QUASI-HOMOMORPHISM RIGIDITY WITH NONCOMMUTATIVE TARGETS

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Abstract. As a strengthening of Kazhdan's property (T) for locally compact groups, property (TT) was introduced by Burger and Monod. In this paper, we add more rigidity and introduce property (TTT). This property is suited for the study of rigidity phenomena for quasi-homomorphisms with noncommutative targets. Partially upgrading a result of Burger and Monod, we will prove that $\mathrm{SL}_n(\mathbb{R})$ with $n \geq 3$ and their lattices have property (TTT). As a corollary, we generalize the well-known fact that every homomorphism from such a lattice into an amenable group or a hyperbolic group has finite image to the extent that it includes a quasi-homomorphism.

1. Introduction

It is proved by Burger and Monod ([BM1, BM2]) that lattices in higher rank Lie groups have property (TT), a property which strengthens Kazhdan's property (T) and implies triviality of quasimorphisms. See [BHV] for a thorough treatment of property (T), and Section 13 in [Mo] for property (TT). The purpose of this paper is to introduce a yet stronger variant of property (TT), which we call property (TTT). Throughout this paper **all groups are assumed to be second countable**.

Definition. Let G be a locally compact group, and consider a Borel map $\mathfrak{b}$ from G into a Hilbert space $\mathcal{H}$, together with a Borel map π from G into the unitary group $\mathcal{U}(\mathcal{H})$. We assume that $\mathfrak{b}$ is *locally bounded*, i.e., it is bounded on every compact subset. The map $\mathfrak{b}$ is a *cocycle* if π is a representation and $\mathfrak{b}$ satisfies $\mathfrak{b}(gh) = \mathfrak{b}(g) + \pi(g)\mathfrak{b}(h)$ for all $g, h \in G$. It is a *quasi-cocycle* if π is a representation and the *defect of* $\mathfrak{b}$ is finite:

$$\sup_{g,h\in G} \|\mathfrak{b}(gh) - \big(\mathfrak{b}(g) + \pi(g)\mathfrak{b}(h)\big)\| < +\infty.$$

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It is a *wq-cocycle* if the defect is finite (and no multiplicativity condition on π). Recall that G has property (T) (resp. (TT)) if every cocycle (resp. quasi-cocycle) on G is bounded. We say G has *property* (TTT) if every wq-cocycle on G is bounded.

The study of property (TTT) is motivated by the following fact. A map $q\colon G \to G'$ is called a *quasi-homomorphism* if it is a continuous map (or just Borel and locally bounded) such that the defect $\{q(gh)^{-1}q(g)q(h) : g,h \in G\}$ is relatively compact in G'. In the case where $G' = \mathbb{R}$, quasi-homomorphisms are often called quasimorphisms and have been studied extensively. (See [Ca].) It is easily seen that the composition $\mathscr{b} \circ q$ of a quasi-homomorphism $q\colon G \to G'$ and a wq-cocycle $\mathscr{b}\colon G' \to \mathcal{H}$ is again a wq-cocycle.

Definition. We say a locally compact group G is *a-TTT-menable* (or G has *property* (h)) if there is a wq-cocycle $\mathscr{b}$ on G which is *proper* in the sense that $\{g \in G : \|\mathscr{b}(g)\| \le C\}$ is relatively compact for every $C > 0$.

Groups with proper cocycles (i.e., a-T-menable groups, also known as groups with Haagerup's property, see [CJV]) are a-TTT-menable. In particular, all amenable groups are a-TTT-menable. All hyperbolic groups are also a-TTT-menable. (See Section 7.E_1 in [Gr]. More explicitly, $\mathscr{b}(g) := q[1,g]$ in the notation of Theorem 10 in [Min] is a proper quasi-cocycle.) From the above discussion, we have the following consequence.

Theorem A. *Let G and G' be locally compact groups such that G with property* (TTT) *and G' a-TTT-menable. Then, every quasi-homomorphism from G into G' has a relatively compact image.*

We will prove that the inclusion of an abelian group A into a semidirect product group $G_0 \ltimes A$ has Kazhdan's relative property (T) if and only if it has relative property (TTT). It follows that the group $\mathrm{SL}_{n\ge3}(\mathbb{R})$ has property (TTT). We then make an extra effort to prove that property (TTT) is inherited to lattices (under a certain condition).

Theorem B. *For any local field $\mathbb{K}$ and $n \ge 3$, the group $\mathrm{SL}_n(\mathbb{K})$ and its lattices have property* (TTT).

Unlike the case of property (T), it is not so straightforward to prove that property (TTT) is inherited to lattices. In the case of property (TT), Burger and Monod ([BM1, BM2]) use bounded cohomology machinery. For property (TTT), we replace their cohomological theorem (injectivity of the L^2-induction map) with the following theorem about a length function on a measured transformation groupoid.

Theorem C. *Let $G \curvearrowright X$ be a measure-preserving action of a locally compact group G on a standard probability space X, and $\ell\colon X \times G \to \mathbb{R}_{\geq 0}$ be a measurable function such that*

$$\ell(x, gh) \leq \ell(x, g) + \ell(g^{-1}x, h)$$

for a.e. $(x, g, h) \in X \times G \times G$. If

$$\operatorname*{ess\text{-}sup}_{g \in G} \int_X \ell(x, g)\, dx < +\infty,$$

then there exists $h \in L^1(X)$ such that

$$\ell(x, g) \leq h(x) + h(g^{-1}x)$$

almost everywhere.

When one views a length function ℓ as a kind of cocycle, this theorem says that if a "cocycle" is bounded in a certain sense, then it is dominated by a "coboundary." A more precise version of this theorem is given as Theorem 8.

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2. Preliminaries on abstract harmonic analysis

In this section, we collect useful facts from abstract harmonic analysis. We refer the reader to [BHV, BO, CH, HR, Pi] for more information. Let G be a locally compact group and denote by λ the left regular representation of G on $L^2(G)$. We extend λ to the Banach algebra $L^1(G)$ by

$$(\lambda(f)\zeta)(x) = \int_G f(g)\zeta(g^{-1}x)\, dg = (f * \zeta)(x).$$

The reduced group C*-algebra $C^*_r(G)$ is defined as the norm-closure of $\lambda(L^1(G))$ in $\mathbb{B}(L^2(G))$. When $G = A$ is abelian, the Fourier transform $L^2(A) \cong L^2(\widehat{A})$ implements a canonical $*$-isomorphism between $C^*_r(A)$ and the C*-algebra $C_0(\widehat{A})$ of all continuous functions on the Pontrjagin dual $\widehat{A}$ that vanish at infinity.

