

Geometry of Schubert varieties and Demazure character formula

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1 Notation

The base field in this note is taken to be the field of complex numbers \mathbb{C} . The varieties are, by definition, quasi-projective, reduced (but not necessarily irreducible) schemes.

Let G be a semisimple, simply-connected, complex algebraic group. A Borel subgroup B is any maximal connected, solvable subgroup; any two of which are conjugate to each other. We will also fix a maximal torus $H \subset B$. The Lie algebras of G , B , and H are given by \mathfrak{g} , \mathfrak{b} , and \mathfrak{h} , respectively. For a fixed B , any subgroup $P \subset G$ containing B is called a *standard parabolic*.

2 Representations of G

Let $R \subset \mathfrak{h}^*$ denote the set of roots of \mathfrak{g} . Recall,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}, \text{ where } \mathfrak{g}_{\alpha} := \{x \in \mathfrak{g} : [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Our choice of B gives rise to R^+ , the set of positive roots, such that

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_{\alpha}.$$

We let $\{\alpha_1, \dots, \alpha_{\ell}\} \subset \mathfrak{h}^*$ be the simple roots and let $\{\alpha_1^{\vee}, \dots, \alpha_{\ell}^{\vee}\} \subset \mathfrak{h}$ be the simple coroots, where $\ell := \dim \mathfrak{h}$ (called the *rank* of \mathfrak{g}).

Elements of $X(H) := \text{Hom}(H, \mathbb{C}^*)$ are called *integral weights*, and can be identified with

$$\mathfrak{h}_{\mathbb{Z}}^* = \{\lambda \in \mathfrak{h}^* : \lambda(\alpha_i^\vee) \in \mathbb{Z}, \forall i\},$$

by taking derivatives. The dominant integral weights $X(H)_+$ are those integral weights $\lambda \in X(H)$ such that $\lambda(\alpha_i^\vee) \geq 0$, for all i .

We let $V(\lambda)$ denote the irreducible G -module with highest weight $\lambda \in X(H)_+$. Then, $V(\lambda)$ has a unique B -stable line such that H acts on this line by λ . This gives a one-to-one correspondence between the set of isomorphism classes of irreducible finite dimensional algebraic representations of G and $X(H)_+$.

3 Tits system

Let $N = N_G(H)$ be the normalizer of H in G , and let $W = N/H$ be the Weyl group, which acts on H by conjugation. For each $i = 1, \dots, \ell$, consider the subalgebra

$$\mathfrak{sl}_2(i) := \mathfrak{g}_{\alpha_i} \oplus \mathfrak{g}_{-\alpha_i} \oplus \mathbb{C} \alpha_i^\vee \subset \mathfrak{g}.$$

There is an isomorphism of Lie algebras $\mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2(i)$, taking $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to \mathfrak{g}_{α_i} , $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ to $\mathfrak{g}_{-\alpha_i}$, and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ to α_i^\vee . This isomorphism gives rise to a homomorphism $SL_2 \rightarrow G$. Let \bar{s}_i denote the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in G . Then, $\bar{s}_i \in N$ and $S = \{s_i\}_{i=1}^\ell$ generates W as a group, where s_i denotes the image of \bar{s}_i under $N \rightarrow N/H$. These $\{s_i\}$ are called *simple reflections*. For details about the Weyl group, see [3, §24,27].

The conjugation action of W on H gives rise to an action on \mathfrak{h} via taking derivatives and also on \mathfrak{h}^* by taking duals. Below are explicit formulae for these induced actions:

$$\begin{aligned} s_j : \mathfrak{h} &\rightarrow \mathfrak{h} & : h &\mapsto h - \alpha_j(h)\alpha_j^\vee \\ s_j : \mathfrak{h}^* &\rightarrow \mathfrak{h}^* & : \beta &\mapsto \beta - \beta(\alpha_j^\vee)\alpha_j. \end{aligned}$$

Theorem 1. *The quadruple (G, B, N, S) forms a Tits system (also called a BN -pair), i.e., the following are true:*

(a) $H = B \cap N$ and S generates W as a group;

(b) B and N generate G as a group;

(c) For every i , $s_i B s_i \not\subseteq B$;

(d) For every $1 \leq i \leq \ell$ and $w \in W$, $(B s_i B)(B w B) \subset (B s_i w B) \cup (B w B)$.

There are many consequences of this theorem. For example, (W, S) is a Coxeter group. In particular, there is a length function on W , denoted by $\ell : W \rightarrow \mathbb{Z}_+$. For any $w \in W$, $\ell(w)$ is defined to be the minimal $k \in \mathbb{Z}_+$ such that $w = s_{i_1} \dots s_{i_k}$ with each $s_{i_j} \in S$. A decomposition $w = s_{i_1} \dots s_{i_k}$ is called a *reduced decomposition* if $\ell(w) = k$.

We also have the *Bruhat-Chevalley ordering*: $v \leq w$ if v can be obtained by deleting some simple reflections from a reduced decomposition of w .

Axiom (d) above can be refined:

$$(B s_i B)(B w B) \subset B s_i w B \text{ if } s_i w > w. \quad (d')$$

Thus, if we have a reduced decomposition $w = s_{i_1} \dots s_{i_k}$, then

$$B w B = (B s_{i_1} B) \dots (B s_{i_k} B), \quad (1)$$

which can be obtained from (d') by inducting on $k = \ell(w)$.

We also have the Bruhat decomposition:

$$G = \bigsqcup_{w \in W} B w B.$$

Theorem 2. *The set of standard parabolics are in one-to-one correspondence with subsets of the set $[\ell] = \{1, \dots, \ell\}$. Specifically, if $I \subset [\ell]$, let*

$$P_I = \bigsqcup_{w \in \langle s_i : i \in I \rangle} B w B,$$

where $\langle s_i : i \in I \rangle$ denotes the subgroup of W generated by the enclosed elements. Then, $I \mapsto P_I$ is the bijection.

Sketch of the proof. By (1) and (d), P_I is clearly a subgroup containing B . Conversely, if $P \supset B$, then, by the Bruhat decomposition,

$$P = \bigsqcup_{w \in S_P} B w B,$$

for some subset $S_P \subset W$. Let I be the following set:

$$\{i \in [\ell] : s_i \text{ occurs in a reduced decomposition of some } w \in S_P\}.$$

From the above (specifically Axiom (d) and (d')), one can prove $P_I = P$. \square

4 A fibration

We begin with a technical theorem.

Theorem 3. *Let F be a closed, algebraic subgroup of G and X be an F -variety. Then, $E = G \times_F X$ is a G -variety, where*

$$G \times_F X := G \times X / \sim \quad \text{with} \quad (gf, x) \sim (g, fx)$$

for all $g \in G$, $f \in F$, and $x \in X$. The equivalence class of (g, x) is denoted by $[g, x]$. Then, G acts on E by:

$$g' \cdot [g, x] = [g'g, x].$$

In particular, $G \times_F \{pt\} = G/F$ is a variety. Furthermore, the map $\pi : E \rightarrow G/F$ given by $[g, x] \mapsto gF$ is a G -equivariant isotrivial fibration with fiber X .

The variety structure on G/F can be characterized by the following universal property: if Y is any variety, then $G/F \rightarrow Y$ is a morphism if and only if the composition $G \rightarrow G/F \rightarrow Y$ is a morphism.

Now, B is a closed subgroup. To see this, we only need to show that \overline{B} is solvable (B being a maximal solvable subgroup, it will follow that $B = \overline{B}$). Since the commutator $G \times G \rightarrow G$ is a continuous map, we have that $[\overline{F}, \overline{F}] \subset \overline{[F, F]}$, for any $F \subset G$. Using this fact and induction, $D_n(\overline{F}) \subset \overline{D_n(F)}$ for all n , where $D_n(F)$ denotes the n -th term in the derived series of F . Since $D_n(B)$ is trivial for large n , $D_n(\overline{B})$ becomes trivial for large n , and \overline{B} is solvable. Thus, G/B is a variety. We wish to give an explicit realization of this variety structure. In the process, we will show that G/B is a projective variety.

