AN ENERGY ESTIMATE FOR THE DIFFERENCE OF SOLUTIONS FOR THE n-DIMENSIONAL EQUATION WITH PRESCRIBED MEAN CURVATURE AND REMOVABLE SINGULARITIES

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Abstract: We derive an energy bound, estimating a weighted Dirichlet integral of two solutions for the nonparametric equation with prescribed mean curvature in n dimensions in terms of the L^1-norm for the difference of their values on the boundary. Furthermore, a similar estimate is established for solutions of the equation $\mathrm{div} F_p(.,\nabla u) = nH(.,u)$, where $F(x,p)$ denotes an elliptic Lagrangian with linear growth in p. These results are used to remove singularities of solutions to these equations.

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§1: Introduction

In this article we consider C^2-solutions $u : \Omega \to \mathbb{R}$ of the equation

$$\mathcal{M}u(x) = nH\Big(x, u(x)\Big)\Big(1 + |\nabla u|^2\Big)^{\frac{3}{2}} \tag{1.1}$$

where

$$\mathcal{M}u := \Big(1 + |\nabla u|^2\Big)\Delta u - \sum_{j,\,k=1}^{n} u_{x_j} u_{x_k} u_{x_j x_k} \tag{1.2}$$

represents the nonlinear minimal surface operator; while ∇ and Δ denote the gradient and the Laplacian, respectively, in $\mathbb{R}^n$ of dimension $n \geq 2$, applied to some function $u = u(x)$ of the variables $x = (x_1, \ldots, x_n)$; and $H = H(x,z) = H(x_1, \ldots, x_n, z)$ is a prescribed function of the $n+1$ variables x, z.
If $u \in C^2(\Omega)$ satisfies (1.1), then its graph $\mathcal{S} := \{(x,z) : \ z = u(x),\ x \in \Omega\}$ represents a surface with the mean curvature $H(x, u(x))$ at the point $(x, u(x))$.

Our central result is an energy estimate for the difference of two solutions for (1.1) on a set $\Omega \backslash K$, where K is a compact subset of the bounded open set Ω in $\mathbb{R}^n$ satisfying $\mathcal{H}^{n-1}(K) = 0$; cf. Section 2. Here $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

In Section 3 we generalize this estimate to solutions u of the elliptic equations

$$\mathrm{div} F_p(.,\nabla u) = nH(.,u) \quad \text{in} \quad \Omega \setminus K \tag{1.3}$$

where $F(x,p)$ grows linearly in p. We note that such an estimate was initially derived by J.C.C. Nitsche [10] in the dimension $n = 2$ for solutions $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ of the p.d.e. $\mathcal{M}u = 0 \quad \text{in} \quad \Omega$.

In Section 4 we show that such estimates can be used to remove singularities of solutions for (1.1) and (1.3) on $\Omega \setminus K$, where K is an **admissible singular subset** of Ω in the sense of Definition 1 below. Results of this type were first proved by L. Bers [2] and R. Finn [4] for the minimal surface equation $\mathcal{M}u = 0 \quad \text{in} \quad \mathbb{R}^2$ – and for related equations.

In $\mathbb{R}^n$ with $n \geq 2$, removability results were established by Nitsche [10] as well as by de Giorgi and Stampacchia [3] for solutions of

$$\mathcal{M}u = 0 \quad \text{in} \quad \Omega \setminus K \quad \text{with} \quad \mathcal{H}^{n-1}(K) = 0.$$

Generalizing the p.d.e. (1.1), interesting results have been attained by Miranda [9], L. Simon [13], and G. Anzellotti [1]; the last authors treated the equation (1.3) as well.

All removability results preceding our investigations were obtained by alternative methods. The present work was initiated during a month in the Hausdorff Research Institute for Mathematics at Bonn in 2008, where we were both invited. We would like to express our sincere gratitude to the director of this institution, Herrn Professor Matthias Kreck, for this wonderful opportunity to mathematical collaboration.

§2: A weighted energy estimate for the difference of two H-graphs

Our central contribution is the following

Theorem 1: *Let Ω be a bounded open set in $\mathbb{R}^n$ with $\partial\Omega \in C^1$, and K be a compact subset of Ω with $\mathcal{H}^{n-1}(K) = 0$. Furthermore, we assume that $H(x,z)$ is a continuous function of $(x,z) \in \overline{\Omega} \times \mathbb{R}$ such that the partial derivative $H_z(x,z)$ exists on $\overline{\Omega} \times \mathbb{R}$, $H_z \in C^0(\overline{\Omega} \times \mathbb{R})$, and*

$$H_z(x,z) \geq 0 \quad \textit{for all} \quad (x,z) \in \overline{\Omega} \times \mathbb{R}. \tag{2.1}$$