Recall that a kernel $\theta\colon G\times G\to\mathbb{C}$ is said to be positive definite if $\sum_{i,j}\theta(x_i,x_j)\xi_i\overline{\xi_j}\geq 0$ for any n and $x_1,\dots,x_n\in G$, $\xi_1,\dots,\xi_n\in\mathbb{C}$. It is well-known that θ is positive definite if and only if there is a map F from G into a Hilbert space $\mathcal{K}$ such that $\theta(x,y)=\langle F(x),F(y)\rangle$. A positive definite kernel θ is said to be *normalized* if $\theta(x,x)=1$ for all $x\in G$. One can normalize $\theta(x,y)=\langle F(x),F(y)\rangle$ by replacing it with $\langle\frac{F(x)}{\|F(x)\|},\frac{F(y)}{\|F(y)\|}\rangle$. We note that if θ is a normalized positive definite kernel, then $\theta(x,y)\approx 1$ implies $\theta(x,z)\approx\theta(y,z)$ uniformly for $z\in G$, or more precisely

$$|\theta(x,z)-\theta(y,z)|\leq\|F(x)-F(y)\|\,\|F(z)\|\leq\sqrt{2}|1-\theta(x,y)|^{1/2}.$$

If θ is a bounded measurable positive definite kernel, then F as above is weakly measurable, i.e., $x\mapsto\langle F(x),v\rangle$ is measurable for every $v\in\mathcal{K}$, and

$$\int\theta(x,y)\xi(x)\overline{\xi(y)}\,dx\,dy=\langle\int\xi(x)F(x)\,dx,\int\xi(y)F(y)\,dy\rangle\geq 0$$

for every $\xi\in L^1(G)$, where $\int\xi(x)F(x)\,dx$ is the unique element in $\mathcal{K}$ such that $\langle\int\xi(x)F(x)\,dx,v\rangle=\int\xi(x)\langle F(x),v\rangle\,dx$ for every $v\in\mathcal{K}$. With this in mind, we say a kernel $\theta\in L^\infty(G\times G)$ is *(essentially) positive definite* if $\int\theta(x,y)\xi(x)\overline{\xi(y)}\,dx\,dy\geq 0$ for every $\xi\in L^1(G)$. A kernel $\theta\in L^\infty(G\times G)$ is positive definite if and only if there is a measurable function P from G into a separable Hilbert space $\mathcal{H}$ such that $\theta(x,y)=\langle P(x),P(y)\rangle$ a.e. Moreover, if θ is continuous in addition, then P is continuous and the previous equality holds everywhere. Indeed, if θ is positive definite, then there are a Hilbert space $\mathcal{H}$ and a bounded linear map $T\colon L^1(G)\to\mathcal{H}$ such that $\langle T\xi,T\eta\rangle=\int\theta(x,y)\xi(x)\overline{\eta(y)}\,dx\,dy$, because the right hand side defines a semi-inner product on $L^1(G)$. But, every bounded linear map $T\colon L^1(G)\to\mathcal{H}$ is represented by $P\in L^\infty(G,\mathcal{H})$ in such a way that $T\xi=\int\xi(x)P(x)\,dx$. It follows that $\|P\|=\|T\|=\|\theta\|_\infty^{1/2}$ and $\theta(x,y)=\langle P(x),P(y)\rangle$ a.e. When θ is continuous, P can be taken continuous. Next, we consider not necessarily positive definite $\theta\in L^\infty(G\times G)$ and define the cb-norm of θ by

$$\|\theta\|_{\mathrm{cb}}=\inf\{\|P\|\,\|Q\|:P,Q\in L^\infty(G,\mathcal{H}),\ \theta(x,y)=\langle P(x),Q(y)\rangle\text{ a.e.}\},$$

where the infimum over empty set is equal to ∞. This norm is also described via positive definite kernels. Let $G^{(2)}=G\sqcup G$ be the disjoint union of two copies of G, and $\iota_{1,2}$ be the inclusion of $G\times G$ into the $(1,2)$-component of $G^{(2)}\times G^{(2)}$. Then, one has

$$\|\theta\|_{\mathrm{cb}}=\inf\{\|\tilde{\theta}\|_\infty:\tilde{\theta}\in L^\infty(G^{(2)}\times G^{(2)})\text{ pos. def., }\tilde{\theta}\circ\iota_{1,2}=\theta\}.$$

Moreover, $\|\theta\|_{\mathrm{cb}}$ coincides with the operator norm viewed as a Schur multiplier on $\mathbb{B}(L^2(G))$. See [Ha] or Section 3.2 of [Sp] for more information.

Now recall G is a group and define $g \cdot \theta$ for $g \in G$ and $\theta \in L^\infty(G \times G)$ by

$$(g \cdot \theta)(x, y) = \theta(g^{-1}x, g^{-1}y),$$

and suppose that θ is continuous, left-invariant, say $\theta(g, h) = \varphi(g^{-1}h)$ for a continuous function φ, and has finite cb-norm. Then, the function φ is a Herz–Schur multiplier. Namely, $m_\theta \colon L^1(G) \ni f \mapsto \varphi f \in L^1(G)$ extends to the reduced group C*-algebra $C^*_r(G)$, which satisfies $\|m_\theta\| \le \|\theta\|_{\rm cb}$. Indeed, given an expression $\theta(g, h) = \langle P(g), Q(h)\rangle$, we define the operator $V_P \colon L^2(G) \to L^2(G, \mathcal{H})$ by $(V_P\zeta)(x) = \zeta(x)P(x^{-1})$. It is clear that $\|V_P\| \le \|P\|$. Likewise for V_Q. We then observe that $\lambda(\varphi f) = V_Q^*(\lambda(f) \otimes 1_{\mathcal{H}})V_P$. (In fact, $\|\theta\|_{\rm cb}$ coincides with the cb-norm of m_θ.) Suppose moreover that $G = A$ is abelian. We denote by 0_A the unit character on A and view it as a character on $C^*_r(A) \cong C_0(\widehat{A})$. Since the liner functional $0_A \circ m_\theta$ on $C^*_r(A) \cong C_0(\widehat{A})$ is bounded, it is given by a finite complex Borel measure μ_θ on $\widehat{A}$ by the Riesz–Markov representation theorem. One has $\|\mu_\theta\| \le \|\theta\|_{\rm cb}$ (actually they coincide), and $\int_{\widehat{A}} \widehat{a}\, d\mu_\theta = \varphi(a)$ for all $a \in A$. Also note that μ_θ is positive if and only if θ is positive definite.

