Characterizing Incentive Compatibility for Convex Valuations

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Abstract. We study implementability in dominant strategies of social choice functions when sets of types are multi-dimensional and convex, sets of outcomes are arbitrary, valuations for outcomes are convex functions in the type, and utilities over outcomes and payments are quasi-linear. Archer and Kleinberg [1] have proven that in case of valuation functions that are linear in the type monotonicity in combination with a local integrability condition are equivalent with implementability. We show that in the case of convex valuation functions one has to require in addition a property called decomposition monotonicity in order to conclude implementability from monotonicity and the integrability condition. Decomposition monotonicity is automatically satisfied in the linear case.

Saks and Yu [9] have shown that for the same setting as in Archer and Kleinberg [1], but finite set of outcomes, monotonicity alone is sufficient for implementability. Later Archer and Kleinberg [1], Monderer [6] and Vohra [10] have given alternative proofs for the same theorem. Using our characterization, we show that the Saks and Yu theorem generalizes to convex valuations. Again, decomposition monotonicity has to be added as a condition.

Keywords: Mechanism design, Social choice theory, Incentive compatibility, Convexity.

1 Introduction

The main goal of mechanism design is to design mechanisms that motivate the agents with private information to choose equilibrium strategies that lead to an implementation of a desired social choice function. In this paper we assume that players have preferences in terms of monetary valuations and that the mechanism designer can use payments to direct agent behavior. Players are assumed to have quasi-linear utilities over outcomes and payments. Whenever the revelation principle holds, the question of implementability reduces then to the existence of a payment rule such that truth telling becomes an equilibrium, in other words, *lying does not pay*.

In this paper we study conditions on the type spaces of the players and on their valuation functions under which there are easily recognizable properties

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that characterize truthfully implementable social choice functions, i.e. functions that can be combined with payments that motivate the players to reveal their true type. In particular, the aim is to have payment free, local characterizations, because with such a characterization implementability can be verified without the need to construct payments. In one-dimensional settings such a condition is monotonicity. In multi-dimensional settings (a generalization of) monotonicity is still necessary, but often not sufficient. The goal is then to identify multi-dimensional settings where monotonicity is sufficient, or, if it is not, to find additional necessary conditions, that in combination with monotonicity become sufficient. A well-known condition of this type in case of convex type spaces is path-independence of a particular vector field (see, e.g., Jehiel, Moldovanu and Stacchetti [4] and Müller, Perea and Wolf [7]). Archer and Kleinberg [1] have shown how to replace this condition by a local condition: for every type there exists an open neighborhood such that path-integrals on triangles within this neighborhood are equal to 0. However, their proof requires linear valuation functions. In this paper we show that for the more general case of convex valuation functions the same local integrability condition is sufficient (in combination with monotonicity), if one makes the additional assumption that the allocation rule is decomposition monotone. Müller et al. [7] have shown that in the linear case decomposition monotonicity is satisfied by all monotone allocation rules, which explains why it does not appear explicitly in the theorem of Archer and Kleinberg. Furthermore, Archer and Kleinberg have to make an assumption on the existence of certain integrals. Our proof shows that even in the convex case the additional assumption is satisfied automatically, thus eliminating it also in their setting.

In case of a finite set of outcomes, convex type spaces, and particular linear valuations, Saks and Yu [9] have shown that monotonicity is a sufficient condition for implementability. In other words, path independence is implied by monotonicity. Later Archer and Kleinberg [1], Monderer [6] and Vohra [10] have given alternative proofs for general linear settings[1]. Using our characterization for convex valuations, we show that the Saks and Yu theorem generalizes to convex valuations, again under the additional assumption of decomposition monotonicity. Thereby we provide yet another, but very short proof for the special case of linear valuations. Our proof differs from the proof in Archer and Kleinberg for the linear case in that it uses as an argument for local implementability a convex generalization of a Lemma by Monderer [6].

