Precluding Collusion in Auctions

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Abstract

Collusive bidding in auctions jeopardizes the revenue to the seller. This paper describes a way to preclude strong and tacit subgroup collusion in a non-repeated auction environment, when cartels can commit to transfer exchange ex post, but not to reallocation. The robustness is attained by an optional assignment rule in a sealed-bid Vickrey auction, which is not applied in equilibrium, but serves as a credible threat to any collusive agreement aimed at generating an extra surplus. In the absence of benefits to collusion the backward-inducing bidders will not engage into collusive negotiations.

1 Introduction

Collusive bidding in auctions jeopardizes the revenue to the seller. The bidding rings can generate positive surpluses at the expense of the auctioneer, even when their commitment power is limited and the interaction is one-shot. The auctioneer however can usually neither prevent, nor punish for collusion, even in the case of detecting and proving it to a third party[1]. This

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[1] See Hendricks, Porter (1989) on detectability of collusion.

fact has motivated the search for auction procedures, that would feature collusion-robustness and so protect the auctioneer against potential losses. The universally robust solution appears unachievable, thus studies would focus a specific model of collusion, which they establish robustness to. The recent consensus in modeling collusion seems to follow the approach of Laffont, Martimort (1997), where the coalitions are capable of full enforcement of their joint decisions, such as the manipulation of reports to the grand mechanism. Put differently, in their model of collusion, after the payoff-relevant information has been disclosed within a ring, infinite punishment can be applied to those deviating from the mutually agreed manipulation.

This paper adopts a different model of collusion. In the first place, it assumes away the infinite punishments, so allowing for moral hazard within the bidding rings. A potentially deviating member puts at stake his agreed share in the collusive surplus, but does not risk to incur a infinite punishment. The joint manipulation of bids can be sustained due to the enforceable ex post monetary transfers. The side contracts over the transfers are signed at the prior negotiation stage and are contingent on the publicly observed outcome of the auction. I study the set of environments where the reallocation of the good is impossible: the cases of public procurement or delivery auctions, as well as of any publicly consumed goods, like advertisement slots or spectrum licenses. Note that in all those instances, the standard auction procedures are prone to collusion if the rings are capable of transfer enforcement.

At the expense of assuming some moral hazard we gain in the reduced amount of knowledge required to construct the collusion-proof mechanism. First, I allow the coalition formation to be unobserved by the auctioneer or mechanism designer, which is typically not the case in the literature. For instance, an influential study by Che, Kim (2009) assigns a respective collusion-proof mechanism to each possible coalition, which implies that their formation (though not necessarily the collusive contract itself) is observed. In contrast, the construction of the present mechanism does not rely on the auctioneer's knowledge as to which bidding rings are formed. The mechanism creates a possibility of a profitable deviation from the collusive agreement within any proper subset of bidders that forms into a coalition. Due to that possibility the mechanism precludes any positive surpluses from the information exchange within coalitions with respect to a non-cooperative play.

The concept of collusion robustness employed in this paper accommodates the possibility of moral hazard and arbitrary coalition formation, and hence differs from the existing concepts. Defining the robustness inherits the complications of modeling cooperative games of asymmetric information. This problematic is extensively surveyed in Forges, Minnelli, Vohra (2002). The first issue is the protocol of coalition formation: the sequence and contents of offers, the selection into a coalition etc. The second issue is the evolution of beliefs underlying the coalition formation. Do the players assume that their counterparts willing to participate count on benefiting from collusion - and update their beliefs accordingly? Do they go one step further, assuming a similar belief update by their counterparts and figuring what their valuations should be for those to remain interested in collusion after the update? The argument would invoke various hierarchies of first and higher order beliefs, tailored to the predictions of further play. Naturally we seek the concept of collusion proofness, which is possibly invariant in specifying both: the protocol and the belief hierarchy.

In line with the previous papers, we take the standpoint, in a sense most general, of a simultaneous decision of a set of players to select into a coalition. To resolve the belief formation issue we model the participation decision so as to be least restrictive about the beliefs on that stage, looking at type profiles in isolation. The participation constraint is then said to be satisfied if for *at least some profile* of valuations in a coalition, its every member is weakly better off in collusion, with at least one *strict* preference.

Our next consideration is the expected behavior within the bidding ring after the coalition formation. Conventionally, we will assume that the expectations formed are those of an equilibrium. The starting point is the correlated equilibrium, which links this paper to the existing collusion literature, which models external coordination of collusive behavior. [2] Since the collusive setting itself implies coalition's deviation, it seems natural to require that this equilibrium is stable against further subgroup deviations. Avoiding additional assumptions on how these subgroup deviations are organized (whether they can write further surplus-sharing contracts and to what extent they can enforce them), I choose to model them as trembles, following Myerson (1986) in his construction of the *acceptable correlated equilibrium.* Myerson's refinement of the correlated equilibrium of Aumann (1974) parallels Selten's (1975) trembling-hand perfection of Nash equilibrium: Myerson confines the equilibrium to stability against coalition deviations occurring with an infinitesimal probability.

[2]Note also that the set of correlated equilibria includes all Nash equilibria.

The collusive game is composed of two stages. The first stage is the communication stage, which is modeled as an imperfect-information game, where the players send messages in an arbitrary sequence. The messages imply commitment to transfers, bargaining and bidding agreement. At the second stage, the coalition members bid in the auction, according to the bidding arrangement. They may use a random coordination device that sends bidding recommendations privately to each of the ring members. Finally, after the auction is run the conditional transfers are made within the bidding ring. At this last stage there is no strategic interaction taking place: the transfers are enforceable and any communication beyond the publicly observed outcome is cheap talk.

We shall say that the strategy profile is an *acceptable equilibrium* of the collusive game, if it satisfies Myerson's (1986) acceptable correlated equilibrium conditions on the second stage, Bayes Nash equilibrium conditions on the first stage, Bayesian belief update on the equilibrium path.

I modify the standard Vickrey auction, so as to make it collusion-proof, by introducing an optional assignment rule. The threat to use the optional rule induces a conflict of interest between the two highest-valuation members of a bidding ring upon the information disclosure that occurs at the negotiation stage. Effectively, the collusion-proof modification of the Vickrey auction has a coalition-proof perfect Bayesian equilibrium, in which the optional rule is not chosen and the players submit their true valuations within the framework of the standard second-price auction.

The next section presents the mechanism in the environment with fully symmetric information within the bidding rings. Section 3 gives a formal treatment and shows that the robustness extends to the asymmetric information environment. Section 4 discusses the results and Section 5 concludes. All the proofs are given in the Appendix.

2 The Modified Vickrey Auction

In the second-price auction of a single good, consider a coalition composed of two or more bidders and the following manipulation of bids: The highest-valuation member submits his valuation truthfully - whereas the rest substantially under-report the lower valuations. This way the coalition reduces the expected second price to be paid by its highest-valuation member if he wins, at no cost to other coalition members. Indeed, conditional on not having the highest valuation, it is a weakly-dominant strategy to withdraw from competition. The incentives to withdraw can be further strengthened, if the leader is able commit to share the extra surplus brought about by the manipulation.

The ring generates a positive extra surplus if, *ex post,* its second-highest valuation was higher than the actual price paid.[3] The probability of this event is almost surely positive *ex interim* and positive *ex ante* - and therefore, there exist positive gains to collusion and losses in the seller's revenue. Let us introduce some notation to give a more formal treatment. Denote $x_{k|S}$ the k^{th} highest value in the set $\{x_i\}_{i\in S}$ of the observations of x, (ignore ties for the moment) and let $(k|S)$ be the identity (index) of the player, with whom value $x_{k|S}$ is associated. b will refer to the bid, and θ to the true valuation. If C is the coalition under consideration ($|C|$ referring to its size), then C's extra surplus is the following, from the three uncertainty perspectives.