We explain how to take a (generalized) limit in dual Banach spaces. Fix a positive linear functional $\mathrm{Lim} \colon \ell_\infty(\mathbb{N}) \to \mathbb{C}$ which extends the usual limit on the convergent sequences. Let $V = (V_*)^*$ be a dual Banach space and (v_n) be a bounded sequence in V. Then, the limit $\mathrm{Lim}_n v_n$ is defined to be the unique element in V that satisfies

$$\langle \mathop{\mathrm{Lim}}_n v_n, \xi\rangle = \mathop{\mathrm{Lim}}_n \langle v_n, \xi\rangle$$

for $\xi \in V_*$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the duality coupling between V and its predual V_*. One has $\|\mathrm{Lim}_n v_n\| \le \mathrm{Lim}_n \|v_n\|$. If T is an operator from V into another dual Banach space $W = (W_*)^*$ that is weak*-continuous, i.e., $T^*\xi := \xi \circ T$ belongs to V_* for every $\xi \in W_*$, then $T(\mathrm{Lim}_n v_n) = \mathrm{Lim}_n T(v_n)$. Indeed,

$$\langle T \mathop{\mathrm{Lim}}_n v_n, \xi\rangle = \langle \mathop{\mathrm{Lim}}_n v_n, T^*\xi\rangle = \mathop{\mathrm{Lim}}_n \langle v_n, T^*\xi\rangle = \mathop{\mathrm{Lim}}_n \langle Tv_n, \xi\rangle = \langle \mathop{\mathrm{Lim}}_n Tv_n, \xi\rangle$$

for every $\xi \in W_*$. Now suppose Y is a σ-finite measure space and (f_n) is a real bounded sequence in $L^\infty(Y) = (L^1(Y))^*$. Then, although $\mathrm{Lim}_n f_n$ may not coincide with the pointwise limit $\mathrm{Lim}_n f_n(y)$, one still has

$$\liminf_n f_n(y) \le (\mathop{\mathrm{Lim}}_n f_n)(y) \le \limsup_n f_n(y)$$

for a.e. $y \in Y$. Indeed, for any $B \subset Y$ with finite measure, one has

$$\int_B \liminf_n f_n\, dy \le \liminf_n \int_B f_n\, dy \le \mathop{\mathrm{Lim}}_n \int_B f_n\, dy = \int_B \mathop{\mathrm{Lim}}_n f_n\, dy.$$

We observe that if $\theta_n \in L^\infty(G \times G)$ have uniformly bounded cb-norm, then $\mathrm{Lim}_n \theta_n \in L^\infty(G \times G)$ has cb-norm at most $\mathrm{Lim}_n \|\theta_n\|_{\mathrm{cb}}$. Indeed, this follows from the fact that $\mathrm{Lim}_n \theta_n$ is positive definite if θ_n are. (In fact, the Banach space $V_2(G)$ of those kernels that have finite cb-norm is a dual Banach space and the natural map from $V_2(G)$ into $L^\infty(G \times G)$ is a weak*-continuous injection.) We finally note that for a continuous kernel θ on G and a closed subgroup $A \leq G$, the restriction of θ to the subgroup A satisfies $\|\theta|_{A\times A}\|_{\mathrm{cb}} \leq \|\theta\|_{\mathrm{cb}}$. Indeed, let $\theta(x,y) = \langle P(x), Q(y)\rangle$ for a.e. $x, y \in G$. Take an approximate unit (f_n) of $L^1(G)$ and let $P_n(x) = \int_G f_n(g)P(g^{-1}x)\,dg$, and likewise for Q_n. Then, P_n and Q_n are continuous and the continuous kernel $\theta_n(x,y) = \langle P_n(x), Q_n(y)\rangle$ has cb-norm at most $\|P\|\,\|Q\|$. The same thing holds for $\theta_n|_{A\times A}$. Taking the limit, we are done.

3. Relationship to other formulations

Property (T) has several equivalent characterizations (see Section 2.12 in [BHV] or Theorem 12.1.7 in [BO]). In this section, we pursue analogous characterizations.

Definition. Let $A \leq G$ be a subgroup of a locally compact group G. We say the pair (G, A) has *relative property* (TTT) if every wq-cocycle on G is bounded on A. We say the pair (G, A) has *relative property* ($\mathrm{T_P}$) or ($\mathrm{T_Q}$) respectively, if for every $\varepsilon > 0$, there exist a compact subset $K \subset G$ and $\delta > 0$ (we will take $\delta < \varepsilon$ for granted) satisfying the following condition.

($\mathrm{T_P}$): If $\theta\colon G\times G \to \mathbb{C}$ is a (normalized) Borel positive definite kernel such that

$$\sup_{g\in G} \|g\cdot\theta - \theta\|_{\mathrm{cb}} < \delta \quad\text{and}\quad \sup_{g^{-1}h\in K} |\theta(g,h) - 1| < \delta,$$

then one has

$$\sup_{x,y\in A} |\theta(x,y) - 1| < \varepsilon.$$

($\mathrm{T_Q}$): If $\pi\colon G \to \mathcal{U}(\mathcal{H})$ is a Borel map and $\xi \in \mathcal{H}$ is a unit vector such that

$$\sup_{g,h\in G} \|\pi(gh)\xi - \pi(g)\pi(h)\xi\| < \delta \quad\text{and}\quad \sup_{g\in K} \|\pi(g)\xi - \xi\| < \delta,$$

then one has

$$\sup_{x\in A} \|\pi(x)\xi - \xi\| < \varepsilon.$$

When (G, G) has relative property ($\mathrm{T_P}$) or ($\mathrm{T_Q}$), we simply say G has *property* ($\mathrm{T_P}$) or ($\mathrm{T_Q}$) respectively.

We remark that the term "Borel" in the definition of property ($\mathrm{T_P}$) can be replaced with "continuous." Indeed, one can replace any Borel positive definite kernel θ with a continuous $\tilde{\theta}$, defined by

$$\tilde{\theta}(x,y) = \frac{1}{|K|^2} \int_{K\times K} \theta(xk, yk')\, dk\, dk',$$

where K is a compact non-negligible subset. Note that $\tilde{\theta}$ is uniformly close to the original θ if $\theta(x, xk) \approx 1$ for all $(x,k) \in G \times K$.

Property ($\mathrm{T_Q}$) is suited for the study of rigidity phenomena in the setting of ε-representations (see [Kaz, BOT]). We will prove that ($\mathrm{T_P}$) $\Rightarrow$ (TTT) $\Rightarrow$ ($\mathrm{T_Q}$), but it is unclear whether they are all equivalent. Relative property ($\mathrm{T_Q}$) implies relative property (T). This fact is not hard to show when A is normal. For the general case, see [Jo].