All our results are stated and proven in terms of single agent models. The characterization result for arbitrary set of outcomes immediately generalizes to a characterization of dominant strategy implementable rules as well as Bayes-Nash

[1] More precisely, Saks and Yu have provided a proof where types are encoded as valuation vectors, with one component for each outcome. Monderer has given a proof for the same encoding, but outcomes are probability vectors over pure outcomes and the range of the social function is finite. Archer and Kleinberg as well as Vohra present a proof for the general setting of linear valuations. The previous proofs can, however, be adopted to the general setting.

implementable rules in the case of multiple agents. The generalization of the Saks and Yu theorem for finite sets of outcomes carries over to dominant strategy implementation in the case of multiple agents.

Organization. Section 2 defines our setting and introduces necessary notation. We prove the main characterization theorem for arbitrary outcome sets and convex valuations in Section 3. In Section 4 we give a short proof of a generalization of the theorem of Saks and Yu [9]. In Section 5 we show how to apply our characterization when outcomes and types are points in $\mathbb{R}^2$ and the valuation for an outcome is the distance to the type.

2 Definitions and Setting

Henceforth we will assume that $T \subseteq \mathbb{R}^d$ $(d \geq 1)$ is a convex set and that $f : T \to A$ is an allocation rule from the set of types T to the set of outcomes A. The valuation for an outcome $a \in A$ of a certain type $t \in T$ is defined by the value $v(a,t)$ given by the function $v : A \times T \to \mathbb{R}$. A *mechanism* is a pair (f,p) of an allocation function f and a payment function $p : T \to \mathbb{R}$. The mechanism is called *truthful* or *incentive compatible* if for all $s,t \in T$ it holds that

$$v(f(s),s) + p(s) \geq v(f(t),s) + p(t), \tag{1}$$

i.e. the utility of a player of type s is always maximized when he reports s. The allocation f is called *truthful* if there exists such a payment function p that makes the mechanism (f,p) truthful. It is our goal to characterize truthful allocation functions without having to provide a p that satisfies (1).

Important concepts in this context are monotonicity and cyclical monotonicity of allocation functions. They can be defined in terms of the absence of negative 2-cycles and negative cycles, respectively, in the type graph T_f, as introduced by Gui et al. [2] and generalized in Archer and Kleinberg [1]. The set of nodes of the type graph is equal to T and every ordered pair of types $s,t \in T$ is connected by a directed edge with edge length either $l_p(s,t)$ or $l_s(s,t)$, which are defined as follows:

$$l_p(s,t) := v(f(s),s) - v(f(t),s), \tag{2}$$

$$l_s(s,t) := v(f(t),t) - v(f(t),s). \tag{3}$$

We call $l_p(s,t)$ and $l_s(s,t)$ the *p-length* and *s-length*, respectively, and use the same terminology for lengths of paths and cycles in the respective graphs.

We can now define monotonicity and cyclical monotonicity for allocation functions.

Definition 1. *An allocation function $f : T \to A$ is called* monotone*, if for all $s,t \in T$ it holds that $l_s(s,t) + l_s(t,s) \geq 0$. f is called* cyclically monotone*, if for all $k \geq 2$ and all $\{s_1,\ldots,s_k\} \subseteq T$, we have that $\sum_{i=1}^{k} l_s(s_i,s_{i+1}) \geq 0$, where indices are taken modulo k.*

The p-length and the s-length are related in the sense that the p-length and the s-length of any cycle in T_f is the same.

Property 1. For every $k \geq 2$ and every subset $\{s_1, \ldots, s_k\} \subseteq T$, we have that $\sum_{i=1}^{k} l_p(s_i, s_{i+1}) = \sum_{i=1}^{k} l_s(s_i, s_{i+1})$, where indices are taken modulo k.

Proof. This property follows from the fact that $l_p(s,t) = l_s(s,t) + v(f(s),s) - v(f(t),t)$ for all $s, t \in T$. □

Note that due to Property 1 monotonicity and cyclical monotonicity could have been defined in terms of l_p as well.

It is due to the following result of Rochet that we will concentrate on cyclically monotone allocation functions in the remainder of this paper.

Theorem 1 (Rochet [8]). *An allocation function $f : T \to A$ is truthful if and only if it is cyclically monotone.*

The simple proof employs the fact that, due to our choice of edge lengths, node potentials in the type graph coincide with payment rules that implement the allocation rule. Node potentials exist if and only if the type graph does not have a negative cycle.