Finally, let $u_1, u_2 \in C^0(\overline{\Omega} \setminus K) \cap C^2(\Omega \setminus K)$ be solutions of the nonparametric equation with prescribed mean curvature

$$\mathcal{M}u_j = nH(.,u_j)\Big(W(\nabla u_j)\Big)^3 \quad \textit{in} \quad \Omega \setminus K \quad , \quad j = 1,2 \quad , \tag{2.2}$$

where $W(\nabla u_j) = \sqrt{1 + |\nabla u_j|^2} =: W_j$.
Then we have the following **weighted energy estimate**

$$\int_{\Omega \setminus K} \mu(u_1,u_2)\Big|\nabla u_1 - \nabla u_2\Big|^2 dx \quad \leq \quad 2\int_{\partial\Omega} |u_1 - u_2| \, d\mathcal{H}^{n-1} \quad , \tag{2.3}$$

where the positive and continuous **weight function** $\mu(u_1,u_2):\Omega\setminus K\to(0,+\infty)$ *is defined by*

$$\mu(u_1,u_2)(x):=\Big(max\{W_1(x),W_2(x)\}\Big)^{-3}\quad for\quad x\in\Omega\setminus K. \tag{2.4}$$

Proof:

1. We may rewrite the two equations (2.2) as

$$\operatorname{div}\Big(\frac{\nabla u_j}{W_j}\Big)=nH(.,u_j)\quad\text{on}\quad\Omega\setminus K\quad,\quad j=1,2 \tag{2.5}$$

(see e.g. Sauvigny [12], Chapter XI, §1). Subtracting these equations (2.5) from each other, we obtain

$$\operatorname{div}\Big(\frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big)=nH(.,u_1)-nH(.,u_2)\quad\text{on}\quad\Omega\setminus K. \tag{2.6}$$

For an arbitrary function $\zeta\in C^0(\overline{\Omega}\setminus K)$ and $M>0$ we define the **truncated function** $[\zeta]_M:\overline{\Omega}\setminus K\to\mathbb{R}$ by

$$[\zeta]_M(x):=\begin{cases}+M, & \zeta(x)\ge M\\ \zeta(x), & -M<\zeta(x)<+M\\ -M, & \zeta(x)\le -M.\end{cases} \tag{2.7}$$

Then we consider the associate truncated function

$$[u_1-u_2]_M:\overline{\Omega}\setminus K\to\mathbb{R}\quad,$$

which belongs to the regularity class

$$C^0(\overline{\Omega}\setminus K)\cap L^\infty(\overline{\Omega}\setminus K)\cap W^{1,2}_{loc}(\Omega\setminus K)$$

but not necessarily to $W^{1,2}(\Omega\setminus K)$. Then, by (2.6), we obtain

$$\begin{aligned}&\operatorname{div}\Big\{[u_1-u_2]_M\cdot\Big(\frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big)\Big\}-\Big\langle\nabla[u_1-u_2]_M\,,\,\frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big\rangle=\\ &[u_1-u_2]_M\cdot\operatorname{div}\Big(\frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big)=\\ &n\cdot[u_1-u_2]_M\cdot\Big\{H(.,u_1)-H(.,u_2)\Big\}\quad\text{on}\quad\Omega\setminus K.\end{aligned} \tag{2.8}$$

For any $x\in\Omega\setminus K$ there exists a number $\vartheta=\vartheta(x)\in(0,1)$ such that

$$H\Big(x,u_1(x)\Big)-H\Big(x,u_2(x)\Big)=H_z\Big(x,u_2(x)+\vartheta[u_1(x)-u_2(x)]\Big)[u_1(x)-u_2(x)]$$

by the mean value theorem. Therefore, the inequality (2.1) implies

$$n\cdot[u_1-u_2]_M\cdot\Big\{H(.,u_1)-H(.,u_2)\Big\}\ge 0\quad\text{on}\quad\Omega\setminus K. \tag{2.9}$$

The statements (2.8) and (2.9) together yield

$$\Big\langle \nabla[u_1-u_2]_M\,,\ \frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big\rangle \le \operatorname{div}\Big\{[u_1-u_2]_M\cdot\Big(\frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big)\Big\} \quad \text{on} \quad \Omega\setminus K \quad \text{for all} \quad M>0. \tag{2.10}$$

2. Since K is compact and $K\subset\Omega$, there is a number $\delta_0>0$ such that

$$K\subset\Omega_\delta := \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>\delta\} \quad \text{for any} \quad \delta\in(0,\delta_0]$$

as well as

$$\partial\Omega_\delta\in C^1 \quad \text{and} \quad \partial\Omega_\delta\to\partial\Omega \quad \text{in} \quad C^1 \quad \text{for} \quad \delta\to 0.$$

Because of $\mathcal{H}^{n-1}(K)=0$, for any $\epsilon>0$ we obtain $N(\epsilon)\in\mathbb{N}$ open balls

$$B_j^\epsilon \subset\subset \Omega_{\delta_0} \quad \text{where} \quad j=1,\dots,N(\epsilon)$$

with the following properties:

$$K\subset B_1^\epsilon\cup\ldots\cup B_{N(\epsilon)}^\epsilon \tag{2.11}$$

and

$$\mathcal{H}^{n-1}\Big(\partial B_1^\epsilon\Big)+\ldots+\mathcal{H}^{n-1}\Big(\partial B_{N(\epsilon)}^\epsilon\Big)<\epsilon. \tag{2.12}$$