	Extra surplus
Ex Post	$max\left\{\theta_{2\|C} - b_{1\|N/C}; 0\right\}$
Interim	$Prob\left(\theta_{2\|C} > \tilde{\theta}_{(1\|N/C)}\right)\mathbb{E}\left(\theta_{2\|C} - \tilde{\theta}_{1\|N/C}\|\theta_{2\|C} > \tilde{\theta}_{1\|N/C}\right)$
Ex Ante	$Prob\left((1\|N),(2\|N)\in C\right)\mathbb{E}\left(\tilde{\theta}_{2\|N} - \tilde{\theta}_{\|C\|+1}\right)$

In order to prevent the collusion, we seek to induce a profitable deviation from the interim collusive equilibrium. We construct a mechanism, in which an informed insider can deviate and make sure that he gets no less than his agreed share in the coalition's surplus.

We shall consider surplus sharing based on each member's ex post contributions. A contribution in this context is the price reduction resulting from the withdrawal of bid. Clearly, the maximal contribution is equal to the coalition's full extra surplus, and it can be pivoted by the second-highest valuation bidder if he enters the coalition last. It seems reasonable to assume that any bidder's reward in a collusive ring does not exceed this maximal contribution

[3]Note that this implies that the winner is among the members of the ring

- otherwise, coalition as a whole would have been necessarily better off without him.[4] The direct implication of this assumption is that collusion can be broken down by allowing a coalition member to grab the whole extra surplus by deviating.

In a private values framework with no reallocation possibility, consider the following modification of the second-price auction. The good is sold to the highest-valuation bidder at the second price as soon as the difference between the first and the second bid is higher than the difference between the second and the third bids. In the latter case the object is allocated to the second-highest bidder, who will pay the third bid to the auctioneer. Put differently, the good is allocated to that of the two highest bidders, who enjoys the larger "revealed surplus" from winning it. In this (non-efficient) auction the second-highest valuation member of a coalition (denoted $(2|C)$), knowing the coalition leader's valuation, can grab the surplus every time it arises. Placing his bid at the average of $\theta_{1|C}$ and his true valuation $\theta_{2|C}$ he gets $\max\left\{\theta_{2|C} - \theta_{1|N/C}; 0\right\}$, which is precisely the ex-post extra surplus of the coalition.

This assignment rule, which we refer to as *gap rule* further on, provokes a conflict of interest between the first two highest-valuation bidders within the coalition. When the second deviates from the collusive manipulation, the first is either deprived of the object he would have won otherwise or pays a price higher than he would have paid, if the second reported truthfully. In turn, the runner-up can be made at most indifferent between obedience to the collusive plan and a deviation. To induce that indifference the leader would have to give the whole surplus to the runner-up, but then the leader does not benefit from collusion whatsoever.

The obvious drawback of the gap rule is its inefficiency - there is a positive probability of assigning the good to a bidder who does not value it most. Thus the auctioneer would not want to apply it unconditionally, but keep it available for a potential deviator. One of the possible ways to integrate the gap option into the auction is the following. Each player is asked to communicate to the seller, in a sealed envelope, his bid and the choice of the assignment rule (second-price or gap rule, as just described). The gap rule is applied if and only if the second-highest bidder has opted for it. In-

[4] One may think of the surplus division according to the Shapley values, which satisfies this assumption with slack.

dependently, if the highest bidder has opted for the gap rule, he pays a fee L.

Provided that L is large enough, and the lower bound of valuations is zero for all bidders, the gap rule will never be played in equilibrium. The gap option is introduced to pose a threat to an eventual collusive agreement: note that any bidder aware that a bid will be placed higher than his own valuation, can apply for the gap rule at no cost. Furthermore, the second-highest valuation bidder of a coalition will always be able to deviate in a way to obtain the whole of the coalition's surplus. As the coalition cannot promise him more, the negotiations are destined to fail.

The next section provides a more formal treatment of this mechanism and shows that no coalition is able to improve upon fair play in the auction.

3 Formalization

In the modified auction collusion fails due to the availability of a secure deviation by a ring's informed member. In the simple setting of the previous section, this possibility relies on the full symmetry of information within a coalition; in particular the exact knowledge of how the leader is going to bid in the auction. Intuition suggests that additional incentive constraints imposed by the information asymmetry would make the collusive outcome even harder to realize. However this intuitive argument becomes less convincing in the regard of the possibility for the coalition's leader to hide his bidding plans and so preclude deviations. To shed light on the actually prevailing effect of information asymmetry, this section replaces the simultaneous information exchange with a general model of communication and studies the arising collusive equilibria.

Additionally, in this section we introduce randomness in two forms: first, the randomization device that coalitions can use for coordination, and second, mixed strategies.

The first result we obtain is that the modified auction has the Vickrey outcome. The second states that it is also robust to collusion, which we define as the fact that no collusive ring can make its members better off in an equilibrium of the collusive game than playing fair.

3.1 Setup

The setup is common knowledge. We consider an auction of a non-transferable good, valued by the seller at zero. The potential buyers form set $N = \{1, .., n\}$, $n > 2$. Every buyer $i \in N$ has a privately known valuation of θ_i euros, modeled as a random draw from a finite set of valuations Θ, with the minimal element $\underline{\theta} \geq 0$. The valuations are independent across bidders, each governed by a probability distribution f_i, with the property $f_i(\theta) > 0$ for all i, $\theta \in \Theta$. Let us denote f_S the joint distribution of types in set $S \subseteq N$ of bidders, $f_S = \prod_{i \in S} f_i$. The utilities are linear in money, hence a bidder, who wins the object and pays p, enjoys $\theta_i - p$ from the auction.

Auction The rules of an auction are defined by three elements: the set of the bidders' actions $\mathcal{A}$, the seller's decision rule (winner and transfers), and the outcome disclosure policy χ.

In the modified auction, each bidder's action comprises the bid and the binary choice of the assignment rule: $\mathcal{A} = \Theta \times \{0; 1\}$,[5] where 0 stands for the choice of the default rule (second-price), and 1 stands for the gap rule; in terms of available actions, the game is fully symmetric. For some non-empty $S \subseteq N$ and the action profile $a \in \mathcal{A}^{|S|}$ denote $bids[a]$ and $ch[a]$, respectively, the vector of bids and assignment rule choices implied by a.

The seller observes the actions of the bidders and transforms them into an outcome in $N \times \mathbb{R}^n$: the assignment of the object and the profile of transfers, such as prices and fines.[6] We shall denote this transformation rule (id, t) : $\mathcal{A}^n \to N \times \mathbb{R}^n$, and assume throughout that the auctioneer can commit to (id, t). For the Vickrey auction, the components of the rule (id, t) are the following:

$$id(a) = (1|N)$$

$$t_i(a) = -\begin{cases} bids[a]_{2|N} & if\ i = (1|N) \\ 0 & otherwise \end{cases},\ i \in N$$

[5] Thus the action set is finite.

[6] The ties are broken by a fair coin flip or a similar "randomization" device.

For the modified auction:

$$id(a) = \begin{cases} (1|N) & if\, G1(a) \\ (2|N) & if\, G2(a) \end{cases}$$

$$t_i(a) = -\begin{cases} bids[a]_{2|N} & if\, G1(a) \wedge (i = (1|N)) \\ bids[a]_{3|N} & if\, G2(a) \wedge (i = (2|N)) \\ 0 & otherwise \end{cases} - \begin{cases} L & if\ (i = (1|N)) \wedge ch[a_i] = 1 \\ 0 & otherwise \end{cases},$$

$$i \in N$$

$G1(a)$ denotes the event (set of action profiles) when $bids[a]_{1|N} - bids[a]_{2|N} \geq bids[a]_{2|N} - bids[a]_{3|N}$, and $G2(a)$ is the complementary event. The map (id, t) from the actions into the outcome is given by the following: the seller assigns the good to the highest-valuation bidder, who pays the bid of the second highest bidder, unless the latter has opted for the gap rule *and* the second gap appeared to be greater than the first gap. In that case the second highest bidder gets the object at the third price. A fine L is imposed the bidder who appears to submit the highest bid and opt for the gap rule.