Theorem 1. *For a pair* (G, A) *as above, one has*

rel. property ($\mathrm{T_P}$) $\Rightarrow$ rel. property (TTT) $\Rightarrow$ rel. property ($\mathrm{T_Q}$).

Proof. We only prove ($\mathrm{T_P}$) $\Rightarrow$ (TTT), and omit the proof of (TTT) $\Rightarrow$ ($\mathrm{T_Q}$) because it is virtually same as the classical one (see Proposition 2.4.5 in [BHV]). Note that we are assuming G is second countable.

Let $\varepsilon = 1/2$ and take (K, δ) which satisfies condition ($\mathrm{T_P}$). We may assume that K is symmetric and contains the unit. Let $b \colon G \to \mathcal{H}$ be a wq-cocycle. Considering realification, we may assume that b is real and π is orthogonal. By scaling b, we may further assume that

$$\sup_{g,h\in G} \|b(gh) - \big(b(g) + \pi(g)b(h)\big)\| \le \delta_0 \quad \text{and} \quad \sup_{g\in K} \|b(g)\| \le \delta_0,$$

where $\delta_0 > 0$ is a sufficiently small number which will be chosen later. We consider the full Fock Hilbert space $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}$, where $\mathcal{H}^{\otimes 0} = \mathbb{R}$, and the exponential map $\mathrm{EXP} \colon \mathcal{H} \to \mathcal{F}$ given by

$$\mathrm{EXP}(\xi) = 1 \oplus \frac{\xi}{\sqrt{1!}} \oplus \frac{\xi\otimes\xi}{\sqrt{2!}} \oplus \frac{\xi\otimes\xi\otimes\xi}{\sqrt{3!}} \oplus \cdots .$$

We define $E \colon \mathcal{H} \to \mathcal{F}$ by $E(\xi) = \exp(-\|\xi\|^2)\,\mathrm{EXP}(\sqrt{2}\xi)$. It follows

$$\langle E(\xi), E(\eta)\rangle = \exp(-\|\xi-\eta\|^2)$$

for all $\xi, \eta \in \mathcal{H}$. In particular, E is a continuous map into the unit sphere of $\mathcal{F}$. Consider the normalized Borel positive definite kernel

$$\theta(x,y) = \langle E(b(x)), E(b(y))\rangle = \exp(-\|b(x) - b(y)\|^2).$$

Since

$$\theta(x,y) = \langle E\big(b(g^{-1}) + \pi(g^{-1})b(x)\big), E\big(b(g^{-1}) + \pi(g^{-1})b(y)\big)\rangle,$$

one has

$$\|g\cdot\theta-\theta\|_{\mathrm{cb}}\leq 2\sup_{x\in G}\|E(\mathfrak{b}(g^{-1}x))-E\big(\mathfrak{b}(g^{-1})+\pi(g^{-1})\mathfrak{b}(x)\big)\|<\delta_1$$

for all $g\in G$, where $\delta_1=2(2-2\exp(-\delta_0^2))^{1/2}$. Also,

$$|\theta(g,h)-1|=1-\exp(-\|\mathfrak{b}(g)-\mathfrak{b}(h)\|^2)<1-\exp(-4\delta_0^2)$$

for all $(g,h)\in G^2$ that satisfy $g^{-1}h\in K$. Thus, if $\delta_0>0$ was chosen sufficiently small, then property $(\mathrm{T_P})$ implies

$$1-\exp(-\|\mathfrak{b}(x)-\mathfrak{b}(1)\|^2)=|\theta(x,1)-1|<\varepsilon=1/2.$$

for all $x\in A$. This means that $\mathfrak{b}$ is bounded on A. □

Corollary 2. *Let $G\curvearrowright X$ be a measure-preserving action on a standard probability space X and $\beta\colon X\times G\to G'$ be a measurable map having the following property: For every compact subset $K\subset G$ and a.e. $x\in X$, the set*

$$\{\beta(x,g):g\in K\}\cup\{\beta(x,gh)^{-1}\beta(x,g)\beta(g^{-1}x,h):g,h\in G\}$$

is relatively compact in G'. If G has property $(\mathrm{T_Q})$ and G' is a-T-menable, then there exists a sequence $X_1\subset X_2\subset\cdots\subset X$ such that $\bigcup X_n$ is co-null in X and

$$\{\beta(x,g):x\in X_n,\ g\in G\text{ such that }g^{-1}x\in X_n\}$$

are relatively compact in G' for all n.

The proof of Corollary 2 will be given at the end of Section 5.

4. Property (TTT) for $SL_n(\mathbb{K})$

In this section, we prove property $(\mathrm{T_P})$ for $SL_n(\mathbb{K})$. We follow a well-established line of the proof for property (T) (see Section 1.4 in [BHV]), also employing ideas from [Bu, Sh1, Sh2].

Let G be a locally compact group and $A\leq G$ be an abelian closed normal subgroup. Then, G acts on the Pontrjagin dual $\widehat{A}$ by the dual action of the conjugate action. This action induces an isometric action of G on the Banach space $\mathcal{M}(\widehat{A})$ of finite regular Borel measures on $\widehat{A}$.

Proposition 3. *Let $G=G_0\ltimes A$ be the semidirect product of a locally compact abelian group A by a continuous action of a locally compact group G_0. Then, the following are equivalent.*

1. *The pair (G,A) has relative property* (T)*.*
2. *The pair (G,A) has relative property* $(\mathrm{T_P})$ *(resp.* (TTT)*,* $(\mathrm{T_Q})$*).*

(3) *For every $\varepsilon > 0$, there exist a compact subset $K \subset G_0 \cup A$ and $\delta > 0$ which have the following property: If μ is a probability measure on $\widehat{A}$ such that $\|g \cdot \mu - \mu\| \leq \delta$ for $g \in K$ and $|1 - \int_{\widehat{A}} \widehat{a}\, d\mu| < \delta$ for $a \in K \cap A$, then $|1 - \int_{\widehat{A}} \widehat{x}\, d\mu| < \varepsilon$ for all $x \in A$.*