We set

$$\Omega[\delta,\epsilon] := \Omega_\delta\setminus\Big\{\overline{B_1^\epsilon}\cup\ldots\cup\overline{B_{N(\epsilon)}^\epsilon}\Big\} \qquad \text{for all} \quad \epsilon>0 \quad \text{and all} \quad 0<\delta\le\delta_0 \tag{2.13}$$

and verify

$$[u_1-u_2]_M\cdot\Big(\frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big)\in C^0\Big(\overline{\Omega[\delta,\epsilon]},\mathbb{R}^n\Big)\cap W^{1,2}\Big(\Omega[\delta,\epsilon],\mathbb{R}^n\Big) \quad .$$

The integration of (2.10) yields

$$\int_{\Omega[\delta,\epsilon]}\Big\langle \nabla[u_1-u_2]_M\,,\ \frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big\rangle\,dx \le \int_{\Omega[\delta,\epsilon]}\operatorname{div}\Big\{[u_1-u_2]_M\cdot\Big(\frac{\nabla u_1}{W_1}-\frac{\nabla u_2}{W_2}\Big)\Big\}\,dx \quad \text{for all} \quad M>0. \tag{2.14}$$

3. A well-known result on Sobolev functions states that

$$\nabla[u_1-u_2]_M(x)=\begin{cases}\nabla u_1(x)-\nabla u_2(x) & \text{on} \quad \Omega_M\\[1ex] 0 & \text{a.e. on} \quad \Big(\Omega\setminus K\Big)\setminus\Omega_M\end{cases} \tag{2.15}$$

introducing the open set

$$\Omega_M := \{x\in\Omega\setminus K :\ |u_1(x)-u_2(x)|<M\}$$

for all values $0 < M < \infty$. Then the upper integral in (2.14) coincides with

$$\mathcal{I}(\delta,\epsilon,M) := \int_{\Omega[\delta,\epsilon]\cap\Omega_M} \Big\langle \nabla u_1 - \nabla u_2 \, , \, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \Big\rangle \, dx \quad , \tag{2.16}$$

and the lower integral in (2.14) is equal to

$$\mathcal{J}(\delta,\epsilon,M) := \int_{\partial\Omega[\delta,\epsilon]} \Big\langle [u_1 - u_2]_M \cdot \Big(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \Big) \, , \, \nu \Big\rangle \, d\mathcal{H}^{n-1} \tag{2.17}$$

via a generalized Gaussian integral theorem. Here $\nu : \partial\Omega[\delta,\epsilon] \to S^{n-1}$ denotes the exterior normal to the domain $\Omega[\delta,\epsilon]$ with the piecewise C^1–boundary

$$\partial\Omega[\delta,\epsilon] = \partial\Omega_\delta \cup \partial\Theta_\epsilon \quad \text{where} \quad \Theta_\epsilon := B_1^\epsilon \cup \ldots \cup B_{N(\epsilon)}^\epsilon .$$

We note the inequality

$$\mathcal{I}(\delta,\epsilon,M) \le \mathcal{J}(\delta,\epsilon,M) \quad \text{for all} \quad 0 < \delta \le \delta_0, \, \epsilon > 0, \, M > 0. \tag{2.18}$$

With the aid of (2.12) we easily estimate

$$\Big| \mathcal{J}(\delta,\epsilon,M) - \int_{\partial\Omega_\delta} \Big\langle [u_1 - u_2]_M \cdot \Big(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \Big) \, , \, \nu \Big\rangle \, d\mathcal{H}^{n-1} \Big| \le \int_{\partial\Theta_\epsilon} M \cdot 2 \cdot \, d\mathcal{H}^{n-1} \le 2M\epsilon \tag{2.19}$$

and obtain

$$\lim_{\epsilon\to 0} \mathcal{J}(\delta,\epsilon,M) = \int_{\partial\Omega_\delta} \Big\langle [u_1 - u_2]_M \cdot \Big(\frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \Big) \, , \, \nu \Big\rangle \, d\mathcal{H}^{n-1} \quad . \tag{2.20}$$

Because of $\partial\Omega_\delta \to \partial\Omega$ in C^1 and $[u_1 - u_2]_M \in C^0(\overline{\Omega} \setminus K)$, we can estimate in the last identity for the limit $\delta \to 0$ as follows:

$$\liminf_{\delta\to 0} \, \inf_{\epsilon\to 0} \mathcal{J}(\delta,\epsilon,M) \le 2 \cdot \int_{\partial\Omega} \Big| u_1 - u_2 \Big| \, d\mathcal{H}^{n-1} \quad \text{for all } M > 0. \tag{2.21}$$