Take any regular $\lambda \in X(H)_+$, so that $\lambda(\alpha_i^\vee) > 0$ for all i . The representation $G \rightarrow \text{Aut}(V(\lambda))$ gives rise to a map

$$\pi : G/B \rightarrow \mathbb{P}V(\lambda), \quad g \mapsto [g \cdot v],$$

since $[v]$ is fixed by B , where v is a highest weight vector of $V(\lambda)$.

Claim. π is a morphism and injective.

Proof. π is a morphism since the composition $G \rightarrow G/B \rightarrow \mathbb{P}V(\lambda)$ is a morphism. To prove injectivity, it suffices to show that the stabilizer of $[v]$ is exactly B . Let P be the stabilizer. Now, $B \subset P$, so P is parabolic and hence $P = P_I$ for some $I \subset [\ell]$. If $I = \emptyset$, then $P = B$. Towards a contradiction, assume $s_i \in P$. Then, s_i stabilizes λ , but

$$s_i(\lambda) = \lambda - \lambda(\alpha_i^\vee)\alpha_i \neq \lambda,$$

since λ is regular. □

We claim $X = \pi(G/B)$ is closed. We will need the following theorem:

Theorem 4 (Borel fixed–point theorem, see §21 in [3]). *Let Z be a projective variety with an action of a solvable group. Then, Z has a fixed point.*

Clearly, \overline{X} is G -stable as a subspace of $\mathbb{P}V(\lambda)$. It follows that $\overline{X} \setminus X$ is G -stable. Thus, $\overline{X} \setminus X$ has a B -fixed point which contradicts the existence of a unique highest weight vector. Thus, $\overline{X} \setminus X = \emptyset$ and X is closed.

Lastly, to show X and G/B are isomorphic varieties, we use the following proposition from algebraic geometry:

Proposition 5 (Theorem A.11 in [1]). *If $f : Y \rightarrow Z$ is a bijective morphism between irreducible varieties and Z is normal, then f is an isomorphism.*

Observe that X is smooth because it is a G -orbit (G takes smooth points to smooth points and any variety has at least one smooth point). In particular, X is normal and $\pi : G/B \rightarrow X$ is an isomorphism.

5 Line bundles on G/B

For any $\lambda \in X(H)$, we define a line bundle $\mathcal{L}(\lambda)$ on G/B . Recall that $B = H \ltimes U$, where $U = [B, B]$ is the unipotent radical. Extend $\lambda : H \rightarrow \mathbb{C}^*$ to $\lambda : B \rightarrow \mathbb{C}^*$ by letting λ map U to 1. Consider $\mathbb{C} = \mathbb{C}_\lambda$ as a B -module, where $b \cdot z = \lambda(b)z$. Then, $\mathcal{L}(\lambda)$ is the line bundle: $\pi : G \times_B \mathbb{C}_{-\lambda} \rightarrow G/B$. Note that λ is made negative in the definition of $\mathcal{L}(\lambda)$.

The space of global sections

$$H^0(G/B, \mathcal{L}(\lambda)) := \{\sigma : G/B \rightarrow G \times_B \mathbb{C}_{-\lambda} : \pi \circ \sigma = \text{id}\}$$

is a G -module, where the G -action is given by

$$(g \cdot \sigma)(g'B) = g\sigma(g^{-1}g'B).$$

Also, this module is finite dimensional since G/B is projective and any cohomology of coherent sheaves on projective varieties is finite dimensional.

6 Borel–Weil theorem

Theorem 6 (Borel–Weil theorem). *If $\lambda \in X(H)_+$, then there is a G -module isomorphism*

$$H^0(G/B, \mathcal{L}(\lambda)) \simeq V(\lambda)^*.$$

Proof. If we pull back the line bundle $\mathcal{L} = \mathcal{L}(\lambda)$ (given by $\pi : G \times_B \mathbb{C}_{-\lambda} \rightarrow G/B$) under $G \rightarrow G/B$, we get the bundle $\hat{\mathcal{L}}$, which is $\hat{\pi} : G \times \mathbb{C}_{-\lambda} \rightarrow G$. We wish to compare sections of these two bundles.

Sections of $\hat{\mathcal{L}}$ are of the form $\sigma(g) = (g, f(g))$, for some map $f : G \rightarrow \mathbb{C}_{-\lambda}$, so we can identify $H^0(G, \hat{\mathcal{L}})$ with $k[G] \otimes \mathbb{C}_{-\lambda}$. There is a B -action on $k[G]$ given by $(b \cdot f)(g) = f(gb)$. Acting diagonally, we get an action on $k[G] \otimes \mathbb{C}_{-\lambda}$. Since $k[G] \otimes \mathbb{C}_{-\lambda}$ is naturally isomorphic to $k[G]$ (make the second coordinate 1), we get a new B -action on $k[G]$ given by

$$(b \cdot f)(g) = \lambda(b)^{-1}f(gb). \tag{2}$$

Use this action to make $H^0(G, \hat{\mathcal{L}})$ a B -module.

Sections of \mathcal{L} are of the form $\sigma(gB) = [g, f(g)]$, for some map $f : G \rightarrow \mathbb{C}_{-\lambda}$. In order to insure that σ is well-defined, we require that for any $b \in B$:

$$[g, f(g)] = [gb, f(gb)] = [g, b \cdot f(gb)] = [g, \lambda(b)^{-1}f(gb)].$$

Therefore, f must have the property that $f(g) = \lambda(b)^{-1}f(gb)$ for all $b \in B$. It follows that

$$\left[H^0(G, \hat{\mathcal{L}}) \right]^B = H^0(G/B, \mathcal{L}).$$

Now, it suffices to show $\left[H^0(G, \hat{\mathcal{L}}) \right]^B \simeq V(\lambda)^*$.

Consider the following two $(G \times G)$ -modules. First, $k[G]$ has a $(G \times G)$ -action given by $((g_1, g_2) \cdot f)(g) = f(g_1^{-1}gg_2)$. Second, acting coordinate-wise, we have:

$$\mathcal{M} := \bigoplus_{\mu \in X(H)_+} V(\mu)^* \otimes V(\mu).$$

It follows from the Peter–Weyl theorem and Tanaka–Krein duality that these are isomorphic as $(G \times G)$ -modules. The explicit isomorphism is $\Phi = \sum_{\mu} \Phi_{\mu} : \mathcal{M} \rightarrow k[G]$, where $\Phi_{\mu} : V(\mu)^* \otimes V(\mu) \rightarrow k[G]$ is given by

$$\Phi_{\mu}(f \otimes v)(g) = f(gv).$$

Furthermore, $k[G] \otimes \mathbb{C}_{-\lambda}$ has a $(G \times B)$ -action given diagonally, where G is forgotten when $G \times B$ acts on the second coordinate $\mathbb{C}_{-\lambda}$, and the action of $G \times B$ on $k[G]$ is the restriction of the $G \times G$ action given above. Since $H^0(G, \hat{\mathcal{L}}) \simeq k[G] \otimes \mathbb{C}_{-\lambda}$ as (left) G -modules, where G acts on $k[G]$ via $(g \cdot f)(x) = f(g^{-1}x)$, for $g, x \in G$ and $f \in k[G]$. Since the action of G on $k[G] \otimes \mathbb{C}_{-\lambda}$ commutes with the B -action given by equation (2), we get an induced G -action on the space of B -invariants:

$$\begin{aligned} \left[H^0(G, \hat{\mathcal{L}}) \right]^B &\simeq [k[G] \otimes \mathbb{C}_{-\lambda}]^B \\ &\simeq \bigoplus_{\mu \in X(H)_+} [V(\mu)^* \otimes V(\mu) \otimes \mathbb{C}_{-\lambda}]^B \\ &\simeq \bigoplus_{\mu \in X(H)_+} V(\mu)^* \otimes [V(\mu) \otimes \mathbb{C}_{-\lambda}]^B \\ &\simeq \bigoplus_{\mu \in X(H)_+} V(\mu)^* \otimes [\mathbb{C}_{\mu} \otimes \mathbb{C}_{-\lambda}]^H \\ &\simeq V(\lambda)^*, \end{aligned}$$

since $\mathbb{C}_{\mu} \otimes \mathbb{C}_{-\lambda}$ will only have H -invariants if $\mu = \lambda$. □

It follows from the next section that the higher cohomology vanishes; that is, for $\lambda \in X(H)_+$ and $i \geq 1$, $H^i(G/B, \mathcal{L}(\lambda)) = 0$.

