal{P}_K(\mathbf{Y})$. Namely, if $\{x_0, x_1, \ldots, x_k\}$ is a path in $\mathcal{C}(\mathbf{Y})$ and Z a vertex in $\mathcal{P}_K(\mathbf{Y})$ then $Y \in \mathbf{Y}$ is a barrier between them if $Y \in \mathbf{Y}_{\boldsymbol{\xi}}(x_i, Z)$ for $i = 0, \ldots, k$. Note that it is possible that $x_i \in \mathcal{C}(Y)$. If neither x_i nor x_j are in $\mathcal{C}(Y)$ then Theorem 2.3 implies that $d_Z(x_i, x_j) < \boldsymbol{\xi}$. If exactly one of the two is in $\mathcal{C}(Y)$ then $d_Z(x_i, x_j) < \boldsymbol{\xi}$ from the inequality on triples. If they are both in $\mathcal{C}(Y)$ then $d_Z(x_i, x_j) = \pi_Z(Y) < \boldsymbol{\xi}$ by Axiom 0.

Proposition 3.6. *Let $\{x_0, x_1, \ldots, x_k\}$ be a path in $\mathcal{C}(\mathbf{Y})$ and Z a vertex in $\mathbf{Y}$ such that $d_{\mathcal{C}(\mathbf{Y})}(x_i, \mathcal{C}(Z)) \geq 3L$ for all i. Then there is a barrier C in $\mathbf{Y}$ between the path and Z. In particular, $d_Z(x_0, x_i) < \boldsymbol{\xi}$.*

Proof. The proof is a word for word repeat of the proof of Proposition 2.8 with Lemma 2.7 replaced with Lemma 3.5 and the upper case X_i replaced with the lower case x_i. □

Remark 3.7. It is not hard to derive Proposition 2.8 from Proposition 3.6. In particular a path in $\mathcal{P}_K(\mathbf{Y})$ that is 3 or more away from a vertex Z can be lifted to path in $\mathcal{C}(\mathbf{Y})$ that is $3L$ away from $\mathcal{C}(Z)$.

Lemma 3.8. *Let x be a vertex in $\mathcal{C}(\mathbf{Y})$, Z a vertex in $\mathcal{P}_K(\mathbf{Y})$ and z a nearest point in $\mathcal{C}(Z)$ to x in $\mathcal{C}(\mathbf{Y})$. Then*

$$d_Z(x, z) \prec 2K.$$

Proof. Let y be the last point in a geodesic from x to z such that $d_{\mathcal{C}(\mathbf{Y})}(z, y) = d_{\mathcal{C}(\mathbf{Y})}(y, \mathcal{C}(Z)) \geq 3L$. Then by Proposition 3.6, $d_Z(x, y) \sim 0$. The case that such y does not exist, i.e., $d_{\mathcal{C}(\mathbf{Y})}(z, x) < 3L$, will be discussed at the end.

If a path in $\mathcal{C}(\mathbf{Y})$ of length at most $kL - 1$ maps to a path in $\mathcal{P}_K(\mathbf{Y})$ then the image path will have length at most $k - 1$. By the way we chose y, $d_{\mathcal{C}(\mathbf{Y})}(z, y) \leq 4L - 1$. Therefore the geodesic from y to z will map to a path of length at most 3 (and at least 1) in $\mathcal{P}_K(\mathbf{Y})$. Let Y and Z'

be the vertices of $\mathcal{P}_K(\mathbf{Y})$ such that $y \in \mathcal{C}(Y)$ and Z' is the last vertex in the path before Z. Since $d(Y, Z') \leq 2$, the coarse triangle inequality implies that $d_Z(Y, Z') \prec 2K$. (We are assuming $Z \neq Z'$ here, but the case $Z = Z'$ is similar and left to the reader.) Since Z' is the last vertex before Z we also have that $z \in \pi_Z(Z')$ and therefore $d_Z(z, y) \prec 2K$. Since $d_Z(x, y) \sim 0$, another application of the coarse triangle inequality then gives $d_Z(x, z) \prec 2K$ as claimed.

Now we are left with the case $d_{\mathcal{C}(\mathbf{Y})}(z, x) < 3L$. If $x \in \mathcal{C}(Z)$, then $z = x$ and there is nothing to prove. Otherwise, letting $y = x$ in the above discussion, we have $d_Z(z, x) \prec 2K$. □

Lemma 3.9. *Let $X, Z \in \mathbf{Y}$, $x \in \mathcal{C}(X)$, $z \in \mathcal{C}(Z)$. If $Y \in \mathbf{Y}_{\mathfrak{E}}(x, z)$ then any path from x to z in $\mathcal{C}(\mathbf{Y})$ contains a vertex w such that*

- $d_{\mathcal{C}(\mathbf{Y})}(w, \mathcal{C}(Y)) < 3L$,
- $d_Y(x, w) \prec K$.

It follows that $d_{\mathcal{C}(\mathbf{Y})}(w, \pi_Y(x)) \prec 3L + 3K$. (A similar statement holds with z in place of x.)

Proof. By Proposition 3.6 every path from x to z must intersect the $3L$-neighborhood of $\mathcal{C}(Y)$ if $Y \neq X, Z$. This is trivially true if $Y = X$ or $Y = Z$. Let w be the first vertex in the path with $d_{\mathcal{C}(\mathbf{Y})}(w, \mathcal{C}(Y)) < 3L$ and let w' be the vertex that precedes it. (If $w = x$ then the lemma holds trivially.) By Proposition 3.6, $d_Y(x, w') \sim 0$. Since w and w' are adjacent in $\mathcal{C}(\mathbf{Y})$ they will map to either adjacent vertices in $\mathcal{P}(\mathbf{Y})$ or the same vertex. In either case $d_Y(w, w') \prec K$ and by the coarse triangle inequality $d_Y(x, w) \prec K$.

Now let $w' \in \mathcal{C}(Y)$ be a nearest point from w to $\mathcal{C}(Y)$. We have $d_{\mathcal{C}(\mathbf{Y})}(w, w') < 3L$. By Lemma 3.8, $d_Y(w', w) \prec 2K$. Therefore $d_{\mathcal{C}(\mathbf{Y})}(w, \pi_Y(x)) \prec 3L + 2K + K$ by the coarse triangle inequality. □

3.3 Hyperbolicity of $\mathcal{C}(\mathbf{Y})$

Theorem 3.10. *Suppose that all $\mathcal{C}(Y)$ for $Y \in \mathbf{Y}$ are quasi-trees in a uniform way, so that there is Δ such that all $\mathcal{C}(Y)$ for $Y \in \mathbf{Y}$ satisfy the bottleneck property with this Δ. Then $\mathcal{C}(\mathbf{Y})$ satisfies the bottleneck property so it is a quasi-tree.*

Proof. Let $x \in \mathcal{C}(X)$ and $z \in \mathcal{C}(Z)$ be given and let $Y_1, Y_2, \cdots, Y_s$ be the elements of $\mathbf{Y}_K(X, Z)$ with indexing reflecting the order. There is a standard path (see the proof of Lemma 3.3) V in $\mathcal{C}(\mathbf{Y})$ from x to z that projects to

$\{X, Y_1, Y_2, \cdots, Y_s, Z\}$ and within each $\mathcal{C}(Y_i)$ (we let $Y_0 = X, Y_{s+1} = Z$) it is a geodesic. We will argue that any path U from x to z comes within a bounded distance from any point on V. This verifies the modified bottleneck property discussed just before Theorem 2.9.