Proof. The implication (2) $\Rightarrow$ (1) is explained in Section 3, and (1) $\Rightarrow$ (3) is proved in [CT, Io]. Now, we assume (3) and prove that the pair (G, A) has relative property $(\mathrm{T_P})$. Let $\varepsilon > 0$ be given and take (K, δ_0) which satisfies condition (3). We assume that K is symmetric and contains a neighborhood of the unit. Let $\theta\colon G \times G \to \mathbb{C}$ be a normalized continuous positive definite kernel such that

$$\sup_{x \in G} \|x \cdot \theta - \theta\|_{\mathrm{cb}} < \delta \quad \text{and} \quad \sup_{g^{-1}h \in K} |\theta(g, h) - 1| < \delta,$$

where $\delta > 0$ is a sufficiently small number which will be chosen later. We will prove that $\sup_{a,b \in A} |\theta(a, b) - 1| < 2\varepsilon$ for every $a, b \in A$. Express θ as $\theta(x, y) = \langle P(x), P(y)\rangle$. Then, one has

$$\|P(x) - P(xg)\|^2 = 2 - 2\Re\theta(x, xg) < 2\delta$$

for all $x \in G$ and $g \in K$. In particular,

$$\sup_{g,h \in K} \|\theta(\,\cdot\, g, \,\cdot\, h) - \theta(\,\cdot\,, \,\cdot\,)\|_{\mathrm{cb}} \leq 3\delta^{1/2}.$$

We will average θ over A to obtain an A-invariant kernel. A similar idea is used in [Mim]. For every $a \in A$ and $g \in K$, define a kernel θ_a^g on A by

$$(\theta_a^g)(x, y) = \frac{1}{|K|^2} \int_{K \times K} \theta(agxg^{-1}k, agyg^{-1}k')\, dk\, dk'.$$

Then, the family $\{\theta_a^g : a \in A,\ g \in K\}$ is (right) equicontinuous and satisfies $\|\theta_a^g - \theta\|_{\mathrm{cb}} \leq \delta + 6\delta^{1/2} =: \delta_1$ (see the last paragraph of Section 2). Take a Følner sequence $L_n \subset A$ and consider the kernel

$$\theta_n^g(x, y) = \frac{1}{|L_n|} \int_{L_n} \theta_a^g(x, y)\, da,$$

where the integration is with respect to the Haar measure of A. We fix a free ultrafilter on $\mathbb{N}$ and denote the associated ultralimit by Lim_n. Then, $\tilde{\theta}^g(x, y) = \mathrm{Lim}_n\, \theta_n^g(x, y)$ is an A-invariant continuous positive definite kernel such that $\|\tilde{\theta}^g - \theta\|_{\mathrm{cb}} \leq \delta_1$ for $g \in K$.

Now, let $\mu_{\tilde{\theta}^1}$ be the measure associated with the A-invariant continuous positive definite kernel $\tilde{\theta}^1$. One has $1 \geq \mu_{\tilde{\theta}^1}(\widehat{A}) = \tilde{\theta}^1(1, 1) \geq 1 - \delta_1$,

$$\sup_{g \in K} \|g \cdot \mu_{\tilde{\theta}^1} - \mu_{\tilde{\theta}^1}\| \leq \sup_{g \in K} \|\tilde{\theta}^g - \tilde{\theta}^1\|_{\mathrm{cb}} \leq 2\delta_1$$

and

$$\sup_{a\in K\cap A}|1-\int_{\widehat{A}}\widehat{a}\,d\mu_{\tilde{\theta}^1}|=\sup_{a\in K\cap A}|1-\tilde{\theta}^1(a,1)|\leq 2\delta_1.$$

Thus, if $\delta>0$ was chosen sufficiently small, then condition (3) implies

$$\sup_{x,y\in A}|\tilde{\theta}^1(x,y)-1|=\sup_{x,y\in A}|1-\int_{\widehat{A}}\widehat{x^{-1}y}\,d\mu_{\tilde{\theta}^1}|\leq\varepsilon.$$

It follows that

$$\sup_{x,y\in A}|\theta(x,y)-1|\leq\varepsilon+\delta_1.$$

This completes the proof. □

We remark that one can prove in a similar manner the following strengthening of Theorem 5.5 in [Sh1].

Proposition 4. *Let G be a locally compact group and A be an abelian closed normal subgroup. Assume that there is no G-invariant finitely additive probability measure defined on the Borel subsets of $\widehat{A}-\{0_A\}$. Then, the pair (G,A) has relative property* $(\mathrm{T_P})$.

Now, we prove one half of Theorem B. Let R be a unital commutative ring. We recall that an elementary matrix means an element in $\mathrm{SL}_n(R)$ of the form $E_{i,j}(r)=I+re_{i,j}$ for some $i\neq j$ and $r\in R$, and $\mathrm{EL}_n(R)$ denotes the subgroup of $\mathrm{SL}_n(R)$ generated by elementary matrices. The group $\mathrm{EL}_n(R)$ is *boundedly elementary generated* if there is a number $l=l(n,R)$ such that every element in $\mathrm{EL}_n(R)$ can be written as a product of at most l elementary matrices. (See [Sh2] or Chapter 4 in [BHV].) Thanks to the Gaussian elimination process, for any field R, one has $\mathrm{EL}_n(R)=\mathrm{SL}_n(R)$ and it is boundedly elementary generated.

Theorem 5. *For any local field $\mathbb{K}$ and $n\geq 3$, the group $\mathrm{SL}_n(\mathbb{K})$ has property* $(\mathrm{T_P})$*. For any finitely generated unital commutative ring R and $n\geq 3$ such that $\mathrm{EL}_n(R)$ is boundedly elementary generated, the discrete group $\mathrm{EL}_n(R)$ has property* $(\mathrm{T_P})$.

Proof. Let $R=\mathbb{K}$ or a finitely generated unital commutative ring, and $G=\mathrm{EL}_n(R)$. The pair $R^2\triangleleft\mathrm{EL}_2(R)\ltimes R^2$ has relative property (T) by Corollary 1.4.13 in [BHV] and by [Sh2]. Thus it has relative property $(\mathrm{T_P})$ as well by Theorem 1. Let $\varepsilon>0$ be arbitrary and take (K_0,δ) which satisfies condition $(\mathrm{T_P})$. For each pair $i\neq j$, there is an embedding $\sigma_{i,j}\colon\mathrm{EL}_2(R)\ltimes R^2\to\mathrm{EL}_n(R)$ such that $E_{i,j}(R)\subset\sigma_{i,j}(R^2)$. Let $K=\bigcup\sigma_{i,j}(K)\subset\mathrm{EL}_n(R)$.