4. In order to estimate the integrand of (2.16), we consider the **density function**

$$W(p) := \sqrt{1 + p^2}, \, p \in \mathbb{R}^n$$

and establish the important

Auxiliary inequality:

$$\Big\langle p' - p'' \, , \, W_p(p') - W_p(p'') \Big\rangle \ge m(p',p'') \cdot \Big| p' - p'' \Big|^2 \quad \text{for all} \quad p', p'' \in \mathbb{R}^n \tag{2.22}$$

where

$$m(p',p'') := \Big(\max\{W(p'), W(p'')\} \Big)^{-3} \quad . \tag{2.23}$$

In order to prove this auxiliary inequality, we set $p = (p_1, \ldots, p_n) \in \mathbb{R}^n$ and calculate

$$W_p(p) = \frac{p}{W(p)} \quad \text{and} \quad W_{pp}(p) = \{W(p)\}^{-3} \cdot \Big((1 + |p|^2)\delta_{jk} - p_j p_k \Big)_{j,k=1,\ldots n} \quad . \tag{2.24}$$

Then we estimate the associate quadratic form of the Hessian matrix by

$$\begin{aligned} \xi \circ W_{pp}(p) \circ \xi^* &= \{W(p)\}^{-3} \cdot \Big\{(1+|p|^2)|\xi|^2 - <\xi, p><p, \xi>\Big\} \\ &\geq \{W(p)\}^{-3} \cdot \Big\{(1+|p|^2)|\xi|^2 - (|\xi| \cdot |p|)^2\Big\} = \{W(p)\}^{-3}|\xi|^2 \end{aligned} \tag{2.25}$$

$$\text{for all} \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n.$$

Now we apply the mean value theorem on the real-valued function

$$\Phi(t) := \Big\langle p' - p'' \,,\, W_p\Big(p'' + t(p' - p'')\Big)\Big\rangle, \quad t \in [0,1] \quad .$$

We obtain an intermediate value $\vartheta = \vartheta(p', p'') \in (0,1)$, such that

$$\begin{aligned} \Big\langle p' - p'' \,,\, W_p(p') - W_p(p'')\Big\rangle &= \Phi(1) - \Phi(0) = \Phi'(\vartheta) \\ &= \Big\langle p' - p'' \,,\, W_{pp}\Big(p'' + \vartheta(p' - p'')\Big) \circ (p' - p'')^*\Big\rangle \\ &= \Big(p' - p''\Big) \circ W_{pp}\Big(p'' + \vartheta(p' - p'')\Big) \circ \Big(p' - p''\Big)^* \\ &\geq \Big\{W\Big(p'' + \vartheta(p' - p'')\Big)\Big\}^{-3} \cdot \Big|p' - p''\Big|^2 \end{aligned} \tag{2.26}$$

holds true, with the aid of the inequality (2.25). Now we recall that the function W is strictly convex on $\mathbb{R}^n$ due to (2.25), such that

$$W(tp' + (1-t)p'') \leq tW(p') + (1-t)W(p'') \leq \max\{W(p'), W(p'')\}, \quad 0 \leq t \leq 1$$

is valid. This implies the auxiliary inequality (2.22), where $m(p', p'')$ is defined by (2.23).

5. When we insert $p' := \nabla u_1(x)$ and $p'' := \nabla u_2(x)$ with $x \in \Omega \setminus K$ into (2.22) and recall (2.4), we arrive at the central estimate

$$\begin{aligned} \Big\langle \nabla u_1 - \nabla u_2 \,,\, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\Big\rangle\Big|_x &= \Big\langle p' - p'' \,,\, W_p(p') - W_p(p'')\Big\rangle \\ \geq m(p', p'') \cdot \Big|p' - p''\Big|^2 &= \mu(u_1, u_2)(x)\Big|\nabla u_1(x) - \nabla u_2(x)\Big|^2, \quad x \in \Omega \setminus K. \end{aligned} \tag{2.27}$$

Now the integrand in (2.16) is nonnegative, and Fatou's lemma is applicable to control the relevant integrals in the – not necessarily monotonic – exhaustion process

$$\Omega[\delta, \epsilon] \to \Omega \setminus K \quad \text{for} \quad \epsilon \to 0 \quad \text{and} \quad \delta \to 0$$

with the null set K. Utilizing (2.18), (2.21), and (2.27), we see that

$$\begin{aligned} \int_{\Omega_M} \mu(u_1, u_2)(x)\Big|\nabla u_1(x) - \nabla u_2(x)\Big|^2 dx &\leq \int_{\Omega_M} \Big\langle \nabla u_1 - \nabla u_2 \,,\, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}\Big\rangle dx \\ \leq \liminf_{\delta \to 0} \liminf_{\epsilon \to 0} \mathcal{I}(\delta, \epsilon, M) &\leq \liminf_{\delta \to 0} \liminf_{\epsilon \to 0} \mathcal{J}(\delta, \epsilon, M) \leq 2 \cdot \int_{\partial\Omega} \Big|u_1 - u_2\Big| \, d\mathcal{H}^{n-1} \end{aligned} \tag{2.28}$$

holds true. With the aid of B. Levi's theorem on monotone convergence we deduce the inequality

$$\int_{\Omega\setminus K} \mu(u_1, u_2) \cdot \left|\nabla u_1 - \nabla u_2\right|^2 dx \leq 2 \int_{\partial\Omega} \left|u_1 - u_2\right| d\mathcal{H}^{n-1} \tag{2.29}$$

for $M \to \infty$. This gives us the desired weighted energy estimate (2.3).