Fix a point $v \in \mathcal{C}(Y_i)$ on V and let $\{x = x_0, x_1, \ldots, x_k = z\}$ be the vertices of an arbitrary path U between x and z. We project the x_j to Y_i and let y_j be points in $\pi_{Y_i}(x_j)$. Note that $d_{\mathcal{C}(Y_i)}(y_j, y_{j+1}) \prec K$ so the y_j form a coarse path in $\mathcal{C}(Y_i)$ from $y_0 = \pi_{Y_i}(x)$ to $y_k = \pi_{Y_i}(z)$. Since $\mathcal{C}(Y_i)$ satisfies the bottleneck property with constant Δ, $d(y_0, \pi_{Y_i}(Y_{i-1})) \sim 0$ and $d(y_k, \pi_{Y_i}(Y_{i+1})) \sim 0$ by the order property, there will be some y_ℓ with $d_{\mathcal{C}(Y_i)}(y_\ell, v) \prec \Delta + K$. Note that if K is sufficiently large then at least one of $d_{Y_i}(x, x_\ell)$ and $d_{Y_i}(z, x_\ell)$ must be large enough to apply Lemma 3.9. Assume it is the former. Applying Lemma 3.9 there exists a vertex $x_{\ell'}$ on the path between x and x_ℓ such that

$$d_{\mathcal{C}(\mathbf{Y})}(x_{\ell'}, \mathcal{C}(Y_i)) < 3L$$

and

$$d_{Y_i}(y_\ell, x_{\ell'}) \prec K$$

since $y_\ell \in \pi_{Y_i}(x_\ell)$. Let $w \in \mathcal{C}(Y_i)$ be the closest point in $\mathcal{C}(\mathbf{Y})$ to $x_{\ell'}$. Then by Lemma 3.8 and the coarse triangle inequality we have

$$d_{Y_i}(w, v) \prec d_{Y_i}(w, x_{\ell'}) + d_{Y_i}(x_{\ell'}, y_\ell) + d_{Y_i}(y_\ell, v) \prec \Delta + 4K$$

and, since $d_{\mathcal{C}(\mathbf{Y})}(w, x_{\ell'}) < 3L$,

$$d_{\mathcal{C}(\mathbf{Y})}(x_{\ell'}, v) \prec \Delta + 4K + 3L.$$

This proves that the bottleneck property holds since $x_{\ell'} \in U$. □

Lemma 3.11. *There exists $K' > 0$ so that the following holds. If $x \in \mathcal{C}(X)$, $z \in \mathcal{C}(Z)$, and $Y \in \mathbf{Y}_{K'}(x, z)$, then every geodesic V in $\mathcal{C}(\mathbf{Y})$ from x to z intersects $\mathcal{C}(Y)$ in a geodesic segment $[v, w]$ and moreover $d_Y(x, v) \prec K'$, $d_Y(z, w) \prec K'$. Y is possibly X or Z.*

Proof. First note that by Lemma 3.1 the intersection, if nonempty, is a geodesic segment (possibly a single point). From Lemma 3.9 it follows that there are points v', w' along V so that $d(v', \pi_Y(x)) \prec 3L + 3K$ and $d(w', \pi_Y(z)) \prec 3L + 3K$. In particular, $d(v', w') \prec 6L + 6K + d_Y(x, z)$.

Assuming the subsegment $[v', w'] \subset V$ is disjoint from $\mathcal{C}(Y)$, we estimate the number of $\mathcal{C}(W)$'s $[v', w']$ has to pass through as being at least $\frac{d_Y(x,z)}{K} - 1$

(the diameter of the projections to Y of the union of two consecutive $\mathcal{C}(W)$'s is at most K). Thus the number of edges of length L the segment passes through is at least $\frac{d_Y(x,z)}{K}$, and we have

$$\frac{Ld_Y(x,z)}{K} \prec 6L + 6K + d_Y(x,z)$$

Since $L/K > 1$ we get a contradiction when $d_Y(x,z)$ is large enough. We have shown that if K' is large enough then $[v',w'] \cap \mathcal{C}(Y) \neq \emptyset$.

Thus $[v',w'] \cap \mathcal{C}(Y)$ is a geodesic segment $[v,w]$. We will argue that v is uniformly close to $\pi_Y(x)$; the argument that w is uniformly close to $\pi_Y(z)$ is symmetric. Let v'' be the vertex on the segment $[x,v] \subset V$ immediately preceding v (if $x = v$ there is nothing to prove). If $d(\pi_Y(x), \pi_Y(v'')) > K'$ we may apply the argument of the preceding paragraph to the geodesic $[x,v'']$ to deduce $[x,v''] \cap \mathcal{C}(Y) \neq \emptyset$, a contradiction. Thus $d(\pi_Y(x), \pi_Y(v'')) \leq K'$ and so $d_Y(x,v) \prec K'$. □

Corollary 3.12. *There is $K' > K$ such that for $x \in \mathcal{C}(X), z \in \mathcal{C}(Z)$*

$$d_{\mathcal{C}(\mathbf{Y})}(x,z) \geq \frac{1}{2} \sum_{W \in \mathbf{Y}_{K'}(x,z)} d_W(x,z)$$

Proof. Let K' be the constant from Lemma 3.11 and assume that $d_Y(x,z) > 6K'$. Then any geodesic from x to z intersects $\mathcal{C}(Y)$ in a segment of length $\succ 4K'$, which is $> 3K'$. The estimate follows after renaming $6K'$ to K'. □

A geodesic metric space is *quasi-convex* if there is $N > 0$ such that for any two geodesic segments $[u,v]$ and $[u',v']$, if $d(u,u') \leq 1$ and $d(v,v') \leq 1$ then $[u',v']$ is contained in the Hausdorff N-neighborhood of $[u,v]$. Note that this implies that if $d(u,u') \leq C$, $d(v,v') \leq C$ then $[u',v']$ is contained in the Hausdorff $(C+1)N$-neighborhood of $[u,v]$.

Also note that if each $\mathcal{C}(Y)$ is quasi-convex with the same constant, then there is a uniform bound on the Hausdorff distance of any two standard paths between any two points in $\mathcal{C}(\mathbf{Y})$.

Lemma 3.13. *Suppose that each $\mathcal{C}(Y)$ is quasi-convex with the same constant N. There is $M > 0$ so that for any x and z, any geodesic from x to z is contained in the Hausdorff M-neighborhood of any standard path (see Definition 3.2) from x to z, and conversely, any standard path from x to z is contained in the Hausdorff M-neighborhood of any geodesic from x to z.*

Proof. If $[v, w]$ is a segment in a standard path U obtained by intersecting with some $\mathcal{C}(W)$, then the endpoints are within uniform distance of any geodesic V from x to z by Lemma 3.9 since $W \in \mathbf{Y}_{\boldsymbol{\xi}}(x, z)$ (the only case the lemma does not apply is when $W = X, Z$ and $W \notin \mathbf{Y}_{\boldsymbol{\xi}}(x, z)$, but then the claim is true with the bound $\boldsymbol{\xi}$). We claim that $[v, w]$ is within uniform distance from V. If $d_W(x, z) \leq K'$, then the length of the geodesic $[v, w]$ is bounded by a constant $\prec 3K'$, therefore $[v, w]$ is within uniform distance from V. If $d_W(x, z) > K'$, then by Lemma 3.11 V intersects $\mathcal{C}(Y)$ in a geodesic segment $[v', w']$ whose endpoints are uniform distance from the endpoints of $[v, w]$. By the uniform quasi-convexity of $\mathcal{C}(Y)$, the claim follows. Thus the standard path U is contained in a uniform neighborhood of the geodesic V.