Suppose that θ is a normalized continuous positive definite kernel on G such that

$$\sup_{g\in G}\|g\cdot\theta-\theta\|_{\mathrm{cb}}<\delta \quad\text{and}\quad \sup_{g^{-1}h\in K}|\theta(g,h)-1|<\delta.$$

Then, by relative property $(\mathrm{T_P})$, one has

$$\sup_{s\in E_{i,j}(R)}|\theta(s,1)-1|<\varepsilon.$$

It follows (see Section 2) that

$$|\theta(gs,1)-\theta(g,1)|^2\leq 2|1-\theta(gs,g)|<2(\delta+\varepsilon)<4\varepsilon$$

for all $g\in G$ and $s\in E_{i,j}(R)$. By bounded generation property, this implies $|1-\theta(x,1)|<2l\varepsilon^{1/2}$ for all $x\in G$. □

Remark. The group $\mathrm{SL}_n(\mathbb{K})$ actually have st.pr.$(\mathrm{T_P})$ in the sense of [Sh1]. Namely, one can take K to be finite rather than compact, and any wq-cocycle on $\mathrm{SL}_n(\mathbb{K})$, which is not assumed locally bounded, is bounded. For the proof, mimic [Sh1], or use [CT].

There is a variant of Mautner's lemma: Let θ be a normalized continuous positive definite kernel on G such that $\|g\cdot\theta-\theta\|_{\mathrm{cb}}<\varepsilon$ for all $g\in G$, and $x,y\in G$ be such that $|\theta(y,1)-1|<\varepsilon$ and $|\theta(y^{-1}xy,1)-1|<\varepsilon$. Then, $|\theta(x,1)-1|<2\varepsilon+4\varepsilon^{1/2}$.

5. Induction and length-like functions

In this section, we prove the remaining half of Theorem B. The proof will involve a general discussion about length-like functions on measured-groupoids.

Theorem 6. *Let G be a locally compact group and $\Gamma\leq G$ be a lattice. Then, G has property $(\mathrm{T_P})$ if and only if Γ has property $(\mathrm{T_P})$.*

Now Theorem B follows from Theorems 5, 6 and 1. We remark that property $(\mathrm{T_P})$ is moreover a measure-equivalence invariant, and the same thing holds for property $(\mathrm{T_Q})$. On the other hand, it is unclear whether property (TTT) is inherited to a lattice unless the lattice is cocompact, because one needs a certain integrability condition to induce wq-cocycles. We do not prove these facts, because we will not (probably ever) need them. For the proof of Theorem 6, we use a random walk technique, in particular double ergodicity of a Poisson boundary, which is also a key ingredient in the proof of the fact that property (TT) is inherited to lattices ([BM1, BM2]). Thus, we fix a symmetric non-degenerate probability measure μ on G, which is absolutely continuous with respect to the Haar measure. Such a measure

μ always exists (because we are assuming that G is second countable). Let V be a coefficient G-module (i.e., V is a dual Banach space on which G acts by weak*-continuous isometries) and $f \in L^\infty(G, V)$. We define

$$(\mu * f)(g) = \int_G s \cdot f(s^{-1}g)\, d\mu(s) \;\text{ and }\; (f * \mu)(g) = \int_G f(gt^{-1})\, d\mu(t).$$

The following is an incarnation of double ergodicity of a Poisson boundary ([Kai]). It is also considered as a noncommutative Choquet–Deny theorem with coefficients (cf. Theorem 1 in [Wi]).

Lemma 7. *Assume V is a separable coefficient G-module and $f \in L^\infty(G, V)$ is such that $\mu * f = f = f * \mu$. Then, there exists a G-invariant vector v_0 in V such that $f = v_0$ almost everywhere.*

Proof. For $m, n \in \mathbb{N}$, define $F_{m,n}\colon (G, \mu)^{\mathbb{Z}} \to V$ by

$$F_{m,n}((g_k)_{k\in\mathbb{Z}}) = g_0^{-1} \cdots g_{-m}^{-1} \cdot f(g_{-m} \cdots g_0 g_1 \cdots g_n).$$

By the martingale convergence theorem, $(F_{m,n})$ converges a.e. as $m, n \to \infty$. The limit function F satisfies $F((g_{k+1})_{k\in\mathbb{Z}}) = g_1^{-1} F((g_k)_{k\in\mathbb{Z}})$, and hence is constant by Theorem 6 in [Kai], say $F = v_0$. Note that v_0 is a G-invariant vector. Since $(F_{m,n})$ is uniformly bounded, for every measurable subsets $B_1, \ldots, B_l \subset G$, one has

$$\int_G f(g)\, d\big((\mu|_{B_1}) * \cdots * (\mu|_{B_l})\big)(g) = \int_{G^{\mathbb{Z}}} F_{m,n}(\boldsymbol{g}) \chi_{B_1}(g_1) \cdots \chi_{B_l}(g_l)\, d\mu^\infty(\boldsymbol{g})$$
$$\to \mu(B_1) \cdots \mu(B_l)\, v_0 \quad \text{as } m, n \to \infty.$$

This implies that $f = v_0$ almost everywhere. □

Theorem 8. *Let $G \curvearrowright X$ be a measure-preserving action on a standard probability space X, $C \geq 1$ and $\ell\colon X \times G \to \mathbb{R}_{\geq 0}$ be a measurable function such that*

$$\ell(x, gh) \leq C(\ell(x, g) + \ell(g^{-1}x, h))$$

for a.e. $(x, g, h) \in X \times G \times G$. Assume that

$$D := \limsup_{n\to\infty} \int_G \int_X \ell(x, g)\, dx\, d\mu^{*n}(g) < +\infty.$$

Then, there exists $h \in L^1(X)$ such that $\|h\|_1 \leq 4C^4 D$ and

$$\ell(x, g) \leq h(x) + h(g^{-1}x)$$

almost everywhere.

Proof. Take $R > 0$ arbitrary. Let $\ell_R = \min(\ell, R)$ and consider

$$f_R := \operatorname*{Lim}_{m,n} \mu^{*m} * \ell_R * \mu^{*n} \in L^\infty(X \times G) \cong L^\infty(G, L^\infty(X)),$$

where the limit is taken along an invariant mean $\mathrm{Lim}_{m,n}$ on $\mathbb{N}^2$ with respect to the weak*-topology on $L^\infty(X \times G)$. (See the discussion in Section 2.) Since the convolutions by μ are weak*-continuous operators on $L^\infty(X \times G)$, one has $\mu * f_R = f_R = f_R * \mu$. Since $L^\infty(X)$ is contained in the separable coefficient G-module $L^2(X)$, Lemma 7 implies that f_R belongs to $L^\infty(X)$ and is G-invariant. We note that

$$(\mu^{*m} * \ell_R * \mu^{*n})(x, g) = \int_{G^2} \ell_R(sx, sgt)\, d\mu^{*m}(s)\, d\mu^{*n}(t).$$

