A DELORME-GUICHARDET THEOREM FOR QUANTUM GROUPS

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Abstract. We prove a Delorme-Guichardet Theorem for discrete quantum groups, expressing property (T) of a quantum group in terms of vanishing of its first cohomology groups. As an application, we show that the first L^2-Betti number of a discrete property (T) quantum group vanishes.

0. Introduction

The notion of property (T) was introduced by Kazhdan in his influential paper [Kaž67] and has since then played a prominent role in a variety of mathematical disciplines, including topology, ergodic theory and operator algebras. Recall, that a discrete group Γ has property (T) if any unitary representation of Γ with almost invariant vectors has a non-zero invariant vector. For details on these definitions and proofs of the classical theorems mentioned below we refer the reader to book [BdlHV08] by Bekka, de la Harpe and Valette. One reason why property (T) is an important notion is that it allows many different descriptions. Firstly, it can be described using the functions on Γ by means of the following theorem.

Theorem (de la Harpe-Valette)**.** *The group Γ has property* (T) *if and only if any sequence of positive definite functions $\varphi_n \colon \Gamma \to \mathbb{C}$ with $\varphi_n(e) = 1$ and converging pointwise to 1, converges uniformly to the constant function 1.*

Property (T) can also be described in terms of the first cohomology of Γ which, among other things, provides a link between property (T) and Serre's property (FA). The precise cohomological description is given by the celebrated Delorme-Guichardet Theorem.

Theorem (Delorme-Guichardet)**.** *The group Γ has property* (T) *if and only if the first group cohomology $H^1(\Gamma, H)$ vanishes for all Hilbert spaces H with a unitary Γ-action.*

Recently Fima introduced the notion of property (T) for the class of discrete quantum groups in the paper [Fim08] in which he also proved that several classical facts (see Theorem 1.7) can be transfered (mutatis-mutandi) from the theory of discrete groups to the more general setting of discrete quantum groups. The

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aim of the present paper is to show how the two classical theorems stated above can be generalized to the quantum group context. If $\hat{\mathbb{G}}$ is a discrete quantum group and $\mathbb{G}$ is its compact dual, we denote by $(\mathrm{Pol}(\mathbb{G}), \Delta, S, \varepsilon)$ the associated Hopf $*$-algebra of matrix coefficients and by $C(\mathbb{G}_{\mathrm{u}})$ the universal C^*-completion of $\mathrm{Pol}(\mathbb{G})$. These objects will be introduced and discussed in greater detail in Section 1 where we also elaborate on Fima's definition of property (T) and the results obtained in [Fim08]. The first main result (Theorem 3.1) of the present paper is a quantum group analogue of the result of de la Harpe and Valette and its precise statement is as follows.

Theorem. *The discrete quantum group* $\hat{\mathbb{G}}$ *has property* (T) *if and only if any net of states* $\varphi_i \colon C(\mathbb{G}_{\mathrm{u}}) \to \mathbb{C}$ *converging pointwise to the counit* ε *converges in the uniform norm.*

Secondly, we prove in Theorem 5.1 the following quantum group version of the Delorme-Guichardet Theorem.

Theorem. *The discrete quantum group* $\hat{\mathbb{G}}$ *has property* (T) *if and only if the following holds: for every* $*$*-representation* $\pi \colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ *on a Hilbert space* H *the first Hochschild cohomology of* $\mathrm{Pol}(\mathbb{G})$ *with values in the bimodule* ${}_\pi H_\varepsilon$ *vanishes.*

The relevant definitions concerning the first Hochschild cohomology are given in Section 4. Along the way, we obtain in Theorem 5.1 a characterization of property (T) in terms of conditionally negative functionals $\psi \colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ that parallels the classical description stating that a discrete group Γ has property (T) if and only if any conditionally negative function $\psi \colon \Gamma \to \mathbb{C}$ is bounded. Finally, using the quantum group version of the Delorme-Guichardet Theorem, we obtain in Corollary 6.1 the following extension of a well known result for groups.

Corollary. *If* $\hat{\mathbb{G}}$ *has property* (T) *then its first* L^2*-Betti number vanishes.*

The paper is organized as follows.

Structure. The first section provides the reader with the necessary background concerning the theory of compact quantum groups and their discrete duals and discusses the definition of property (T) for discrete quantum groups. In Section 2 we show how property (T) can be described in terms of the dual compact quantum group and use this description to give a spectral interpretation of property (T). Section 3 is devoted to the proof of the characterization of property (T) in terms of convergence of states on the associated universal C^*-algebra, and in Section 4 the proof of the quantum Delorme-Guichardet Theorem is given. In the sixth and final section we show how the results obtained can be used to derive information about the L^2-invariants of the quantum group in question.

Notation. Throughout the paper, the symbol $\odot$ will be used to denote algebraic tensor products while the symbol $\bar{\otimes}$ will be used to denote tensor products in the category of Hilbert spaces or in the category of von Neumann algebras. All tensor products between C^*-algebras are assumed minimal/spatial and these will be denoted by the symbol $\otimes$. All Hilbert spaces are assumed to be complex and their inner products are assumed linear in the first variable. Furthermore, $*$-representations of unital algebras on Hilbert spaces will always be assumed to be unit-preserving.

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1. Preliminaries on quantum groups

We choose here the approach to compact quantum groups developed by Woronowicz in [Wor87a], [Wor87b] and [Wor98]. Thus, a compact quantum group $\mathbb{G}$ consists of a (not necessarily commutative) separable, unital C^*-algebra $C(\mathbb{G})$ together with a unital, coassociative $*$-homomorphism $\Delta\colon C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ satisfying a certain density condition. The map Δ is referred to as the comultiplication. Such a quantum group possesses a unique Haar state; i.e. a state $h\colon C(\mathbb{G}) \to \mathbb{C}$ such that

$$(\mathrm{id} \otimes h)\Delta a = h(a)1 = (h \otimes \mathrm{id})\Delta(a) \text{ for every } a \in C(\mathbb{G}).$$

The GNS-construction applied to the Haar state yields a separable Hilbert space $L^2(\mathbb{G})$ together with a representation $\lambda\colon C(\mathbb{G}) \to B(L^2(\mathbb{G}))$ and a linear map $\Lambda\colon C(\mathbb{G}) \to L^2(\mathbb{G})$ with dense image. In general, the Haar state need not be faithful and hence λ might have a kernel. We denote by $C(\mathbb{G}_{\mathrm{red}})$ the image $\lambda(C(\mathbb{G}))$. This C^*-algebra inherits a quantum group structure from $\mathbb{G}$ and the comultiplication Δ_{red} on $C(\mathbb{G}_{\mathrm{red}})$ is implemented by the so-called multiplicative unitary $W \in B(L^2(\mathbb{G})\bar{\otimes}L^2(\mathbb{G}))$ given by

$$W^*(\Lambda(a) \otimes \Lambda(b)) = \Lambda \otimes \Lambda(\Delta(b)(a \otimes 1)).$$

The statement that W implements Δ_{red} means that $\Delta_{\mathrm{red}}(\lambda(a)) = W^*(1\otimes\lambda(a))W$ for every $a \in C(\mathbb{G})$. One notes that the right hand side of this formula makes sense for any $a \in B(L^2(\mathbb{G}))$ and it turns out that the enveloping von Neumann algebra $L^\infty(\mathbb{G}) = C(\mathbb{G}_{\mathrm{red}})''$ is turned into a compact von Neumann algebraic quantum group (see [KV03]) when endowed with this map as comultiplication.

A unitary corepresentation of $\mathbb{G}$ on a Hilbert space H is of a unitary element $u \in \mathcal{M}(\mathcal{K}(H) \otimes C(\mathbb{G}))$ such that $(\mathrm{id} \otimes\Delta)u = u_{(12)}u_{(13)}$. Here $\mathcal{K}(H)$ denotes the compact operators on H, $\mathcal{M}(\mathcal{K}(H) \otimes C(\mathbb{G}))$ is the multiplier algebra of $\mathcal{K}(H) \otimes C(\mathbb{G})$ and the subscripts (12) and (13) is the standard leg-numbering convention. Representation theoretic notions from the theory of compact groups, such as direct sums, tensor products, intertwiners and irreducibility, have natural counterparts in the corepresentation theory for compact quantum groups. In particular the following important theorem holds true.

Theorem 1.1 (Woronowicz). *Any irreducible unitary corepresentation of $\mathbb{G}$ is finite dimensional and an arbitrary unitary corepresentation decomposes as a direct sum irreducible ones.*

We denote by $\mathrm{Irred}(\mathbb{G})$ the set of equivalence classes of irreducible, unitary corepresentations of $\mathbb{G}$. The separability assumption on $C(\mathbb{G})$ together with the quantum Peter-Weyl theorem [Wor98] ensures that $\mathrm{Irred}(\mathbb{G})$ is a countable set. We label its elements by an auxiliary countable set I and choose for each $\alpha \in I$ a Hilbert space H^α and a concrete representative $u^\alpha \in B(H^\alpha) \otimes C(\mathbb{G})$. Abusing notation slightly, we shall often identify the index α with the corresponding class of u^α. Fix an orthonormal basis $\{e_1, \dots, e_{n_\alpha}\}$ for H^α and consider the corresponding functionals $\omega_{ij}\colon B(H^\alpha) \to \mathbb{C}$ given by $\omega_{ij}(T) = \langle Te_i | e_j\rangle$. The matrix coefficients of u^α, relative to the chosen basis, are then defined as

$$u_{ij}^\alpha = (\omega_{ij} \otimes \mathrm{id})u^\alpha \in C(\mathbb{G}).$$

It turns out that these matrix coefficients are linearly independent and that their linear span constitutes a dense $*$-subalgebra $\mathrm{Pol}(\mathbb{G})$ of $C(\mathbb{G})$. Furthermore, the comultiplication descends to a comultiplication $\Delta\colon \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G}) \odot \mathrm{Pol}(\mathbb{G})$ and with this comultiplication $\mathrm{Pol}(\mathbb{G})$ becomes a Hopf $*$-algebra; i.e. there exists an antipode $S\colon \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ as well as a counit $\varepsilon\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ satisfying the standard Hopf $*$-algebra relations [KS97]. The fact that $\mathrm{Pol}(\mathbb{G})$ is spanned by matrix coefficients arising from finite dimensional, unitary corepresentations of $\mathbb{G}$ also ensures that the relation

$$\|a\|_u = \sup\{\|\pi(x)\| \mid \pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H) \text{ a } *\text{-representation}\}$$

defines a C^*-norm $\|\cdot\|_u$ on $\mathrm{Pol}(\mathbb{G})$. The C^*-completion of $\mathrm{Pol}(\mathbb{G})$ with respect to this norm is called the universal C^*-algebra associated with $\mathbb{G}$ and is denoted $C(\mathbb{G}_{\mathrm{u}})$. By definition of $\|\cdot\|_u$, the comultiplication extends to a comultiplication $\Delta_{\mathrm{u}}\colon C(\mathbb{G}_{\mathrm{u}}) \to C(\mathbb{G}_{\mathrm{u}}) \otimes C(\mathbb{G}_{\mathrm{u}})$ turning $C(\mathbb{G}_{\mathrm{u}})$ into a compact quantum group. Note that the $*$-representations of $C(\mathbb{G}_{\mathrm{u}})$ are in one-to-one correspondence with the $*$-representations of $\mathrm{Pol}(\mathbb{G})$ via restriction/extension.