7 Borel–Weil–Bott theorem

Let ρ be half the sum of the positive roots. Since G is simply-connected, $\rho \in X(H)_+$. Also, ρ has the property that $\rho(\alpha_i^{\vee}) = 1$ for all i . We will need a shifted action of the Weyl group on \mathfrak{h}^* given by:

$$w \star \lambda = w(\lambda + \rho) - \rho.$$

Theorem 7 (Borel–Weil–Bott). *If $\lambda \in X(H)_+$ and $w \in W$, then*

$$H^p(G/B, \mathcal{L}(w \star \lambda)) = \begin{cases} V(\lambda)^* & \text{if } p = \ell(w) \\ 0 & \text{if } p \neq \ell(w) \end{cases}.$$

Before we prove this theorem, we need to establish a number of results. For any i , let P_i denote the minimal parabolic subgroup $P_i = B \sqcup Bs_iB$. In what follows, if M is a B -module, the notation $H^p(G/B, M)$ is the p -th sheaf cohomology for the sheaf of sections of the bundle $G \times_B M \rightarrow G/B$.

Lemma 8. *If M is a P_i -module, then $H^p(G/B, M \otimes \mathbb{C}_\mu) = 0$, for all $p \geq 0$ and any $\mu \in X(H)$ such that $\mu(\alpha_i^\vee) = 1$.*

Proof. Apply the Leray–Serre spectral sequence to the fibration $G/B \rightarrow G/P_i$ with fiber P_i/B and the vector bundle on G/B corresponding to the B -module $M \otimes \mathbb{C}_\mu$. Thus,

$$E_2^{p,q} = H^p(G/P_i, H^q(P_i/B, M \otimes \mathbb{C}_\mu)) \implies H^*(G/B, M \otimes \mathbb{C}_\mu).$$

If we can show $E_2^{p,q} = 0$, then we are done.

It suffices to show $H^q(P_i/B, M \otimes \mathbb{C}_\mu)$ vanishes for all $q \geq 0$. By the next exercise, we have

$$H^q(P_i/B, M \otimes \mathbb{C}_\mu) \simeq M \otimes H^q(P_i/B, \mathbb{C}_\mu),$$

since M is a P_i -module by assumption. Since $P_i/B \simeq SL_2(i)/B(i) \simeq \mathbb{P}^1$, where $SL_2(i)$ is the subgroup of P_i with Lie algebra $\mathfrak{sl}_2(i)$ and $B(i)$ is the standard Borel subgroup of $SL_2(i)$, we have that

$$H^q(P_i/B, \mathbb{C}_\mu) \simeq H^q(\mathbb{P}^1, \mathcal{O}(-\mu(\alpha_i^\vee))) = H^q(\mathbb{P}^1, \mathcal{O}(-1)),$$

which is known to be zero (for example, [2, Ch. III, Theorem 5.1]). \square

Exercise 9. For any closed subgroup $F \subset G$, if M is a G -module, then $G \times_F M \rightarrow G/F$ is a trivial vector bundle.

Proposition 10. *If for some i , $\mu \in X(H)$ has the property that $\mu(\alpha_i^\vee) \geq -1$, then for all $p \geq 0$,*

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+1}(G/B, \mathcal{L}(s_i \star \mu)).$$

Proof. First, consider the case where $\mu(\alpha_i^\vee) \geq 0$. Let $X_i := P_i/B \simeq \mathbb{P}^1$ and $\mathcal{H} := H^0(X_i, \mathcal{L}(\mu + \rho))$. It can easily be seen (by using the definition of the action of P_i on \mathcal{H}) that the action of the unipotent radical U_i of P_i is trivial on \mathcal{H} . Moreover, P_i/U_i is isomorphic with the subgroup $\widehat{SL_2(i)}$ of G generated by $SL_2(i)$ and H . Thus, by the Borel-Weil theorem for $G = \widehat{SL_2(i)}$, we get $\mathcal{H} \simeq V_i(\mu + \rho)^*$, as $\widehat{SL_2(i)}$ -modules, where $V_i(\mu + \rho)$ is the irreducible $\widehat{SL_2(i)}$ -module with highest weight $\mu + \rho$. (Even though we stated the Borel-Weil theorem for semisimple, simply-connected groups, the same proof gives the result for any connected, reductive group.) Thus, we have the weight space decomposition (as H -modules):

$$\mathcal{H} \simeq V_i(\mu + \rho)^* = \bigoplus_{j=0}^{(\mu+\rho)(\alpha_i^\vee)} \mathbb{C}_{-(\mu+\rho)+j\alpha_i}.$$

There is a short exact sequence of B -modules:

$$0 \longrightarrow K \longrightarrow \mathcal{H} \longrightarrow \mathbb{C}_{-(\mu+\rho)} \longrightarrow 0,$$

where K , by definition, is the kernel of the projection. Tensoring with \mathbb{C}_ρ , we get the following exact sequence of B -modules:

$$0 \longrightarrow K \otimes \mathbb{C}_\rho \longrightarrow \mathcal{H} \otimes \mathbb{C}_\rho \longrightarrow \mathbb{C}_{-\mu} \longrightarrow 0.$$

Passing to the long exact cohomology sequence, we get:

$$\begin{aligned} \cdots \rightarrow H^p(G/B, \mathcal{H} \otimes \mathbb{C}_\rho) \rightarrow H^p(G/B, \mathbb{C}_{-\mu}) \rightarrow \\ H^{p+1}(G/B, K \otimes \mathbb{C}_\rho) \rightarrow H^{p+1}(G/B, \mathcal{H} \otimes \mathbb{C}_\rho) \rightarrow \cdots \end{aligned}$$

By the previous lemma, $H^p(G/B, \mathcal{H} \otimes \mathbb{C}_\rho) = 0$ for all p . Thus,

$$H^p(G/B, \mathcal{L}(\mu)) = H^p(G/B, \mathbb{C}_{-\mu}) \simeq H^{p+1}(G/B, K \otimes \mathbb{C}_\rho). \quad (3)$$

Consider another short exact sequence of B -modules:

$$0 \longrightarrow \mathbb{C}_{-s_i(\mu+\rho)} \longrightarrow K \longrightarrow M \longrightarrow 0,$$

where M is just the cokernel of the inclusion. In particular, as H -modules,

$$M = \bigoplus_{j=1}^{(\mu+\rho)(\alpha_i^\vee)-1} \mathbb{C}_{-(\mu+\rho)+j\alpha_i},$$

so it may be regarded as a P_i -module. Then, as B -modules, we can tensor with \mathbb{C}_ρ to arrive at the following exact sequence:

$$0 \longrightarrow \mathbb{C}_{-s_i \star \mu} \longrightarrow K \otimes \mathbb{C}_\rho \longrightarrow M \otimes \mathbb{C}_\rho \longrightarrow 0.$$

Again, passing to the long exact sequence, we see:

$$\begin{aligned} \cdots \rightarrow H^p(G/B, M \otimes \mathbb{C}_\rho) \rightarrow H^{p+1}(G/B, \mathbb{C}_{-s_i \star \mu}) \rightarrow \\ H^{p+1}(G/B, K \otimes \mathbb{C}_\rho) \rightarrow H^{p+1}(G/B, M \otimes \mathbb{C}_\rho) \rightarrow \cdots \end{aligned}$$

By the previous lemma, $H^p(G/B, M \otimes \mathbb{C}_\rho) = 0$ for all p . Thus,

$$H^{p+1}(G/B, \mathcal{L}(s_i \star \mu)) = H^{p+1}(G/B, \mathbb{C}_{-s_i \star \mu}) \simeq H^{p+1}(G/B, K \otimes \mathbb{C}_\rho). \quad (4)$$

Combining equations (3) and (4), we get the proposition in the case where $\mu(\alpha_i^\vee) \geq 0$.