In this paper we focus on settings where T is a convex set, and where for all outcomes $a \in A$ the function $v(a,.) : T \to \mathbb{R}$ is convex. That is, for all $s, t \in T$ and $\lambda \in [0,1]$, $v(a, (1-\lambda)s + \lambda t) \leq (1-\lambda)v(a,s) + \lambda v(a,t)$. Almost all previous literature focused on linear valuation functions. In this case we may identify A with a set of vectors in R^k such that $v(a,t) = a \cdot t$. Saks and Yu [9] choose to present their theorem in the model where $T \subset \mathbb{R}^A$ and $v(a,t) = t_a$, that is, outcomes are unit vectors. Monderer [6] has chosen the same model for T but allowed outcomes to be lotteries over unit vectors. Allocation rules in his model where restricted to those with finite range. Archer and Kleinberg [1], Müller et al. [7] and Vohra [10] allow for arbitrary linear functions. For finite A and linear valuations there are almost no differences between the linear models, for infinite A the canonical representation might move us to infinitely dimensional type spaces. However, even for finite A there is a fundamental difference between linear and convex valuation functions: Moving from a model with convex T and convex valuations to the canonical model may result in a non-convex set in $\mathbb{R}^A$. All previous theorems with linear valuations do not apply on non-convex sets of types. Therefore our results apply to a strictly larger domain of settings. However, one needs an additional condition on the setting which we define next.

Definition 2 (Müller et al. [7]). *Let T be convex. An allocation function $f : T \to A$ is called* decomposition monotone*, if for all $s, t \in T$ and all $\lambda \in [0,1]$ and we have that:*

$$l_p(s,t) \geq l_p(s, (1-\lambda)s + \lambda t) + l_p((1-\lambda)s + \lambda t, t). \tag{4}$$

It is easy to see that we could have used s-lengths rather than p-lengths in this definition. Müller et al. [7] have shown that for linear valuation functions any monotone allocation rule is decomposition monotone. In the full version of the paper we provide an example that this does not generalize to convex valuation functions.

3 Characterizing Incentive Compatibility

In this section we will prove our main theorem that characterizes cyclically monotone allocation functions for convex valuations and arbitrary outcome sets. We start with two Lemmas that relate s-lengths of edges in the type graph to path integrals on line segments. We will denote by $L_{s,t} := \{s + \lambda(t-s) : \lambda \in [0,1]\}$ the line segment between two types $s, t \in T$.

Recall that a vector $\nabla \in \mathbb{R}^d$ is a subgradient of a function $h : \mathbb{R}^d \to \mathbb{R}$ at t if $h(s) \geq h(t) + \nabla \cdot (s-t)$ for all $s \in T$. For every $t \in T$ allocation function f defines a convex function $v(f(t), .) : T \to \mathbb{R}$, $s \mapsto v(f(t), s)$. We denote the set of subgradients of $v(f(t), .)$ at $s = t$ by $\partial f(t)$. We assume that $\partial f(t) \neq \varnothing$ on T.[2]

We can now define a vector field $\nabla f : T \to \mathbb{R}^d$ by selecting for each $t \in T$ an element from $\partial f(t)$. Any such vector field satisfies for all $s, t \in T$

$$v(f(t), s) \geq v(f(t), t) + \nabla f(t) \cdot (s-t). \tag{5}$$

We summarize a couple of properties of $\nabla f(t)$ in the following lemma. They are key to the proof of our main theorem.

Lemma 1. *Let $s, t \in T$ and assume that $f : T \to A$ is monotone. Moreover, define $g : [0,1] \to \mathbb{R}$ by $g(\lambda) = \nabla f(s + \lambda(t-s)) \cdot (t-s)$. Then the following hold:*

1. $\nabla f(s) \cdot (t-s) \leq l_s(s,t) \leq \nabla f(t) \cdot (t-s)$,
2. *g is non-decreasing, and*
3. $\nabla f(s) \cdot (t-s) \leq \int_{L_{s,t}} \nabla f(\sigma) \cdot d\sigma \leq \nabla f(t) \cdot (t-s)$.