Now we show that the geodesic V is contained in a uniform neighborhood of the standard path U. Let $\mathbf{Y}_K(x, z) = \{Y_1, Y_2, \cdots, Y_k\}$ and let $i_1 < i_2 < \cdots < i_s$ be the indices of those Y_i with $d_{Y_i}(x, z) > K'$, where K' is large (at least as large as in Lemma 3.11, but in fact a bit larger, see below). Then $V \cap \mathcal{C}(Y_{i_j})$ is an interval I_{i_j} and the intervals $I_{i_1}, I_{i_2}, \cdots, I_{i_s}$ occur along V in order of their indices (if I_{i_j} occurs after $I_{I_{j+1}}$ apply Lemma 3.11 to the subsegment of V that starts with I_{i_j} to get a contradiction – this is where we need K' to be larger by $\boldsymbol{\xi}$ than in Lemma 3.11). Let I'_{i_j} be the geodesic segment $\mathcal{C}(Y_{i_j}) \cap U$. Since U is a standard path, the endpoints of I'_{i_j} are within distance $\prec \boldsymbol{\xi}$ from $\pi_{Y_{i_j}}(x), \pi_{Y_{i_j}}(z)$, respectively. Also, by Lemma 3.11, the endpoints of I_{i_j} are within distance $\prec K'$ from $\pi_{Y_{i_j}}(x), \pi_{Y_{i_j}}(z)$, respectively. Therefore, I_{i_j} and I'_{i_j} are contained in a uniform neighborhood of each other by the uniform quasi-convexity of $\mathcal{C}(Y)$. It suffices to argue that each complementary interval in V and the corresponding (with respect to the order) complementary interval in U are contained in a uniform neighborhood of each other.

Let J be one such complementary interval, say between I_{i_j} and $I_{i_{j+1}}$. The corresponding interval J' in U is between I'_{i_j} and $I'_{i_{j+1}}$. We already know the endpoints of J and J' are uniformly close. Note that $Y_i \in \mathbf{Y}_{\boldsymbol{\xi}}(Y_{i_j}, Y_{i_{j+1}})$ for $i_j < i < i_{j+1}$, so applying Lemma 3.9 again to J we find that each endpoint r_m of each segment of J' in the standard path within some $\mathcal{C}(Y_i)$ is within uniform distance of some point R_m on J. (The bound is perhaps worse than $3L + 3K$ since the endpoints of J and J' do not exactly coincide, but they are uniformly close, which is enough.) Index the points r_m in order in which they occur along the standard path, and note that we do not know that the corresponding points R_m appear in linear order along J. However, since $d(r_m, r_{m+1})$ is uniformly bounded (by $L + K'$), it follows that $d(R_m, R_{m+1})$

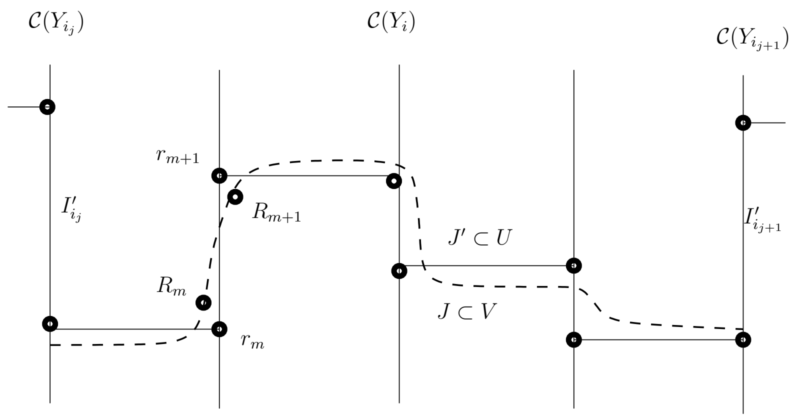


Figure 3: J is the dashed line

is uniformly bounded. Moreover, the first point R_1 and the last point R_n are within a uniform distance of the corresponding endpoints of J. It follows that the R_m's cut J into segments of bounded length and also r_m's cut J' into segments of bounded length, therefore J and J' are contained in a uniform neighborhood of each other, and the lemma follows.

The extremal cases, when J contains an endpoint of V, differs only in notation and is left to the reader. □

Remark 3.14. A similar argument shows that $\mathcal{C}(\mathbf{Y})$ is quasi-convex.

Recall that a geodesic metric space is δ-hyperbolic if for any three points x, y, z any geodesic $[x, z]$ is contained in the δ-neighborhood of the union $[x, y] \cup [y, z]$ of any two geodesics joining x to y and y to z. A space is hyperbolic if it is δ-hyperbolic for some δ.

Theorem 3.15. *Assume that each $\mathcal{C}(Y)$ is δ-hyperbolic with the same δ. Then $\mathcal{C}(\mathbf{Y})$ is hyperbolic.*

Proof. Let x, y, z be three vertices of $\mathcal{C}(\mathbf{Y})$. Recall that δ-hyperbolic spaces are quasi-convex, with the constant depending only on δ. Thus Lemma 3.13 applies and it suffices to show that a standard path U from x to z is contained in a uniform neighborhood of the union of two geodesics $[x, y]$ and $[y, z]$.

Let $W \in \mathbf{Y}_K(x, z)$. We claim that $[\pi_W(x), \pi_W(z)]$ is contained in a uniform neighborhood of $[x, y] \cup [y, z]$. First consider the case when $d_W(x, y) > \boldsymbol{\xi}$, $d_W(y, z) > \boldsymbol{\xi}$. Then a geodesic $[\pi_W(x), \pi_W(y)] \subset \mathcal{C}(W)$ is contained in a uniform neighborhood of $[x, y]$ by Lemma 3.9 and Lemma 3.11

(see the first paragraph of the proof of Lemma 3.13). Likewise, a geodesic $[\pi_W(y), \pi_W(z)]$ is contained in a uniform neighborhood of $[y, z]$. Since $\mathcal{C}(W)$ is δ-hyperbolic, $[\pi_W(x), \pi_W(z)]$ is contained in the δ-neighborhood of $[\pi_W(x), \pi_W(y)] \cup [\pi_W(y), \pi_W(z)]$ and consequently in a uniform neighborhood of $[x, y] \cup [y, z]$.

Now suppose that $d_W(x, y) \leq \boldsymbol{\xi}$. Since $W \in \mathbf{Y}_K(x, z)$, it follows $d_W(y, z) > \boldsymbol{\xi}$. Again by Lemmas 3.9 and 3.11 we have that $[\pi_W(y), \pi_W(z)]$ is in a uniform neighborhood of $[y, z]$. By quasi-convexity, it follows from $d_{\mathcal{C}(\mathbf{Y})}(\pi_W(x), \pi_W(y)) \leq \boldsymbol{\xi}$ that $[\pi_W(x), \pi_W(z)]$ is contained in a uniform neighborhood of $[\pi_W(y), \pi_W(z)]$ and hence of $[y, z]$. The case when $d_W(y, z) \leq \boldsymbol{\xi}$ is handled symmetrically.

By the definition of a standard path and the uniform quasi-convexity of $\mathcal{C}(Y)$, a standard path U from x to z is contained in a uniform neighborhood of the union of $[\pi_W(x), \pi_W(z)]$ for all W with $W \in \mathbf{Y}_K(x, z)$ (see the proof of Lemma 3.13). Therefore it follows that U is contained in a uniform neighborhood of $[x, y] \cup [y, z]$. □

3.4 Group action and WWPD

Now assume that G is a group that acts on the set $\mathbf{Y}$, that for each $Y \in \mathbf{Y}$ we have a geodesic metric space $\mathcal{C}(Y)$ and projections π_Y satisfying the axiom of Section 3.1, and that G preserves this structure, i.e. there are isometries $F_g^Y : \mathcal{C}(Y) \to \mathcal{C}(g(Y))$ so that

- $F_{g'}^{g(Y)} F_g^Y = F_{g'g}^Y$ for all $g, g' \in G$, $Y \in \mathbf{Y}$, and
- $\pi_Y(X) = \pi_{g(Y)}(g(X))$ for all $g \in G$ and $X, Y \in \mathbf{Y}$.

Then projection distances are preserved, i.e. $d^\pi_{g(A)}(g(B), g(C)) = d^\pi_A(B, C)$ for all $A, B, C \in \mathbf{Y}$ and $g \in G$, and therefore G acts naturally on $\mathcal{C}(\mathbf{Y})$. To simplify notation, we will denote the isometry F_g^Y simply by $g : \mathcal{C}(Y) \to \mathcal{C}(g(Y))$.

We defined WPD for group actions in Section 2.5. Here we define a weaker property, WWPD, to allow for elements with large centralizers. We restrict ourselves to actions on hyperbolic spaces. For a motivation, see Remark 2.15.