For the case that $\mu(\alpha_i^\vee) = -1$, we have that $s_i \star \mu = \mu$, so the statement reduces to proving that $H^p(G/B, \mathcal{L}(\mu)) = 0$, for all p . In this case, $K = 0$. From the isomorphism $\mathcal{H} \otimes \mathbb{C}_\rho \simeq \mathbb{C}_{-\mu}$, we conclude $H^p(G/B, \mathcal{L}(\mu)) \simeq H^p(G/B, \mathcal{H} \otimes \mathbb{C}_\rho)$ which vanishes by the previous lemma. \square

Corollary 11. *If $\mu \in X(H)_+$ and $w \in W$, then for all $p \in \mathbb{Z}$, as G -modules:*

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(w)}(G/B, \mathcal{L}(w \star \mu)).$$

Proof. We induct on $\ell(w)$. Assume the above for all $v \in W$ such that $\ell(v) < \ell(w)$, and write $w = s_i v$ for some $v < w$. Then,

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(v)}(G/B, \mathcal{L}(v \star \mu)).$$

Now $(v \star \mu)(\alpha_i^\vee) = (\mu + \rho)(v^{-1} \alpha_i^\vee) - 1 \geq -1$, since $v^{-1} \alpha_i^\vee$ is a positive coroot and $\mu + \rho$ is dominant. So, applying Proposition 10, we get:

$$H^p(G/B, \mathcal{L}(\mu)) \simeq H^{p+\ell(v)+1}(G/B, \mathcal{L}(s_i \star (v \star \mu))) = H^{p+\ell(w)}(G/B, \mathcal{L}(w \star \mu)),$$

which is our desired result. \square

We are now ready to prove the Borel–Weil–Bott theorem.

Proof of the Borel–Weil–Bott theorem. From the above corollary,

$$H^p(G/B, \mathcal{L}(w \star \lambda)) \simeq H^{p-\ell(w)}(G/B, \mathcal{L}(\lambda)).$$

We claim that $H^j(G/B, \mathcal{L}(\lambda)) = 0$ if $j \neq 0$. Indeed, if $j < 0$, this is true. Let w_0 denote the unique longest word in the Weyl group, so that $\ell(w_0) = \dim(G/B)$. If $j > 0$, then by Corollary 11,

$$H^j(G/B, \mathcal{L}(\lambda)) \simeq H^{j+\dim(G/B)}(G/B, \mathcal{L}(w_0 \star \lambda)) = 0.$$

This implies

$$H^p(G/B, \mathcal{L}(w \star \lambda)) = \begin{cases} H^0(G/B, \mathcal{L}(\lambda)) & \text{if } p = \ell(w) \\ 0 & \text{if } p \neq \ell(w) \end{cases},$$

which is our desired result, by the Borel–Weil theorem. \square

Exercise 12. Show that for any μ not contained in $W \star (X(H)_+)$, $H^p(G/B, \mathcal{L}(\mu)) = 0$, for all $p \geq 0$.

8 Schubert varieties

For any $w \in W$, let $X_w := \overline{BwB/B} \subset G/B$ denote the corresponding *Schubert variety*. This variety is projective and irreducible of dimension $\ell(w)$. By the Bruhat decomposition, we have the following decomposition of X_w :

$$X_w = \bigsqcup_{v \leq w} BvB/B.$$

9 Bott–Samelson–Demazure–Hansen variety

Let \mathfrak{W} be the set of all ordered sequences $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$, $n \geq 0$, of simple reflections, called *words*. For any such word, define the *Bott–Samelson–Demazure–Hansen variety* as follows: if $n = 0$ (thus, \mathfrak{w} is the empty sequence), $Z_{\mathfrak{w}}$ is a point. For $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$, with $n \geq 1$, define

$$Z_{\mathfrak{w}} = P_{i_1} \times \cdots \times P_{i_n} / B^n,$$

where the product group B^n acts on $P_{\mathfrak{w}} := P_{i_1} \times \cdots \times P_{i_n}$ from the right via:

$$(p_1, \dots, p_n) \cdot (b_1, \dots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n).$$

This action is free and proper. The group P_{i_1} (in particular, B) acts on $Z_{\mathfrak{w}}$ via its left multiplication on the first factor.

Lemma 13. $Z_{\mathfrak{w}}$ is a smooth projective variety.

Sketch of the proof. Induct on the length of \mathfrak{w} , where length refers to the number of terms in the sequence. Let \mathfrak{v} be the last $n - 1$ terms in the sequence \mathfrak{w} , so that $\mathfrak{w} = (s_{i_1}) \cup \mathfrak{v}$, where order is preserved when taking the union.

Let

$$\pi : Z_{\mathfrak{w}} \simeq P_{i_1} \times_B Z_{\mathfrak{v}} \longrightarrow Z_{(s_{i_1})} = P_{i_1}/B \simeq \mathbb{P}^1$$

be the map $[p_1, \dots, p_n] \mapsto p_1 B$. This map has fiber $Z_{\mathfrak{v}}$ and since it is a fibration, we get that $Z_{\mathfrak{w}}$ is smooth. Furthermore, $Z_{\mathfrak{w}}$ is complete since \mathbb{P}^1 is complete and the fibers of π are complete by induction.

Furthermore, it is a trivial fibration restricted to $\mathbb{P}^1 \setminus \{x\}$, for any $x \in \mathbb{P}^1$. Hence, projectivity follows from the Chevalley–Kleiman criterion asserting that a smooth complete variety is projective if and only if any finite set of points is contained in an affine open subset. \square

There is a map $\xi : \mathfrak{W} \rightarrow W$ given by $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n}) \mapsto s_{i_1} \cdots s_{i_n}$. For any $\mathfrak{w} \in \mathfrak{W}$, we say \mathfrak{w} is reduced if $s_{i_1} \cdots s_{i_n}$ is a reduced decomposition of $\xi(\mathfrak{w})$.

For $\mathfrak{w} \in \mathfrak{W}$, consider the map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \rightarrow G/B$ given by $[p_1, \dots, p_n] \mapsto p_1 \cdots p_n B$.

Lemma 14. If \mathfrak{w} is reduced, then $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}}) = X_{\xi(\mathfrak{w})}$. Moreover, $\theta_{\mathfrak{w}}$ is a desingularization of $X_{\xi(\mathfrak{w})}$; that is, it is birational and proper.

If \mathfrak{w} is not reduced, then $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}})$ is NOT equal to $X_{\xi(\mathfrak{w})}$ in general.

Sketch of the proof. The open subset of $Z_{\mathfrak{w}}$ given by

$$(Bs_{i_1}B) \times \cdots \times (Bs_{i_n}B)/B^n$$

maps isomorphically to the open cell BwB/B by (1) of Section (3). \square

10 A fundamental cohomology vanishing theorem

Let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ be an arbitrary word. For any j , $1 \leq j \leq n$, define $\mathfrak{w}(j) = (s_{i_1}, \dots, \widehat{s_{i_j}}, \dots, s_{i_n})$. The variety

$$Z_{\mathfrak{w}(j)} = P_{i_1} \times \cdots \times \widehat{P_{i_j}} \times \cdots \times P_{i_n} / B^{n-1}$$

embeds into $Z_{\mathfrak{w}}$ by:

$$[p_1, \dots, p_{j-1}, p_{j+1}, \dots, p_n] \mapsto [p_1, \dots, p_{j-1}, 1, p_{j+1}, \dots, p_n].$$

Denote also by $Z_{\mathfrak{w}(j)}$ the images of these maps. These are divisors in $Z_{\mathfrak{w}}$.

For $\lambda \in X(H)_+$, let $\mathcal{L}_{\mathfrak{w}}(\lambda) = \theta_{\mathfrak{w}}^*(\mathcal{L}(\lambda))$ be the pull back of $\mathcal{L}(\lambda)$ under the map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \rightarrow G/B$. We state the following fundamental theorem without proof.