Choose a subset $A \subset \{g \in G : \int_X \ell(x, g)\, dx \le 2D\}$ of measure 1 (it is not difficult to see that the latter set has infinite measure). One has

$$\begin{aligned}
\|f_R\|_{L^1(X)} &= \int_{X \times A} f_R(x, g)\, d(x, g) \\
&= \operatorname*{Lim}_{m,n} \int_{X \times A} (\mu^{*m} * \ell_R * \mu^{*n})(x, g)\, d(x, g) \\
&\le \operatorname*{Lim}_{m,n} C^2 \int_{X \times A} \int_{G^2} \begin{pmatrix} \ell(sx, s) + \ell(x, g) \\ +\ell(g^{-1}x, t) \end{pmatrix} d\mu^{*m}(s)\, d\mu^{*n}(t)\, d(x, g) \\
&\le 4C^2 D.
\end{aligned}$$

We note that f_R is monotone increasing in R and define

$$h(x) = \frac{1}{2} C^2 \lim_{R \to \infty} f_R(x) + C^2 \liminf_{n \to \infty} \int_G \ell(x, s) + \ell(s^{-1}x, s^{-1})\, d\mu^{*n}(s).$$

By Fatou & Fubini, $h \in L^1(X)$ with $\|h\|_1 \le 4C^4 D$. Moreover, one has

$$\begin{aligned}
\ell_R(x, g) &= \liminf_{m,n} \int_{G^2} \ell_R(x, g)\, d\mu^{*m}(s)\, d\mu^{*n}(t) \\
&\le C^2 \liminf_{m,n} \int_{G^2} \begin{pmatrix} \ell_R(x, s^{-1}) + \ell_R(sx, sgt) \\ +\ell_R(t^{-1}g^{-1}x, t^{-1}) \end{pmatrix} d\mu^{*m}(s)\, d\mu^{*n}(t) \\
&\le C^2 \liminf_{m,n} \begin{pmatrix} \int_G \ell(x, s)\, d\mu^{*m}(s) + (\mu^{*m} * \ell_R * \mu^{*n})(x, g) \\ + \int_G \ell(t^{-1}g^{-1}x, t^{-1})\, d\mu^{*n}(t) \end{pmatrix} \\
&\le h(x) + h(g^{-1}x)
\end{aligned}$$

for a.e. $(x, g) \in X \times G$ and $R > 0$. This completes the proof. □

Now, let $\Gamma \leq G$ be a lattice. By rescaling, we assume that $X = G/\Gamma$ is a probability G-space. Choose a Borel lifting $\sigma\colon X \to G$ and denote by $\beta\colon X \times G \to \Gamma$ the associated cocycle given by

$$\beta(x,g) = \sigma(x)^{-1} g \sigma(g^{-1}x).$$

It satisfies the cocycle relation

$$\beta(x,gh) = \beta(x,g)\beta(g^{-1}x,h) \text{ and } \beta(x,g)^{-1} = \beta(g^{-1}x, g^{-1}).$$

We note the following fact, which has its own interest and can be used to prove that property (TTT) is inherited to cocompact lattices.

Corollary 9. *Let $\Gamma \leq G$ be a lattice and $\ell\colon \Gamma \to \mathbb{R}_{\geq 0}$ be a function such that $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in \Gamma$, and let*

$$L(g) := \int_{G/\Gamma} \ell(\beta(x,g))\, dx \in [0, +\infty].$$

If L is essentially bounded, then ℓ is bounded.

Proof. We consider the function $\ell(\beta(x,g))$ on the groupoid $X \times G$, where $X = G/\Gamma$. By Theorem 8, there is $h \in L^1(X)$ such that $\|h\|_1 \leq 4\|L\|_\infty$ and $\ell(\beta(x,g)) \leq h(x) + h(g^{-1}x)$. Let $X_0 = \{x : h(x) \leq 5\|L\|_\infty\}$, which is non-negligible. Then, for every $s \in \Gamma$ and a.e. $x, y \in X_0$, one has

$$\ell(s) = \ell(\beta(x, \sigma(x)s\sigma(y)^{-1})) \leq h(x) + h(y) \leq 10\|L\|_\infty.$$

This completes the proof. $\square$

Proof of Theorem 6. First, we suppose that G has property $(\mathrm{T_P})$ and prove Γ has the same. Let $\varepsilon > 0$ be given and take (K, δ) which satisfies condition $(\mathrm{T_P})$ for G. We may assume that the lifting $\sigma\colon X \to G$ is regular in the sense that it maps a compact subset of $X = G/\Gamma$ to a relatively compact subset of G. Choose a compact subset $X_0 \subset X$ whose measure is at least $1 - \delta/4$, and let $F = \{\beta(x,g) : x \in X_0,\ g \in K\}$, which is a finite subset in Γ. We will prove that $(F, \delta/2)$ satisfies condition $(\mathrm{T_P})$ for Γ. To do so, let $\theta\colon \Gamma \times \Gamma \to \mathbb{C}$ be a normalized positive definite kernel such that

$$\sup_{s \in \Gamma} \|s \cdot \theta - \theta\|_{\mathrm{cb}} < \delta/2 \quad \text{and} \quad \sup_{s^{-1}t \in F} |\theta(s,t) - 1| < \delta/2.$$

We induce θ from Γ to G by defining

$$\tilde{\theta}(g,h) = \int_X \theta(\beta(x,g), \beta(x,h))\, dx.$$

Then, $\tilde{\theta}$ is a normalized Borel positive definite kernel such that

$$\sup_{g\in G}\|g\cdot\tilde{\theta}-\tilde{\theta}\|_{\mathrm{cb}}\leq\sup_{g\in G}\int_X\|\beta(x,g)\cdot\theta-\theta\|\,dx<\delta/2$$

and, since $\beta(x,g)^{-1}\beta(x,h)=\beta(g^{-1}x,g^{-1}h)\in F$ for $x\in gX_0$ and $g^{-1}h\in K$,

$$\sup_{g^{-1}h\in K}|\tilde{\theta}(g,h)-1|\leq\sup_{g^{-1}h\in K}\int_{gX_0}|\theta(\beta(x,g),\beta(x,h))-1|\,dx+\delta/2<\delta.$$

It follows from property $(\mathrm{T_P})$ that

$$\sup_{g\in G}|\tilde{\theta}(g,1)-1|<\varepsilon.$$

We express θ as $\theta(s,t)=\langle P(s),P(t)\rangle$ and define $\ell\colon X\times G\to\mathbb{R}_{\geq0}$ by

$$\ell(x,g)=\|P(\beta(x,g))-P(1)\|+\delta^{1/2}.$$

Since $\|P(st)-P(s)\|^2<\|P(t)-P(1)\|^2+\delta$ for all $s,t\in\Gamma$, one has

$$\begin{aligned}\ell(x,gh)&\leq\ell(x,g)+\|P(\beta(x,g)\beta(g^{-1}x,h))-P(\beta(x,g))\|\\&\leq\ell(x,g)+\ell(g^{-1}x,h).\end{aligned}$$