Example 1.2. The fundamental example, on which the general definition is modeled, is obtained by considering a compact, second countable, Hausdorff topological group G and its commutative C^*-algebra $C(G)$ of continuous, complex valued functions. In this case the comultiplication is the Gelfand dual of the multiplication map $G \times G \to G$ and the Haar state is given by integration against the unique Haar probability measure μ on G. The GNS-space therefore identifies with $L^2(G,\mu)$ and the representation λ with the action of $C(G)$ on $L^2(G,\mu)$ by pointwise multiplication. Similarly, the von Neumann algebra identifies with $L^\infty(G,\mu)$ and the Hopf $*$-algebra $\mathrm{Pol}(G)$ is the subalgebra of $C(G)$ generated by matrix coefficients arising from irreducible, unitary representations of G. The antipode in $\mathrm{Pol}(G)$ is the Gelfand dual of the inversion map and the counit is given by evaluation at the identity element in G.

In the previous example there is no real difference between the reduced and universal version of the compact quantum group. The next example, however, will illustrate this difference more clearly.

Example 1.3. Consider a countable, discrete group Γ. Denote by $C^*_{\mathrm{red}}(\Gamma)$ its reduced group C^*-algebra acting on $\ell^2(\Gamma)$ via the left regular representation and define a comultiplication on group elements by $\Delta_{\mathrm{red}}\gamma = \gamma\otimes\gamma$. This turns $C^*_{\mathrm{red}}(\Gamma)$ into a compact quantum group whose Haar state is given by the natural trace on $C^*_{\mathrm{red}}(\Gamma)$. Hence the GNS-space and GNS-representation can be identified, respectively, with $\ell^2(\Gamma)$ and the left regular representation, and the enveloping von Neumann algebra is therefore nothing but the group von Neumann algebra $\mathscr{L}(\Gamma)$. Each element in Γ is a one-dimensional corepresentation for this quantum group and the Hopf $*$-algebra therefore identifies with the complex group algebra $\mathbb{C}\Gamma$. Thus, the universal C^*-algebra is, by definition, equal to the maximal group C^*-algebra $C^*_{\mathrm{u}}(\Gamma)$.

Remark 1.4. The three C^*-algebras $C(\mathbb{G}), C(\mathbb{G}_{\mathrm{red}})$ and $C(\mathbb{G}_{\mathrm{u}})$, together with their comultiplications, can be thought of as "different pictures of the same quantum group", each having its advantages and disadvantages. For instance, whereas the the Haar state is always faithful on $C(\mathbb{G}_{\mathrm{red}})$ this is in general not the case on $C(\mathbb{G}_{\mathrm{u}})$ and, conversely, the counit is always well defined on all of $C(\mathbb{G}_{\mathrm{u}})$ but not necessarily on $C(\mathbb{G}_{\mathrm{red}})$. The latter difference is the fundamental observation leading to the notion of (co)-amenability for quantum groups as studied in [BMT01].

Any compact quantum group $\mathbb{G}$ has a dual quantum group $\hat{\mathbb{G}}$ of so-called discrete type. As in the compact case, $\hat{\mathbb{G}}$ comes with both a C^*-algebra $c_0(\hat{\mathbb{G}})$ and a von Neumann algebra $\ell^\infty(\hat{\mathbb{G}})$ defined, respectively, as

$$c_0(\hat{\mathbb{G}}) = \bigoplus_{\alpha\in I}^{c_0} B(H^\alpha) \quad\text{and}\quad \ell^\infty(\hat{\mathbb{G}}) = \prod_{\alpha\in I}^{\ell^\infty} B(H^\alpha).$$

In the discrete picture we will primarily be working with the von Neumann algebra $\ell^\infty(\hat{\mathbb{G}})$. This algebra is endowed with a natural comultiplication $\hat{\Delta}\colon \ell^\infty(\hat{\mathbb{G}}) \to \ell^\infty(\hat{\mathbb{G}})\bar{\otimes}\ell^\infty(\hat{\mathbb{G}})$ arising from the quantum group structure on $\mathbb{G}$. Since $c_0(\hat{\mathbb{G}})$ is a direct sum of finite dimensional C^*-algebras we have isomorphisms

$$\ell^\infty(\hat{\mathbb{G}})\bar{\otimes}B(H) \simeq \mathcal{M}(c_0(\hat{\mathbb{G}})\otimes B(H)) \simeq \prod_{\alpha\in I}^{\ell^\infty} B(H^\alpha)\otimes B(H)$$

for any Hilbert space H. For an element $T \in \ell^\infty(\hat{\mathbb{G}})\bar{\otimes}B(H)$ we will denote by $(T^\alpha)_{\alpha\in I}$ the corresponding element in $\prod_{\alpha\in I}^{\ell^\infty} B(H^\alpha)\otimes B(H)$ and in the sequel we will freely identify T and $(T^\alpha)_{\alpha\in I}$. By a unitary corepresentation of $\hat{\mathbb{G}}$ on a Hilbert space H we shall mean a unitary operator $V \in \ell^\infty(\hat{\mathbb{G}})\bar{\otimes}B(H)$ satisfying

$$(\hat{\Delta}\otimes \mathrm{id})V = V_{(13)}V_{(23)}.$$

Consider the unitary $V_{\mathrm{u}} = (u^\alpha)_{\alpha\in I} \in \prod_{\alpha\in I}^{\ell^\infty} B(H_\alpha)\otimes C(\mathbb{G}_{\mathrm{u}})$. This unitary encodes the duality between $\mathbb{G}$ and $\hat{\mathbb{G}}$ in the following sense: for every unitary corepresentation $V \in \ell^\infty(\hat{\mathbb{G}})\bar{\otimes}B(H)$ of $\hat{\mathbb{G}}$ there exists a unique $*$-representation $\pi_V\colon C(\mathbb{G}_{\mathrm{u}}) \to B(H)$ such that

$$(\mathrm{id}\otimes\pi_V)u^\alpha = V^\alpha \quad \text{for each } \alpha\in I.$$

Conversely, every $*$-representation $\pi\colon C(\mathbb{G}_{\mathrm{u}}) \to B(H)$ defines a unitary corepresentation of $\hat{\mathbb{G}}$ on H by the above relation. For detailed introductions to compact quantum groups and their discrete duals we refer the reader to the survey articles [MVD98] and [KT99] as well as the original articles [VD96] and [Wor98].

The notion of property (T) was recently introduced in the quantum group setting by Fima in [Fim08] and the definition is as follows.

Definition 1.5 (Fima). *Let $\hat{\mathbb{G}}$ be a discrete quantum group and consider a unitary corepresentation $V = (V^\alpha)_{\alpha\in I} \in \ell^\infty(\hat{\mathbb{G}})\bar{\otimes}B(H)$ of $\hat{\mathbb{G}}$ on a Hilbert space H.*

(i) *A vector $\xi\in H$ is said to be invariant if $V^\alpha(\eta\otimes\xi) = \eta\otimes\xi$ for all $\alpha\in I$ and $\eta\in H^\alpha$.*

(ii) *For a finite, non-empty subset $E\subseteq \mathrm{Irred}(\mathbb{G})$ and a $\delta>0$ a vector $\xi\in H$ is called (E,δ)-invariant if*

$$\|V^\alpha(\eta\otimes\xi)-\eta\otimes\xi\| < \delta\|\eta\|\|\xi\| \quad \text{for all } \alpha\in E \text{ and all } \eta\in H^\alpha,$$

and V is said to have almost-invariant vectors if it has an (E,δ)-invariant vector for each finite, non-empty $E\subseteq\mathrm{Irred}(\mathbb{G})$ and each $\delta>0$.

(iii) *The discrete quantum group $\hat{\mathbb{G}}$ is said to have property* (T) *if any unitary corepresentation of $\hat{\mathbb{G}}$ with almost invariant vectors has a non-zero invariant vector.*

Remark 1.6. For notational smoothness we will adopt the convention that, unless explicitly specified otherwise, subsets E of $\mathrm{Irred}(\mathbb{G})$ are always both finite and non-empty.

The main results in [Fim08] are summarized in the following theorem.

Theorem 1.7 (Fima). *If $\hat{\mathbb{G}}$ is a discrete quantum group with property* (T) *then the following holds.*

(i) *The quantum group is automatically a Kac-algebra; i.e. the Haar state $h\colon C(\mathbb{G})\to\mathbb{C}$ is a trace.*

(ii) *The discrete quantum group is finitely generated; i.e. the corepresentation category* $\mathrm{Corep}(\mathbb{G})$ *of the compact dual is a finitely generated tensor category.*

(iii) *The quantum group allows Kazhdan pairs; i.e. for every finite subset* $E \subseteq \mathrm{Irred}(\mathbb{G})$ *generating the corepresentation category and containing the trivial corepresentation there exists a* $\delta > 0$ *such that whenever* V *is a unitary corepresentation of* $\hat{\mathbb{G}}$ *having an* (E, δ)*-invariant vector, then* V *has a non-trivial invariant vector.*

Moreover, if $\mathbb{G}$ *is an infinite, compact quantum group such that* $L^\infty(\mathbb{G})$ *is a factor then* $\hat{\mathbb{G}}$ *has property* (T) *iff* $L^\infty(\mathbb{G})$ *is a type* II_1*-factor with property* (T) *in the sense of Connes-Jones* [CJ85].

In the following section we show how property (T) can be expressed using $*$-representations of the Hopf $*$-algebra $\mathrm{Pol}(\mathbb{G})$.

2. Property (T) from the dual point of view

In this section we reformulate property (T) for discrete quantum groups in terms of their compact duals and use this description to give a spectral characterization of property (T). In the compact setting it is natural to consider the following notions of invariance and almost invariance.

Definition 2.1. *Let* $\pi \colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ *be a* $*$*-representation. A vector* $\xi \in H$ *is said to be invariant if* $\pi(a)\xi = \varepsilon(a)\xi$ *for all* $a \in \mathrm{Pol}(\mathbb{G})$*. If a non-zero invariant vector exists,* π *is said to contain the counit. For a subset* $E \subseteq \mathrm{Irred}(\mathbb{G})$ *and* $\delta > 0$ *a vector* $\xi \in H$ *is said to be* (E, δ)*-invariant if*

$$\|\pi(u^\alpha_{ij})\xi - \varepsilon(u^\alpha_{ij})\xi\| < \delta\|\xi\|,$$

for all $\alpha \in E$ *and* $i, j \in \{1, \ldots, n_\alpha\}$*. The representation* π *is said to have almost invariant vectors if it allows a non-zero* (E, δ)*-invariant vector for every* $E \subseteq \mathrm{Irred}(\mathbb{G})$ *and every* $\delta > 0$*.*

Remark 2.2. Since the set $\{u^\alpha_{ij} \mid \alpha \in I, 1 \leq i, j \leq n_\alpha\}$ spans $\mathrm{Pol}(\mathbb{G})$ linearly it is not difficult to see that a representation $\pi \colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ has almost invariant vectors iff there exists a sequence of unit vectors $\xi_n \in H$ such that

$$\lim_{n\to\infty} \|\pi(a)\xi_n - \varepsilon(a)\xi_n\| = 0$$

for every $a \in \mathrm{Pol}(\mathbb{G})$.

The following proposition contains the translation of property (T) from the discrete to the compact picture.