Theorem 15 (Theorem 8.1.8 in [1]). *Let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ be a word and let $1 \leq p \leq q \leq n$ be such that $(s_{i_p}, \dots, s_{i_q})$ is reduced. Then, for any $\lambda \in X(H)_+$ and $r > 0$,*

$$H^r \left(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}} \left(- \sum_{j=p}^q Z_{\mathfrak{w}(j)} \right) \otimes \mathcal{L}_{\mathfrak{w}}(\lambda) \right) = 0.$$

Also, $H^r(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = 0$.

Corollary 16. *For any word $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$, $\lambda \in X(H)_+$, and j such that $1 \leq j \leq n$, the map*

$$H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \rightarrow H^0(Z_{\mathfrak{w}(j)}, \mathcal{L}_{\mathfrak{w}(j)}(\lambda))$$

is surjective.

Proof. Consider the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{Z_{\mathfrak{w}}}(-Z_{\mathfrak{w}(j)}) \longrightarrow \mathcal{O}_{Z_{\mathfrak{w}}} \longrightarrow \mathcal{O}_{Z_{\mathfrak{w}(j)}} \longrightarrow 0,$$

where $\mathcal{O}_{Z_{\mathfrak{w}}}(-Z_{\mathfrak{w}(j)})$ is identified with the ideal sheaf of $Z_{\mathfrak{w}(j)}$ inside $Z_{\mathfrak{w}}$. Since $\mathcal{L}_{\mathfrak{w}}(\lambda)$ is locally free, we may tensor the above sequence to get the exact sequence:

$$0 \longrightarrow \mathcal{O}_{Z_{\mathfrak{w}}}(-Z_{\mathfrak{w}(j)}) \otimes \mathcal{L}_{\mathfrak{w}}(\lambda) \longrightarrow \mathcal{L}_{\mathfrak{w}}(\lambda) \longrightarrow \mathcal{L}_{\mathfrak{w}(j)}(\lambda) \longrightarrow 0.$$

Passing to the long exact sequence and applying Theorem 15 gives us our desired result. \square

11 Geometry of Schubert varieties

In this section we show that Schubert varieties are normal, have rational singularities, and are Cohen-Macaulay.

Theorem 17 (Zariski's Main Theorem, see [2], Chap. III, Corollary 11.4 and its proof). *If $f : X \rightarrow Y$ is a birational projective morphism between irreducible varieties and X is smooth, then Y is normal if and only if $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

Lemma 18 (Lemma A.32 in [1]). *If $f : X \rightarrow Y$ is a surjective morphism between projective varieties and \mathcal{L} is an ample line bundle on Y such that $H^0(Y, \mathcal{L}^{\otimes d}) \rightarrow H^0(X, (f^*\mathcal{L})^{\otimes d})$ is an isomorphism for all large d , then $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

For any $w \in W$, choose a reduced decomposition $w = s_{i_1} \cdots s_{i_n}$, with each $s_{i_j} \in S$, and take $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$. Then, $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \rightarrow X_w$ is a desingularization. By the previous two results, to prove the normality of X_w , it suffices to find an ample line bundle $\mathcal{L}(\lambda)$ such that for all large d ,

$$H^0(X_w, \mathcal{L}(\lambda)^{\otimes d}) \simeq H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)^{\otimes d}).$$

In fact, $\mathcal{L}(\lambda)$ is ample on G/B if and only if λ is a dominant, regular weight (this claim is easy to prove from the results in Section 4). Since the restriction of ample line bundles are ample, in order to show that X_w is normal, it suffices to prove the following theorem:

Theorem 19. *If $\lambda \in X(H)_+$ and $w \in W$, then $H^0(X_w, \mathcal{L}(\lambda)) \rightarrow H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ is an isomorphism.*

Before we give the proof, we recall the following useful lemma:

Lemma 20 (Projection formula, Exercise 8.3 of Chap. III in [1]). *If $f : X \rightarrow Y$ is any morphism, η is a vector bundle on Y , \mathcal{S} is a quasi-coherent sheaf on X , then for all i :*

$$R^i f_*(\mathcal{S} \otimes f^*\eta) \simeq (R^i f_*\mathcal{S}) \otimes \eta.$$

Proof of the theorem. This map is clearly injective since $Z_{\mathfrak{w}} \twoheadrightarrow X_w$. Choose a reduced decomposition of the longest element $w_0 \in W$, $w_0 = s_{i_1} \cdots s_{i_N}$, each $s_{i_j} \in S$, $N = \dim(G/B) = |R^+|$, and let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_N})$. Introduce the

following notation: for $0 \leq j \leq N$, let $w_j = s_{i_1} \cdots s_{i_j}$ and $\mathfrak{w}_j = (s_{i_1}, \dots, s_{i_j})$. Consider the following diagram:

$$\begin{array}{ccc}
Z_{\mathfrak{w}_N} & \xrightarrow{\theta_{\mathfrak{w}_N}} & X_{w_N} = G/B \\
\uparrow & & \uparrow \\
Z_{\mathfrak{w}_{N-1}} & \xrightarrow{\theta_{\mathfrak{w}_{N-1}}} & X_{w_{N-1}} \\
\uparrow & & \uparrow \\
Z_{\mathfrak{w}_{N-2}} & \xrightarrow{\theta_{\mathfrak{w}_{N-2}}} & X_{w_{N-2}} \\
\uparrow & & \uparrow \\
\vdots & & \vdots
\end{array}$$

In this diagram, the horizontal arrows are surjective and the vertical arrows (which are the canonical inclusions) are injective. Passing to global sections, we get:

$$\begin{array}{ccc}
H^0(Z_{\mathfrak{w}_N}, \mathcal{L}_{\mathfrak{w}_N}(\lambda)) & \longleftarrow & H^0(X_{w_N}, \mathcal{L}(\lambda)) \\
\downarrow & & \downarrow \\
H^0(Z_{\mathfrak{w}_{N-1}}, \mathcal{L}_{\mathfrak{w}_{N-1}}(\lambda)) & \longleftarrow & H^0(X_{w_{N-1}}, \mathcal{L}(\lambda)) \\
\downarrow & & \downarrow \\
H^0(Z_{\mathfrak{w}_{N-2}}, \mathcal{L}_{\mathfrak{w}_{N-2}}(\lambda)) & \longleftarrow & H^0(X_{w_{N-2}}, \mathcal{L}(\lambda)) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

In this diagram, the horizontal arrows are of course injective and the vertical arrows on the left are surjective by Corollary 16. Furthermore, by Lemma 20 (with $\mathcal{S} = \mathcal{O}_{Z_{\mathfrak{w}_N}}$ and $\eta = \mathcal{L}(\lambda)$) and Theorem 17, the top horizontal arrow is an isomorphism. By a standard diagram chase, all of the horizontal arrows are isomorphisms.

Since $w_0 = w(w^{-1}w_0)$ and $\ell(w^{-1}w_0) = \ell(w_0) - \ell(w)$, a reduced decomposition of w_0 can always be obtained so that the first $\ell(w)$ terms of the decomposition give the word \mathfrak{w} . This completes the proof. \square

Thus, using Theorem 17 and Lemma 18, we get the following:

Corollary 21. *Any Schubert variety X_w is normal.*

Corollary 22. *For any $v \leq w$ and $\lambda \in X(H)_+$, the restriction map*

$$H^0(X_w, \mathcal{L}(\lambda)) \rightarrow H^0(X_v, \mathcal{L}(\lambda))$$

is surjective.