Moreover,

$$\int_X\ell(x,g)\,dx\leq\left(\int_X\|P(\beta(x,g))-P(1)\|^2\,dx\right)^{1/2}+\delta^{1/2}<3\varepsilon^{1/2}$$

for all $g\in G$. By Theorem 8, there is $h\in L^1(X)$ such that $\|h\|_1\leq12\varepsilon^{1/2}$ and $\ell(x,g)\leq h(x)+h(g^{-1}x)$ a.e. Let $X_0=\{x:h(x)<13\varepsilon^{1/2}\}$, which is non-negligible. Then, for every $s\in\Gamma$ and a.e. $x,y\in X_0$, one has

$$|1-\theta(s,1)|\leq\frac{1}{2}\|P(s)-P(1)\|^2\leq\frac{1}{2}\ell(x,\sigma(x)s\sigma(y)^{-1})^2<100\varepsilon.$$

This proves that Γ has property $(\mathrm{T_P})$. We just mention that the proof of measure-equivalence invariance of property $(\mathrm{T_P})$ is similar to above.

Next, we suppose Γ has property $(\mathrm{T_P})$ and prove G has the same. Let $\varepsilon>0$ be given and take (F,δ) which satisfies condition $(\mathrm{T_P})$ for Γ. We take a compact subset $K\subset G$ such that $F\subset K$ and $|K\cap\sigma(X)|>1-\varepsilon$. We will prove that (K,δ) satisfies condition $(\mathrm{T_P})$ for G. To do so, let $\theta\colon G\times G\to\mathbb{C}$ be a normalized continuous positive definite kernel such that

$$\sup_{g\in G}\|g\cdot\theta-\theta\|_{\mathrm{cb}}<\delta\quad\text{and}\quad\sup_{g^{-1}h\in K}|\theta(g,h)-1|<\delta.$$

Then, property $(\mathrm{T_P})$ implies

$$\sup_{s\in\Gamma}|\theta(s,1)-1|<\varepsilon.$$

It follows (see Section 2) that for any $g \in G$, $y \in X$ and $s \in \Gamma$, one has

$$\begin{aligned}|\theta(g\sigma(y)s, g) - 1| &< \delta + |\theta(\sigma(y)s, \sigma(y)) - 1| + \sqrt{2}|\theta(\sigma(y), 1) - 1|^{1/2} \\ &< 2\delta + \varepsilon + \sqrt{2}|\theta(\sigma(y), 1) - 1|^{1/2}.\end{aligned}$$

Hence,

$$|1 - \int_X \theta(g\sigma(x)\beta(x, g^{-1}), g)\, dx| < 6\varepsilon^{1/2}$$

for every $g \in G$. On the other hand, since $g\sigma(x)\beta(x, g^{-1}) = \sigma(gx)$,

$$\int_X \theta(g\sigma(x)\beta(x, g^{-1}), g)\, dx = \int_X \theta(\sigma(gx), g)\, dx \approx_{\varepsilon+(2\delta)^{1/2}} \theta(1, g).$$

Therefore, $|\theta(g, 1) - 1| \leq 10\varepsilon^{1/2}$ for all $g \in G$. This completes the proof. □

Proof of Corollary 2. Let $\varepsilon > 0$ be given and take (K, δ) which satisfies condition $(\mathrm{T_Q})$. Let $\pi' \colon G' \to \mathcal{U}(\mathcal{H})$ be a C_0 unitary representation which has approximately G'-invariant unit vectors ξ_n. (See Theorem 2.1.1 in [CJV]). We consider $\pi \colon G \to \mathcal{U}(L^2(X, \mathcal{H}))$ defined by

$$(\pi(g)\xi)(x) = \pi'(\beta(x, g))\xi(g^{-1}x),$$

(see the remark at the end of this proof) and let

$$D_n(x) = \sup_{g,h\in G} \|\xi_n - \pi'\big(\beta(x, gh)^{-1}\beta(x, g)\beta(g^{-1}x, h)\big)\xi_n\|.$$

We view $\xi_n \in \mathcal{H}$ as constant vectors in $L^2(X, \mathcal{H})$. Since $D_n(x) \leq 2$ and $D_n(x) \to 0$ for a.e. $x \in X$ by assumption, one has

$$\sup_{g,h\in G} \|\pi(gh)\xi_n - \pi(g)\pi(h)\xi_n\|^2 \leq \int_X D_n(x)^2\, dx \to 0,$$

and

$$\sup_{g\in K} \|\pi(g)\xi_n - \xi_n\|^2 = \sup_{g\in K} \int_X \|\pi'(\beta(x, g))\xi_n - \xi_n\|^2\, dx \to 0.$$

Hence by property $(\mathrm{T_Q})$, there is n such that $\xi = \xi_n$ satisfies

$$\sup_{g\in G} \|\pi(g)\xi - \xi\| < \varepsilon \text{ and } \int_X D(x)\, dx < \varepsilon.$$

Then, $\ell(x, g) = \|\pi'(\beta(x, g))\xi - \xi\| + D(x)$ satisfies $\ell(x, gh) \leq 2\ell(x, g) + \ell(g^{-1}x, h)$ and $\sup_g \int_X \ell(x, g)\, dx \leq 2\varepsilon$. Hence, by Theorem 8, there is $h \in L^1(X)$ such that $\|h\|_1 \leq 2^7\varepsilon$ and $\ell(x, g) \leq h(x) + h(g^{-1}x)$ a.e. Then, $X' = \{x : h(x) < 1/4\}$ has measure at least $1 - 2^9\varepsilon$. Since π' is a C_0-representation, $\{\beta(x, g) : x, g^{-1}x \in X'\}$ is relatively compact in G'.

Remark. The map π, defined as above, is in general Haar measurable instead of Borel measurable. To fix this problem, either go through all proofs in this paper with measurable maps and ess-sup in place of Borel maps and sup, or take an ad hoc measure as follows: there is a null set N such that π is Borel on $G \setminus N$. Let K be any compact neighborhood of G. By the Lusin–Novikov uniformization theorem, one can find a Borel map $t\colon G \to K$ such that $gt_g^{-1}, t_g \in G \setminus N$ for all $g \in G$. Now, replace $\pi(g)$ with $\pi(gt_g^{-1})\pi(t_g)$, which is a Borel map. □

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