Proposition 2.3. *Let* $\hat{\mathbb{G}}$ *be a discrete quantum group and consider a unitary corepresentation* $V \in \ell^\infty(\hat{\mathbb{G}})\bar{\otimes}B(H)$ *as well as the the corresponding* $*$*-representation* $\pi_V \colon \mathrm{Pol}(\mathbb{G}) \to B(H)$*. Let furthermore* $E \subseteq \mathrm{Irred}(\mathbb{G})$ *and* $\delta > 0$ *be given and define* $K_E = \max\{n_\alpha \mid \alpha \in E\}$*. Then the following holds.*

(i) *A vector* $\xi \in H$ *is* V*-invariant if and only if it is* π_V*-invariant.*
(ii) *If* $\xi \in H$ *is* (E, δ)*-invariant for* V *then it is also* (E, δ)*-invariant for* π_V*.*
(iii) *If* $\xi \in H$ *is* (E, δ)*-invariant for* π_V *then it is* $(E, K_E\delta)$*-invariant for* V*.*

Thus, V *has almost-invariant vectors if and only if* π_V *has almost invariant vectors. In particular, the discrete quantum group* $\hat{\mathbb{G}}$ *has property* (T) *iff any* $*$*-representation* $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ *with almost invariant vectors has a non-zero invariant vector.*

Proof. Consider the fixed basis $\{e_1, \dots, e_{n_\alpha}\}$ of H_α and the corresponding functionals $e_i'\colon H_\alpha \to \mathbb{C}$ and $\omega_{ij}\colon B(H_\alpha) \to \mathbb{C}$ given, respectively, by

$$e_i'(x) = \langle x | e_i \rangle \quad \text{and} \quad \omega_{ij}(T) = \langle T e_i | e_j \rangle.$$

A vector $\xi \in H$ is V-invariant exactly when $V^\alpha(e_j \otimes \xi) = e_j \otimes \xi$ for any $\alpha \in I$ and $j \in \{1, \dots, n_\alpha\}$. This in turn holds iff

$$(e_i' \otimes \mathrm{id})V^\alpha(e_j \otimes \xi) = e_i'(e_j)\xi \quad \text{for all } \alpha \in I \text{ and } i, j \in \{1, \dots, n_\alpha\}.$$

Keeping in mind that $\varepsilon(u_{ij}^\alpha) = \delta_{ij}$, the above equation translates into

$$\pi_V(u_{ij}^\alpha)\xi = \varepsilon(u_{ij}^\alpha)\xi \quad \text{for all } \alpha \in I \text{ and } i, j \in \{1, \dots, n_\alpha\},$$

which is equivalent to ξ being π_V-invariant since the u_{ij}^α's constitute a linear basis for $\mathrm{Pol}(\mathbb{G})$. This proves (i). To prove (ii), fix $E \subseteq \mathrm{Irred}(\mathbb{G})$ and $\delta > 0$ and assume that $\xi \in H$ is an (E, δ)-invariant unit vector for V. Then for each $\alpha \in E$ and $i, j \in \{1, \dots, n_\alpha\}$ we have

$$\|\pi_V(u_{ij}^\alpha)\xi - \varepsilon(u_{ij}^\alpha)\xi\| = \|(e_i' \otimes \mathrm{id})[V^\alpha(e_j \otimes \xi) - e_j \otimes \xi]\| \leq \|V^\alpha(e_j \otimes \xi) - e_j \otimes \xi\| < \delta,$$

as desired. To prove (iii), assume that $\xi \in H$ is an (E, δ)-invariant unit vector for π_V. For $j \in \{1, \dots, n_\alpha\}$ we now get

$$\begin{aligned}
\|V^\alpha(e_j \otimes \xi) - e_j \otimes \xi\|^2 &= \sum_{i=1}^{n_\alpha} \|(e_i' \otimes \mathrm{id})[V^\alpha(e_j \otimes \xi) - e_j \otimes \xi]\|^2 \\
&= \sum_{i=1}^{n_\alpha} \|\pi_V(u_{ij}^\alpha)\xi - \varepsilon(u_{ij}^\alpha)\xi\|^2 < n_\alpha \delta^2.
\end{aligned}$$

Hence for $\eta = \sum_{i=1}^{n_\alpha} \eta_i e_i \in H_\alpha$ we get by Hölder's inequality

$$\begin{aligned}
\|V^\alpha(\eta \otimes \xi) - \eta \otimes \xi\| &\leq \sum_{i=1}^{n_\alpha} |\eta_i| \|V^\alpha(e_i \otimes \xi) - e_i \otimes \xi\| \\
&< \|\eta\|_1 \sqrt{n_\alpha}\delta \\
&\leq \|\eta\|_2 n_\alpha \delta,
\end{aligned}$$

which shows that ξ is $(E, K_E\delta)$-invariant for V. □

Similarly, the existence of Kazhdan pairs also translates to the dual picture.

Corollary 2.4. *Let* $\hat{\mathbb{G}}$ *have property* (T) *and let* $E \subseteq \mathrm{Irred}(\mathbb{G})$ *be a finite subset containing the trivial corepresentation* 1 *which generates the corepresentation category of* $\mathbb{G}$. *Then there exists* $\delta > 0$ *such that any representation* $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ *having an* (E, δ)*-invariant vector has a non-zero invariant vector.*

Proof. Assume that $\hat{\mathbb{G}}$ has property (T) and let $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ be given. Denote by V the corresponding corepresentation of $\hat{\mathbb{G}}$ on H and choose $\delta > 0$ such that (E, δ) is a Kazhdan pair for V. If we put $K_E = \max\{n_\alpha \mid \alpha \in E\}$ and $\xi \in H$ is an $(E, K_E^{-1}\delta)$-invariant vector for π then, by Proposition 2.3 (iii), ξ is an (E, δ)-invariant vector for V. Hence V allows a non-zero invariant vector which is then also invariant for π by Proposition 2.3 (i). □

Recall from Theorem 1.7 that a discrete property (T) quantum group is automatically finitely generated. Consider now any finitely generated, discrete quantum group $\hat{\mathbb{G}}$ and let $E \subseteq \mathrm{Irred}(\mathbb{G})$ be a finite generating set for $\mathrm{Corep}(\mathbb{G})$ containing the trivial corepresentation. For each $\alpha \in E$ and $i, j \in \{1, \dots, n_\alpha\}$ define $x_{ij}^\alpha = u_{ij}^\alpha - \varepsilon(u_{ij}^\alpha)1$ and put $x = \sum_{\alpha \in E, i,j} x_{ij}^{\alpha *} x_{ij}^\alpha$. Property (T) can then be read of the element x by means of the following result.

Theorem 2.5. *The discrete quantum group* $\hat{\mathbb{G}}$ *has property* (T) *if and only if zero is not in the spectrum of* $\pi(x)$ *for any representation* $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ *not containing the counit.*

Proof. Assume that $\hat{\mathbb{G}}$ has property (T) and let $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ be a representation such that $\pi(x)$ is not bounded away from zero. Then there exists a sequence $(\xi_k)_{k \in \mathbb{N}}$ in the unit ball of H such that $\pi(x)\xi_k \to 0$. Hence

$$\begin{aligned} 0 &= \lim_k \langle \pi(x)\xi_k \,|\, \xi_k \rangle \\ &= \lim_{k\to\infty} \sum_{\alpha \in E} \sum_{i,j=1}^{n_\alpha} \langle \pi(x_{ij}^\alpha)^* \pi(x_{ij}^\alpha)\xi_k \,|\, \xi_k \rangle \\ &= \lim_{k\to\infty} \sum_{\alpha \in E} \sum_{i,j=1}^{n_\alpha} \|\pi(u_{ij}^\alpha)\xi_k - \varepsilon(u_{ij}^\alpha)\xi_k\|^2. \end{aligned}$$

For suitable $\delta > 0$ the pair (E, δ) is a Kazhdan pair for $\hat{\mathbb{G}}$ and therefore the above convergence forces π to have a non-trivial invariant vector; hence π contains ε. Conversely, assume that $\hat{\mathbb{G}}$ does not have property (T). Then there exists a representation $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ with almost invariant vectors, but without non-zero invariant vectors. In particular we may find a sequence of unit vectors $(\xi_k)_{k \in \mathbb{N}}$ in H such that

$$\lim_{k\to\infty} \sum_{\alpha \in E} \sum_{i,j=1}^{n_\alpha} \|\pi(u_{ij}^\alpha)\xi_k - \varepsilon(u_{ij}^\alpha)\xi_k\|^2 = 0.$$

On the other hand we have that

$$\sum_{\alpha\in E}\sum_{i,j=1}^{n_\alpha}\|\pi(u_{ij}^\alpha)\xi_k-\varepsilon(u_{ij}^\alpha)\xi_k\|^2=\langle\pi(x)\xi_k\,|\,\xi_k\rangle=\|\pi(x)^{\frac{1}{2}}\xi_k\|^2,$$

and hence zero is in the spectrum of $\pi(x)^{\frac{1}{2}}$. Thus π is a representation not containing ε such that $\pi(x)^{\frac{1}{2}}$, and hence also $\pi(x)$, is not invertible.

□

Remark 2.6. The above spectral characterization of property (T) should be compared with the Kesten condition for coamenability [Ban99], which states that $\mathbb{G}$ is coamenable iff zero is in the spectrum of $\lambda(x)$. Theorem 2.5 is an extension of a result for groups due to de la Harpe, Robertson and Valette [dlHRV93].

As in the classical case, we also get a version of property (T) with "continuity constants".

Proposition 2.7. *A discrete quantum group* $\hat{\mathbb{G}}$ *has property* (T) *if and only if the following holds: For every* $\delta>0$ *there exists* $E_0\subseteq\mathrm{Irred}(\mathbb{G})$ *and* $\delta_0>0$ *such that any representation* $\pi\colon\mathrm{Pol}(\mathbb{G})\to B(H)$ *with an* $(E_0,\delta_0\delta)$*-invariant vector* $\xi\in H$ *has an invariant vector* $\eta\in H$ *such that* $\|\xi-\eta\|<\delta\|\xi\|$.

The proof of the proposition is basically identical to the corresponding proof in the group case [BdlHV08, 1.1.19], but we include it here for the sake of completeness.

Proof. Assume that $\hat{\mathbb{G}}$ has property (T) and choose a Kazhdan pair (E_0,δ_0) for $\hat{\mathbb{G}}$. Let $\pi\colon\mathrm{Pol}(\mathbb{G})\to B(H)$ and $\delta>0$ be given and denote by $P\in B(H)$ the projection onto the closed subspace of invariant vectors. Assume furthermore that ξ is an $(E_0,\delta_0\delta)$-invariant vector and decompose ξ as $\xi=\xi'+\xi''$ with $\xi'=P\xi$ and $\xi''=(1-P)\xi$. Since $P(H)^\perp$ does not have non-zero invariant vectors and (E_0,δ_0) is a Kazhdan pair there must exist $\beta\in E_0$ and $k,l\in\{1,\dots,n_\beta\}$ such that

$$\|\pi(u_{kl}^\beta)\xi''-\varepsilon(u_{kl}^\beta)\xi''\|\geq\delta_0\|\xi''\|.$$

Using that ξ is $(E_0,\delta_0\delta)$-invariant we get

$$\delta_0\delta\|\xi\|>\|\pi(u_{kl}^\beta)\xi-\varepsilon(u_{kl}^\beta)\xi\|=\|\pi(u_{kl}^\beta)\xi''-\varepsilon(u_{kl}^\beta)\xi''\|\geq\delta_0\|\xi''\|,$$

and hence that $\delta\|\xi\|>\|\xi''\|$. Putting $\eta=\xi'$ we get

$$\|\xi-\eta\|=\|\xi-\xi'\|=\|\xi''\|<\delta\|\xi\|$$

as desired. The opposite implication is obvious.