Q.e.d.

Remark 1: The result of Theorem 1 can easily be generalized to *weak* solutions of

$$\operatorname{div}\Big(W(\nabla u)^{-1}\,\nabla u\Big) = nH(.,u).$$

§3: A weighted energy estimate for the difference of two nonparametric extremals for a Cartan functional

Denote by $y = (y_1, \dots, y_{n+1})$ the points in $\mathbb{R}^{n+1}$, and consider the smooth mappings $Y : \Omega \to \mathbb{R}^{n+1}$ on a domain $\Omega \subset \mathbb{R}^n$ given by

$$Y = (Y_1(v), \dots, Y_{n+1}(v)), \quad v = (v_1, \dots, v_n) \in \Omega.$$

We interpret any such Y as an n-dimensional surface of codimension 1 in $\mathbb{R}^{n+1}$. Geometrically this is justified if the associate exterior product in normal direction $Q : \Omega \to \mathbb{R}^{n+1}$ defined by

$$Q(v) := Y_{v_1} \wedge \ldots \wedge Y_{v_n}\Big|_v, \quad v \in \Omega$$

does not vanish.

Consider a real-valued continuous function $G(y, q)$ of the variables $(y, q) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ satisfying

$$G(y, tq) = tG(y, q) \quad \text{for all} \quad t > 0 \tag{3.1}$$

and

$$G(y, q) > 0 \tag{3.2}$$

for any $y = (y_1, \dots, y_{n+1})$, $q = (q_1, \dots, q_{n+1}) \in \mathbb{R}^{n+1}$. With G we associate a **Cartan functional** or alternatively a **parametric variational integral**

$$\mathcal{G}(Y) := \int_\Omega G\Big(Y(v), Q(v)\Big)\, dv \tag{3.3}$$

defined for mappings $Y : \overline{\Omega} \to \mathbb{R}^{n+1} \in C^1\Big(\overline{\Omega}\,, \mathbb{R}^{n+1}\Big)$. The area integral

$$\mathcal{A}(Y) := \int_\Omega A\Big(Q(v)\Big)\, dv \tag{3.4}$$

represents the prototype of a Cartan functional with the Lagrangian

$$A(q) := |q|, \quad q \in \mathbb{R}^{n+1}. \tag{3.5}$$

We easily verify the identities

$$A_{q_\alpha\, q_\beta}(q) = |q|^{-3} \cdot \Big(|q|^2 \cdot \delta_{\alpha\,\beta} - q_\alpha q_\beta \Big) \quad \text{for all} \quad q \neq 0 \quad \text{and} \quad \alpha,\, \beta = 1, \ldots n+1 \quad . \tag{3.6}$$

The **Lagrangian** $G(y, q)$ of $\mathcal{G}$ is called **elliptic**, if there exists a real number $\lambda > 0$ such that $G(y, q) - \lambda A(q)$ represents a convex function of $q \in \mathbb{R}^{n+1}$ for any $y \in \mathbb{R}^{n+1}$. Suppose that $G(y, .)$ is of class C^2 on $\mathbb{R}^{n+1} \setminus \{0\}$ for any $y \in \mathbb{R}^{n+1}$. Then G is elliptic if and only if the following condition is fulfilled:

$$\begin{gathered} \sum_{\alpha,\,\beta=1}^{n+1} G_{q_\alpha\, q_\beta}(y, q)\eta_\alpha \eta_\beta \geq \lambda \cdot |q|^{-3} \cdot \Big(|q|^2 \cdot |\eta|^2 - < q, \eta >^2 \Big) \\ \text{for all} \quad \eta = (\eta_1, \ldots \eta_{n+1}) \in \mathbb{R}^{n+1} \quad \text{and any} \quad y \in \mathbb{R}^{n+1},\, q \in \mathbb{R}^{n+1} \setminus \{0\} \quad . \end{gathered} \tag{3.7}$$

Now we consider nonparametric surfaces $Y : \overline{\Omega} \to \mathbb{R}^{n+1}$ given by real-valued functions $u \in C^1(\overline{\Omega})$ on a domain $\Omega \subset \mathbb{R}^n$ as follows:

$$Y(x) = (x, u(x)), \quad x \in \overline{\Omega}. \tag{3.8}$$

The associate exterior product in normal direction $Q : \overline{\Omega} \to \mathbb{R}^{n+1}$ is given by

$$Q(x) = (-\nabla u(x), 1), \quad x \in \overline{\Omega} \quad , \tag{3.9}$$

and we see

$$\mathcal{G}(Y) := \int_\Omega G\Big(x, u(x), -\nabla u(x), 1\Big)\, dx \quad . \tag{3.10}$$