Given the auction rule (id, t), we can define the implied utility over the actions $u_i : \mathcal{A}^n \times \Theta \rightarrow \mathbb{R}_+$ as follows:[7]

$$u_i\,(a; \theta_i) = \theta_i\,(id(a) = i) + t_i(a)$$

By the choice of the auction designer, both the bidders' actions and the outcomes (in a broad sense) of the auction may or may not be public. In the case of maximal revelation all the bids and choices are non-anonymous, however the collusive concerns will discourage the seller to go for the maximal revelation (see Appendix, page 23, for a discussion). If the auctioneer cannot conceal the identity of the winner and the price paid for it, the minimal disclosure policy preferred by the auctioneer is given by $\chi : A^n \rightarrow N \times \mathbb{R}_+$, where $\chi(a) = (id(a), -t_{id(a)}(a))$. The policy χ is a coarsening (but not a

[7] I don't index the utility function u_i by the particular auction rule $(\mathcal{A}, (id, t), \chi)$, presuming that it does not create confusion.

distortion) of the information contained in the outcome of the auction. We will label the image of χ by Ω and its typical element $\omega \in \Omega$. Ω can be referred to as the space of publicly observed outcomes.

Before proceeding to collusion proofness let us first show that the optional rule does not affect the behavior in the auction when it is played individually.

Proposition 1 For any type distribution profile f_N, there exists an L, such that the profile of Vickrey actions constitutes a Bayesian equilibrium of Γ.

The proof consists in showing that no *individual* deviations from the $a_N^0(\theta)$ can be profitable. Hence the option of the gap rule does not distort the incentive for the truthful bidding, and the Vickrey outcome occurs in the equilibrium of the modified auction, absent of collusion.

Collusion Proofness Let us say that an auction procedure $(\mathcal{A}, (id, t), \chi)$ *fails* against collusion by a group of bidders, if the collusive game with efficient allocation has at least one equilibrium (within a class) that makes at least one member of the group better off without hurting the other members. The welfare comparison is with respect to the Vickrey outcome, provided that it is an equilibrium outcome under fair play.[8] The collusion proofness is then defined by the negation of failure: There is no group of colluders, against which the auction fails.

The collusion is modeled as a joint deviation of a coalition that results in a subgame of side negotiations, manipulation of bids and surplus sharing. If Γ is the entire game associated with the auction $(\mathcal{A}, (id, t), \chi)$, let $\Gamma(S)$ designate the collusive subgame following the deviation of coalition S. Consistent with the backward induction, the decision to deviate by S relies on the payoff expectation in the equilibrium of the respective $\Gamma(S)$. We will have to employ some more notation to state the formal definition.

First, let us introduce a label for the particular action that involves bidding the true valuation and choosing the second-price rule $(\theta, 0) \equiv a^0(\theta)$: the *Vickrey action*. Further let $a_S^0(\theta_S)$ denote the Vickrey action profile played by the bidders in $S \subseteq N$ given their types $\theta_S \in \Theta^{|S|}$. The profile $a_N^0(\theta_N)$

[8]The proposition on page 10 shows that it is the case for the modified auction.

is what I refer to as *fair play*. Let $u_i^{S,Eq.}(\theta_S)$ denote the expected utility of $i \in S$ in the equilibrium $Eq.$ of the side game within S with types θ_S. The equilibrium concept is discussed in the following subsection (page 11) after the side game is presented. I shall refer to the relevant class as the admissible[9] collusive equilibria.

The collusion-proofness is equivalent to the following.

$\forall S \subseteq N \ \forall \theta_S \in \Theta^{|S|}$ and for all admissible equilibria (Eq.) with efficient collusion:[10]

$$\exists i \in S \left(u_i^{S,Eq.}(\theta_S) > \mathbb{E}_{\tilde{\theta}_{N/S}} u_i \left(a_N^0 \left(\theta_S, \tilde{\theta}_{N/S} \right); \theta_i \right) \right) \Rightarrow$$

$$\Rightarrow \exists j \in S \left(u_j^{S,Eq.}(\theta_S) < \mathbb{E}_{\tilde{\theta}_{N/S}} u_j \left(a_N^0 \left(\theta_S, \tilde{\theta}_{N/S} \right); \theta_j \right) \right)$$

In prose, for any group of bidders with any valuation profile the following statement holds: If in an equilibrium of the collusive game one player is strictly better off than under fair play, then another one is strictly worse off in that equilibrium. This is exactly the negation of the existence of a group of bidders in which one member can be made better off in collusion and the participation constraint of the others is satisfied.[11]

The class of admissible equilibria referred to in our definition of robustness, imply collusive efficiency in the following sense: the leader's conditional winning probability is not reduced compared to the fair play.

3.2 Collusive Game

Consider the collusive game $\Gamma(S)$ that follows the formation of an arbitrary coalition S. Two assumptions allow us to restrict the game $\Gamma(S)$ to the members of S without loss of generality. First the bidders in S keep their

[9]"Admissible" is used in the common sense and does not refer to the admissibility in the decision theory, as in e.g. DeGroot, Morris (2004).

[10]$\mathbb{E}_{\tilde{\theta}_{N/S}}$ denotes the mathematical expectation with respect to the residual types $\tilde{\theta}_{N/S}$.

[11]For a singleton coalition, the condition is satisfied trivially, because the premise is false: for $S = \{i\}$ $u_i^{S,Eq.}(\theta_S) = \mathbb{E}_{\tilde{\theta}_{N/S}} u_i \left(a_N^0 \left(\theta_S, \tilde{\theta}_{N/S} \right); \theta_i \right)$.

deviation secret from the rest and so assume that N/S play the default equilibrium strategies. Second, due to the independent valuations assumption,[12] the beliefs over the valuations of the bidders in N/S are not affected by the messages sent within coalition S. Hence the consistency of beliefs in $\Gamma(S)$ is not affected by its restriction to S. I let $\Gamma(S)$ denote the *restricted* collusive game further.

In our two-stage model of the game the cartel members first negotiate and agree on the bidding manipulation and transfers. Then at the second stage they (may) use a randomization device to coordinate in their manipulation. There is perfect commitment to transfers in the outcome compatible with the manipulation.

3.2.1 Model of the game

At the communication stage player i of type θ_i chooses a message from M_i. For the coalition S of size s, denote the message space by $M_S = \underset{i\in S}{\times} M_i$ with a typical element m (the subscript S is dropped for brevity). Player i's mixed communication strategy is given by

$$\sigma_i : \Theta \to \Delta(M_i)$$

so that $\sigma_i(m_i\,|\theta_i)$ is the probability that i chooses m_i. Let $\sigma(m\,|\theta) = \prod\limits_{i\in S}\sigma_i(m_i\,|\theta_i)$, where $m = (m_i)_{i\in S}$, $\theta = (\theta_i)_{i\in S}$. Assume that the communication within a coalition is all-inclusive, in the sense that all players observe all messages m (and know that the others observe m, and know that the others know *etc*).

Communication m specifies a bidding arrangement - the correlated strategy

$$\alpha_m : \mathcal{A}^s \to \Delta^{|\mathcal{A}|\times s}$$

([13]) and the transfers in every public outcome possible given $\alpha_m(\cdot)$.[14] Let the vector-valued function of the side transfers implied by communication m be

[12] Otherwise the communication within a subgroup may shed light on the outsiders' valuations. The colluders' messages are revealing of their private information, which, in case there is correlation, has further implications on the values of the outsiders.