Proof. The first property follows immediately from monotonicity and the definitions of $l_s(s,t)$ and $\nabla f(s)$. For the second property let $0 \leq \lambda_1 < \lambda_2 \leq 1$, and let $r_1 = s + \lambda_1(t-s)$ and $r_2 = s + \lambda_2(t-s)$. Then, by using monotonicity and property 1, we get that

$$\begin{aligned}0 &\leq l_s(r_1, r_2) + l_s(r_2, r_1)\\ &\leq \nabla f(r_2) \cdot (r_2 - r_1) + \nabla f(r_1) \cdot (r_1 - r_2)\\ &= (\lambda_2 - \lambda_1)(g(\lambda_2) - g(\lambda_1)),\end{aligned}$$

i.e. $g(\lambda_2) \geq g(\lambda_1)$ and the second property is proven.

Since g is non-decreasing, g is integrable on $[0,1]$ and

$$\begin{aligned}\int_0^1 g(\lambda) d\lambda &= \int_0^1 \nabla f(s + \lambda(t-s)) \cdot (t-s) d\lambda\\ &= \int_{L_{s,t}} \nabla f(\sigma) \cdot d\sigma.\end{aligned}$$

[2] It is well-known that any convex function on T has a subgradient in all t in the interior of T. We will need the existence also on the boundary of T.

Thus the line integral of ∇f along the path $L_{s,t}$ is well defined and finite[3].

Also, we have that

$$g(0) \leq \int_0^1 g(\lambda)d(\lambda) \leq g(1).$$

If we replace $g(0)$ and $g(1)$ with their respective values, the third property follows. □

In the following we denote for $s_1, s_2, s_3 \in T$, all three distinct, by $\blacktriangle_{s_1,s_2,s_3}$ the convex hull of s_1, s_2, s_3 and let $\triangle_{s_1,s_2,s_3}$ be the path describing the boundary of $\blacktriangle_{s_1,s_2,s_3}$, i.e $L_{s_1,s_2} \cup L_{s_2,s_3} \cup L_{s_3,s_1}$, with direction $s_1 \to s_2 \to s_3 \to s_1$. The following lemma will establish the relation between the line integral of any selection from the subgradient and the s-lengths in the type graph of f.

Lemma 2. *Let $s, t \in T$ and assume that $f : T \to A$ is monotone. For every $n \geq 1$ we let $S_n = \sum_{i=0}^{n-1} l_s(r_i^n, r_{i+1}^n)$, where $r_k^n := s + \frac{k}{n}(t - s)$ for $0 \leq k \leq n$. Then*

$$\lim_{n\to\infty} S_n = \int_{L_{s,t}} \nabla f(\sigma) \cdot d\sigma.$$

Proof. Fix $n \geq 1$. According to Lemma 1 we have that for $0 \leq i \leq n-1$

$$\nabla f(r_i^n) \cdot (r_{i+1}^n - r_i^n) \leq l_s(r_i^n, r_{i+1}^n) \leq \nabla f(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n).$$

If we sum up the inequalities we get that

$$\sum_{i=0}^{n-1} \nabla f(r_i^n) \cdot (r_{i+1}^n - r_i^n) \leq S_n \leq \sum_{i=0}^{n-1} \nabla f(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n).$$

For every $n \in \mathbb{N}$ we define $L_n := \sum_{i=0}^{n-1} \nabla f(r_i^n) \cdot (r_{i+1}^n - r_i^n)$ and $U_n := \sum_{i=0}^{n-1} \nabla f(r_{i+1}^n) \cdot (r_{i+1}^n - r_i^n)$. Since ∇f is line-integrable on the path $L_{s,t}$ we have that

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} U_n = \int_{L_{s,t}} \nabla f(\sigma) \cdot d\sigma.$$

Furthermore, since $L_n \leq S_n \leq U_n$, we conclude that

$$\lim_{n\to\infty} S_n = \int_{L_{s,t}} \nabla f(\sigma) \cdot d\sigma.$$

□

We are now ready to prove our main theorem of this section.