Definition 3.16. Let G act on a δ-hyperbolic metric space X. We say $g \in G$ is a *WWPD element* if

(1) $\langle g \rangle$ has an unbounded orbit in X,

(2) there is $x \in X$, a subgroup $N(g) \subset G$ with $g \in N(g)$ and a constant $B > 0$ such that

- for $h \in G - N(g)$ the projection of $h(< g > x)$ to $< g > x$ has diameter $\leq B$,
- there is a homomorphism $N(g) \to Q(g)$ to a virtually cyclic group $Q(g)$ whose kernel fixes every $g^k(x)$, $k \in \mathbb{Z}$.

Remark 3.17. This definition is not independent of the choice of x. The set of translates of the g-orbit of x is again "discrete" as in the definition of WPD, but this time we allow a big group that fixes the whole orbit pointwise.

Proposition 3.18. *Suppose each $\mathcal{C}(Y)$ is δ-hyperbolic so that $\mathcal{C}(\mathbf{Y})$ is hyperbolic. Let $g \in G$ so that $g(Y) = Y$ and denote by $K_{\mathcal{C}(Y)}$ the kernel of the action of $Stab_G(Y)$ on $\mathcal{C}(Y)$. Assume that $g : \mathcal{C}(Y) \to \mathcal{C}(Y)$ is a WPD element for the action of $Stab_G(Y)/K_{\mathcal{C}(Y)}$ on $\mathcal{C}(Y)$. Then g is a WWPD element for the action of G on $\mathcal{C}(\mathbf{Y})$. If moreover $Stab_G(Y)$ is virtually cyclic then g is a WPD element for the action of G on $\mathcal{C}(\mathbf{Y})$.*

Proof. We take $N(g) := Stab_G(Y)$ and $Q(g) := Stab_G(Y)/K_{\mathcal{C}(Y)}$ with the obvious quotient map $N(g) \to Q(g)$, and we choose $x \in \mathcal{C}(Y)$. If $h \in G - N(g)$ then h moves the orbit $< g > x$ to another $\mathcal{C}(Y')$ and the projection to $\mathcal{C}(Y)$ is uniformly bounded (see Axiom (0), Lemma 3.8 and Corollary 3.12). Therefore (WWPD) is immediate. For the moreover part, note that under the assumption on $Stab_G(Y)$, i.e., $< g >$ has finite index in this group, the pointwise stabilizer of the g-orbit of x is finite. □

Examples 3.19. Let Γ be a discrete group of isometries of $\mathbb{H}^n$ and $\mathbf{Y}$ the collection of translates of finitely many axes of elements γ_i of Γ, as in Example 2.1(1). Then each γ_i is a WPD element of $\mathcal{C}(\mathbf{Y})$, where $\mathcal{C}(Y) \cong \mathbb{R}$ is the axis Y. Similar conclusions hold in examples (2) and (4). In example (3) elements g that are pseudo-Anosov when restricted to a subsurface $Y \in \mathbf{Y}$ (or Dehn twists when Y is an annulus) will be WWPD elements for the action of the mapping class group on $\mathcal{C}(\mathbf{Y})$.

3.5 Asymptotic dimension

Let $\mathcal{X}$ be a metric space. Recall that $\text{asdim}(\mathcal{X}) \leq n$ provided for every $R > 0$ there is a covering of $\mathcal{X}$ by bounded sets such that every R-ball intersects at most $(n + 1)$ of these sets. We say that a collection $\{\mathcal{X}_\alpha\}$ of metric spaces has $\text{asdim}(\mathcal{X}_\alpha) \leq n$ *uniformly*, if for every R one can choose coverings of $\mathcal{X}_\alpha$ by uniformly bounded sets so that every R-ball in any $\mathcal{X}_\alpha$ intersects at most $(n + 1)$ of these sets.

Recall that a function $f : \mathcal{X} \to \mathcal{Y}$ between metric spaces is a *coarse embedding* if there are constants A, B and a function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(t) \to \infty$ as $t \to \infty$ such that

$$\Phi(d_{\mathcal{X}}(x, x')) \leq d_{\mathcal{Y}}(f(x), f(x')) \leq A\ d_X(x, x') + B$$

In that case $\operatorname{asdim}(\mathcal{X}) \leq \operatorname{asdim}(\mathcal{Y})$. In particular, a finitely generated group has a well defined asymptotic dimension, independent of the choice of a generating set. As mentioned in the introduction, It is also a fact (see [Gro93]) that unbounded trees have asymptotic dimension 1.

We will need the following theorems. A general reference for asymptotic dimension is [BD08]; for the original definition and interesting discussion see Gromov's article [Gro93].

Bell-Dranishnikov's Hurewicz Theorem [BD06]. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a Lipschitz map with $\mathcal{X}$ a geodesic space. Suppose that for every R the family $\{F_y = f^{-1}(B(y, R)) \mid y \in \mathcal{Y}\}$ has* $\operatorname{asdim}(F_y) \leq n$ *uniformly. Then* $\operatorname{asdim}(\mathcal{X}) \leq \operatorname{asdim}(\mathcal{Y}) + n$.

Union Theorem. *Let $\mathcal{X} = \cup \mathcal{X}_\alpha$ and assume that* $\operatorname{asdim}(\mathcal{X}_\alpha) \leq n$ *uniformly. Also assume that for every $R > 0$ there is a subset $\mathcal{Y}_R \subset X$ such that* $\operatorname{asdim}(\mathcal{Y}_R) \leq n$ *and the sets $\mathcal{X}_\alpha \setminus \mathcal{Y}_R$ and $\mathcal{X}_\beta \setminus \mathcal{Y}_R$ are R-separated for $\alpha \neq \beta$ (i.e. $d(x, y) > R$ for any $x \in \mathcal{X}_\alpha \setminus \mathcal{Y}_R$ and $y \in \mathcal{X}_\beta \setminus \mathcal{Y}_R$). Then* $\operatorname{asdim}(\mathcal{X}) \leq n$. *Furthermore the uniformity constants for* $\operatorname{asdim}(\mathcal{X})$ *only depend on the uniformity constants for $\mathcal{X}_\alpha$ and $\mathcal{Y}_R$.*

Remark 3.20. The uniformity statement is not in [BD08] but is easily seen from the proof.

Product Theorem. $\operatorname{asdim}(\mathcal{X} \times \mathcal{Y}) \leq \operatorname{asdim}(\mathcal{X}) + \operatorname{asdim}(\mathcal{Y})$.

3.6 $\mathcal{C}(\mathbf{Y})$ has finite asymptotic dimension

We would like to show that $\mathcal{C}(\mathbf{Y})$ has finite asymptotic dimension under the assumption that the asymptotic dimension of the $\mathcal{C}(Y)$ are uniformly bounded. To do so we will apply the Bell-Dranishnikov Hurewicz Theorem to the map from $\mathcal{C}(\mathbf{Y})$ to $\mathcal{P}_K(\mathbf{Y})$. The theorem is most natural to apply when the pre-image of balls are a Hausdorff neighborhood of the pre-image of a point. This is not the case in our situation and we need the following technical lemma to deal with this issue.