[13] $\Delta = \left\{ x \in \mathbb{R}_+^{|\mathcal{A}|\times s} \,\middle|\, x_l \geq 0, \sum\limits_l x_l = 1,\ l = 1, ..., |\mathcal{A}| \times s \right\}$.

[14] Public outcome ω is *possible* given $\alpha(\cdot|m)$ of S, if $\exists\theta_{N/S} \in \Theta^{n-s}$, $\exists a \in \mathcal{A}^s\, |\alpha(a|m) > 0$: $\left(\omega = \chi(a, a^0_{N/S}(\theta_{N/S}))\right)$

denoted $T_m : \Omega \to \mathbb{R}^s$, so that $T_{mi}(\omega)$, $i \in S$ refers to the transfer received by bidder i when the outcome of the auction is ω. The transfers are conditional on the public information and balanced, $\sum_i T_{mi}(\omega) \leq 0$.

The set-up can feature type-dependent message spaces, $M_i(\theta_i)$, whereby the types are "partially provable" in the terms of Lipman, Seppi (1995). For instance when for some bidder i $M_i(\theta_i)/M_i\left(\tilde{\theta}_i\right) \neq \emptyset$ whenever $\theta_i > \tilde{\theta}_i$ implies that every higher-valuation type can credibly distinguish himself from a lower-valuation type (of the same player). Such message spaces would be a relevant formalization of the possibility of "burning money" in front of the other bidders, or any other sort of provable high valuations, for example, the complementarity of the good at sale to a good already owned by the bidder.

Consider the member i of S having valuation θ_i. His payoff in the restricted game $\Gamma(S)$ is the expected (over the residual types) utility level from the auction plus the side transfer, defined for every action-message profile $a \in \mathcal{A}^s$, $m \in M_S$:

$$u_i^S(a, m; \theta_i) = \sum_{v \in \Theta^{n-s}} f_{N/S}(v)\left(u_i\left(a, a^0_{N/S}(v); \theta_i\right) + T_{mi}\left(\chi(a, a^0_{N/S}(v))\right)\right)$$

,

where $a, a^0_{N/S}(v)$ is the combination of actions, where S play $a \in \mathcal{A}^s$ and N/S play the Vickrey profile $a^0(v) \in \mathcal{A}^{n-s}$.

Now fix $(\sigma^*, \alpha^*) \equiv Eq.$, an admissible collusive equilibrium of $\Gamma(S)$. Then we can define for each $\theta \in \Theta^s$ $u_i^{S,Eq.}(\theta) = \sum_{a,m} u_i^S(a, m; \theta_i)\,\alpha^*_m(a)\sigma^*(m\,|\theta)$, the utility expected by i in equilibrium. Recall that the payoffs $u_i^{S,Eq.}(\theta)$, $i \in S$ are taken into consideration by the coalition S contemplating a deviation. The relevant equilibrium class is studied next.

3.2.2 Equilibrium

Bidding Stage Coherent with backward induction, we start the analysis from the bidding stage. Recall that the preceding stage of negotiations boils down to the publicly observed communication $m \in M_S$. Upon completing the negotiation process the bidding ring uses a random coordination device to prompt their actions in the simultaneous-move game. The correlated strategy $\alpha_m(a)$ is a *correlated equilibrium* of the bidding stage, if a recommendation generated according to α is rationally followed by the colluders. Subscript m emphasizes that the bidding agreement is reached at the stage of negotiations.

Myerson's acceptability refines the correlated equilibrium notion in the following way: it imposes stability against further subgroup deviations that occur with small probabilities. First Myerson introduces ε-correlated strategy, which parallels the totally mixed strategy notion of Selten's. An ε-correlated strategy α^{ε} is defined as a probability distribution over $\mathcal{A}^{|S|} \times (\underset{C\subseteq S}{\cup} \mathcal{A}^{|C|})$. The value of $\alpha^{\varepsilon}(a, e_C)$, $a \in \mathcal{A}^{|S|}$, $e_C \in \mathcal{A}^{|C|}$ has the interpretation of the probability that given recommendation a sub-coalition $C \subseteq S$ will tremble and deviate to e_C. So for instance $\alpha^{\varepsilon}(a, \emptyset)$ is the probability of obedience to a. A ε-correlated strategy has the properties:

1. $\varepsilon\alpha^{\varepsilon}(a, e_C) \geq (1-\varepsilon)\alpha^{\varepsilon}\left(a, e_{C\cup\{i\}}\right)$, $\forall i \in S$, $\forall a \in \mathcal{A}^{|S|}$, $\forall C \subseteq S/i$, $\forall e_C \in \mathcal{A}^{|C|}$
2. $\alpha^{\varepsilon}(a, e_C) > 0 \Rightarrow \alpha^{\varepsilon}\left(a, e_{C\cup\{i\}}\right) > 0$, $\forall i \in S$, $\forall a \in \mathcal{A}^{|S|}$, $\forall C \subseteq S/i$, $\forall e_{C\cup\{i\}} \in \mathcal{A}^{|C|+1}$

After any recommendation profile and a possible tremble by C, the conditional probability of a player i also trembling is positive (property 2); this conditional probability however dos not exceed ε (property 1). An appealing feature of an ε-correlated strategy is in the declining probabilities for deviations of larger groups; that makes it conceptually resemble the proper equilibria in Myerson (1978), which assign higher probabilities to actions which are more profitable, as opposed to Selten's uniform trembling across actions.

The correlated strategy $\alpha^{\varepsilon}(a, e_C)$ is an ε-correlated equilibrium, if the incentive constraint for the obedience is satisfied, for all $i \in S$, a_i, m, θ_i :

$$a_i \in \underset{e_i \in \mathcal{A}}{argmax} \sum_{a_{-i}, C, e_C} \eta(a, e_C)\, u_i^S\left(\left(a_{S/C\cup\{i\}}, e_{C\cup\{i\}}\right), m; \theta_i\right)$$

(under summation: $a_{-i} \in A^{s-1}$, $C \subseteq S/\{i\}$, $e_C \in A^{|C|}$)

if any action played with a positive probability is a best response to (the mix of) the actions taken by the other colluders, Finally, the correlated equilibrium $\alpha_m^*(\cdot)$ is *acceptable* in the sense of Myerson (1986), if there exists an ε-correlated equilibrium α^ε, such that $\alpha^\varepsilon(a,\emptyset)$ tends to $\alpha_m^*(\cdot)$ as ε goes to zero, for all a. Rephrasing the original interpretation, the acceptable correlated equilibrium is a correlated equilibrium in which obedient behavior by every member of the ring could still be rational when each member has positive but infinitesimal probability of trembling.

The acceptable correlated equilibrium notion is applied to collusive games in order to account for the possibility of further deviations by subgroups of the ring. This approach is an alternative to the coalition formation modeling, as represented by Bloch (1996), Ray, Vohra (1999), and is arguably preferable in the case of collusion in auctions, since it avoids the specification of the underlying bargaining process within smaller subgroups. The specification of the bargaining process would come ad hoc, given, in particular, the information asymmetry in the beginning of the game. Thus instead of tailoring the collusion robustness to a particular subcoaliton formation process, I adopt a less specific view as that of Myerson's equilibrium.

Two results of Myerson (1986) will be used in this paper. First, any perfect equilibrium in the sense of Selten (1975) is an acceptable correlated equilibrium. Second, every acceptable equilibrium includes only actions that are "acceptable",[15] which are in turn undominated actions, in the weak-dominance sense.[16]

Belief Update The belief update $\beta_i : \Theta \to [0;1]$, occurring after the rounds of communication, follows the Bayes rule on the equilibrium path:

$$\beta_i(\theta_i | m_i) = \frac{\sigma_i^*(m_i | \theta_i) f_i(\theta_i)}{\sum\limits_{\widetilde{\theta}_i \in \Theta} \sigma_i^*\left(m_i \middle| \widetilde{\theta}_i\right) f_i\left(\widetilde{\theta}_i\right)}$$

where σ^* is the equilibrium message profile.