Theorem 2. *Let $T \subseteq \mathbb{R}^d$ be convex. Assume that for every fixed $a \in A$ the function $v(a, .) : T \to \mathbb{R}$ is convex and has non-empty sets of subgradients on T. Assume further that $f : T \to A$ is monotone and decomposition monotone. Then the following are equivalent:*

[3] Archer and Kleinberg [1] make the assumption that the allocation function is locally path integrable in order to get this property. In fact our way of defining ∇f releases us from this assumption.

(1) f is cyclically monotone.

(2) for every $t \in T$ there exists an open neighborhood $U(t) \subseteq \mathbb{R}^d$, $t \in U(t)$, such that for all $s_1, s_2, s_3 \in U(t) \cap T$, all three distinct:

$$\int_{\Delta_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma = 0$$

(3) for all $s_1, s_2, s_3 \in T$, all three distinct:

$$\int_{\Delta_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma = 0$$

(4) for all $k \geq 3$ and every $\{s_1, \ldots, s_k\} \subseteq T$ and $P = \bigcup_{i=1}^{k} L_{s_i,s_{i+1}}$:

$$\int_P \nabla f(\sigma) \cdot d\sigma = 0$$

Proof. **(1) ⇒ (2)** This implication follows immediately from a result in Krishna and Maenner [5]. We provide an elementary proof on the basis of type graphs. Consider any $s_1, s_2, s_3 \in T$, all three distinct. Let ϵ be an arbitrary positive number. From Lemma 2 we get that for $i = 1, 2, 3$ there exist N_i such that for all $n \geq N_i$ we have that

$$S_n^i < \int_{L_{s_i,s_{i+1}}} \nabla f(\sigma) \cdot d\sigma + \frac{1}{3}\epsilon. \tag{6}$$

Now let $N = \max\{N_1, N_2, N_3\}$. For $n \geq N$ it holds that

$$S_n^1 + S_n^2 + S_n^3 < \int_{\triangle_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma + \epsilon.$$

Since f is cyclically monotone,

$$S_n^1 + S_n^2 + S_n^3 \geq 0,$$

and thus

$$0 \leq \int_{\triangle_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma + \epsilon.$$

Since ϵ is an arbitrary positive number we can conclude:

$$\int_{\triangle_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma \geq 0.$$

If we started with $\triangle_{s_1,s_3,s_2}$ we would conclude that

$$\int_{\triangle_{s_1,s_3,s_2}} \nabla f(\sigma) \cdot d\sigma \geq 0.$$

Since $\int_{\triangle_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma = -\int_{\triangle_{s_1,s_3,s_2}} \nabla f(\sigma) \cdot d\sigma$ we also get that

$$\int_{\triangle_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma \leq 0,$$

and thus $\int_{\triangle_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma = 0$.

(2) ⇒ (3) Let $s_1, s_2, s_3 \in T$. Since $\blacktriangle_{s_1,s_2,s_3}$ is closed and bounded it is compact. According to our assumption, for every point in $\blacktriangle_{s_1,s_2,s_3}$ there is an open neighborhood such that the integral of ∇f along every triangle in the intersection of the neighborhood and T is zero. The Lebesgue Number Lemma implies that there is a $\delta > 0$ such that every subset of $\blacktriangle_{s_1,s_2,s_3}$ of diameter less than δ is contained in at least one of these neighborhoods. In particular, if we subdivide $\blacktriangle_{s_1,s_2,s_3}$ into triangles $\blacktriangle^1, \blacktriangle^2, ..., \blacktriangle^M$ each of which having diameter less than δ, and orient the boarders $\triangle^j$ consistently with $\triangle_{s_1,s_2,s_3}$, we get

$$0 = \sum_{j=1}^{M} \int_{\triangle^j} \nabla f(\sigma) \cdot d\sigma.$$

In this formula, the path-integral of ∇f along $\triangle_{s_1,s_2,s_3}$ appears exactly once. All path-integrals of sides of $\triangle^j$ which are not contained in $\triangle_{s_1,s_2,s_3}$ appear exactly once in each direction of these sides, and cancel each other out. Therefore we have

$$\int_{\triangle_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma = \sum_{j=1}^{M} \int_{\triangle^j} \nabla f(\sigma) \cdot d\sigma = 0.$$