Lemma 3.21. *Fix a vertex Y in $\mathcal{P}_K(\mathbf{Y})$. Given $R > 0$ and vertices X and Z with $d(X, Y) = d(Z, Y) = m$ and $x \in \mathcal{C}(X)$, $z \in \mathcal{C}(Z)$ with $d_{\mathcal{C}(\mathbf{Y})}(x, z) < R$ there exist a vertex W with $d(W, Y) = m - 1$ and $d_{\mathcal{C}(\mathbf{Y})}(x, W) < R + 2mL + \xi$.*

Proof. By Lemma 3.1 we have $d^\pi_X(x,z) \le R$. Since $d(X,Y) = d(Z,Y) = m$ there is a path $X = X_0, X_1, \cdots, X_N = Z$ in $\mathcal{P}_K(\mathbf{Y})$ of length $N \le 2m$ with $d(X_1, Y) = m-1$. By Lemma 3.1, for adjacent vertices in $\mathcal{P}_K(\mathbf{Y})$ we have $d^\pi_X(X_i, X_{i+1}) < d_{\mathcal{C}(\mathbf{Y})}(X_i, X_{i+1}) = L$ so the triangle inequality implies that $d^\pi_X(X_1, Z) < (2m-1)L$ and

$$\begin{aligned} d^\pi_X(x, X_1) &< d^\pi_X(x, Z) + d^\pi_X(Z, X_1) \le d^\pi_X(x,z) + \xi + d^\pi_X(Z, X_1) \\ &< R + (2m-1)L + \xi. \end{aligned}$$

By the definition of $d^\pi_X(x, X_1)$ the distance in $\mathcal{C}(X)$ from x to any point $\pi_X(X_1)$ is not more than $d^\pi_X(x, X_1)$. Furthermore there is an edge in $\mathcal{C}(\mathbf{Y})$ from any point in $\pi_X(X_1)$ to $\mathcal{C}(X_1)$ of length L and therefore the distance from x to $\mathcal{C}(X_1)$ in $\mathcal{C}(\mathbf{Y})$ is less than $R + 2mL + \xi$. Setting $W = X_1$ the lemma is proved. □

Theorem 3.22. *If the metric spaces $\mathcal{C}(Y)$ for $Y \in \mathbf{Y}$ have asymptotic dimension uniformly bounded by n then $\mathcal{C}(\mathbf{Y})$ has asymptotic dimension $\le n+1$.*

Proof. Consider the projection map $p : \mathcal{C}(\mathbf{Y}) \to \mathcal{P}_K(\mathbf{Y})$. The target is a quasi-tree so its asymptotic dimension is ≤ 1. We will verify the conditions of Bell-Dranishnikov's Hurewicz Theorem for p. Let B_m denote the ball of radius m in $\mathcal{P}_K(\mathbf{Y})$ (centered at some vertex). We will prove by induction on m that $\operatorname{asdim}(p^{-1}(B_m)) \le n$. Uniformity is not an issue since all of our choices of constants will be independent of the vertex in $\mathcal{P}_K(\mathbf{Y})$. When $m = 0$ this is true by definition of n.

Now suppose $\operatorname{asdim}(p^{-1}(B_m)) \le n$ and we will argue $\operatorname{asdim}(p^{-1}(B_{m+1})) \le n$. To that end, we write

$$p^{-1}(B_{m+1}) = \bigcup_{Y \in B_{m+1}} p^{-1}(Y)$$

and check that the hypotheses of the Union Theorem hold. Each $p^{-1}(Y)$ has asdim $\le n$ by definition of n.

Let R be given and set

$$\mathcal{Y}_R = N_{\tilde{R}}(p^{-1}(B_m))$$

the Hausdorff $\tilde{R}$-neighborhood of $p^{-1}(B_m)$, where

$$\tilde{R} = R + (2m+2)L + \xi.$$

By induction, $p^{-1}(B_m)$, and hence $\mathcal{Y}_R$, have asdim $\leq n$. If X and Z are distinct vertices at distance $m+1$ from the center of B_{m+1} then by Lemma 3.21, $p^{-1}(X) - \mathcal{Y}_R$ and $p^{-1}(Z) - \mathcal{Y}_R$ are R-separated. It now follows from Bell-Dranishnikov's Hurewicz Theorem that

$$\operatorname{asdim}(\mathcal{C}(\mathbf{Y})) \leq n+1.$$

□

Question 3.23. *Is* $\operatorname{asdim}(\mathcal{C}(\mathbf{Y})) \leq n$*?*

4 Mapping class group

4.1 Curve complexes

We will apply our previous work to a collection of curve graphs of a subsurface of a fixed surface Σ, as in the work of Masur and Minsky [MM00]. We begin by recalling the definition of the curve graph and projections. We follow an approach that is not standard but is convenient.

Let Σ be a compact orientable surface with boundary such that $\chi(\Sigma) < 0$. Let $\mathcal{C}_0(\Sigma)$ be the set of homotopy classes of simple closed curves and properly embedded simple arcs that are not peripheral or boundary compressible. We then define the *curve graph*, $\mathcal{C}(\Sigma)$, to be the 1-complex obtained by attaching an edge to disjoint closed curves or arcs in $\mathcal{C}_0(\Sigma)$. We could also attach higher dimensional simplices but the resulting complex is quasi-isometric to its 1-skeleton so we stop at the curve graph.

Remark 4.1. The graph we have constructed is often called the *curve and arc graph*. The usual curve graph is quasi-isometric to the curve and arc graph and so we will use the less cumbersome name of curve graph. We also note that in the usual definition of the curve graph there are exceptional cases, the punctured torus and the sphere with 3 or 4 punctures, where the graph needs to be defined differently. One advantage of the curve-arc graph is that one definition works for all cases.

We also note that if Σ is a 3-punctured sphere then $\mathcal{C}(\Sigma)$ is bounded and we could ignore such subsurfaces. However there is also no harm in including them.

We now define projections between curve graphs of essential (i.e. connected, boundary components essential and nonperipheral) subsurfaces of Σ. If Y and Z are essential subsurfaces, we can only define the projection of $\mathcal{C}(Z)$ to $\mathcal{C}(Y)$ if ∂Z intersects Y essentially. We then define $\pi_Y(Z)$ by taking the intersection of ∂Z with Y and identifying homotopic curves and arcs. If z is vertex in $\mathcal{C}(Z)$ then we define $\pi_Y(z) = \pi_Y(Z)$.

We will also need the curve graph for an annulus or simple closed curve. The definition here has a somewhat different flavor although once we make the definition we can use it just as we do for the other curve complexes. The simplest way to define the curve graph is to fix a complete hyperbolic metric on the interior of Σ. If γ is an essential non-peripheral simple closed curve let X_γ be the annular cover of Σ to which γ lifts. Let $\mathcal{C}_0(\gamma)$ be the set of complete geodesics in X_γ that cross the core curve and we form $\mathcal{C}(\gamma)$ by attaching an edge to vertices that represent disjoint geodesics. It is easy to check that distance in $\mathcal{C}(\gamma)$ is intersection number plus one and that $\mathcal{C}(\gamma)$ is quasi-isometric to $\mathbb{Z}$.

We now define projections to and from $\mathcal{C}(\gamma)$. If Y is an essential subsurface such that ∂Y intersects γ let $\pi_\gamma(Y)$ be those components of the pre-image of the geodesic representatives of ∂Y in X_γ that intersect the core curve. If β is a simple closed curve that intersects γ we similarly define $\pi_\gamma(\beta)$ where we replace the ∂Y with β. Finally if γ intersects Y essentially then define $\pi_Y(\gamma)$ by restricting γ to Y.

With these definitions in hand we will not distinguish between essential subsurfaces and simple closed curves.

The following lemma (without the explicit bound) was proved by Behrstock [Beh06] using the Masur-Minsky theory of hierarchies [MM00]. For a simple proof due to Leininger that produces the explicit bound below see [Man10, Man].

We say that subsurfaces X and Y overlap if $\partial X \cap \partial Y \neq \emptyset$ (this means that ∂X and ∂Y cannot be made disjoint by a homotopy). Note that in that case $\pi_X(Y)$ and $\pi_Y(X)$ are defined.

Lemma 4.2. *Let X, Y and Z be overlapping subsurfaces. If*

$$d^\pi_X(Y, Z) > 10$$

then

$$d^\pi_Y(X, Z) < 10.$$

We also have a finiteness statement. See [MM00].

Lemma 4.3. *There is $K_0 > 0$ such that given subsurfaces X and Y there are only finitely many subsurfaces Z with $d^\pi_Z(X, Y) > K_0$.*

The following theorem of Bell-Fujiwara [BF08] is crucial for our approach.