[15] An action a_i for player i is acceptable, if for every $\varepsilon > 0$ there exists some ε-correlated equilibrium η, such that $\sum\limits_{a_{-i}} \eta(a,\emptyset) > 0$.

[16] The first result is Theorem 1 in Myerson (1986); the second combines Theorem 2, Theorem 3 and the remark that follows.

Negotiation Stage The equilibrium condition for σ^* states that for all i, $m_i \in M_i$ such that $\sigma_i^*(m_i|\theta_i) > 0$,

$$m_i \in \underset{m_i' \in M_i}{argmax} \sum_{\theta_{-i},m_{-i},a} f_{-i}(\theta_{-i}) \prod_{j=S/i} \sigma_j^*(m_j|\theta_j) \cdot \alpha^*_{m_i',m_{-i}}(a)\, u_i^S(a, m_i', m_{-i}; \theta_i)$$

(under summation: $\theta_{-i} \in \Theta_{-i}$, $m_{-i} \in M_{-i}$, $a \in \mathcal{A}^s$).

Strategies (σ^*, α^*) together with beliefs β satisfying the above conditions, constitute an *admissible equilibrium* of the two-stage collusive game.

3.3 Results

Classic results on efficient collusion in the first-price (McAfee, McMillan, 1987), and the second-price auctions (Mailath, Zemsky, 1989) do not extend to the disobedience cases. For the case of the first-price auction for instance, a non-designated bidder could be better off bidding actively and eventually winning the object (the mechanism prescribes the designated bidder to bid the reserve price) than obeying the mechanism and getting his side transfer[17]. However, as I demonstrate in subsection 3.3.2, disobedience in coations does not rule out efficient collusion, and so the use of an enhanced procedures is justified.

3.3.1 Collusion Proofness

The collusion proofness of the modified auction procedure is obtained by showing that the admissible collusive equilibria fail to induce participation. In particular, the transfer paid to the coalition's runner-up must compensate his gain from deviation to applying for the gap rule (as described in section 2), which coincides with the ex post collusive extra surplus. Overall, the eventually generated surplus is insufficient to preclude deviations, and thus any equilibrium is incompatible with the balanced budget restriction.

Proposition 2 The modified Vickrey auction is robust to collusion.

[17] the side transfer equals $\frac{1}{n-1}$ of the designated winner's bid in the knock-out pre-auction, in the symmetric environment.

The proof of the proposition, given in the appendix, shows that Myerson's refinement implies that the coalition's highest-valuation bidder will play the Vickrey strategy at the second stage, or bid higher than his true valuation - but never aim at winning the auction by the gap rule. By backward induction, the coalition's runner-up has an available deviation, that gives him a surplus no less than he get from the leader, unless the latter gives up the whole surplus that he can get in collusion. This violates the participation constraint, implying that a coalition then cannot do better than bidding fairly. An additional slack is given by the fact that an deviation may remain unobserved, in which case everyone should be paid the same transfer as in case of obedience.

Myerson's (1986) results give us an simple necessary condition for the admissible collusive equilibrium. This equilibrium does not involve actions that are unacceptable (see the footnotes on page 15). Weakly dominated actions are unacceptable and may not be part of equilibrium. Any leader's action that features the choice of the gap rule or[18] bidding less than his proper valuation is weakly dominated, and thus can be ruled out. The restricted set of leader's actions provides the runner-up with a secure the action yielding him the whole collusive surplus, along the lines of Section 2, page 6.

In the case of collusion by any proper subset of N the elimination of the leader's actions is *not* driven by weak dominance; instead, it is solely driven by the non-reduction of the leaders conditional winning probability. In the case of grand coalition ($S = N$) the elimination of the leader's actions is a little more intricate and involves an iterated argument.

The availability of deviation is due to the revelation of the leader's identity. This revelation is inevitable given the nature of commitment in the bidding ring. The collusive process at the pre-auction stage involves the negotiation of transfers contingent - without the explicit specification of these transfers commitment cannot be achieved. Given the impossibility of reallocation the agreement on the manipulation of bids necessarily reveals the information that can be used for a deviation.

[18] And/or

3.3.2 Failure of the Standard Auction

In the beginning of Section 2 of the present paper we consider a manipulation that creates a non-negative surplus to any bidding ring in the Vickrey auction. The manipulation comes down to the non-winning members bidding the minimal price (zero) and the leader bidding his true valuation. The extra surplus generated by this manipulation can be divided between the members of the bidding rings via monetary transfers that are agreed at the negotiation stage. It is not straightforward however that the possibility of sharing surplus implies the failure of the Vickrey auction against collusion; recall that the auctioneer can choose to conceal a bulk of information so as to make deviations unobservable. Let us verify that the Vickrey auction is indeed non-robust to collusion in the sense of our definition.

To establish non-robustness by the definition we search for a triplet S, θ_S, $Eq.$: a coalition, a profile of valuations and a collusive equilibrium in which the coalition members are not worse off than playing fair, with at least one strictly preferring the collusive outcome. Subsection 3.1 defined this as the failure of the auction procedure against coalition S of type θ_S. For a sufficient equilibrium condition we refer to Theorem 1 in Myerson (1986), which states that acceptability is implied by trembling-hand perfection. Hence non-robustness is obtained with a relevant trembling hand perfect equilibrium in the bidding stage. An example of an admissible collusive equilibrium for the grand coalition equilibrium is described in the appendix (page 27). The construction is an application of Mailath, Zemsky (1989) results to the environment without commitment to actions. In this equilibrium, the ring members adopt the Mailath-Zemsky transfers and obey the sole bidding recommendation, in which the leader bids his true valuation and the rest withdraw from the competition.

4 Discussion

4.1 Moral Hazard in Coalitions

Following the seminal papers on collusion, the previous literature assumes away any moral hazard within cartels; the agents strictly follow an eventual pre-play agreement. Obviously, the robustness of an auction to collusion under the obedience assumption carries over to the case of its absence. The problem is, however, that the collusion-robustness in a fully obedient environment is achieved at the expense of assuming away such possibilities, as tacit collusion and collusion within certain subgroups. For example, Graham, Marshall (1987), find that the losses caused by collusive bidding can be mitigated by a reserve price, whose optimal value depends on the size of the coalition - and hence is not defined when collusion is tacit. Che and Kim (2009), design a procedure which is collusion-proof[19] under the assumption that a grand coalition cannot form (or if with some positive probability no trade is optimal). I claim that in a non-repeated auction case, the designer's concern should be rather the tacit collusion within any subgroups than unbounded commitment in cartels.

In collusion research in general, assuming away moral hazard in cartels can be innocuous in cases, when any player's action is verifiable (observable) and punishable by the coalition. To certain dynamic collusive frameworks, such as production cartels, verifiability and the possibility to punish are inherent. But in one-shot auctions, where the verifiability of bids is left at discretion of the auction designer, full obedience seems too much to require. Sealed bidding creates scope for unobservable deviations, and hence impedes punishment in coalitions. There are hence no strong incentives for exact compliance with the cartels joint decisions beyond outcome-contingent monetary transfers (unless one assumes intrinsic motivation).[20]

Further, moral hazard within cartels seems plausible not only in non-repeated interaction cases, but also if stakes in a particular auction are substantial compared to what cooperation can bring in the future.

4.2 Information Disclosure

When choosing between the sealed-bid (static) and the open (dynamic) versions of the second-price auction, the collusion-concerned seller should bear

[19] In terms of Che, Kim (2009) a "collusion-proof" mechanism is one that replicates the revenue of the seller in a non-collusive environment.