□

In particular we get the following convenient description of property (T).

Corollary 2.8. *A discrete quantum group* $\hat{\mathbb{G}}$ *has property* (T) *if and only if the following holds: For any* $\delta > 0$ *there exist* $E_0 \subseteq \mathrm{Irred}(\mathbb{G})$ *and* $\delta_0 > 0$ *such that any representation* $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ *with an* (E_0, δ_0)*-invariant unit vector* $\xi \in H$ *has an invariant vector* $\eta \in H$ *such that* $\|\xi - \eta\| < \delta$.

Proof. Assume $\hat{\mathbb{G}}$ has property (T) and let $\delta > 0$ be given. Assume without loss of generality that $\delta \leq 1$ and choose a Kazhdan pair (E_0, δ_0') for $\hat{\mathbb{G}}$. Put $\delta_0 = \delta_0'\delta$. Then (E_0, δ_0) is also a Kazhdan pair and from (the proof of) Proposition 2.7 we get that the pair (E_0, δ_0) satisfies the claim. □

3. Property (T) in terms of states on the universal C^*-algebra

As already mentioned in the introduction, a discrete group Γ has property (T) exactly when every sequence of normalized, positive definite functions on Γ converging pointwise to 1 actually converges uniformly to 1. Recall that a function $\varphi\colon \Gamma \to \mathbb{C}$ is called normalized if $\varphi(e) = 1$ and positive definite if

$$\sum_{i=1}^{n} \bar{\alpha}_i \alpha_j \varphi(\gamma_i^{-1}\gamma_j) \geq 0 \quad \text{for all } n \in \mathbb{N}, \gamma_1, \ldots, \gamma_n \in \Gamma \text{ and } \alpha_1, \ldots, \alpha_n \in \mathbb{C}.$$

Hence, there is a one-to-one correspondence between normalized, positive definite functions on Γ and states on the universal group C^*-algebra $C^*_{\mathrm{u}}(\Gamma)$. Having this correspondence in mind, the following theorem generalizes the classical result.

Theorem 3.1. *A discrete quantum group* $\hat{\mathbb{G}}$ *has property* (T) *if and only if any net of states on* $C(\mathbb{G}_{\mathrm{u}})$ *converging pointwise to the counit* ε *converges in the uniform norm.*

Here *the uniform norm* is the norm on the state space of $C(\mathbb{G}_{\mathrm{u}})$ given by

$$\|\varphi\| = \sup\{|\varphi(a)| \mid \|a\|_{\mathrm{u}} \leq 1\},$$

and convergence in this norm will often be referred to as uniform convergence. The fact that property (T) implies the convergence property was proved independently by Fima (private communication) in the dual picture. For the proof of Theorem 3.1 we need the following lemma.

Lemma 3.2. *Let* $\delta > 0$ *and* $\pi\colon C(\mathbb{G}_{\mathrm{u}}) \to B(H)$ *be a* $*$*-representation. If* $\xi \in H$ *is a unit vector such that* $\|\pi(v)\xi - \varepsilon(v)\xi\| \leq \delta$ *for every unitary* $v \in C(\mathbb{G}_{\mathrm{u}})$ *then there exists an invariant vector* $\eta \in H$ *such that* $\|\xi - \eta\| \leq \delta$.

The proof of Lemma 3.2 is inspired by the corresponding proof for (pairs of) groups which can be found in [Jol05]. For the proof, and throughout the rest of the paper, we denote the unitary group of $C(\mathbb{G}_{\mathrm{u}})$ by $\mathcal{U}$.

Proof of Lemma 3.2. Denote by C the closed, convex hull of the set

$$\Omega = \{\pi(v)\xi\varepsilon(v^*) \mid v \in \mathcal{U}\}.$$

For any element $\eta = \sum_{k=1}^{n} t_k \pi(v_k)\xi\varepsilon(v_k^*)$ in the convex hull of Ω we have

$$\begin{aligned}\|\xi - \eta\| &= \|\sum_{k=1}^{n} t_k(\pi(v_k)\xi\varepsilon(v_k^*) - \xi)\| \\ &\leq \sum_{k=1}^{n} t_k \|\pi(v_k)\xi\varepsilon(v_k^*) - \xi\| \\ &= \sum_{k=1}^{n} t_k \|\pi(v_k)\xi - \varepsilon(v_k)\xi\| \leq \delta,\end{aligned}$$

and hence $\|\xi - \eta\| \leq \delta$ for any $\eta \in C$. Now let $\eta \in C$ be the unique element of minimal norm [KR83, 2.2.1]. For every $v \in \mathcal{U}$ we have

$$\begin{aligned}\pi(v)\Omega &= \pi(v)\{\pi(u)\xi\varepsilon(u^*) \mid u \in \mathcal{U}\} \\ &= \pi(v)\{\pi(v^*u)\xi\varepsilon(u^*v) \mid u \in \mathcal{U}\} \\ &= \Omega\varepsilon(v),\end{aligned}$$

and hence $\pi(v)C = C\varepsilon(v)$. Since $\pi(v)\eta$ is the element of minimal norm in $\pi(v)C$ and $\eta\varepsilon(v)$ is the ditto element in $C\varepsilon(v)$ we conclude that $\pi(v)\eta = \eta\varepsilon(v)$ for every $v \in \mathcal{U}$. But since the elements in $\mathcal{U}$ span $C(\mathbb{G}_{\mathrm{u}})$ linearly the vector η is invariant. □

Proof of Theorem 3.1. Assume first that $\hat{\mathbb{G}}$ has property (T) and consider any net $(\varphi_\lambda)_{\lambda\in\Lambda}$ of states on $C(\mathbb{G}_{\mathrm{u}})$ converging pointwise to ε. Denote by $(H_\lambda, \pi_\lambda, \xi_\lambda)$ the GNS-triple associated with φ_λ. A straight forward calculation reveals that

$$|\varphi_\lambda(a) - \varepsilon(a)|^2 = \|\pi_\lambda(a)\xi_\lambda - \varepsilon(a)\xi_\lambda\|^2 - (\varphi_\lambda(a^*a) - \varphi_\lambda(a^*)\varphi_\lambda(a)) \tag{1}$$

for any $a \in C(\mathbb{G}_{\mathrm{u}})$. Note also, that the Cauchy-Schwartz inequality implies that $\varphi_\lambda(a^*a) - \varphi_\lambda(a^*)\varphi_\lambda(a) \geq 0$ and that this quantity converges to zero. Hence $\lim_\lambda \|\pi_\lambda(a)\xi_\lambda - \varepsilon(a)\xi_\lambda\| = 0$ for every $a \in C(\mathbb{G}_{\mathrm{u}})$. Let $\delta > 0$ be given. Since $\hat{\mathbb{G}}$ has property (T), Corollary 2.8 allows us to find a Kazhdan pair (E_0, δ_0) such that any representation with an (E_0, δ_0)-invariant unit vector ξ has an invariant vector η such that $\|\xi - \eta\| \leq \frac{\delta}{2}$. We now claim that the representation

$$\pi = \oplus_{\lambda\in\Lambda}\pi_\lambda \colon \mathrm{Pol}(\mathbb{G}) \to B(\oplus_{\lambda\in\Lambda}^{\ell^2} H_\lambda)$$

is of this type. To see this, denote by $\tilde{\xi_\lambda}$ the image of ξ_λ under the natural embedding of H_λ into $H = \oplus_{\mu\in\Lambda} H_\mu$ and note that

$$\|\pi(a)\tilde{\xi_\lambda} - \varepsilon(a)\tilde{\xi_\lambda}\| = \|\pi_\lambda(a)\xi_\lambda - \varepsilon(a)\xi_\lambda\| \underset{\lambda}{\longrightarrow} 0$$

for any $a \in C(\mathbb{G}_{\mathrm{u}})$. In particular we get an $\lambda_0 \in \Lambda$ such that

$$\forall \lambda \geq \lambda_0 \ \forall \alpha \in E_0 \ \forall i, j \in \{1, \ldots, n_\alpha\} : \|\pi(u_{ij}^\alpha)\tilde{\xi_\lambda} - \varepsilon(u_{ij}^\alpha)\tilde{\xi_\lambda}\| < \delta_0,$$

and we may therefore find an invariant vector $\tilde{\eta}_\lambda \in H$ such that $\|\tilde{\xi}_\lambda - \tilde{\eta}_\lambda\| \leq \frac{\delta}{2}$ for all $\lambda \geq \lambda_0$. The equation (1) now gives

$$\begin{aligned}|\varphi_\lambda(a) - \varepsilon(a)| &\leq \|\pi_\lambda(a)\xi_\lambda - \varepsilon(a)\xi_\lambda\| \\ &= \|\pi(a)\tilde{\xi}_\lambda - \varepsilon(a)\tilde{\xi}_\lambda\| \\ &= \|\pi(a)(\tilde{\xi}_\lambda - \tilde{\eta}_\lambda) + \varepsilon(a)(\tilde{\eta}_\lambda - \tilde{\xi}_\lambda)\| \\ &\leq \|\pi(a)\|\|\tilde{\xi}_\lambda - \tilde{\eta}_\lambda\| + |\varepsilon(a)|\|\tilde{\xi}_\lambda - \tilde{\eta}_\lambda\|^2 \\ &\leq \delta\|a\|_{\mathrm{u}}\end{aligned}$$

for every $\lambda \geq \lambda_0$. Hence $(\varphi_\lambda)_{\lambda\in\Lambda}$ converges uniformly to ε as desired.

Assume, conversely, that $\hat{\mathbb{G}}$ does not have property (T) and choose an increasing sequence of subsets $E_n \subseteq \mathrm{Irred}(\mathbb{G})$ with union $\mathrm{Irred}(\mathbb{G})$. This is possible since $C(\mathbb{G})$ is assumed separable and $\mathrm{Irred}(\mathbb{G})$ therefore is a countable set. By Corollary 2.8 we can find $\delta > 0$ such that for any $n \in \mathbb{N}$ there exists a Hilbert space H_n and a representation $\pi_n \colon \mathrm{Pol}(\mathbb{G}) \to B(H_n)$ which has an $(E_n, \frac{1}{n})$-invariant unit vector ξ_n, but such that any invariant vector is a least δ away from ξ_n. Define $\varphi_n \colon C(\mathbb{G}_{\mathrm{u}}) \to \mathbb{C}$ by $\varphi_n(a) = \langle \pi_n(a)\xi_n | \xi_n\rangle$. Just as above we get that each φ_n satisfies the equation (1) and by construction of the E_n's it follows that $\lim_n \|\pi_n(a)\xi_n - \varepsilon(a)\xi_n\| = 0$ for any $a \in \mathrm{Pol}(\mathbb{G})$. Hence $(\varphi_n)_{n\in\mathbb{N}}$ converges pointwise to ε on $\mathrm{Pol}(\mathbb{G})$ and a standard approximation argument shows that the pointwise convergence then holds on all of $C(\mathbb{G}_{\mathrm{u}})$. Since there are no non-zero invariant vectors within distance $\frac{\delta_0}{2}$ from ξ_n, Lemma 3.2 provides us with a $v_n \in \mathcal{U}$ such that

$$\|\pi(v_n)\xi_n - \varepsilon(v_n)\xi_n\| > \frac{\delta_0}{2}$$

Using again the equation (1) we see that

$$|\varphi_n(v_n) - \varepsilon(v_n)|^2 + (1 - |\varphi_n(v_n)|^2) \geq \frac{\delta_0^2}{4},$$

proving that the convergence can not be uniform. □

The proof of Theorem 3.1 shows that we can get a bit closer to the classical formulation in that we can replace nets with sequences.