Theorem 4.4. *Every curve graph has finite asymptotic dimension.*

If we apply this result to Theorem 3.22 we have:

Theorem 4.5. *Let* $\mathbf{Y}$ *be a collection of subsurfaces that pairwise overlap. Then* $\mathcal{C}(\mathbf{Y})$ *has finite asymptotic dimension.*

4.2 Partitioning subsurfaces into finitely many collections

We would like to apply our construction of the projection complex to subsurfaces and their associated curve complexes. To do so we need to partition the set of all subsurfaces into finite many collections where any two subsurfaces in the same collection overlap.

Lemma 4.6. *There is a coloring* $\phi : \mathcal{C}(\Sigma)^{(0)} \to F$ *of the set of simple closed curves on* Σ *with a finite set* F *of colors so that if* a, b *span an edge then* $\phi(a) \neq \phi(b)$.

Proof. Let T be the set of all connected double covers of Σ. If a is a simple closed curve in Σ define a function f_a on the set T as follows. For a double cover $\tilde{\Sigma} \to \Sigma$ define $f_a(\tilde{\Sigma})$ as 0 if a does not lift to $\tilde{\Sigma}$, and otherwise as the set $\{\alpha, \beta\}$ of homology classes in $H_1(\tilde{\Sigma}; \mathbb{Z}_2)$ determined by the two lifts of a.

The set F of colors is the set of all such functions – it is clearly finite.

We now show that if a, b are disjoint nonparallel simple closed curves, then $f_a \neq f_b$.

We will use the following construction of double covers. Let $\mathcal{C}$ be a non-separating collection of disjoint simple closed curves and properly embbeded arcs in Σ. Then $\mathcal{C}$ determines a double cover $\tilde{\Sigma} \to \Sigma$ by cutting along $\mathcal{C}$ and gluing cross-wise two copies of the resulting surface (equivalently, the associated index two subgroup is given by curves that intersect $\mathcal{C}$ in an even number of points). In particular for any a we can find a cover $\tilde{\Sigma} \to \Sigma$ where a lifts by applying the above construction to a non-separating curve or properly embedded arc that is disjoint from a. If a represents a non-trivial homology class and b represents a differently class then $f_a(\tilde{\Sigma}) \neq f_b(\tilde{\Sigma})$. Therefore we can assume that a and b are homologous.

For each component S of $\Sigma \backslash (a \cup b)$ whose boundary is contained in $a \cup b$ choose a simple curve c such that $S \backslash c$ is connected and let $\mathcal{C}$ be the union of such curves. There is at least one and at most three such components so $\mathcal{C}$ contains between one and three curves. Note that the curve c exists since S will have one or two boundary components and can't be a disk or annulus. Therefore S must have positive genus and hence contain a simple curve that doesn't separate S. Let $\tilde{\Sigma}$ be the double cover associated to $\mathcal{C}$ by the construction above. If $f_a(\tilde{\Sigma}) = f_b(\tilde{\Sigma})$ then there will be lifts $\tilde{a}$ and $\tilde{b}$ of a and b that bound a surface $\tilde{S} \subset \tilde{\Sigma}$ such that $\tilde{S}$ doesn't contain either of the other lifts of a and b. Then the restriction of the covering map to $\tilde{S}$ will be a homeomorphism and its image will contain a component of $\mathcal{C}$. This is a contradiction so we must have $f_a(\tilde{\Sigma}) \neq f_b(\tilde{\Sigma})$. □

Lemma 4.7. *There is a finite index subgroup G of the mapping class group $MCG(\Sigma)$ (where Σ is closed) such that every element of G preserves the colors from the proof of Lemma 4.6.*

Proof. The group $Aut(\pi_1(\Sigma))$ lifts to an action on the union of connected double covers of Σ. Let Γ be the subgroup of $Aut(\pi_1(\Sigma))$ that fixes the $\mathbb{Z}_2$-homology of this union. This will be finite index subgroup of $Aut(\pi_1(\Sigma))$ so its image G in $Out(\pi_1\Sigma) \cong MCG(\Sigma)$ will have finite index in $MCG(\Sigma)$ and will fix the colors form Lemma 4.6. □

Proposition 4.8. *Let Σ be a compact surface with (possibly empty) boundary. Let $\mathbf{Y}$ be the collection connected incompressible subsurfaces of Σ that are not the sphere with 3 boundary components. Then $\mathbf{Y}$ can be written as a finite disjoint union*

$$\mathbf{Y}^1 \sqcup \mathbf{Y}^2 \sqcup \cdots \sqcup \mathbf{Y}^k$$

so that

- *the boundaries of any two surfaces in any $\mathbf{Y}^i$ intersect, and*
- *there is a subgroup $G < MCG(\Sigma)$ of finite index that preserves each $\mathbf{Y}^i$: if $W \in \mathbf{Y}^i$ and $g \in G$ then $g(W) \in \mathbf{Y}^i$.*

Proof. The mapping class group acts on $\mathbf{Y}$ and there are finitely many orbits under the action. Let G be the subgroup given by Lemma 4.7. Since G has finite index in $MCG(\Sigma)$, the action of G on $\mathbf{Y}$ also has finitely many orbits. These orbits are our $\mathbf{Y}^i$ and by definition are invariant under the G-action.

We now show that if $W_0 \neq W_1$ are in $\mathbf{Y}^i$ then they have intersecting boundary. There is a $g \in G$ such that $W_0 = g(W_1)$. Since g preserves the colors if the W_0 and W_1 don't have intersecting boundary then g must fix $\partial W_0 = \partial W_1$ and W_0 must be the complement of W_1. By assumption the W_i are not spheres with three boundary components. They are also not annuli for if they were then we would have $W_0 = W_1$. In particular W_0 must contain an non-peripheral simple closed curve γ. Since $g(\gamma)$ will be disjoint from γ it will have a different color. As G fixes the colors this is a contradiction. □

Here is a perhaps unexpected application of our construction.

Theorem 4.9. *(i) Let f be a Dehn twist in the curve γ on Σ. There is a finite index subgroup $\Gamma \subset MCG(\Sigma)$ and an action of Γ on a quasi-tree such that any power f^k of f, $k \neq 0$, that belongs to Γ is a hyperbolic isometry.*

(ii) If Σ has even genus g and γ separates into two subsurfaces of genus $g/2$ then we may take $\Gamma = MCG(\Sigma)$.

(iii) In these actions, there is a bound to the diameter of the projection of a fixed quasi-axis of f^k to any non-parallel translate.

By contrast, semisimple actions of mapping class groups on $CAT(0)$ spaces always have the property that Dehn twists are elliptic (see [Bri]). From (i) it follows that a Dehn twist has linear growth in the word length of Γ, therefore in $MCG(\Sigma)$ (known by [FLM01]).

Proof. If Γ is the subgroup of Proposition 4.8 or if γ is as in (ii) and $\Gamma = MCG(\Sigma)$ then the Γ-orbit of γ consists of pairwise intersecting curves. Let $\mathbf{Y}$ be this orbit and consider the action of G on the quasi-tree of curve complexes $\mathcal{C}(\mathbf{Y})$. Since each curve complex $\mathcal{C}(g\gamma)$ is quasi-isometric to a line (and they are all isometric to each other), it follows from Theorem 3.10 that $\mathcal{C}(\mathbf{Y})$ is a quasi-tree. Since a nontrivial power of f acts as a hyperbolic isometry on $\mathcal{C}(\gamma)$ the claim follows. The last statement is a consequence, for example, of Corollary 3.12. □

Here is another related application. Recall [Gro87] that the *Rips complex*, $R_d(\mathcal{G})$ of a graph $\mathcal{G}$ is a complex whose vertices are the vertices of $\mathcal{G}$ and a simplex is a collection of vertices whose pairwise distance is $\leq d$. Rips has shown that if $\mathcal{G}$ is δ-hyperbolic then for d sufficiently large, $R_d(\mathcal{G})$ is contractible. It has been hoped that with the same assumptions, for d sufficiently large $R_d(\mathcal{G})$ is $CAT(0)$. The quasi-tree given by (ii) gives a counter-example to this conjecture, at least for infinite valence graphs.