(3) ⇒ (4) Consider $\{s_1, ..., s_k\} \subseteq T$ and $P = \bigcup_{i=1}^{k} L_{s_i,s_{i+1}}$. P can be decomposed into the following triangles:

$$\triangle_{s_1,s_2,s_3}, \triangle_{s_1,s_3,s_4}, ..., \triangle_{s_1,s_{k-1},s_k}$$

According to our assumption the integral of ∇f along every triangle is zero. By a similar argument as before we get

$$\int_P \nabla f(\sigma) \cdot d\sigma = \int_{\triangle_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma + ... + \int_{\triangle_{s_1,s_{k-1},s_k}} \nabla f(\sigma) \cdot d\sigma = 0.$$

(4) ⇒ (1) Let $k \geq 2$ and $\{s_1, s_2, ..., s_k\} \in T$. Let ϵ be an arbitrary positive number. According to Lemma 2, for every $1 \leq j \leq k$ we have:

$$\exists N_j \text{ such that} \quad \forall n : n \geq N_j \quad S_n^j \geq \int_{L_{s_j,s_{j+1}}} \nabla f(\sigma) \cdot d\sigma - \frac{\epsilon}{k},$$

where $S_n^j = \sum_{i=0}^{n-1} l_s(r_{i,j}^n, r_{i+1,j}^n)$ and $r_{i,j}^n = s_j + \frac{i}{n}(s_{j+1} - s_j)$ for all $0 \leq i \leq n$. Since f is decomposition monotone,

$$S_n^j \leq l_s(s_j, s_{j+1}).$$

So for every $1 \leq j \leq k$

$$\int_{L_{s_j,s_{j+1}}} \nabla f(\sigma) \cdot d\sigma \leq S_n^j + \frac{\epsilon}{k} \leq l_s(s_j, s_{j+1}) + \frac{\epsilon}{k}.$$

If we sum up all these inequalities we get that

$$0 = \int_P \nabla f(\sigma) \cdot d\sigma = \sum_{j=1}^{k} \int_{L_{s_j,s_{j+1}}} \nabla f(\sigma) \cdot d\sigma \leq \sum_{j=1}^{k} l_s(s_j, s_{j+1}) + \epsilon.$$

Therefore

$$\sum_{j=1}^{k} l_s(s_j, s_{j+1}) \geq -\epsilon.$$

Since the last inequality holds for every $\epsilon > 0$, f is cyclically monotone. □

Archer and Kleinberg [1] prove a similar characterization for the case that valuations $v(a,t)$ are linear in t. In particular, we use their approach to show that (2) implies (3). Obviously, in this case $\nabla f(.) = v(f(t), .)$, and it is sufficient to relate path lengths in the type graph to path integrals of $v(f(t), .)$. When applied to their special case of linear valuations, our proof shows that it is not necessary to make the explicit assumption that $v(f(t), .)$ is path integrable.

In Heydenreich et al. [3] it is shown that for any implementable rule f revenue equivalence holds if and only if $dist_p(s,t) = -dist_p(t,s)$ in T_f, where $dist_p(s,t)$ is defined as the infimum over all p-lengths of paths from s to t. By the relation between p-lengths and s-lengths, the same characterization can be stated in terms of distances with respect to s-lengths. From Lemma 2 it follows that

$$dist_s(s,t) \leq \int_{L_{s,t}} \nabla f(\sigma) \cdot d\sigma,$$

and therefore $dist_s(s,t) + dist_s(t,s) \leq 0$. By cyclical monotonicity we get

$$dist_s(s,t) + dist_s(t,s) = 0.$$

This proves:

Corollary 1 (Revenue Equivalence). *If T, v and f satisfy the assumptions of Theorem 2 and f is implementable, then any two payments that implement f differ by at most a constant.*

4 A Generalization of Saks and Yu

We will now prove a generalization of the result of Saks and Yu [9] to convex valuation functions.