Corollary 3.3. *The discrete quantum group $\hat{\mathbb{G}}$ has property* (T) *iff any sequence of states on $C(\mathbb{G}_{\mathrm{u}})$ converging pointwise to the counit converges in the uniform norm.*

Proof. If $\hat{\mathbb{G}}$ has property (T) the desired conclusion follows from Theorem 3.1. If, on the other hand, $\hat{\mathbb{G}}$ does not have property (T) the proof of Theorem 3.1 shows how to construct a sequence of states converging pointwise, but not uniformly, to the counit. □

4. Cocycles and conditionally negative functions

The Delorme-Guichardet Theorem for groups, stated in the introduction, expresses property (T) in terms of vanishing of the first cohomology of the group in question. In order to prove a quantum version of this result we first introduce the relevant notion of cohomology for our purposes.

Definition 4.1. *Let $\hat{\mathbb{G}}$ be a a discrete quantum group and let $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ be a $*$-representation. A 1-cocycle for the representation π is a linear map $c\colon \mathrm{Pol}(\mathbb{G}) \to H$ satisfying*

$$c(ab) = \pi(a)c(b) + c(a)\varepsilon(b),$$

for all $a, b \in \mathrm{Pol}(\mathbb{G})$. A 1-cocycle c is called inner if there exists $\xi \in H$ such that $c(a) = \pi(a)\xi - \xi\varepsilon(a)$ for all $a \in \mathrm{Pol}(\mathbb{G})$. The set of cocycles $Z^1(\mathrm{Pol}(\mathbb{G}), H)$ is naturally a complex vector space in which the set of inner cocycles $B^1(\mathrm{Pol}(\mathbb{G}), H)$ constitutes a subspace, and the first cohomology $H^1(\mathrm{Pol}(\mathbb{G}), H)$ of $\mathrm{Pol}(\mathbb{G})$ with coefficients in H is then defined as the space of cocycles modulo the inner ones. Finally, a cocycle c is called real if

$$\langle c(S(y^*)) \,|\, c((Sx)^*)\rangle = \langle c(x) \,|\, c(y)\rangle \quad \textit{for all } x, y \in \mathrm{Pol}(\mathbb{G}).$$

Remark 4.2. Note that a cocycle $c\colon \mathrm{Pol}(\mathbb{G}) \to H$ is nothing but a derivation into H where H is considered a $\mathrm{Pol}(\mathbb{G})$-bimodule with left action given by π and right action given by the counit ε. This is the reason why we from time to time write the scalar action via ε on the right. Using the standard description of the first Hochschild cohomology in terms of derivations [Lod98], we see that $H^1(\mathrm{Pol}(\mathbb{G}), H)$ is exactly the first Hochschild cohomology of $\mathrm{Pol}(\mathbb{G})$ with coefficients in the bimodule ${}_{\pi}H_{\varepsilon}$. Throughout the paper, we shall only make use of the *first* Hochschild cohomology group and in the sequel the term cocycle will therefore be used to mean 1-cocycle.

The following lemma gives an alternative description of the space of inner cocycles and is a modified version of a result in [Pet09].

Lemma 4.3. *If $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ is a representation and $c\colon \mathrm{Pol}(\mathbb{G}) \to H$ is a cocycle then c is inner if and only if it is bounded with respect to the norm $\|\cdot\|_u$ on $C(\mathbb{G}_{\mathrm{u}})$.*

Proof. First note that both π and ε extend to $C(\mathbb{G}_{\mathrm{u}})$ by definition of the universal norm. It is clear that an inner cocycle is bounded so assume, conversely, that c extends to $C(\mathbb{G}_{\mathrm{u}})$. We denote the extensions of π, ε and c by the same symbols and define

$$X = \{c(u)\varepsilon(u^*) \mid u \in \mathcal{U}\},$$

where $\mathcal{U}$ as before denotes the unitary group of $C(\mathbb{G}_{\mathrm{u}})$. Since X is a bounded set in the Hilbert space H there is a unique Chebyshev center [BdlHV08, 2.2.7]; i.e. there exists a unique $\xi_0 \in H$ minimizing the function

$$H \ni \xi \mapsto \sup\{\|x - \xi\| \mid x \in X\} \in \mathbb{R}.$$

Consider now the affine isometric action of $\mathcal{U}$ on H given by $\alpha(v)(\xi) = \pi(v)\xi + c(v)$. Then for any $v \in \mathcal{U}$ we have that $\alpha(v)\xi_0$ is the Chebyshev center for $\alpha(v)X$ and that $\xi_0\varepsilon(v)$ is the Chebyshev center for $X\varepsilon(v)$. On the other hand

$$\begin{aligned}
\alpha(v)X &= \alpha(v)\{c(u)\varepsilon(u^*) \mid u \in \mathcal{U}\} \\
&= \alpha(v)\{c(v^*u)\varepsilon(u^*v) \mid u \in \mathcal{U}\} \\
&= \{\pi(v)c(v^*u)\varepsilon(u^*v) + c(v) \mid u \in \mathcal{U}\} \\
&= \{\pi(v)[\pi(v^*)c(u) + c(v^*)\varepsilon(u)]\varepsilon(u^*v) + c(v) \mid u \in \mathcal{U}\} \\
&= \{c(u)\varepsilon(u^*)\varepsilon(v) + \pi(v)c(v^*)\varepsilon(v) + c(v) \mid u \in \mathcal{U}\} \\
&= \{c(u)\varepsilon(u^*)\varepsilon(v) - c(v)\varepsilon(v^*)\varepsilon(v) + c(v) \mid u \in \mathcal{U}\} \\
&= \{c(u)\varepsilon(u^*)\varepsilon(v) \mid u \in \mathcal{U}\} \\
&= X\varepsilon(v),
\end{aligned}$$

and hence $\alpha(v)\xi_0 = \xi_0\varepsilon(v)$ for any $v \in \mathcal{U}$. Thus $c(v) = \pi(v)(-\xi_0) - (-\xi_0)\varepsilon(v)$ for any $v \in \mathcal{U}$ and since the elements in $\mathcal{U}$ span $C(\mathbb{G}_{\mathrm{u}})$ linearly we conclude that c is inner.

□

The notion of cocycles on a discrete group Γ is intimately linked (see e.g. section 2.10 in [BdlHV08]) to the notion of conditionally negative functions. Recall, that a function $\psi\colon \Gamma \to \mathbb{R}$ is called conditionally negative if $\psi(\gamma) = \psi(\gamma^{-1})$ for every $\gamma \in \Gamma$ and if ψ, for any finite subset $\{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$, furthermore satisfies

$$\sum_{i=1}^{n} \bar{\alpha}_i \alpha_j \psi(\gamma_i^{-1}\gamma_j) \leq 0 \text{ for all } \alpha_1, \dots, \alpha_n \in \mathbb{C} \text{ with } \sum_{i=1}^{n} \alpha_i = 0.$$

Generalizing this to quantum groups we arrive at the following definition.

Definition 4.4. *A functional $\psi\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ is said to be conditionally negative if $\psi(x^*x) \leq 0$ for all $x \in \ker(\varepsilon)$. Moreover, ψ is called normalized if $\psi(1) = 0$ and hermitian if $\psi(x^*) = \overline{\psi(x)}$ for all $x \in \mathrm{Pol}(\mathbb{G})$.*

The conditionally negative, normalized and hermitian functionals are also called *infinitesimal generators* because of the following quantum group version of Schönberg's Theorem whose proof can be found in [Sch90].

Theorem 4.5 (Schürmann)**.** *A functional $\psi\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ is conditionally negative, normalized and hermitian if and only if $\varphi_t = \exp(-t\psi)\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ is a positive and unital functional for every $t \geq 0$.*

Here positivity of the map φ_t simply means that $\varphi_t(x^*x) \geq 0$ for every $x \in \mathrm{Pol}(\mathbb{G})$. Note that [BMT01] Theorem 3.3 states that such functionals automatically extend to states on $C(\mathbb{G}_{\mathrm{u}})$. Perhaps the definition of the φ_t's require a bit of explanation. For two functionals $\mu, \omega\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ their convolution product

$\omega \star \mu$ is defined as $(\omega \otimes \mu)\Delta$. For a single functional ψ, the co-semisimplicity of $\mathrm{Pol}(\mathbb{G})$ makes the series

$$\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \psi^{\star k}(x)$$

convergent for each $x \in \mathrm{Pol}(\mathbb{G})$ and its sum is denoted $\exp(-t\psi(x))$. For an infinitesimal generator ψ, the family φ_t defined above is actually a 1-parameter convolution semigroup of states on $C(\mathbb{G}_{\mathrm{u}})$ converging pointwise to the counit; i.e. for all $s, t \geq 0$ we have $\varphi_t \star \varphi_s = \varphi_{t+s}$, $\varphi_0 = \varepsilon$ and for every $x \in \mathrm{Pol}(\mathbb{G})$ we have $\varepsilon(x) = \lim_{t\to 0} \varphi_t(x)$. Such 1-parameter semigroups of states on C^*-bialgebras have been studied by Lindsay and Skalski in [SL09] where it is also proved that if

$$\lim_{t\to 0} \|\varphi_t - \varepsilon\| = 0,$$

i.e. if the convergence is uniform, then the infinitesimal generator ψ is bounded with respect to the norm $\|\cdot\|_{\mathrm{u}}$.

In the classical case, a group cocycle $c\colon \Gamma \to {}_{\pi}H$ gives rise to an infinitesimal generator $\psi\colon \Gamma \to \mathbb{C}$ by setting $\psi(\gamma) = \|c(\gamma)\|^2$; for quantum groups we have the following analogues result.

Theorem 4.6 (Vergnioux). *Let $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ be a $*$-representation and $c\colon \mathrm{Pol}(\mathbb{G}) \to H$ a cocycle. Then $\psi\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ defined by*

$$\psi(x) = \langle c(x_{(1)}) | c(Sx_{(2)}^*)\rangle$$

is linear and satisfies

$$\psi(x^*y) = -\langle c((Sx)^*) | c(S(y^*))\rangle - \langle c(y)|c(x)\rangle \quad \text{for all } x, y \in \ker(\varepsilon). \tag{2}$$

If furthermore c is a real cocycle then ψ is an infinitesimal generator; i.e. ψ is conditionally negative, normalized and hermitian.

In the definition of ψ we made use of the so-called Sweedler notation, writing $x_{(1)} \otimes x_{(2)}$ for Δx. We shall use this notation without further elaboration in the following and refer the reader to [KS97] for a detailed treatment. Theorem 4.6 is due to R. Vergnioux and the author would like to express his gratitude to Vergnioux for communicating it and for allowing its appearance in the present paper. Since the result is not published elsewhere we include Vergnioux's proof, but before doing so a bit of notation is needed.