Proof. By the above proof, $H^0(G/B, \mathcal{L}(\lambda)) \rightarrow H^0(X_v, \mathcal{L}(\lambda))$ is surjective and hence so is $H^0(X_w, \mathcal{L}(\lambda)) \rightarrow H^0(X_v, \mathcal{L}(\lambda))$. \square

An irreducible projective variety Y has *rational singularities* if for some desingularization $f : X \rightarrow Y$ we have that $f_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^i f_*\mathcal{O}_X = 0$ for all $i > 0$. This definition does not depend on a choice of desingularization. (In characteristic $p > 0$, we also need to assume that $R^i f_*\kappa_X = 0$, for the canonical bundle κ_X .) To prove that X_w has rational singularities, we use the following theorem of Kempf:

Theorem 23 (Lemma A.31 in [1]). *Let $f : X \rightarrow Y$ be a morphism of projective varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume there exists an ample line bundle \mathcal{L} on Y such that $H^i(X, (f^*\mathcal{L})^{\otimes d}) = 0$ for all $i > 0$ and all large d . Then, $R^i f_*\mathcal{O}_X = 0$ for $i > 0$.*

Corollary 24. *Any Schubert variety X_w has rational singularities.*

Proof. It suffices to prove $H^i(Z_w, \mathcal{L}_w(d\lambda)) = 0$ for all large d , for all $i > 0$, and some regular $\lambda \in X(H)_+$, which follows from Theorem 15. \square

We recall the following general theorem:

Theorem 25 (Lemma A.38 in [1]). *Any projective variety which has rational singularities is Cohen-Macaulay.*

Thus, we get:

Corollary 26. *X_w is Cohen-Macaulay.*

Another consequence of having rational singularities (which we will use later) is given in the following two results.

Proposition 27. *Let Y be a projective variety with rational singularities. Then, for any desingularization $f : X \rightarrow Y$ and any vector bundle η on Y , $H^i(Y, \eta) \rightarrow H^i(X, f^*\eta)$ is an isomorphism for $i \geq 0$.*

Proof. Applying the Leray-Serre spectral sequence, we have

$$E_2^{p,q} = H^p(Y, R^q f_* f^* \eta) \implies H^*(X, f^* \eta).$$

By the projection formula (with $\mathcal{S} = \mathcal{O}_X$),

$$R^q f_*(\mathcal{O}_X \otimes f^* \eta) \simeq \eta \otimes (R^q f_* \mathcal{O}_X).$$

Since Y has rational singularities, $R^q f_* \mathcal{O}_X = 0$ for $q > 0$. Therefore, $E_2^{p,q} = 0$ for $q > 0$, and hence $H^p(Y, \eta) \simeq E_2^{p,0}$ for all p , and the result follows. \square

Corollary 28. *For any $\lambda \in X(H)$ and $i \geq 0$,*

$$H^i(X_w, \mathcal{L}(\lambda)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)),$$

for any reduced word \mathfrak{w} with $\xi(\mathfrak{w}) = w$. In particular, for any $\lambda \in X(H)_+$, $H^i(X_w, \mathcal{L}(\lambda)) = 0$ if $i > 0$.

Proof. By Corollary 24 and Proposition 27, $H^i(X_w, \mathcal{L}(\lambda)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$, which vanishes by Theorem 15 for $\lambda \in X(H)_+$. \square

12 Demazure modules

Let $w \in W$ and $\lambda \in X(H)_+$. The Demazure module $V_w(\lambda) \subset V(\lambda)$ is the B -submodule defined by $V_w(\lambda) = \mathcal{U}(\mathfrak{b}) \cdot V(\lambda)_{w\lambda}$, where $\mathcal{U}(\mathfrak{b})$ is the enveloping algebra of \mathfrak{b} and $V(\lambda)_{w\lambda}$ is the weight space of $V(\lambda)$ with weight $w\lambda$. Observe that $V(\lambda)_{w\lambda}$ is one-dimensional. The formal character of $V_w(\lambda)$ is defined by

$$\text{ch } V_w(\lambda) = \sum_{\mu \in X(H)} \dim(V_w(\lambda)_\mu) e^\mu.$$

If $w = w_0$, then $V_w(\lambda) = V(\lambda)$. Therefore, $\text{ch } V_{w_0}(\lambda)$ is given by the Weyl character formula.

For an arbitrary $\mathfrak{w} \in \mathfrak{W}$, we need to introduce the Demazure operators $D_{\mathfrak{w}}$. For each simple reflection s_i , let $D_{s_i} : \mathbb{Z}[X(H)] \rightarrow \mathbb{Z}[X(H)]$ be the \mathbb{Z} -linear map given by:

$$D_{s_i}(e^\mu) = \frac{e^\mu - e^{s_i \mu - \alpha_i}}{1 - e^{-\alpha_i}}.$$

Given $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n}) \in \mathfrak{W}$, define $D_{\mathfrak{w}} : \mathbb{Z}[X(H)] \rightarrow \mathbb{Z}[X(H)]$ by

$$D_{\mathfrak{w}} = D_{s_{i_1}} \circ \dots \circ D_{s_{i_n}}.$$

In what follows, we will also need $*$: $\mathbb{Z}[X(H)] \rightarrow \mathbb{Z}[X(H)]$ given by

$$* e^{\mu} = e^{-\mu},$$

and extended \mathbb{Z} -linearly.

Theorem 29. *For any reduced word \mathfrak{w} ,*

$$\text{ch } V_{\xi(\mathfrak{w})}(\lambda) = D_{\mathfrak{w}}(e^{\lambda}).$$

Proof. The first step is to show $V_w(\lambda)^* \simeq H^0(X_w, \mathcal{L}(\lambda))$.

By the Borel–Weil theorem, $V(\lambda)^* \simeq H^0(G/B, \mathcal{L}(\lambda))$. The isomorphism $\phi : V(\lambda)^* \rightarrow H^0(G/B, \mathcal{L}(\lambda))$ is explicitly given by $\phi(f)(gB) = [g, f(gv_{\lambda})]$, where v_{λ} is a highest weight vector in $V(\lambda)$.

By Corollary 22, the restriction $H^0(G/B, \mathcal{L}(\lambda)) \rightarrow H^0(X_w, \mathcal{L}(\lambda))$ is surjective. Let ϕ_w denote the composition

$$V(\lambda)^* \rightarrow H^0(G/B, \mathcal{L}(\lambda)) \rightarrow H^0(X_w, \mathcal{L}(\lambda)).$$

We compute the kernel of ϕ_w ; i.e., find all $f \in V(\lambda)^*$ such that $\phi_w(f)$ is the zero section. It suffices to check that $\phi_w(f) = 0$ on BwB/B , since BwB/B is a dense open subset of X_w . For $f \in V(\lambda)^*$,

$$\begin{aligned} \phi_w(f) = 0 &\iff f(BwB \cdot v_{\lambda}) = 0 \\ &\iff f(B \cdot v_{w\lambda}) = 0 \\ &\iff f \text{ vanishes on } V_w(\lambda). \end{aligned}$$

Thus, $\ker \phi_w = \{f \in V(\lambda)^* : f|_{V_w(\lambda)} = 0\}$; that is, we have the following exact sequence:

$$0 \longrightarrow \left(\frac{V(\lambda)}{V_w(\lambda)} \right)^* \longrightarrow V(\lambda)^* \longrightarrow H^0(X_w, \mathcal{L}(\lambda)) \longrightarrow 0.$$

Therefore, $H^0(X_w, \mathcal{L}(\lambda))^* \simeq V_w(\lambda)$, which completes the first step.