Theorem 3. *Let $T \subseteq \mathbb{R}^d$ be convex and let $|A|$ be finite. Assume that for every fixed $a \in A$ the function $v(a,.) : T \to \mathbb{R}$ is continuous, convex, and has non-empty sets of subgradients on T. Assume further that $f : T \to A$ is monotone and decomposition monotone. Then f is cyclically monotone.*

This is indeed a generalization of the above mentioned result, since in the case of linear valuation functions every monotone allocation rule is also decomposition monotone [7].

Proof. In order to show that f is cyclically monotone, we will show that condition (2) of Theorem 2 holds for f.

Let $t \in T$. For all $a \in A$ let $D_a := \overline{f^{-1}(a)}$, where $\overline{X}$ denotes the topological closure of a set $X \subseteq \mathbb{R}^d$. Moreover, for all $a \in A$, let $\varepsilon_a(t) := \inf_{x \in D_a} \|x - t\|_2$. Then, for each $a \in A$ we have that $t \in D_a$ if and only if $\varepsilon_a(t) = 0$.

We show first that for each $t \in T$ there exists a neighborhood $U(t)$ of t such that $t \in D_a$ for all $a \in f(U(t))$. Set $A(t) := \{a \in A : \varepsilon_a(t) = 0\}$. As $t \in D_{f(t)}$, we have that $A(t) \neq \emptyset$ and $t \in \bigcap_{a \in A(t)} D_a$. If $A(t) = A$ we let $U(t) = \mathbb{R}^d$, otherwise let $\varepsilon = \min\{\varepsilon_a(t) : a \in A \setminus A(t)\}$. Note that $\varepsilon > 0$. Define $U(t) = \{x \in \mathbb{R}^d : \|x - t\|_2 < \varepsilon\}$.

Next we generalize a lemma by Monderer [6] stating that monotonicity of f on some type set S together with $\bigcap_{a \in f(S)} D_a \neq \varnothing$ implies cyclical monotonicity on S. We prove its generalization to convex valuation for $S = U(t) \cap T$.

For this let $\{s_1, \ldots, s_k\} \subseteq U(t) \cap T$ for some $k \geq 3$. Let us fix $1 \leq i \leq k$. Since $t \in D_{f(s_{i+1})}$, there is a sequence $(t_j)_{j \in \mathbb{N}}$, such that $f(t_j) = f(s_{i+1})$ for every $j \in \mathbb{N}$ and $\lim_{j \to \infty} t_j = t$. Note that

$$\begin{aligned} l_p(s_i, s_{i+1}) &= v(f(s_i), s_i) - v(f(s_{i+1}), s_i) = v(f(s_i), s_i) - v(f(t_j), s_i) \\ &\geq v(f(s_i), t_j) - v(f(t_j), t_j) = v(f(s_i), t_j) - v(f(s_{i+1}), t_j). \end{aligned}$$

Using that $v(a,t)$ is continuous in t we get

$$l_p(s_i, s_{i+1}) \geq v(f(s_i), t) - v(f(s_{i+1}), t).$$

Hence

$$\sum_{i=1}^{k} l_p(s_i, s_{i+1}) \geq \sum_{i=1}^{k} v(f(s_i), t) - v(f(s_{i+1}), t) = 0,$$

and f is cyclically monotone when restricted to $U(t) \cap T$.

Finally, we use Theorem 2 [(1) $\Rightarrow$ (3)] to conclude that for all $s_1, s_2, s_3 \in U(t) \cap T$, all three distinct, we have that $\int_{\Delta_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma = 0$. □

5 An Example

In this section we will show an example for an allocation rule to which our result can be applied. Before we give the example, we first give a general class of (non-linear) convex valuation functions that can be used in different contexts.

In this setting we restrict ourselves to the case when the type space as well as the outcome space are a subset of $\mathbb{R}^d$. The valuation functions we consider arise from norms on $\mathbb{R}^d$. This will implicitly mean that agents value those outcomes more that are farther away from their own type. Imagine, for example, that a state council has to decide upon the site for a new garbage dump, and the different communities (agents) want to be as far away as possible from the proposed site.