[20] Assuming away moral hazard helps model collusion employing the "pure asymmetric information" formulation of the revelation principle (Myerson, 1982). The efficient collusion schemes, which are direct revelation mechanisms, found in those papers fail when moral hazard is present.

in mind that revealing as little information as possible makes the eventual cartel agreements less enforceable. Coarsening the outcome partition widens the range of unobservable deviations from the agreed manipulations of bids. Therefore a sealed-bid auction should be preferred.

I assume that the privacy of bids can be protected at virtually no cost to the seller. Thus he is advised, in this simple framework, to disclose as little information about the auction's outcome, as possible. This way the enforcement of ex-post collusive contracts is deterred.

However even if moral hazard within cartels is conceivable, the sealed-bid auction may not completely discourage collusion. Suppose a potential bidder for a good is convinced that some contestant's willingness to pay is higher. In that case, he is at least not worse off bidding less, than he supposed to bid before the information arrived. If on top, the high-valuation contestant promises to pay him, conditional on the winning the auction, a part of the surplus generated by the underbidding, the bidder will comply.

4.3 Information Requirements versus Revenue Maximization

Narrowing the base of the bidder-specific information employed in the construction of the collusion-proof auction, I have intended to improve its practical applicability. This is precisely the reason why the present construction, although motivated by the seller's revenue considerations, departs from a non-optimal second-price (Vickrey) auction. Inheriting the basic features of the Vickrey auction, the present mechanism can be qualified as detail-free and thus applicable to fairly uncertain environments; in particular, it is robust to tacit collusion, suitable for arbitrary beliefs, and invariant in the number of bidders.[21]

The gap rule is perhaps most appealing because it's detail-free. The *a priori* value distribution and thus the players beliefs do not affect the best reply strategy of the coalition's runner-up. Hence the auctioneer does not have to know the exact distributions, interdependence patterns or the beliefs of the bidders to apply the gap rule. An auction designer who is sure to have more information can reconsider the revenue maximization and extend the idea of

[21] As long as the distribution supports are the same across bidders

this mechanism to the respective knowledge environment; for instance, having the commonly known distribution of valuations at hand, he can depart from the Myerson's optimal auction (1981) instead of the Vickrey.

Note that we departed from the auction type, which is especially vulnerable to collusive manipulations, the Vickrey auction. The manipulability is a major obstacle to the practical use of the Vickrey auction that otherwise has many desirable properties (see Ausubel, Milgrom, 2006). Solving the manipulability problem of the Vickrey auction without change to its core properties gives hope to make it more appealing to the practitioners.

On the other side, the procedure seemingly requires an excessive amount of trust to the auctioneer. Indeed, with the bids being secret, the selection of the winner is unverifiable. To resolve such doubts one may use encryption techniques[22] to preclude the seller from falsifying the price the winner will have to pay. Alternatively, in the simplest instance, the seller is asked to present the respective bid from the pull of received envelopes, publicly collected.

4.4 No reallocation possibility

The applicability is restricted to no-reallocation environments, which guarantees that the bidders within a coalition will have sufficient information to deviate. Otherwise, if the good at sale can be reallocated, the coalition could introduce a blind rotation of bids, so that a member of a ring would not know the identity of the supposed winner. If the blind bid rotation is impossible for other reasons, then the impossibility of reallocation is not essential and the modified procedure can be employed in the more general environment.

5 Summary

The paper contributes to the collusion proof auction design, focusing on the environments in which the coalitions can induce collusive bidding by the outcome-contingent transfers, whereas the good cannot be reallocated. For this setup I introduce the notion of robustness to collusion, which the

[22] See Izmalkov, Micali, Lepinski (2005) for a description.

standard Vickrey auction fails satisfy, and describe a modified procedure that is shown to be robust.

A mechanism is defined as robust to collusion, if no welfare-improving manipulation of bids emerges after a simultaneous deviation of a coalition. The welfare in deviation is taken as an equilibrium prediction for the two-stage (negotiations and bidding) collusive game that follows. The equilibrium notion for the two-stage collusive game combines subgame perfection, consistency of beliefs and Myerson's (1986) acceptable correlated equilibrium at the bidding stage. Myerson's refinement of correlated equilibrium rules out strategies that are not stable to infinitesimal trembles by sub-coalitions. Our collusion-proofness notion then postulates that there must be no set of bidders that can collude in the acceptable equilibrium whereby the participation, efficiency and budget balance constraints are satisfied.

The auction procedure described in the paper is robust to collusion in the above sense. Employing the seller's ability to keep the bids (but not the winner's identity) secret, we enlarge the scope of non-detectable deviations from the collusive bidding agreement. The increased possibility of moral hazard provides grounds for a collusion-proof mechanism. Collusion-proofness is achieved due to an optional assignment rule in a sealed-bid Vickrey auction, that allows the second-highest bidder to win and pay the third price if his revealed surplus is high enough. This optional rule will not be applied in equilibrium, but will create a potential conflict of interest within any bidding ring - so that no surplus-generating manipulation of bids will be sustained. Hence, backward-inducing bidders will not find it profitable to enter into collusive agreements.

A Appendix

Some additional notation used in the appendix.

Recall that $a_i \in \Theta \times \{0, 1\}$, so I will refer to the two components of action a_i, as $bid[a_i]$, the bid implied by action a_i and $ch[a_i]$, the choice of the assignment rule, where $ch[a_i] = 0$ is for the default (second-price), and $ch[a] = 1$ is for the gap rule.

Comment on Information Disclosure

Note that the transformation (id, t) coarsens the information contained in the actions. The auctioneer, who observes the action profile $a \in A$, chooses, with respect to the existing constraints, the system $(F_i)_{i \in N}$ of algebras over A that describes the bidders' knowledge after the play in the following way: For all a the outcomes in $\cap_{F_i^k \ni a} F_i^k$ (where F_i^k are the elements of algebra F_i) are indistinguishable from a from the point of view of i (in terms of Wilson they belong to the same event as a). Since all communication after the end of the auction is cheap talk, the only common knowledge within a coalition S amounts to $\wedge_{i \in S} F_i \equiv F_S$. Consider a set $S \subseteq N$, $s \geq 2$, an algebra F_S and (a strictly finer) algebra $\tilde{F}_S \supsetneq F_S$. Suppose coalition S can sustain some surplus-generating manipulation of bids using F_S-measurable transfers. If its knowledge is $\tilde{F}_S$, it can use these transfers, or generate even higher surplus with some $\tilde{F}_S$-measurable transfers. Since the collusive surplus translates directly into the auctioneer's loss, he prefers F_S to $\tilde{F}_S$, and in general, the coarsest possible algebra for *any* conceivable coalition. This implies the coarsest possible $\wedge_{i \in N} F_i$, or the least public disclosure.

Proof of Proposition 1

The proof relies on two facts, the bounded support and the positive probability of winning at valuation $\underline{\theta}$. It provides a sufficient value of the fine L.