Corollary 4.10. *There exist infinite diameter, infinite valence graphs that are quasi-isometric to trees but whose Rips complex is never $CAT(0)$.*

Proof. Let $\mathcal{G}$ be the quasi-tree given by (ii) of Theorem 4.9. Then $MCG(\Sigma)$ acts on $\mathcal{G}$ with the Dehn twist about the curve γ acting hyperbolically. Then $MCG(\Sigma)$ will act on $R_d(\mathcal{G})$ for all d and the Dehn twist will still act hyperbolically. Therefore, by Bridson's theorem [Bri], $R_d(\mathcal{G})$ is not $CAT(0)$. □

4.3 Embedding MCG into a finite product of $\mathcal{C}(\mathbf{Y})$'s

Fix a set of finite generators for $MCG(\Sigma)$ and for all $g \in MCG(\Sigma)$ let $|g|$ be the word length norm. We need the following proposition. Recall that a finite collection of simple closed curves is *binding* if every nonperipheral curve intersects at least one curve in α. If W is any subsurface and $g \in MCG(\Sigma)$, the restrictions $\pi_W(\alpha)$ and $\pi_W(g(\alpha))$ are nonempty and we denote by $d^\pi_W(\alpha, g(\alpha))$ the diameter of their union in the curve complex of W.

Proposition 4.11. *Let α be a finite binding collection of simple closed curves on Σ. Given any $B > 0$ there exists a $C > 0$ such that if $|g| > C$ then there is a subsurface W such that $d^\pi_W(\alpha, g(\alpha)) > B$.*

Proof. Fix a hyperbolic metric on Σ. When we discuss the Hausdorff limit of a sequence of curves we assume that they have been realized by hyperbolic geodesics in this metric.

Assume that the lemma is false. Then there exists a sequence of g_i such that $|g_i| \to \infty$ but $d^\pi_W(\alpha, g_i(\alpha)) \leq B$ for all subsurfaces W. We pass to a subsequence (which we don't relabel) such that $g_i(c)$ has a Hausdorff limit for each curve c in α (see e.g. [CB88] for basic facts about Hausdorff convergence in the lamination space). There are then three possibilities:

- If the Hausdorff limits are all simple closed curves then the sequences $g_i(c)$ must become constant. However there are only finitely many elements of $MCG(\Sigma)$ that have the same image on a set of binding curves. This contradicts $|g_i| \to \infty$.

- Fix a c in α and let λ be the Hausdorff limit of $g_i(c)$. Also assume that there is a minimal component λ_Y of λ that fills a non-annular subsurface Y. Let c' be a curve in α that intersects Y. We will modify an argument of F. Luo (see [MM99]) to show that $d^\pi_Y(g_i(c), c') \to \infty$. If $d_{\mathcal{C}(Y)}(\pi_Y(c'), \pi_Y(g_i(c)))$ is bounded we can pass to a subsequence where the distance is constant. For each i let $x_i \in \mathcal{C}(Y)$ be adjacent to $\pi_Y(g_i(c))$ but closer to $\pi_Y(c')$. We can pass to another subsequence such that x_i converges in the Hausdorff topology to a lamination λ'. As the x_i and $\pi_Y(g_i(c))$ are disjoint λ' and λ_Y can't intersect and since λ_Y fills Y this implies that $\lambda' = \lambda_Y$, perhaps with some isolated leaves added. We can repeat this until we have a sequence in $\mathcal{C}(Y)$ disjoint from $\pi_Y(c')$ that converges to the filling lamination λ_Y (plus isolated leaves). This is a contradiction so we must have $d_Y(g_i(c), c') \to \infty$.

- The final case is when the Hausdorff limit λ isn't a collection of simple curves but doesn't have a component that fills a non-annular subsurface. In this case there must be a leaf of λ that spirals around a simple closed curve β. Let c' be a curve in α that intersects β. Again fix a hyperbolic metric on Σ. We also fix an annular neighborhood X of β. Then $d^{\pi}_X(g_i(c), c') = i_X(g_i(c), c')$. Since λ spirals around β we have $i_X(g_i(c), c') \to \infty$ and therefore $d^{\pi}_X(g_i(c), c') \to \infty$.

$\square$

Let Γ be the subgroup of $MCG(\Sigma)$ from Proposition 4.8 and let $\mathbf{Y}^1, \cdots, \mathbf{Y}^k$ be the orbits of subsurfaces under Γ. Note that by construction one of the collections consists of the single surface Σ. Let

$$\Pi = \mathcal{C}(\mathbf{Y}^1) \times \mathcal{C}(\mathbf{Y}^2) \times \cdots \times \mathcal{C}(\mathbf{Y}^k)$$

be the product of complexes of curve complexes. Then $MCG(\Sigma)$ acts on Π. For elements in Γ the coordinates are fixed while other elements will permute them.

Define $\Psi : MCG(\Sigma) \to \Pi$ by choosing a base vertex as the image of 1 and extending the map equivariantly. Note that one of the factors in the target is just the curve complex $\mathcal{C}(\Sigma)$. We put the l_1-metric on the product space Π. By construction Ψ is Lipschitz.

Proposition 4.12. *Ψ is a coarse embedding.*

Proof. We will show that the restriction of Ψ to Γ is a coarse embedding. This will imply the proposition.

Say the basepoint has $\mathcal{C}(\mathbf{Y}^i)$-coordinate equal to a curve γ_i in a surface W_i, and in the special factor $\mathcal{C}(\Sigma)$ the coordinate is a curve γ. We may choose the binding set α to contain γ, the γ_i and the boundary components of the W_i's.

Note that for all subsurfaces W the diameter of $\pi_W(\alpha)$ in $\mathcal{C}(W)$ is bounded by a fixed constant $D > 0$. For example we could choose D to be one plus the number of intersection points.

Fix some $B > 0$ and let C be the constant given by Proposition 4.11 with respect to α and $B+2D$. We'll show that if $|g| > C$ then $d_\Pi(\Psi(id), \Psi(g)) > B$ which implies that Ψ is a coarse embedding.

By Proposition 4.11 there exists a subsurface W such that $d^{\pi}_W(\alpha, g(\alpha)) > C$. The subsurface W is in one of the collections $\mathbf{Y}^i$. Since $\pi_W(\gamma_i)$ and

$\pi_W(g(\gamma_i))$ are contained in $\pi_W(\alpha)$ and $\pi_W(g(\alpha))$ and the latter have diameter bounded by D we have $d^{\pi}_W(\gamma_i, g(\gamma_i)) \geq d(\alpha, g(\alpha)) - 2D$. By Proposition 3.1 we then have

$$\begin{aligned} d_\Pi(\Psi(id), \Psi(g)) &\geq d_{\mathcal{C}(\mathbf{Y}^i)}(\gamma_i, g(\gamma_i)) \\ &\geq d^{\pi}_W(\gamma_i, g(\gamma_i)) \\ &\geq \pi_W(\alpha, g(\alpha)) - 2D \\ &\geq B \end{aligned}$$

and the proposition is proved. □

It is also true that Ψ is a quasi-isometric embedding. We will not need this stronger result, but we include the proof since it may be of independent interest.

Theorem 4.13. *Ψ is a quasi-isometric embedding.*

Proof. The proof uses the remarkable Masur-Minsky formula [MM00], which asserts that

$$|g| \sim \sum_W \{\{d_W(\alpha, g(\alpha)\}\}_M$$

where $g \in MCG(\Sigma)$, $|g|$ is the word-norm of g with respect to any fixed finite generating set for $MCG(\Sigma)$, $\sim$ is coarse equivalence, i.e. each side is bounded by a linear function of the other, $\{\{x\}\}_M = x$ if $x > M$ and otherwise it is 0, the sum is taken over all subsurfaces of Σ, α is a fixed finite binding set of curves in Σ, and $d_W(\alpha, g(\alpha))$ is the distance in the curve complex of W between the projections of a curve in α and a curve in $g(\alpha)$ (we must choose a curve that has a projection; choosing a different such curve changes the distance by a bounded amount), and M is a sufficiently large constant. By enlarging M or K' from Corollary 3.12 we may assume that $M = K'$. The two estimates combine to give that $|g|\ leqAd(\Psi(1), \Psi(g)) + B$ for universal constants A, B. The reverse bound follows from the fact that Ψ is Lipschitz. □

Theorem 4.14. *Let Σ be a compact orientable surface with (possibly empty) boundary. Then* $\operatorname{asdim}(MCG(\Sigma)) < \infty$.