Notation 4.7. The dual of the Hilbert space H is denoted H^{op} and the inner product will considered as a linear map $\langle\cdot|\cdot\rangle\colon H \odot H^{\mathrm{op}} \to \mathbb{C}$. For $\xi \in H$ we denote by $\xi^{\mathrm{op}} \in H^{\mathrm{op}}$ the dual element $\langle\cdot|\xi\rangle$ and for $T \in B(H)$ we denote by $T^{\mathrm{op}} \in B(H^{\mathrm{op}})$ the operator $T^{\mathrm{op}}\xi^{\mathrm{op}} = (T\xi)^{\mathrm{op}}$. The symbol m will denote both the multiplication map $\mathrm{Pol}(\mathbb{G}) \odot \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ as well as the action $\pi(\mathrm{Pol}(\mathbb{G})) \odot H \to H$. The antipode in $\mathrm{Pol}(\mathbb{G})$ is denoted by S and as usual we denote the counit by ε. Recall that in the general (i.e. non-Kac) case $S^2 \neq \mathrm{id}$, but the relation $S(S(x^*)^*) = x$

holds always. We will often consider the $*$-operation as a self-map of $\mathrm{Pol}(\mathbb{G})$ and may therefore write $*(a)$ instead of a^*; the above relation involving the antipode may then be written as $S * S * = \mathrm{id}$. Likewise, we will consider $\xi \mapsto \xi^{\mathrm{op}}$ as a map $\mathrm{op}\colon H \to H^{\mathrm{op}}$ an write $\mathrm{op}(\xi)$ in stead of ξ^{op} whenever convenient. By σ we will denote the flip-map on $\mathrm{Pol}(\mathbb{G}) \odot \mathrm{Pol}(\mathbb{G})$ as well as the flip-map on $H \odot H$. Similarly, $\sigma_{(13)}$ will denote the map on a three-fold tensor product which flips the first and the third leg and leaves the middle leg untouched. In the following we shall furthermore make use of the abundance of relations valid in a Hopf $*$-algebra without further reference. These may be found in any standard book on Hopf $*$-algebras; for instance [KS97].

For the proof of Theorem 4.6 we will need a small lemma concerning the interplay between the cocycle and the antipode.

Lemma 4.8 (Vergnioux)**.** *For $x \in \mathrm{Pol}(\mathbb{G})$ with $\varepsilon(x) = 0$ we have*

$$c(Sx) = -\pi(Sx_{(1)})c(x_{(2)}); \tag{3}$$

$$c(x) = -\pi(x_{(1)})c(Sx_{(2)}); \tag{4}$$

$$c((Sx)^*) = -\pi((Sx_{(2)})^*)c(x^*_{(1)}). \tag{5}$$

Proof. The equation $m(S \otimes \mathrm{id})\Delta x = \varepsilon(x)1$ gives

$$\begin{aligned} 0 &= c(m(S \otimes \mathrm{id})\Delta x) \\ &= c((Sx_{(1)})x_{(2)}) \\ &= \pi(Sx_{(1)})c(x_{(2)}) + c(Sx_{(1)})\varepsilon(x_{(2)}) \\ &= \pi(Sx_{(1)})c(x_{(2)}) + c(S((\mathrm{id} \otimes \varepsilon)\Delta x)) \\ &= \pi(Sx_{(1)})c(x_{(2)}) + c(Sx), \end{aligned}$$

proving equation (3). In the same manner, the equation (4) follows from the formula $m(\mathrm{id} \otimes S)\Delta x = \varepsilon(x)1$. The equation (5) follows from (4) and the formula $\Delta S = (S \otimes S)\sigma\Delta$:

$$\begin{aligned} c((Sx)^*) &= -\pi(((Sx)^*)_{(1)})c(S(((Sx)^*)_{(2)})) \\ &= \pi((Sx_{(2)})^*)c(S((Sx_{(1)})^*)) \\ &= \pi((Sx_{(2)})^*)c(x^*_{(1)}). \end{aligned}$$

This concludes the proof of the lemma. ☐

Proof of Theorem 4.6. We may write $\psi = \langle \cdot | \cdot \rangle c \otimes (\mathrm{op}\ c\ S\ *)\Delta$ which shows that ψ is well defined and linear. It is first proved that ψ satisfies equation (2). Using the cocycle condition we get

$$\begin{aligned}\psi(x^*y) &= \langle\cdot|\cdot\rangle c\otimes(\mathrm{op}\ c\ S\ *)\Delta(x^*y)\\ &= \langle c(x^*_{(1)}y_{(1)})\,|\,c((Sx_{(2)})S(y^*_{(2)}))\rangle\\ &= \langle \pi(x^*_{(1)})c(y_{(1)})\,|\,\pi(Sx_{(2)})c(S(y^*_{(2)}))\rangle + \langle \pi(x^*_{(1)})c(y_{(1)})\,|\,c(Sx_{(2)})\varepsilon(S(y^*_{(2)}))\rangle\\ &+ \langle c(x^*_{(1)})\varepsilon(y_{(1)})\,|\,\pi(Sx_{(2)})c(S(y^*_{(2)}))\rangle + \langle c(x^*_{(1)})\varepsilon(y_{(1)})\,|\,c(Sx_{(2)})\varepsilon(S(y^*_{(2)}))\rangle.\end{aligned}$$

We now treat the four terms one by one. Since $\varepsilon(x)=0$, the first term vanishes:

$$\begin{aligned}\langle \pi(x^*_{(1)})c(y_{(1)})\,|\,\pi(Sx_{(2)})c(S(y^*_{(2)}))\rangle &= \langle c(y_{(1)})\,|\,\pi(m(\mathrm{id}\otimes S)\Delta x)c(S(y^*_{(2)}))\rangle\\ &= \langle c(y_{(1)})\,|\,\pi(\varepsilon(x)1)c(S(y^*_{(2)}))\rangle\\ &= 0.\end{aligned}$$

Using the formula (3) and the fact that $\varepsilon S=\varepsilon$, the second term becomes

$$\begin{aligned}\langle \pi(x^*_{(1)})c(y_{(1)})\,|\,c(Sx_{(2)})\varepsilon(S(y^*_{(2)}))\rangle &= \langle \varepsilon(y_{(2)})c(y_{(1)})\,|\,\pi(x_{(1)})c(Sx_{(2)})\rangle\\ &= -\langle c((\mathrm{id}\otimes\varepsilon)\Delta y)\,|\,c(x)\rangle\\ &= -\langle c(y)\,|\,c(x)\rangle.\end{aligned}$$

Similarly, using the formula (5) the third term becomes

$$\begin{aligned}\langle c(x^*_{(1)})\varepsilon(y_{(1)})\,|\,\pi(Sx_{(2)})c(S(y^*_{(2)}))\rangle &= \langle \pi((Sx_{(2)})^*)c(x^*_{(1)})\,|\,\varepsilon(y^*_{(1)})c(S(y^*_{(2)}))\rangle\\ &= -\langle c((Sx)^*)\,|\,c((\varepsilon\otimes\mathrm{id})(S\otimes S)\Delta(y^*))\rangle\\ &= -\langle c((Sx)^*)\,|\,c((\varepsilon\otimes\mathrm{id})\sigma\Delta(S(y^*)))\rangle\\ &= -\langle c((Sx)^*)\,|\,c(S(y^*))\rangle.\end{aligned}$$

We are therefore done if we can show that the fourth term vanishes. This follows from the assumption $\varepsilon(y)=0$ and the following calculation:

$$\begin{aligned}\langle c(x^*_{(1)})\varepsilon(y_{(1)})\,|\,c(Sx_{(2)})\varepsilon(S(y^*_{(2)})))\rangle &= \varepsilon(y_{(1)})\varepsilon(y_{(2)})\langle c(x^*_{(1)})\,|\,c(Sx_{(2)})\rangle\\ &= \varepsilon((\mathrm{id}\otimes\varepsilon)\Delta y)\langle c(x^*_{(1)})\,|\,c(Sx_{(2)})\rangle\\ &= 0.\end{aligned}$$

Hence ψ satisfies (2). Assume now that c is a real cocycle. The equation (2) then gives that $\psi(x^*x)=-2\|c(x)\|^2$ whenever $x\in\ker(\varepsilon)$ which shows that ψ is conditionally negative. Since $c(1)=0$ it is clear that ψ is normalized. That ψ is hermitian is seen by the following calculation.

$$\begin{aligned}
\overline{\psi(x^*)} &= \overline{\langle\cdot|\cdot\rangle(c\otimes(\mathrm{op}\ c\ S\ *))x^*_{(1)}\otimes x^*_{(2)}}\\
&= \overline{\langle c(x^*_{(1)})|c(Sx_{(2)})\rangle}\\
&= \langle c(Sx_{(2)})|c(x^*_{(1)})\rangle\\
&= -\langle \pi(S(x_{(2)_{(1)}}))c(x_{(2)_{(2)}})|c(x^*_{(1)})\rangle && \text{(by (3))}\\
&= -\langle c(x_{(2)_{(2)}})|\pi(S(x_{(2)_{(1)}})^*)c(x^*_{(1)})\rangle\\
&= -\langle\cdot|\cdot\rangle c(x_{(2)_{(2)}})\otimes(\pi((Sx_{(2)_{(1)}})^*)^{\mathrm{op}}c(x^*_{(1)})^{\mathrm{op}}\\
&= -\langle\cdot|\cdot\rangle c\otimes[m((\mathrm{op}\ \pi\ *\ S)\otimes(\mathrm{op}\ c\ *))](x_{(2)_{(2)}}\otimes x_{(2)_{(1)}}\otimes x_{(1)})\\
&= -\langle\cdot|\cdot\rangle c\otimes[m((\mathrm{op}\ \pi\ *\ S)\otimes(\mathrm{op}\ c\ *))]\sigma_{(13)}(x_{(1)}\otimes x_{(2)_{(1)}}\otimes x_{(2)_{(2)}})\\
&= -\langle\cdot|\cdot\rangle c\otimes[m((\mathrm{op}\ \pi\ *\ S)\otimes(\mathrm{op}\ c\ *))]\sigma_{(13)}(x_{(1)_{(1)}}\otimes x_{(1)_{(2)}}\otimes x_{(2)})\\
&= -\langle c(x_{(2)})|\pi((S(x_{(1)_{(2)}}))^*)c(x_{(1)}{}^*_{(1)})\rangle\\
&= \langle c(x_{(2)})|c((Sx_{(1)})^*)\rangle && \text{(by (5))}\\
&= \langle c(S(S(x^*_{(2)})^*))|c((Sx_{(1)})^*)\rangle\\
&= \langle c(x_{(1)})|c(S(x^*_{(2)}))\rangle && (c\text{ real})\\
&= \psi(x).
\end{aligned}$$

This concludes the proof of Theorem 4.6. □

5. The Delorme-Guichardet Theorem

Using the material gathered in the previous sections we are now able to prove our main result.

Theorem 5.1. *For a discrete quantum group* $\hat{\mathbb{G}}$ *the following are equivalent.*

(i) $\hat{\mathbb{G}}$ *has property* (T).

(ii) $\hat{\mathbb{G}}$ *is Kac and every normalized, hermitian, conditionally negative functional* $\psi\colon \mathrm{Pol}(\mathbb{G})\to\mathbb{C}$ *is bounded with respect to* $\|\cdot\|_{\mathrm{u}}$.