Fix player i, and let $P(v) = \prod_{j \neq i} \left(\sum_{w<v} f_j(w) \right) + \sum_{k=2}^{n} \frac{1}{k} \sum_{C \subseteq N, \#C = k} \prod_{j \in C} f_j(v)$, $v \in \Theta$, the winning probability under the second-price rule, and

$$G(v) = \sum_{j \neq i} \sum_{w>v} \left(f_j(w) \prod_{k \neq j,i} \left(\sum_{x<2v-w} f_j(x) \right) \right) + \sum_{k=2}^{n} \frac{1}{k} \sum_{C \subseteq N, \#C = k} \prod_{j \in C} f_j(v)$$

$v \in \Theta$, the winning probability under the gap rule. Then, the second-price rule is preferred by player i having valuation v if:

$$\max_b \left\{ \left(v - v_{(1)}^e(b) \right) P(b) \right\} \geq \max_b \left\{ G(b) \left(v - v_{(2)}^e(b) \right) - L_i P(b) \right\}$$

where $v^e_{(1)}(\cdot), v^e_{(2)}(\cdot)$ are the expected prices to pay conditional on winning under the second-price and the gap rule, respectively. Since in the second-price auction truthful bidding is a dominant strategy, this is equivalent to: $\left(v - v^e_{(1)}(v)\right) P(v) \geq G(b^*(v)) \left(v - v^e_{(2)}(b^*(v))\right) - L_i P(b^*(v))$, where $b^*(v)$ is an optimal bid under deviation to opting for the gap rule. Now, setting L_i equal to $\underset{v \in \Theta_i}{max} \left\{ \frac{G(b^*(v))}{P(b^*(v))} \left(v - v^e_{(2)}(b^*(v))\right) - \frac{P(v)}{P(b^*(v))} \left(v - v^e_{(1)}(v)\right) \right\}$ yields the demanded inequality. The existence of maximum is guaranteed by the boundedness of the value set for $\left(v - v^e_{(m)}(b)\right)$, $m = 1, 2$, $\forall v$, $\forall b \in \Theta_i$ and $P(b) > 0$ $\forall b \in \Theta_i$.

Hence, provided the sufficient fine $L = \underset{i=1,\ldots,n}{max} \{L_i\}$, the dominant action $a^0_i(\theta_i)$ in Γ is opting for the standard (second-price) rule and bidding the true valuation θ_i.

Proof of Proposition 2

Outline. First, we show that the efficiency requirement has implications on the equilibrium message spaces. Looking at the second-stage equilibria we eliminate those that include weakly dominated actions, relying on Myerson's characterization of the acceptable correlated equilibria.[23] Finally, for remaining equilibria, we demonstrate an available deviation, that puts a constraint on the transfers. This constraint is incompatible with budget balance and the participation constraint.

1. stage: negotiations

Consider the subgame following the communication m. Let $\Theta^*_i(m_i)$ be the set of types of player i that send message m_i with a positive probability in equilibrium (σ^*, α^*), and $\underline{\theta^*_i}(m_i)$ and $\overline{\theta^*_i}(m_i)$ be the minimal and the maximal elements of $\Theta^*_i(m_i)$. By the requirement of collusive efficiency, the identity of the highest-valuation bidder in the coalition is revealed during the negotiation, implying that $\underline{\theta^*_{1|S}}(m_{1|S}) > \overline{\theta^*_j}(m_j)$, $\forall j \neq (1|S)$. Given that the highest bid among the $\overline{N/S}$ takes any value in Θ with a positive probability,

[23] Myerson (1986) shows that acceptable correlated equilibria can not include unacceptable actions, whereas any weakly dominated action is unacceptable.

no reallocation possibility and the acceptability refinement, I show next that if $\alpha^*_m(a) > 0$, then $bid[a_{1|S}] \geq \theta_{1|S}$,and $ch[a_{1|S}] = 0$, in order to demonstrate an available deviation further.

2. stage: actions

Fix coalition $S \subseteq N$, message profile $m \in M_S$, and acceptable correlated equilibrium $\alpha^*_m(\cdot)$. First we introduce some additional notation.

$P^*_{1|S} = \{p\,|\exists a \in \mathcal{A}^s,\ s.t.\, \alpha^*_m(a) > 0,\ Prob((1|S), p) > 0\}$ - the range of prices paid by $(1|S)$ with positive probability, given the collusive equilibrium α^*.

The proof relies on Myerson's proposition that if a is such that $\alpha^*_m(a) > 0$, then a does not involve weakly dominated actions.

Note that weak dominance rules out manipulations that involve opting for the gap rule: meaning that if $a \in \mathcal{A}^s$ is such that $\alpha^*_m(a) > 0$, then $ch[a_i] = 0$ for all $i \in S$. Due to the impossibility of reallocation we only consider equilibria in which for the actions played with positive probability it holds necessarily that$\underset{j\in S}{max}\{bid[a_j]\} = bid[a_i] \Rightarrow i = 1|S$.

Consider first the coalition of all bidders, $S = N$. In this case the coalition's joint action fully determines the outcome (by efficiency, they will not induce ties).

Suppose $a \in \mathcal{A}^n$, $\alpha^*_m(a) > 0$, is such that $bid[a_{1|S}] < \theta_{1|S}$. For a to be an acceptable equilibrium, action $a_{1|S}$ has to be undominated - in particular, by action $a^o(\theta_{1|S})$. Observe that for all the residual bidding profiles $\tilde{a}_{-1|S}$, such that $max\{bid[\tilde{a}_{-1|S}]\} \notin P^*_{1|S}$ and all $bid[\tilde{a}_{-1|S}] < \theta_{1|S}$, $(1|S)$ is strictly better off taking $a^o(\theta_{1|S})$. (Recall that, by assumption, there is no commitment to out-of-equilibrium transfers). Next, for profiles $\tilde{a}_{-1|S}$, such that $bid[\tilde{a}_j] \geq \theta_{1|S}$, for some $j \notin N/(1|S)$, yield the same payoff whether $a_{1|S}$ or $a^o(\theta_{1|S})$ is taken ($1|S$ does not win).

Thus, for $a_{1|S}$ not to be weakly dominated by $a^o(\theta_{1|S})$, it has to be the case that for some profile $\tilde{a}_{-1|S}$, such that $max\{bid[\tilde{a}_{-1|S}]\} \in P^*_{1|S}$ and all $bid[\tilde{a}_{-1|S}] < \theta_{1|S}$, the following strict inequality holds: $\theta_{1|S} - p + T(1|S, p) < 0$, where $p = max\{bid[\tilde{a}_{-1|S}]\}$. Since $p \in P^*_{1|S}$, it must be that $\exists a' \in \mathcal{A}^n$, such that $\alpha^*_m(a') > 0$ and $max\{bid[a'_{-1|S}]\} = p$. But then $a'_{1|S}$ (note, $a'_{1|S} > p$) should be undominated. Repeating the argument for $a'_{1|S}$ and p', we conclude that there must exist $a" \in \mathcal{A}^n$, such that $\alpha_m^*(a") > 0$ and $max\{bid[a'_{-1|S}]\} = p'$. Since there may only be a finite number of equilibria, we arrive at the point when the strict inequality of type $\theta_{1|S} - p + T(1|S,p) < 0$ for some $p \in P_{1|S}^*$ can no longer hold. Thus the whole argument chain fails and $a_{1|S}$ to be weakly dominated (by $a^o(\theta_{1|S})$); it can not be involved in an equilibrium.

We have shown that, for $S = N$, whenever $\alpha_m^*(a) > 0$, a is such that $bid[a_{1|S}] \geq \theta_{1|S}$.

Now consider collusion within a proper subset S of N. The efficiency requirement rules out all $a \in \mathcal{A}^s$ that decrease the winning probability of $1|S$. Any profile such that $bid[a_{1|S}] < \theta_{1|S}$ implies a probability loss $\sum\limits_{bid[a_{1|S}] < p \leq \theta_{1|S}} f_{(1|N/S)}(p)$, where $f_{(1|N/S)}(\cdot)$ is the distribution of $max_{i \in N/S}\{\theta_i\}$ (the first-order statistic), strictly positive in any point within the interval. Thus $\alpha_m^*(a) = 0$ whenever $bid[a_{1|S}] < \theta_{1|S}$.