Proof. If $\chi(\Sigma) > 0$ then $MCG(\Sigma)$ is finite and $\operatorname{asdim}(MCG(\Sigma)) = 0$. If the $\chi(\Sigma) = 0$ is a torus, $MCG(\Sigma)$ is virtually free and hence $\operatorname{asdim}(MCG(\Sigma)) = 1$. Assume $\chi(\Sigma) < 0$. By the Product Theorem and Theorem 3.22 it follows that $\operatorname{asdim}(\Pi) < \infty$. Proposition 4.12 then implies that $\operatorname{asdim}(MCG(\Sigma)) < \infty$. □

Let Σ be a possibly punctured closed surface and $\mathcal{T}(\Sigma)$ its Teichmüller space equipped with the Teichmüller metric.

Theorem 4.15. $\operatorname{asdim}(MCG(\Sigma)) \leq \operatorname{asdim}(\mathcal{T}(\Sigma)) < \infty$.

Since $MCG(\Sigma)$ acts on $\mathcal{T}(\Sigma)$ properly discontinuously we have $\operatorname{asdim}(MCG(\Sigma)) \leq \operatorname{asdim}(\mathcal{T}(\Sigma))$. The proof of the second inequality will use the following facts. When γ is a curve in Σ and $\epsilon > 0$ denote by $Thin_\epsilon(\Sigma, \gamma)$ the subset of $\mathcal{T}(\Sigma)$ where γ has hyperbolic length $< \epsilon$.

(A) **Minsky's Product Theorem.** [Min96] If ϵ is small enough, $Thin_\epsilon(\Sigma, \gamma)$ is quasi-isometric to the product $\mathcal{T}(\Sigma/\gamma) \times Z$ where Z is a horoball in hyperbolic plane and Σ/γ denotes the surface obtained from Σ by cutting open along γ and crushing the boundary components to punctures (if γ is separating this Teichmüller space is the product of Teichmüller spaces of the components).

(B) For every $R > 0$ there is $\epsilon_0 > 0$ such that whenever γ and γ' intersect then $Thin_{\epsilon_0}(\Sigma, \gamma)$ and $Thin_{\epsilon_0}(\Sigma, \gamma')$ are R-separated.

Statement (B) follows easily from Kerckhoff's Theorem [Ker80], or indeed from (A).

Proof of Theorem 4.15. The proof is by induction on the complexity of the surface, which is the dimension of $\mathcal{T}(\Sigma)$. Induction starts with the case of 2-dimensional Teichmüller space (hyperbolic plane) when asymptotic dimension is 2.

For the inductive step, note that (A) and the Product Theorem for asymptotic dimension immediately imply that thin parts have finite asymptotic dimension. Write the collection of all curves on Σ as a finite disjoint union $C_1 \sqcup C_2 \sqcup \cdots \sqcup C_k$ so that curves in the same collection intersect. It was shown that this is possible for closed Σ in Lemma 4.6, but the punctured case follows quickly from the closed case (e.g. blow up the punctures to boundary components and double).

Consider the subsets

$$Thick = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_k = \mathcal{T}(\Sigma)$$

where X_i is the subset of $\mathcal{T}(\Sigma)$ consisting of hyperbolic surfaces with the property that if γ is a curve with length $< \epsilon$ then $\gamma \in C_1 \cup \cdots \cup C_i$, and Thick consists of hyperbolic surfaces with no essential curves of length $< \epsilon$. Let N

be chosen so that $\text{asdim}(MCG(\Sigma)) \leq N$ and so that $\text{asdim}(Thin_\epsilon(\Sigma, \gamma)) \leq N$ for every curve γ. We will argue by induction on i that $\text{asdim}(X_i) \leq N$.

When $i = 0$ this follows from the fact that X_0 (the thick part) is quasi-isometric to $MCG(\Sigma)$. Suppose $\text{asdim}(X_i) \leq N$.

Now write

$$X_{i+1} = X_i \cup \bigcup_{\gamma \in C_{i+1}} Y^i_\gamma$$

where Y^i_γ is the set of hyperbolic structures in $Thin_\epsilon(\Sigma, \gamma)$ where every curve shorter than ϵ is either equal to γ or belongs to $C_1 \cup \cdots \cup C_i$. We will check the conditions of the Union Theorem.

Let $R > 0$ be given, let ϵ_0 be as in (B) (we may assume that $\epsilon_0 < \epsilon$). Define

$$Y_R = X_i \cup \bigcup_{\gamma \in C_{i+1}} Z^i_\gamma$$

where Z^i_γ is the set of hyperbolic structures where γ has length in the interval $[\epsilon_0, \epsilon)$ and any curve of length $< \epsilon$ is either γ or belongs to $C_1 \cup \cdots \cup C_i$. By (B) the sets $Y^i_\gamma \backslash Y_R$ are R-separated and each set is contained in $Thin_\epsilon(\Sigma, \gamma)$ and the latter sets have asdim $\leq N$ uniformly, since there are only finitely many isometry types of such sets. Therefore we only need to argue that $\text{asdim}(Y_R) \leq N$. But Y_R is contained in a Hausdorff neighborhood of X_i, as follows easily from Minsky's Product Theorem. That $\text{asdim}(X_i) \leq N$ is the inductive hypothesis. □

A variation of the argument also shows that Teichmüller space equipped with Weil-Petersson metric has finite asymptotic dimension. Denote this space by $\mathcal{T}_{WP}(\Sigma)$. Let $\mathcal{P}(\Sigma)$ be the *pants complex* for Σ, where a vertex is represented by a pants decomposition of Σ and an edge corresponds to a pair of pants decompositions that differ in only one curve in each, and the two curves intersect minimally (one or two points, depending on whether their removal produces a complementary component which is a punctured torus or a 4-punctured sphere). There is a natural coarse map $\Upsilon : \mathcal{P}(\Sigma) \to \mathcal{T}_{WP}(\Sigma)$ that sends a pants decomposition to the (bounded) set consisting of hyperbolic metrics where the curves in the decomposition have length bounded by a Bers constant. Brock [Bro03, Bro02] proved that Υ is an equivariant quasi-isometry.

Theorem 4.16. $\text{asdim}(\mathcal{T}_{WP}(\Sigma)) = \text{asdim}(\mathcal{P}(\Sigma)) < \infty$.

Proof. Consider an orbit map $MCG(\Sigma) \to \mathcal{P}(\Sigma)$ and define a (pseudo) metric on $MCG(\Sigma)$ by restricting the one from $\mathcal{P}(\Sigma)$ (some pairs of points may have distance 0). Since the action of $MCG(\Sigma)$ on the pants complex has finitely many orbits of simplices, $MCG(\Sigma)$ with this metric, d, is quasi-isometric to the pants complex. There is a Masur-Minsky estimate for the distance between 1 and $g \in MCG(\Sigma)$ (see the discussion in [MM00, Section 8]):

$$d(1,g) \sim \sum_W \{\{d_W(\alpha, g(\alpha)\}\}_M$$

where W runs over subsurfaces which are *not* annuli. We have an action of $MCG(\Sigma)$ on

$$\Pi = \mathcal{C}(\mathbf{Y}^1) \times \mathcal{C}(\mathbf{Y}^2) \times \cdots \times \mathcal{C}(\mathbf{Y}^k)$$

as before, where we delete all annuli from the $\mathbf{Y}^i$'s. The orbit map is a quasi-isometric embedding (with respect to the new metric on $MCG(\Sigma)$) by exactly the same argument as before. The theorem follows. □

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