(ii) *For every* $*$*-representation* $\pi\colon \mathrm{Pol}(\mathbb{G})\to B(H)$ *the first cohomology group* $H^1(\mathrm{Pol}(\mathbb{G}),H)$ *vanishes.*

Proof. We first prove (i)⇒(ii). Assume therefore that $\hat{\mathbb{G}}$ has property (T) and let $\psi\colon \mathrm{Pol}(\mathbb{G})\to\mathbb{C}$ be normalized, conditionally negative and hermitian. By exponentiation (see section 4) we get a 1-parameter family of states on $C(\mathbb{G}_u)$ converging pointwise to the counit and by Theorem 3.1 the convergence has to be uniform. Applying [SL09] Proposition 2.3, this implies that the infinitesimal generator ψ is bounded. That $\hat{\mathbb{G}}$ is Kac follows from Theorem 1.7.

Next we prove (ii)⇒(iii). Assume therefore that $\hat{\mathbb{G}}$ is Kac and that every infinitesimal generator is bounded, and let a representation $\pi\colon \mathrm{Pol}(\mathbb{G})\to B(H)$

as well as a cocycle $c\colon \mathrm{Pol}(\mathbb{G}) \to H$ be given. By Lemma 4.3 it suffices to show that c is bounded with respect to the norm $\|\cdot\|_{\mathrm{u}}$. Since $\hat{\mathbb{G}}$ is assumed Kac the antipode $S\colon \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{G})$ is $*$-preserving. The $*$-representation π therefore gives rise to a dual $*$-representation $\pi^{\mathrm{op}}\colon \mathrm{Pol}(\mathbb{G}) \to B(H^{\mathrm{op}})$ on the dual Hilbert space H^{op} given by $\pi^{\mathrm{op}}(a)\xi^{\mathrm{op}} = (\pi(Sa^*)\xi)^{\mathrm{op}}$ and the map $c^{\mathrm{op}}\colon \mathrm{Pol}(\mathbb{G}) \to H^{\mathrm{op}}$ given by $c^{\mathrm{op}}(a) = (c(Sa^*))^{\mathrm{op}}$ is a cocycle for this representation. Furthermore, it is easy to check that

$$\mathrm{Pol}(\mathbb{G}) \ni a \overset{\delta}{\longmapsto} (c(a), c^{\mathrm{op}}(a)) \in H \oplus H^{\mathrm{op}}$$

is a real $(\pi \oplus \pi^{\mathrm{op}})$-cocycle which is bounded if and only if c is bounded. It therefore suffices to treat the case where the cocycle c is real. In this case, Theorem 4.6 provides us with a conditionally negative functional $\psi\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ such that $\psi(x^*x) = -2\|c(x)\|^2$ for all $x \in \ker(\varepsilon)$. By assumption the functional ψ is bounded with respect to $\|\cdot\|_{\mathrm{u}}$ so for $x \in \ker(\varepsilon)$ we get

$$\|x\|_{\mathrm{u}}^2\|\psi\| \geq |\psi(x^*x)| = 2\|c(x)\|^2,$$

and hence the restriction c_0 of c to $\ker(\varepsilon)$ extends boundedly to a map $\tilde{c}_0$ on the closure J of $\ker(\varepsilon)$ inside $C(\mathbb{G}_{\mathrm{u}})$. The ideal J is exactly the kernel of the extension of the counit $\varepsilon\colon C(\mathbb{G}_{\mathrm{u}}) \to \mathbb{C}$ and therefore the map

$$C(\mathbb{G}_{\mathrm{u}}) \ni a \mapsto \tilde{c}_0(a - \varepsilon(a)1) \in H$$

is bounded and extends c.

The proof of the implication (iii)⇒(i) is a minor modification of the corresponding proof for groups in [BdlHV08]. Assume therefore the vanishing of all the first cohomology groups and fix a $*$-representation $\pi\colon \mathrm{Pol}(\mathbb{G}) \to B(H)$ on a Hilbert space H without non-zero invariant vectors. We then need to show that π can not have invariant vectors either. Define, for a finite subset $E \subseteq \mathrm{Irred}(\mathbb{G})$, a seminorm on $Z^1(\mathrm{Pol}(\mathbb{G}), H)$ by

$$\|c\|_E = \sup\{\|c(u_{ij}^{\alpha})\| \mid \alpha \in E, 1 \leq i,j \leq n_\alpha\}.$$

Since the matrix coefficients span $\mathrm{Pol}(\mathbb{G})$ linearly, it is a routine to check that $Z^1(\mathrm{Pol}(\mathbb{G}), H)$ becomes a Frechet space when endowed with the topology arising from this family of seminorms. The claim now is that π does not have almost invariant vectors iff $B^1(\mathrm{Pol}(\mathbb{G}), H)$ is closed in $Z^1(\mathrm{Pol}(\mathbb{G}), H)$. Clearly the claim implies the desired result since the vanishing of $H^1(\mathrm{Pol}(\mathbb{G}), H)$ in particular implies that $B^1(\mathrm{Pol}(\mathbb{G}), H)$ is closed. To prove the claim, consider the map $\Phi\colon H \to B^1(\mathrm{Pol}(\mathbb{G}), H)$ mapping a vector ξ to the corresponding inner cocycle. Then Φ is linear, continuous and surjective and since π is assumed to have no non-zero fixed vectors it follows that Φ is also injective. Assume first that π does not have almost invariant vectors either. Then there exists $E_0 \subseteq \mathrm{Irred}(\mathbb{G})$ and $\delta_0 > 0$ such that

$$\|\Phi(\xi)\|_{E_0} = \sup\{\|\pi(u_{ij}^{\alpha})\xi - \varepsilon(u_{ij}^{\alpha})\xi\| \mid \alpha \in E_0, 1 \leq i,j \leq n_\alpha\} \geq \delta_0\|\xi\| \quad (6)$$

for all $\xi \in H$. Let c be in the closure of $B^1(\mathrm{Pol}(\mathbb{G}), H)$ and choose a sequence $(\xi_n)_{n\in\mathbb{N}}$ such that $\Phi(\xi_n)$ converges to c in the Frechet topology. Then, in particular, $(\Phi(\xi_n))_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the seminorm $\|\cdot\|_{E_0}$ and by (6) this implies that $(\xi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H; hence it has a limit point $\xi \in H$. By continuity of Φ we conclude that $c = \Phi(\xi)$ and therefore c is inner and $B^1(\mathrm{Pol}(\mathbb{G}), H)$ closed. Conversely, assume that $B^1(\mathrm{Pol}(\mathbb{G}), H)$ is closed and therefore a sub-Frechet space in $Z^1(\mathrm{Pol}(\mathbb{G}), H)$. Then the open mapping theorem [Rud73, 2.12] implies that $\Phi\colon H \to B^1(\mathrm{Pol}(\mathbb{G}), H)$ is bi-continuous and thus bounded away from zero in at least one of the seminorms $\|\cdot\|_{E_0}$. Hence there exists a $\delta_0 > 0$ such that

$$\sup\{\|\pi(u_{ij}^\alpha)\xi - \varepsilon(u_{ij}^\alpha)\xi\| \mid \alpha \in E_0, 1 \leq i,j \leq n_\alpha\} = \|\Phi(\xi)\|_{E_0} \geq \delta_0\|\xi\|$$

for all $\xi \in H$, and therefore π can not have almost invariant vectors. □

6. An application to L^2-invariants

In this section we prove the vanishing of the first L^2-Betti number for a discrete quantum group $\hat{\mathbb{G}}$ with property (T). The corresponding statement for groups was known to Gromov (see [Gro93]), but the first detailed proof was given by Bekka and Valette in [BV97]. The modern algebraic approach to L^2-invariants developed by Lück [Lüc02] (see also [Tho08a], [TP09], [Tho08b], [Sau05],[Far96]) makes this statement easier to prove and it can, for instance, easily be deduced from [TP09] Theorem 2.2. In this section we show how their argument passes through to quantum groups. Before doing so, we briefly remind the reader of the necessary definitions concerning L^2-Betti numbers for groups and quantum groups.

For a discrete group Γ, its L^2-Betti numbers can be described/defined in purely algebraic terms (see [Lüc02]) as $\beta_p^{(2)}(\Gamma) = \dim_{\mathscr{L}(\Gamma)} \mathrm{Tor}_p^{\mathbb{C}\Gamma}(\mathscr{L}(\Gamma), \mathbb{C})$ where $\dim_{\mathscr{L}(\Gamma)}(-)$ us Lück's extended Murray-von Neumann dimension. For a discrete quantum group $\hat{\mathbb{G}}$ of Kac type it is therefore natural to define its L^2-Betti numbers as

$$\beta_p^{(2)}(\hat{\mathbb{G}}) = \dim_{L^\infty(\mathbb{G})} \mathrm{Tor}_p^{\mathrm{Pol}(\mathbb{G})}(L^\infty(\mathbb{G}), \mathbb{C}),$$

where $\dim_{L^\infty(\mathbb{G})}(-)$ is the Murray-von Neumann dimension arising from the tracial Haar state. These L^2-Betti numbers have been studied in [CHT09], [Ver09], [Kye08b] and [Kye08a][1], and the aim of the present section is to prove the following.

Corollary 6.1. *If $\hat{\mathbb{G}}$ has property* (T) *then* $\beta_1^{(2)}(\hat{\mathbb{G}}) = 0$.

Note that if $\hat{\mathbb{G}}$ has property (T) then $\mathbb{G}$ is automatically of Kac type, and its Haar state $h\colon L^\infty(\mathbb{G}) \to \mathbb{C}$ is therefore a trace, so that the first L^2-Betti number

[1]Note that the notation $\beta_p^{(2)}(\mathbb{G})$ is used in [Kye08b] and [Kye08a].

makes sense. As mentioned above, the proof of Corollary 6.1 follows the lines of the corresponding proof in [TP09]. During the proof we will have to consider dimensions of both right and left modules for $L^\infty(\mathbb{G})$, and to avoid confusion we will let $\dim_{L^\infty(\mathbb{G})}(X)$ denote the dimension of a left module X whereas $\dim_{L^\infty(\mathbb{G})^{\mathrm{op}}}(Y)$ will denote the dimension of a right module Y.