The **nonparametric Lagrangian** $F(x, z, p)$ associated with $G(y, q)$ appears as

$$F(x, z, p) := G\Big(x, z, -p, 1\Big) \quad \text{for} \quad (x, z, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \quad , \tag{3.11}$$

defining the **variational integral**

$$\mathcal{F}(u) := \int_\Omega F\Big(x, u(x), \nabla u(x)\Big)\, dx \quad \text{for} \quad u \in C^1(\overline{\Omega}) \tag{3.12}$$

with $\mathcal{F}(u) = \mathcal{G}(Y)$ and $Y(x) = (x, u(x))$. We readily evaluate

$$\begin{gathered} F_{p_j}(x, z, p) = -G_{p_j}\Big(x, z, -p, 1\Big), \quad F_{p_j\, p_k}(x, z, p) = G_{p_j\, p_k}\Big(x, z, -p, 1\Big) \\ \text{for all} \quad (x, z, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \quad \text{and} \quad j, k = 1, \ldots n \quad . \end{gathered} \tag{3.13}$$

Now we insert the variables $\eta = (\xi, 0) = (\xi_1, \ldots, \xi_n, 0) \in \mathbb{R}^n \times \mathbb{R}$ and the vector

$$q = (-p, 1) = (-p_1, \ldots, -p_n, 0) \in \mathbb{R}^{n+1} \quad \text{with} \quad |q| = \sqrt{1 + |p|^2} = W(p)$$

into the ellipticity condition (3.7), and we obtain the **nonparametric ellipticity condition**

$$\begin{gathered} \sum_{j,k=1}^{n} F_{p_j\, p_k}(x, z, p)\xi_j \xi_k \geq \lambda \cdot W(p)^{-3} \cdot \Big((1 + |p|^2) \cdot |\xi|^2 - < p, \xi >^2 \Big) \\ \text{for all} \quad \xi = (\xi_1, \ldots \xi_n) \in \mathbb{R}^n \quad \text{and any} \quad (x, z, p) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \quad . \end{gathered} \tag{3.14}$$

Because of $< p\,,\,\xi >^2 \leq |p|^2 \cdot |q|^2$ this leads to the inequality

$$\Big\langle \xi\,,\, F_{pp}(x,z,p)\circ\xi^* \Big\rangle \geq \lambda\cdot W(p)^{-3}\cdot|\xi|^2$$
$$\text{for all} \quad \xi = (\xi_1,\dots\xi_n)\in\mathbb{R}^n \quad \text{and any} \quad (x,z,p)\in\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n \quad , \tag{3.15}$$

reducing to the estimate (2.25) when we replace F_{pp} with W_{pp} . As in Part 4. of the proof for Theorem 1, we can deduce from (3.15) the following

General auxiliary inequality:
The nonparametric elliptic Lagrangian $F(x,z,p)$ from (3.11) satisfies

$$\Big\langle p'-p''\,,\, F_p(x,z,p')-F_p(x,z,p'')\Big\rangle \geq \lambda\cdot m(p',p'')\cdot\Big|p'-p''\Big|^2 \quad \text{for all} \quad p',p''\in\mathbb{R}^n$$
$$\text{and any} \quad (x,z)\in\overline{\Omega}\times\mathbb{R} \qquad \text{with} \quad m(p',p'') := \Big(\max\{W(p'),W(p'')\}\Big)^{-3} \quad . \tag{3.16}$$

Now we shall derive the analogue of Theorem 1 from Section 1 for solutions u of the inhomogeneous Euler equation of the functional (3.12), where we have to restrict ourselves to Lagrangians independent of the height variable z, namely

$$\operatorname{div} F_p\Big(x,\nabla u(x)\Big) = nH\Big(x,u(x)\Big) \tag{3.17}$$

with

$$\operatorname{div} F_p\Big(x,\nabla u(x)\Big) = \sum_{j=1}^n \frac{\partial}{\partial x_j}\Big[F_{p_j}\Big(x,\nabla u(x)\Big)\Big] \quad . \tag{3.18}$$

Suppose that $u_1,u_2\in C^0(\overline{\Omega}\setminus K)\cap C^2(\Omega\setminus K)$ are two solutions of (3.17). Subtracting these two equations, we obtain

$$\operatorname{div}\Big\{F_p\Big(x,\nabla u_1(x)\Big)-F_p\Big(x,\nabla u_2(x)\Big)\Big\}$$
$$= nH\Big(x,u_1(x)\Big)-nH\Big(x,u_2(x)\Big) \qquad \text{on} \quad \Omega\setminus K. \tag{3.19}$$

If we specialize F to W then

$$F_p\Big(.,\nabla u_j\Big) = W_p(\nabla u_j) = \frac{\nabla u_j}{W_j} \quad \text{with} \quad W_j = W(\nabla u_j) \quad \text{for} \quad j=1,2 \tag{3.20}$$

appears, and (3.19) reduces to the equation (2.6). Then we repeat the reasoning in the proof of Theorem 1 and replace the auxiliary inequality (2.22) with the general auxiliary inequality (3.16). Since the Lagrangian is independent of z, no variation in this argument is necessary. Therefore, the combination of (3.19) with (3.16) yields the following