Now, take a reduced decomposition of $w = s_{i_1} \dots s_{i_n}$ and let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$. The map $Z_{\mathfrak{w}} \rightarrow X_w$ is B -equivariant and by Corollary 28, $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \simeq$

$H^i(X_w, \mathcal{L}(\lambda))$ for all i as B -modules (for any $\lambda \in X(H)$). Therefore, their characters coincide; that is,

$$\text{ch } H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = \text{ch } H^i(X_w, \mathcal{L}(\lambda)).$$

Consider the Euler–Poincaré characteristic:

$$\chi_H(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) := \sum_i (-1)^i \text{ch } H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) \in \mathbb{Z}[X(H)].$$

Since $\text{ch } H^0(X_w, \mathcal{L}(\lambda)) = \chi_H(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ for $\lambda \in X(H)_+$, it suffices to show:

$$\chi_H(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = *D_{\mathfrak{w}}(e^\lambda).$$

In fact, we will prove a stronger result which is given as the next theorem. \square

Theorem 30. *For a B -module M , let $G \times_B M \rightarrow G/B$ be the associated vector bundle. Denote its pull back to $Z_{\mathfrak{w}}$ (for any word \mathfrak{w}) under the morphism $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \rightarrow G/B$ by $\theta_{\mathfrak{w}}^* M$. Then,*

$$\chi_H(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* M) = *D_{\mathfrak{w}}(*\text{ch } M).$$

Proof. We induct on the length n of $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$. The Leray spectral sequence for the fibration $Z_{\mathfrak{w}} \rightarrow Z_{\mathfrak{w}(n)}$ which maps $[p_1, \dots, p_n] \mapsto [p_1, \dots, p_{n-1}]$, with fibers $\mathbb{P}^1 \simeq P_{i_n}/B$, takes the form

$$E_2^{p,q} = H^p \left(Z_{\mathfrak{w}(n)}, \theta_{\mathfrak{w}(n)}^* (H^q(P_{i_n}/B, \theta_{s_{i_n}}^* M)) \right),$$

and converges to $H^{p+q}(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* M)$. From this we see that

$$\chi_H(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* M) = \chi_H(Z_{\mathfrak{w}(n)}, \theta_{\mathfrak{w}(n)}^* (\chi_H(P_{i_n}/B, \theta_{s_{i_n}}^* M))).$$

By induction,

$$\begin{aligned} \chi_H(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* M) &= *D_{\mathfrak{w}(n)} \left(* \chi_H(P_{i_n}/B, \theta_{s_{i_n}}^* M) \right) \\ &= *D_{\mathfrak{w}(n)} (* * D_{s_{i_n}} (*\text{ch } M)), \text{ by the next exercise} \\ &= *D_{\mathfrak{w}}(*\text{ch } M). \end{aligned}$$

\square

Combining Theorem 30 for $M = \mathbb{C}_\lambda$ and Corollary 28, we get the following:

Corollary 31. *For any reduced word \mathfrak{w} , the operator $D_{\mathfrak{w}}$ depends only upon $\xi(\mathfrak{w})$.*

Exercise 32. Show $\chi_H(P_{i_n}/B, \mathbb{C}_\mu) = *D_{s_{i_n}}(e^{-\mu})$ and conclude that

$$\chi_H(P_{i_n}/B, \theta_{s_{i_n}}^* M) = *D_{s_{i_n}}(*\text{ch } M).$$

13 Verma modules

For any $\lambda \in \mathfrak{h}^*$, define the *Verma module*

$$M(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda,$$

where \mathfrak{h} acts on \mathbb{C}_λ by the action of \mathfrak{h} via the weight λ and the nil-radical $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$ acts trivially. Then, $M(\lambda)$ is a \mathfrak{g} -module under left multiplication by elements of \mathfrak{g} on the $\mathcal{U}(\mathfrak{g})$ factor. If $\lambda \in \mathfrak{h}^* \setminus X(H)$, then \mathbb{C}_λ is only a representation of \mathfrak{b} and not of B .

Exercise 33. Show

$$\text{ch } M(\lambda) = e^\lambda \prod_{\beta \in R^+} (1 - e^{-\beta})^{-1}.$$

14 BGG resolution

Let $\lambda \in X(H)_+$ and $N = |R^+|$. We define a resolution of the form:

$$0 \rightarrow \mathcal{F}_N \rightarrow \cdots \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 = M(\lambda) \rightarrow V(\lambda) \rightarrow 0, \quad (5)$$

where

$$\mathcal{F}_p := \bigoplus_{\substack{v \in W \\ \ell(v)=p}} M(v \star \lambda).$$

Fix one non-zero vector $1_\lambda \in \mathbb{C}_\lambda$, then $1 \otimes 1_\lambda \in M(\lambda)$ maps to v_λ , a highest weight vector in $V(\lambda)$. This map extends to

$$x \otimes 1_\lambda \mapsto x \cdot v_\lambda,$$

for each $x \in \mathcal{U}(\mathfrak{g})$. Up to a scalar, this is the unique \mathfrak{g} -module map $M(\lambda) \rightarrow V(\lambda)$. To define the maps $\delta_j : \mathcal{F}_{j+1} \rightarrow \mathcal{F}_j$, we first recall the following theorem:

Theorem 34 (Theorem 9.2.3 in [1]). *Let $\lambda \in X(H)_+$ and $v, w \in W$. If $w \not\geq v$, then*

$$\text{Hom}_{\mathfrak{g}}(M(w \star \lambda), M(v \star \lambda)) = 0.$$

If $w \geq v$, then it is one-dimensional.

We now define \mathfrak{g} -module maps $\mathcal{F}_{v,w} : M(w \star \lambda) \rightarrow M(v \star \lambda)$. If $w \not\prec v$, then $\mathcal{F}_{v,w} = 0$. For any $w \in W$, take any non-zero \mathfrak{g} -module map $i_w : M(w \star \lambda) \rightarrow M(\lambda)$.

Exercise 35. For any $\lambda, \mu \in \mathfrak{h}^*$, a \mathfrak{g} -module map $M(\lambda) \rightarrow M(\mu)$ is injective if non-zero.

When $w > v$ and $\ell(w) = \ell(v) + 1$, we write $w \leftarrow v$. There exists a unique choice of non-zero \mathfrak{g} -maps $\mathcal{F}_{v,w} : M(w \star \lambda) \rightarrow M(v \star \lambda)$ for every $w \leftarrow v$ satisfying

$$i_v \circ \mathcal{F}_{v,w} = i_w.$$

Define

$$\delta_p = \sum_{\substack{\ell(w)=p+1 \\ w \leftarrow v}} \epsilon_{v,w} \mathcal{F}_{v,w} : \mathcal{F}_{p+1} \rightarrow \mathcal{F}_p,$$

where $\epsilon_{v,w} \in \{\pm 1\}$ are chosen satisfying the following result due to Bernstein–Gelfand–Gelfand.

Lemma 36 (Lemma 9.2.2 in [1]). *There is a choice of $\epsilon : \{v \rightarrow w\} \rightarrow \{\pm 1\}$ satisfying the following: for any square as below consisting of elements $v, w, x, y \in W$*

$$\begin{array}{ccc} v & \longrightarrow & x \\ \downarrow & & \downarrow \\ y & \longrightarrow & w \end{array},$$

we have

$$\epsilon_{v,x} \epsilon_{x,w} \epsilon_{v,y} \epsilon_{y,w} = -1.$$

The following is the celebrated Bernstein–Gelfand–Gelfand (BGG, for short) resolution.

Theorem 37. *The above sequence (5) is a resolution of $V(\lambda)$ for any $\lambda \in X(H)_+$.*

We will give two applications of this powerful result.

15 Application: Weyl character formula

Corollary 38 (Weyl character formula). *If $\lambda \in X(H)_+$, then*

$$\text{ch } V(\lambda) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \star \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \star 0}}.$$

Proof. From the exactness of complex (5), we easily see that

$$\text{ch } V(\lambda) = \sum_{p \geq 0} (-1)^p \text{ch } \mathcal{F}_p = \sum_{w \in W} (-1)^{\ell(w)} \text{ch } M(w \star \lambda).$$

In particular, by Exercise 33,

$$1 = \text{ch } V(0) = \sum_{w \in W} (-1)^{\ell(w)} \text{ch } M(w \star 0) = \left(\sum_{w \in W} (-1)^{\ell(w)} e^{w \star 0} \right) \prod_{\beta \in R^+} (1 - e^{-\beta})^{-1},$$

which implies

$$\prod_{\beta \in R^+} (1 - e^{-\beta}) = \sum_{w \in W} (-1)^{\ell(w)} e^{w \star 0}.$$

Therefore,

$$\begin{aligned} \text{ch } V(\lambda) &= \sum_{w \in W} (-1)^{\ell(w)} \text{ch } M(w \star \lambda) \\ &= \left(\sum_{w \in W} (-1)^{\ell(w)} e^{w \star \lambda} \right) \prod_{\beta \in R^+} (1 - e^{-\beta})^{-1} \\ &= \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w \star \lambda}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w \star 0}}. \end{aligned}$$

□

16 Application: Kostant's theorem on \mathfrak{n} -homology

Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ denote the nil-radical of the Borel \mathfrak{b} . Then, by definition,

$$H_i(\mathfrak{n}^-, V(\lambda)) := \text{Tor}_i^{\mathcal{U}(\mathfrak{n}^-)}(\mathbb{C}, V(\lambda)),$$

where $\mathfrak{n}^- := [\mathfrak{b}^-, \mathfrak{b}^-]$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \bigoplus_{\beta \in R^+} \mathfrak{g}_{-\beta}$. Since H normalizes \mathfrak{n}^- and $V(\lambda)$ is a \mathfrak{g} -module, $H_i(\mathfrak{n}^-, V(\lambda))$ is canonically an H -module. Now, we prove the following theorem due to Kostant as a consequence of the BGG resolution.