Lemma 3. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and let $T \subseteq \mathbb{R}^d$ be convex. Then for any fixed $a \in \mathbb{R}^d$ the valuation function defined by $v(a,t) = \| a - t \|$ is continuous and convex in t.*

We now come to the example in which we use the Euclidean norm for our valuation functions. Suppose $T = A = [0,1]^2$ and define an allocation rule on T by $f(t_1,t_2) = (1-t_1, 1-t_2)$ for every $(t_1,t_2) \in T$. For every t and a in $[0,1]^2$ we define the valuation function $v(a,t)$ as the Euclidean distance of these two points in the plane: $v(a,t) = \|a-t\| = \sqrt{(a_1-t_1)^2 + (a_2-t_2)^2}$. In order to have monotonicity, for every s and t in $[0,1]^2$ we must have $\|f(s),s\| + \|f(t),t\| \geq \|f(t),s\| + \|f(s),t\|$. This fact, however, follows easily from the triangle inequality, as the line segments from s to $f(s)$ and from t to $f(t)$ always cross in the "midpoint" $(1/2,1/2)$ of T (cf. Fig. 1 (left)).

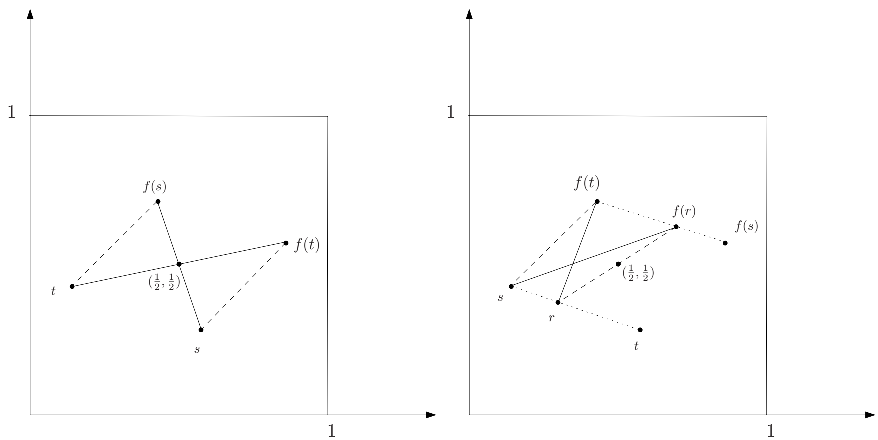

Fig. 1. Left: The segment $(t, f(t))$ passes through $(\frac{1}{2}, \frac{1}{2})$ for every $t \in T$. Right: Segments $(s, f(r))$ and $(r, f(t))$ always cross each other.

Decomposition monotonicity of f can be shown similarly (cf. Fig. 1 (right)). Let us now verify the condition from Theorem 2 that will ensure that f is implementable. According to our definition $\nabla f : [0,1]^2 \to [0,1]^2$ is

$$\nabla f(t_1,t_2) = \begin{cases} \left(\frac{-2(1-2t_1)}{\sqrt{(1-2t_1)^2+(1-2t_2)^2}}, \frac{-2(1-2t_2)}{\sqrt{(1-2t_1)^2+(1-2t_2)^2}}\right) & (t_1,t_2) \neq (\frac{1}{2},\frac{1}{2}) \\ (-2,0) & (t_1,t_2) = (\frac{1}{2},\frac{1}{2}). \end{cases}$$

For every s and t in $[0,1]^2$ we get that

$$\int_{L_{s,t}} \nabla f(\sigma) \cdot d\sigma = \sqrt{(1-2t_1)^2 + (1-2t_2)^2} - \sqrt{(1-2s_1)^2 + (1-2s_2)^2}.$$

Since the integral depends only on the end points of $L_{s,t}$ we can conclude that $\int_{\Delta_{s_1,s_2,s_3}} \nabla f(\sigma) \cdot d\sigma = 0$ for all $s_1, s_2, s_3 \in [0,1]^2$. Therefore, according to Theorem 2, f is implementable.

6 Conclusions

In this paper we have presented results about the truthfulness of social choice functions when the valuation functions are assumed to be convex rather than the previously used concept of linear valuations.

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