Proof. Denote by $M(\mathbb{G})$ the $*$-algebra of closed, densely defined (potentially unbounded) operators affiliated with $L^\infty(\mathbb{G})$. This is a self-injective and von Neumann regular ring and tensoring $L^\infty(\mathbb{G})$-modules with $M(\mathbb{G})$ is a flat and dimension preserving functor [Rei01]. Therefore

$$\begin{aligned}\beta_1^{(2)}(\hat{\mathbb{G}}) &= \dim_{L^\infty(\mathbb{G})} M(\mathbb{G}) \odot_{L^\infty(\mathbb{G})} \operatorname{Tor}_1^{\mathrm{Pol}(\mathbb{G})}(L^\infty(\mathbb{G}), \mathbb{C})\\ &= \dim_{L^\infty(\mathbb{G})} \operatorname{Tor}_1^{\mathrm{Pol}(\mathbb{G})}(M(\mathbb{G}), \mathbb{C}).\end{aligned}$$

By [Tho08a] Corollary 3.4 we have

$$\dim_{L^\infty(\mathbb{G})} \operatorname{Tor}_1^{\mathrm{Pol}(\mathbb{G})}(M(\mathbb{G}), \mathbb{C}) = \dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} \operatorname{Hom}_{M(\mathbb{G})}(\operatorname{Tor}_1^{\mathrm{Pol}(\mathbb{G})}(M(\mathbb{G}), \mathbb{C}), M(\mathbb{G})),$$

and using the self-injectiveness of $M(\mathbb{G})$ (see e.g. [Tho08a] Theorem 3.5 and its proof) we get an isomorphism of right $M(\mathbb{G})$-modules

$$\operatorname{Hom}_{M(\mathbb{G})}(\operatorname{Tor}_1^{\mathrm{Pol}(\mathbb{G})}(M(\mathbb{G}), \mathbb{C}), M(\mathbb{G})) \simeq \operatorname{Ext}^1_{\mathrm{Pol}(\mathbb{G})}(\mathbb{C}, M(\mathbb{G})).$$

By considering the bar-resolution of the trivial $\mathrm{Pol}(\mathbb{G})$-module $\mathbb{C}$, one sees that $\operatorname{Ext}^1_{\mathrm{Pol}(\mathbb{G})}(\mathbb{C}, M(\mathbb{G}))$ may also be computed as the first Hochschild cohomology $H^1(\mathrm{Pol}(\mathbb{G}), M(\mathbb{G}))$ where $M(\mathbb{G})$ carries the natural left action of $\mathrm{Pol}(\mathbb{G})$ and right action given by the counit. We therefore arrive at the formula

$$\beta_1^{(2)}(\hat{\mathbb{G}}) = \dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} H^1(\mathrm{Pol}(\mathbb{G}), M(\mathbb{G})).$$

Since $\hat{\mathbb{G}}$ has property (T), Theorem 5.1 implies that $H^1(\mathrm{Pol}(\mathbb{G}), L^2(\mathbb{G}))$ vanishes and we are therefore done if we can prove that

$$\dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} H^1(\mathrm{Pol}(\mathbb{G}), L^2(\mathbb{G})) = \dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} H^1(\mathrm{Pol}(\mathbb{G}), M(\mathbb{G})).$$

To see this, we consider the following diagram of right $L^\infty(\mathbb{G})$-modules:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B^1(\mathrm{Pol}(\mathbb{G}), L^\infty(\mathbb{G})) & \longrightarrow & Z^1(\mathrm{Pol}(\mathbb{G}), L^\infty(\mathbb{G})) & \longrightarrow & H^1(\mathrm{Pol}(\mathbb{G}), L^\infty(\mathbb{G})) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^1(\mathrm{Pol}(\mathbb{G}), L^2(\mathbb{G})) & \longrightarrow & Z^1(\mathrm{Pol}(\mathbb{G}), L^2(\mathbb{G})) & \longrightarrow & H^1(\mathrm{Pol}(\mathbb{G}), L^2(\mathbb{G})) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B^1(\mathrm{Pol}(\mathbb{G}), M(\mathbb{G})) & \longrightarrow & Z^1(\mathrm{Pol}(\mathbb{G}), M(\mathbb{G})) & \longrightarrow & H^1(\mathrm{Pol}(\mathbb{G}), M(\mathbb{G})) & \longrightarrow & 0
\end{array}$$

The rows in this diagrams are exact by definition and the two first columns clearly consist of inclusions. We now prove that

$$\dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} B^1(\mathrm{Pol}(\mathbb{G}), L^\infty(\mathbb{G})) = \dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} B^1(\mathrm{Pol}(\mathbb{G}), M(\mathbb{G})); \tag{7}$$

$$\dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} Z^1(\mathrm{Pol}(\mathbb{G}), L^\infty(\mathbb{G})) = \dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} Z^1(\mathrm{Pol}(\mathbb{G}), M(\mathbb{G})), \tag{8}$$

and the result then follows from additivity [Lüc02] of the dimension function $\dim_{L^\infty(\mathbb{G})^{\mathrm{op}}}(-)$. To prove the equality (7), notice that the first column identifies with the inclusions $L^\infty(\mathbb{G}) \subseteq L^2(\mathbb{G}) \subseteq M(\mathbb{G})$ so it suffices to see that

$$\dim_{L^\infty(\mathbb{G})^{\mathrm{op}}} M(\mathbb{G})/L^\infty(\mathbb{G}) = 0.$$

By [Sau05] Theorem 2.4, it is enough to see that for every $\xi \in M(\mathbb{G})$ and every $\delta > 0$ there exists a projection $p \in L^\infty(\mathbb{G})$ such that $h(p) \geq 1-\delta$ and $\xi p \in L^\infty(\mathbb{G})$. But this follows from the fact all the spectral projections of the absolute value of ξ are in $L^\infty(\mathbb{G})$. To prove the equality (8), consider a cocycle $c\colon \mathrm{Pol}(\mathbb{G}) \to M(\mathbb{G})$ and a $\delta > 0$. Again by [Sau05], we have to find a projection $p \in L^\infty(\mathbb{G})$ such that $h(p) \geq 1 - \delta$ and $c(-)p \in Z^1(\mathrm{Pol}(\mathbb{G}), L^\infty(\mathbb{G}))$. For this, consider again the set of matrix coefficients

$$\{u^\alpha_{ij} \mid \alpha \in I, 1 \leq i,j \leq n_\alpha\}.$$

Since $C(\mathbb{G})$ is assumed separable, this set is at most countable so we may choose a sequence of numbers $\delta^\alpha_{ij} > 0$ such that

$$\sum_{\alpha\in I}\sum_{i,j=1}^{n_\alpha} \delta^\alpha_{ij} \leq \delta.$$

For each u^α_{ij} we have $c(u^\alpha_{ij}) \in M(\mathbb{G})$ and hence we can find a projection $p^\alpha_{ij} \in L^\infty(\mathbb{G})$ such that $c(u^\alpha_{ij})p^\alpha_{ij} \in L^\infty(\mathbb{G})$ and such that $h(p^\alpha_{ij}) \geq 1 - \delta^\alpha_{ij}$. Let p be the infimum of all these projections. We then have

$$h(1-p) = h(1 - \bigwedge_{\alpha,i,j} p^\alpha_{ij})) = h(\bigvee_{\alpha,i,j} p^\alpha_{ij}) \leq \sum_{\alpha,i,j} h(p^\alpha_{ij}) \leq \delta.$$

Since the set $\{u^\alpha_{ij} \mid \alpha \in I, 1 \leq i,j \leq n_\alpha\}$ spans $\mathrm{Pol}(\mathbb{G})$ linearly we also have that $c(-)p$ is a cocycle with values in $L^\infty(\mathbb{G})$. □

Turning things around, vanishing of the first L^2-Betti number may be turned into an honest vanishing of cohomology result.

Corollary 6.2. *If $\hat{\mathbb{G}}$ is non-amenable and of Kac type with $\beta^{(2)}_1(\hat{\mathbb{G}}) = 0$ then $H^1(\mathrm{Pol}(\mathbb{G}), L^2(\mathbb{G}))$ vanishes.*

Recall that a discrete quantum group $\hat{\mathbb{G}}$ is said to be amenable if the counit $\varepsilon\colon \mathrm{Pol}(\mathbb{G}) \to \mathbb{C}$ extends to a bounded character on $C(\mathbb{G}_{\mathrm{red}})$. A detailed study of this notion may be found in [BMT01] and [Tom06]. The proof Corollary 6.2 is again a modification of the corresponding proof in [TP09].

Proof. Since $\beta^{(2)}_1(\hat{\mathbb{G}}) = 0$ we have[2] that

$$\dim_{L^\infty(\mathbb{G})} \mathrm{Tor}^{\mathrm{Pol}(\mathbb{G})}_1(M(\mathbb{G}), \mathbb{C}) = 0,$$

[2] See e.g. the beginning of the proof of Corollary 6.1.

and by [Tho08a] Corollary 3.3 this implies vanishing of the dual $M(\mathbb{G})$-module

$$\mathrm{Hom}_{M(\mathbb{G})}(\mathrm{Tor}_1^{\mathrm{Pol}(\mathbb{G})}(M(\mathbb{G}),\mathbb{C}),M(\mathbb{G})).$$

As in the proof of Corollary 6.1 we have an isomorphism of right $M(\mathbb{G})$-modules

$$\mathrm{Hom}_{M(\mathbb{G})}(\mathrm{Tor}_1^{\mathrm{Pol}(\mathbb{G})}(M(\mathbb{G}),\mathbb{C}),M(\mathbb{G}))\simeq H^1(\mathrm{Pol}(\mathbb{G}),M(\mathbb{G})),$$

and hence every cocycle with values in $M(\mathbb{G})$ is inner. Denote by $J\in B(L^2(\mathbb{G}))$ the modular conjugation arising from the tracial state h and recall that for each $x\in L^2(\mathbb{G})$ the operator $L(x)^0\colon \Lambda a\mapsto J\lambda(a)^*Jx$ is pre-closed and its closure $L(x)$ is affiliated with $L^\infty(\mathbb{G})$. We therefore have $L^2(\mathbb{G})$ embedded into $M(\mathbb{G})$ via the map $L\colon x\mapsto L(x)$ which is easily seen to bee an embedding of $\mathrm{Pol}(\mathbb{G})$-bimodules. For a given cocycle $c\colon \mathrm{Pol}(\mathbb{G})\to L^2(\mathbb{G})$ we can therefore find an affiliated operator $\xi\in M(\mathbb{G})$ such that $L(c(a))=\lambda(a)\xi-\xi\varepsilon(a)$ for every $a\in\mathrm{Pol}(\mathbb{G})$. Choose an increasing sequence of projections $p_n\in L^\infty(\mathbb{G})$ such that $\xi p_n\in L^\infty(\mathbb{G})$ for every $n\in\mathbb{N}$ and such that $(p_n)_{n\in\mathbb{N}}$ converges in the strong operator topology to 1. For each $n\in\mathbb{N}$ and $a\in\mathrm{Pol}(\mathbb{G})$ we now have

$$L(Jp_nJ(c(a)))=L(c(a))p_n=\lambda(a)(\xi p_n)-(\xi p_n)\varepsilon(a),$$

and evaluating the operators on $\Lambda(1)$ we get

$$Jp_nJ(c(a))=\lambda(a)\Lambda(\xi p_n)-\Lambda(\xi p_n)\varepsilon(a).$$

But also $(Jp_nJ)_{n\in\mathbb{N}}$ converges in the strong operator topology to 1 and hence

$$c(a)=\lim_{n\to\infty}\lambda(a)\Lambda(\xi p_n)-\Lambda(\xi p_n)\varepsilon(a),$$

which proves that c is the pointwise limit of a sequence inner cocycles with values in $L^2(\mathbb{G})$. But since $\hat{\mathbb{G}}$ is non-amenable, the left regular representation λ can not have almost invariant vectors and the space $B^1(\mathrm{Pol}(\mathbb{G}),L^2(\mathbb{G}))$ is therefore closed in the Frechet topology on $Z^1(\mathrm{Pol}(\mathbb{G}),L^2(\mathbb{G}))$ introduced and studied in the proof of Theorem 5.1. This topology is exactly the topology of pointwise convergence and we conclude that $c\in Z^1(\mathrm{Pol}(\mathbb{G}),L^2(\mathbb{G}))$ is inner. □

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