Theorem 39. *As H -modules,*

$$H_i(\mathfrak{n}^-, V(\lambda)) \simeq \bigoplus_{\substack{w \in W \\ \ell(w)=i}} \mathbb{C}_{w\star\lambda}.$$

Proof. By the PBW-theorem, $M(\lambda)$ is free as a $\mathcal{U}(\mathfrak{n}^-)$ -module. Specifically, as $\mathcal{U}(\mathfrak{n}^-)$ -modules, we have:

$$\mathcal{U}(\mathfrak{g}) \simeq \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b}).$$

Thus, as $\mathcal{U}(\mathfrak{n}^-)$ -modules,

$$M(\lambda) = (\mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{b})) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_{\lambda} = \mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda}.$$

Since any Verma module is free as a $\mathcal{U}(\mathfrak{n}^-)$ -module, by the BGG resolution (5) and the definition of Tor, we get that

$$\mathrm{Tor}_i^{\mathcal{U}(\mathfrak{n}^-)}(\mathbb{C}, V(\lambda)) = H_i(\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_*).$$

To compute $H_i(\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_*)$, observe

$$\begin{aligned} \mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_j &= \bigoplus_{\ell(w)=j} \mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} (\mathcal{U}(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_{w\star\lambda}) \\ &= \bigoplus_{\ell(w)=j} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}_{w\star\lambda} \\ &= \bigoplus_{\ell(w)=j} \mathbb{C}_{w\star\lambda}. \end{aligned}$$

But all of the \mathfrak{h} -module maps

$$\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_j = \bigoplus_{\ell(w)=j} \mathbb{C}_{w\star\lambda} \xrightarrow{1 \otimes \delta_{j-1}} \bigoplus_{\ell(v)=j-1} \mathbb{C}_{v\star\lambda} = \mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_{j-1}$$

are zero, since for $\lambda \in X(H)_+$ we have

$$v \star \lambda = w \star \lambda \iff v = w.$$

Thus, $H_i(\mathbb{C} \otimes_{\mathcal{U}(\mathfrak{n}^-)} \mathcal{F}_*) = \bigoplus_{\ell(w)=i} \mathbb{C}_{w\star\lambda}$, proving the theorem. \square

17 Grothendieck–Cousin complex

Our aim is to realize/prove the BGG-resolution geometrically.

Let X be a variety and let it be filtered by closed (but not necessarily irreducible) sub-varieties

$$X = X_0 \supset X_1 \supset X_2 \supset \cdots .$$

Let \mathcal{S} be a coherent sheaf (a vector bundle is enough for our purposes) on X . Then, there exists a complex (called the *Grothendieck–Cousin complex*) as follows:

$$0 \rightarrow H^0(X, \mathcal{S}) \rightarrow H_{X_0/X_1}^0(X, \mathcal{S}) \rightarrow H_{X_1/X_2}^1(X, \mathcal{S}) \rightarrow H_{X_2/X_3}^2(X, \mathcal{S}) \rightarrow \cdots .$$

For $Z \subset Y \subset X$ closed, $H_{Y/Z}^i(X, \mathcal{S})$ is the cohomology with support (see Appendix B of [1]).

Theorem 40 (Kempf). *The above complex is exact if the following properties hold:*

- (1) X is a Cohen–Macaulay irreducible variety,
- (2) \mathcal{S} is a vector bundle,
- (3) the maps $X_i \setminus X_{i+1} \rightarrow X$ are affine morphisms for all i (i.e., inverse images of affine open subsets are affine) and $X_i \setminus X_{i+1}$ are affine varieties,
- (4) codimension of each irreducible component of X_i in X is at least i ,
- (5) $H^n(X, \mathcal{S}) = 0$ if $n > 0$.

In our case, take $X = G/B$ and $X_i = \bigcup_{\ell(v) \geq i} X^v$, where $X^v = \overline{B^- v B / B} = \overline{w_0 B w_0 v B / B} = w_0 X_{w_0 v}$, where B^- is the subgroup of G with Lie algebra \mathfrak{b}^- . Take $\mathcal{S} = \mathcal{L}(\lambda)$ for $\lambda \in X(H)_+$.

Since G/B is smooth, it is Cohen–Macaulay, and property (1) follows. Of course, (2) is given. Since X_w is of dimension $\ell(w)$, property (4) follows. Property (5) follows from the Borel–Weil–Bott theorem, Theorem 7. Finally,

$$X_i \setminus X_{i+1} = \left(\bigcup_{\ell(v)=i} X^v \right) \setminus \left(\bigcup_{\ell(w) \geq i+1} X^w \right) = \bigsqcup_{\ell(v)=i} B^- v B / B,$$

which is affine since $BvB/B \simeq \mathbb{A}^{\ell(v)}$. If we check that the inclusion $\varphi : B^-vB/B \rightarrow G/B$ is an affine morphism, then (3) will be verified.

Let $U_{R^+ \cap vR^-}$ be the subgroup of G with the Lie algebra

$$\bigoplus_{\beta \in R^+ \cap vR^-} \mathfrak{g}_\beta.$$

Then, the map $U_{R^+ \cap vR^-} \rightarrow BvB/B$, $g \mapsto gvB$, is a biregular isomorphism. Now, identifying BvB/B with $U_{R^+ \cap vR^-}$ as above, we get a biregular isomorphism

$$BvB/B \times B^-vB/B \simeq vB^-B/B \simeq U^-, \quad (6)$$

under $(g, x) \mapsto gx$, where $U^- := [B^-, B^-]$. For any affine open subset V of G/B , by the following exercise, $(vB^-B/B) \cap V$ is an affine open subset of vB^-B/B . But, B^-vB/B is an affine closed subset of vB^-B/B by the above isomorphism (6). Thus, $V \cap (B^-vB/B)$ is a closed subset of affine $V \cap (vB^-B/B)$ and hence $V \cap (B^-vB/B)$ is an affine open subset of B^-vB/B . This establishes (3).

Exercise 41. If U, V are affine open in any variety Y , then $U \cap V$ is affine.

Theorem 42 (Lemma 9.3.5 and Proposition 9.3.7 in [1]). *As \mathfrak{g} -modules, for any $p \geq 0$,*

$$H_{X_p/X_{p+1}}^p(G/B, \mathcal{L}(\lambda)) \simeq \bigoplus_{\ell(w)=p} M(w \star \lambda)^\vee,$$

where $^\vee$ denotes the restricted dual.

Thus, in our case the Grothendieck–Cousin complex becomes the resolution (due to Kempf)

$$0 \longrightarrow V(\lambda)^* \longrightarrow \mathcal{F}_0^\vee \longrightarrow \mathcal{F}_1^\vee \longrightarrow \cdots \longrightarrow \mathcal{F}_N^\vee \longrightarrow 0,$$

which is dual to the BGG resolution.

18 Remarks

We have not given any historical comments. The interested reader can find them in sections 8.C and 9.C of [1].

References

- [1] Kumar, Shrawan. *Kac-Moody Groups, their Flag Varieties and Representation Theory*. Boston, Birkhäuser, 2002.
- [2] Hartshorne, Robin. *Algebraic Geometry*. New York: Springer-Verlag, 1977.
- [3] Humphreys, James. *Linear Algebraic Groups*. New York: Springer-Verlag, 1975.