Theorem 2: *Let $F(x,p)$ be a Lagrangian on $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$ as described above subject to the nonparametric ellipticity condition with the constant $\lambda>0$. Additionally, we assume that K is a compact subset of a bounded open set Ω in $\mathbb{R}^n$ with $\partial\Omega\in C^1$ such that $\mathcal{H}^{n-1}(K)=0$ holds true. Furthermore, we assume that $H(x,z)$ is a continuous function of $(x,z)\in\overline{\Omega}\times\mathbb{R}$ where the partial derivative $H_z(x,z)$ exists on $\overline{\Omega}\times\mathbb{R}$, $H_z\in C^0(\overline{\Omega}\times\mathbb{R})$, and*

$$H_z(x,z)\geq 0 \quad \textit{for all} \quad (x,z)\in\overline{\Omega}\times\mathbb{R}. \tag{3.21}$$

Finally, we consider solutions $u_1,u_2\in C^0(\overline{\Omega}\setminus K)\cap C^2(\Omega\setminus K)$ of the equation

$$div\, F_p\Big(.,\nabla u_j\Big)=nH\Big(.,u_j\Big) \quad in \quad \Omega\setminus K \quad , \quad j=1,2. \tag{3.22}$$

Then we have the following **general weighted energy estimate**

$$\int_{\Omega\setminus K}\mu(u_1,u_2)\Big|\nabla u_1-\nabla u_2\Big|^2\,dx \quad\leq\quad \frac{2}{\lambda}\cdot\int_{\partial\Omega}|u_1-u_2|\,d\mathcal{H}^{n-1} \tag{3.23}$$

with

$$\mu(u_1,u_2)(x):=\Big(max\{W_1(x),W_2(x)\}\Big)^{-3} \quad for \quad x\in\Omega\setminus K. \tag{3.24}$$

§4: Removable singularities

As an immediate consequence of Section 2 we establish the following uniqueness result.

Theorem 3: *Suppose that Ω, K, H, and u_1,u_2 satisfy the assumptions of Theorem 1, assume that $\Omega\setminus K$ is connected, and require*

$$u_1(x)=u_2(x) \quad for\ all \quad x\in\partial\Omega \quad . \tag{4.1}$$

Then we have

$$u_1\equiv u_2 \quad on \quad \overline{\Omega}\setminus K \quad . \tag{4.2}$$

Proof: The weighted energy estimate (2.3) of Theorem 1 implies

$$\nabla u_1\equiv\nabla u_2 \quad \text{on} \quad \Omega\setminus K. \tag{4.3}$$

Since $\Omega\setminus K$ represents a domain, we deduce

$$u_1-u_2\equiv\text{const} \quad \text{on} \quad \Omega\setminus K. \tag{4.4}$$

When we employ the assumptions $u_1-u_2\in C^0(\overline{\Omega}\setminus K)$ and $K\subset\subset\Omega$, the boundary condition $u_1=u_2$ on $\partial\Omega$ yields the identity (4.2). Q.e.d.

Remark 2: If, for a given $u_1:\overline{\Omega}\setminus K\to\mathbb{R}$, we can find a function $u_2\in C^0(\overline{\Omega})\cap C^2(\Omega)$ which solves the boundary value problem

$$\mathcal{M}u_2=nH\Big(.,u_2\Big)\Big(1+|\nabla u_2|^2\Big)^{\frac{3}{2}} \quad \text{in} \quad \Omega \qquad \text{with} \qquad u_2(x)=u_1(x) \quad \text{on} \quad \partial\Omega \quad , \tag{4.5}$$

where Ω, K, H, u_1,u_2 satisfy the assumptions of Theorem 1, then $u_1\equiv u_2$ on $\overline{\Omega}\setminus K$ follows. This means the singularities of u_1 are removed, and the function u_1 is extended to a regular C^2-solution of the equation $\mathcal{M}u=nH\Big(.,u\Big)\Big(1+|\nabla u|^2\Big)^{\frac{3}{2}}$ on Ω.

In order to apply Theorem 3 to the question which singularities can be removed in the way described in Remark 2, we need the following

Definition 1: Let Ω be a bounded connected open set in $\mathbb{R}^n$, i.e. a bounded domain in $\mathbb{R}^n$. A subset K of Ω is called an **admissible singular subset**, if K is compact and the following **covering property** holds true: For each $\epsilon > 0$ we find $N = N(\epsilon) \in \mathbb{N}$ open balls B_j with $j = 1, \ldots, N(\epsilon)$ which are mutually disjoint due to

$$\overline{B_j} \cap \overline{B_k} = \emptyset \quad \text{for all} \quad j,k \in \{1,\ldots,N\} \quad \text{with} \quad j \neq k \tag{4.6}$$

and satisfy the inclusion

$$K \subset B_1 \cup \ldots \cup B_N \tag{4.7}$$

as well as the following estimate

$$\sum_{j=1}^{N} \mathcal{H}^{n-1}\Big(\partial B_j\Big) < \epsilon \tag{4.8}$$

for the total area of the spheres ∂B_j.