Deviation

Fix a coalition $S \subseteq N$ and equilibrium $(\sigma^*, \alpha^*) \equiv Eq.$. We shall say that a deviation from $\alpha_m^*(a)$ is *detected* given a public outcome ω, if there is no such $\theta \in \Theta^{n-s}$, and $a \in \{a' \in \mathcal{A}^s \,|\alpha_m^*(a') > 0\}$, $\omega \neq \chi(a, a_{N/S}^0(\theta))$. Fix a type profile $\theta_N \in \Theta^n$. Consider $(2|S)$ deviating to action $a_{2|S}^d$: $ch[a_{2|S}^d] = 1$, $bid[a_{2|S}^d] = \left(\theta_{2|S} + \underline{\theta_{1|S}^*}(m_{1|S})\right)/2$. Whenever detected, this deviation yields the ex post payoff $\overline{\theta_{2|S}} - \theta_{1|N/S}$. For any $\theta_{1|S} \in \Theta_{1|S}^*(m_{1|S})$ observe that $\theta_{(1|S)} > bid[a_{2|S}^d]$, and in case $bid[a_{2|S}^d] - \theta_{1|N/S} > \theta_{1|S} - bid(a_{2|S}^d)$, (and hence $bid(a_{2|S}^d) - \theta_{1|N/S} > 0$ and there is positive surplus to collusion) $(2|S)$ wins the object and hence gets $\theta_{2|S} - \theta_{1|N/S}$. All other outcomes do not reveal of his deviation.

The obedience to α_m^* then requires the total transfer to $(2|S)$ expectationally equivalent to $\left[\theta_{2|S} - \theta_{1|N/S}\right]^+$; $(2|S)$ will not accept a smaller transfer. Observe that during the game all ring members hold the same beliefs about N/S types, due to independence. This implies that $(1|S)$ gives away all extra surplus compared to the fair-play payoff in expectation. His participation constraint is thus violated,

$$u_{1|S}^{S,Eq.}(\theta_S) < \mathbb{E}_{\tilde{\theta}_{N/S}} u_{1|S}\left(a_N^0\left(\theta_S, \tilde{\theta}_{N/S}\right); \theta_{1|S}\right).$$

We have shown that the transfers sustaining the second-stage obedience are incompatible with jointly the budget balance and the participation constraints. Thus the mechanism is robust to collusion.

Proof of Claim 3.3.2

For the simplicity of exposition let$\underline{\theta} = 0$.

ACE = admissible collusive equilibrium

THPE = trembling hand perfect equilibrium

Consider coalition $S = N$ with $\theta_{1|S} > \theta_{2|S} > 0$, and arbitrary valuations of the followers. The following is an ACE:

1. At the negotiation stage the equilibrium communication is m ($\sigma^*(m) = 1$), that specifies α^*_m, T_m as below;
2. The beliefs reflect the true state of valuations on the equilibrium path, coincide with the prior otherwise;
3. At the bidding stage, the recommendation α^*_m is obeyed.

$\alpha^*_m(\overline{a}) = 1$, where $\overline{a} \in \mathcal{A}^n$ implies that the leader $(1|S)$ bids his true valuation and the rest bid zero.

Proceed backwards. The bidding manipulation α^*_m, that assigns probability 1 to a single action profile $\overline{a}$, is a THPE. Definition 8.5A in Fudenberg, Tirole (1991) characterizes THPE as the limit of an ε- constrained equilibria sequence. (The totally mixed strategy profile α^ε is an ε- constrained equilibrium, if α^ε_i solves $\underset{\alpha_i}{max} u_i(\alpha_i, \alpha^\varepsilon_{-i}(a))$ subject to $\alpha_i(a_i) \geq \varepsilon(a_i)$, $\forall a_i$ for some sequence $\{\varepsilon(a_i)\}_{a_i,i}$, where $0 < \varepsilon(a_i) < \varepsilon$.)

Thus we are looking for a sequence of α^ε such that $\alpha^*_m = \underset{\varepsilon \to 0}{lim} \alpha^\varepsilon$.

Lemma Each player's action in $\overline{a}$ is a best reply to the perfectly mixed actions of the others, for a sufficiently small ε. Consider player i who is prescribed to bid zero: suppose there exists an action $a_i > 0$, which is a better reply. Taking this action i looses his share of the surplus for sure $(\chi(a_i, \overline{a}_{-i}) \neq \chi(\overline{a}))$, so he is not worse off bidding his true valuation θ_i

rather than a_i. Bidding the true valuation yields a positive surplus with a probability less than $(|\Theta| \cdot \varepsilon)^s$ (the number of possible bids less than θ_i is less than $|\Theta|$; each of the actions is played with probability less than ε in the ε-constrained equilibrium, and mixing is independent). Clearly there is an $\hat{\varepsilon}$ small enough that the upper bound for the surplus $(|\Theta| \cdot \hat{\varepsilon})^s \cdot \theta_i$ is lower than $T_{mi}(\overline{a})$ times the respective probability, which is decreasing in ε. Hence there is no better reply than bidding 0 for $\varepsilon \leq \hat{\varepsilon}$. For player $(1|S)$ the non-existence of a reply better than $\theta_{1|S}$, for small enough ε, is obvious.

Thus there is a sequence of α^ε, in which the ε- constraint is exactly satisfied: each player chooses action $\overline{a}_i$ as soon as possible. Combining the Lemma with the fact that $\overline{a}$ is a Nash equilibrium we obtain that $\alpha_m^* = \lim_{\varepsilon \to 0} \alpha^\varepsilon$, and hence α_m^* is a THPE. By Theorem 1 in Myerson (1986) α_m^* is then an acceptable correlated equilibrium.

Now that we established a secure prediction in the bidding stage, we can study the one-stage game of communication. The revelation principle (Myerson 1979) states that if the equilibrium exists it can be replicated in a direct side mechanism which is truthful. As a starting point consider a direct revelation mechanism. (Note that the side game is in general *not* a DRM). The DRM will collect valuation submissions ($m \in \Theta^s$) from the members of the ring and assign bidding recommendation α_m^*, and transfers T_m. Mailath and Zemsky (1989) show that the truth-telling incentive compatibility in the DRM is achieved when the expected *extra* gain a bidder enjoys from the collusion does not depend on his valuation. This is captured by the transfers T_m - a replication of Theorem 3 of their paper.

$T_{mi}(\omega) = -\sum_{v \leq \theta_i} vP_i(v) + \frac{1}{s-1}\sum_{j \neq i}\sum_{v \leq \theta_j} vP_j(v) + c_i$, $\omega = \chi(\overline{a})$

$T_{mi}(\omega) = 0$, $\omega \neq \chi(\overline{a})$

where $P_i\left(v\right) = \prod_{j \neq i}\left(\sum_{w<v} f_j\left(w\right)\right) + \sum_{k=2}^{n} \frac{1}{k} \sum_{C \subseteq N, \#C=k} \prod_{j \in C} f_j\left(v\right)$,

$\sum_i c_i = 0$, the constants $(c_i)_i$ can be chosen so as to satisfy the participation constraints.

Assuming that $|M_i| \geq |\Theta|$, $i \in S$ (to be understood as: the set of messages has at least as many members as the type set; this holds in particular is

M_i is larger than finite) we can replicate the DRM equilibrium in the richer communication game by choosing for each i an injection μ_i from Θ to M_i, such that $\sigma_i^*(m_i|\theta_i) = 1$ iff $m_i = \mu(\theta_i)$; α_m^* and T_m as above if $m \in \times_i \mu_i(\Theta)$, and $T_{mi} \equiv -\overline{\theta}$, any α_m if $m_i \notin \mu_i(\Theta)$. The beliefs reflect the true state in equilibrium and coincide with the prior out of it. Strategy profile σ_i^* is an equilibrium of the communication stage given the obedience to α_m^*.

Thus $(\sigma^*\alpha^*)$ the admissible collusive equilibrium satisfying the participation constraints for coalition $S = N$ with $\theta_{1|S} > \theta_{2|S} > 0$.

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