Remark 3: If K constitutes an admissible singular subset of Ω, then $\mathcal{H}^{n-1}(K) = 0$ and $\Omega \setminus K$ is a domain in $\mathbb{R}^n$. Therefore, our Theorem 3 applies to Ω and K. However, the basic Theorem 1 can be established via the Gaussian integral theorem only for C^1-domains in this situation.

Remark 4: Let B be a ball in $\mathbb{R}^n$ of radius $r(B) > 0$. Then we calculate

$$\mathcal{H}^{n-1}(\partial B) = \text{area}(\partial B) = \omega_n \cdot [r(B)]^{n-1} \quad \text{with} \quad \omega_n = 2 \cdot \frac{\sqrt{\pi}^{\,n}}{\Gamma(\frac{n}{2})} \quad . \tag{4.9}$$

Hence any ball B_j of a covering $\{B_1, \ldots, B_N\}$ of K satisfying (4.6)–(4.8) has a radius $r(B_j)$ such that

$$r(B_j) < \Big(\frac{\epsilon}{\omega_n}\Big)^{\frac{1}{n-1}} =: R(\epsilon) \quad . \tag{4.10}$$

Since $\text{dist}(K, \partial\Omega) > 0$ for $K \subset\subset \Omega$ is valid, we conclude: If K is an admissible singular subset of Ω, then there exists a number $\epsilon_0 > 0$ such that any covering $\{B_1, \ldots, B_N\}$ of K described in Definition 1 fulfills

$$B_j \subset\subset \Omega \quad \text{for} \quad 1 \leq j \leq N \quad \text{for all} \quad 0 < \epsilon \leq \epsilon_0 \quad . \tag{4.11}$$

We are now prepared to establish the following removability result.

Theorem 4: *At first, we suppose that H, $H_z \in C^0(\overline{\Omega} \times \mathbb{R})$ and $H_z(x,z) \geq 0$ on $\overline{\Omega} \times \mathbb{R}$ holds true. Secondly, we assume that there exists a number $\rho(\Omega) > 0$ such that the boundary value problem*

$$\mathcal{M}\zeta = nH\Big(.,\zeta\Big)\Big(1 + |\nabla\zeta|^2\Big)^{\frac{3}{2}} \quad in \quad B \qquad with \qquad \zeta(x) = \varphi(x) \quad on \quad \partial B \tag{4.12}$$

has a solution $\zeta \in C^0(\overline{B}) \cap C^2(B)$ for any ball $B \subset\subset \Omega$ of radius $r(B) \leq \rho(\Omega)$ and any boundary function $\varphi \in C^0(\partial B)$.
Then we have: If K is an admissible singular subset of Ω and $u \in C^2(\Omega \setminus K)$ is a solution of

$$\mathcal{M}u = nH(.,u)\Big(1 + |\nabla u|^2\Big)^{\frac{3}{2}} \tag{4.13}$$

on $\Omega \setminus K$, then u can be extended to a function of class $C^2(\Omega)$ satisfying (4.13) on Ω.

Proof: Choose $\epsilon > 0$ so small that $\epsilon \le \epsilon_0$ and $R(\epsilon) \le \rho(\Omega)$ is fulfilled. Let $B_1, \ldots, B_{N(\epsilon)}$ denote a covering of the singular set K belonging to $\epsilon > 0$, as described in Definition 1. Then the inclusion $B_j \subset\subset \Omega$ and the existence of a solution to the Dirichlet problem

$$\mathcal{M}\zeta_j = nH\Big(.,\zeta_j\Big)\Big(1+|\nabla\zeta_j|^2\Big)^{\frac{3}{2}} \quad \text{in} \quad B_j \qquad \text{with} \qquad \zeta_j(x) = u(x) \quad \text{on} \quad \partial B_j \tag{4.14}$$

is ascertained for $j = 1, \ldots, N$. The sets $K_j := K \cap B_j$ are compact and $B_j \setminus K_j$ represent domains. Applying Theorem 3 to B_j, K_j, H and u, ζ_j we infer

$$\zeta_j(x) \equiv u(x) \quad \text{on} \quad B_j \setminus K_j \quad \text{for} \quad 1 \le j \le N \quad , \tag{4.15}$$

which implies the assertion. Q.e.d.

Applications of Theorem 4:

1. If H is a constant function, the existence of a number $\rho(\Omega) > 0$ is established in Theorem 15.11 of the treatise by D. Gilbarg and N. Trudinger [8].

2. If $H(x)$ is a Lipschitz continuous function on $\overline{\Omega}$ and does not depend on z, then the existence of a number $\rho(\Omega) > 0$ follows from M. Giaquinta [6].

3. For a smooth function $H(x, z)$ satisfying $H_z(x, z) \ge 0$ the existence of a number $\rho(\Omega) > 0$ was derived by C. Gerhardt [5], provided that we merely require the solvability of the Dirichlet problem (4.12) for sufficiently smooth boundary values $\varphi : \partial B \to \mathbb{R}$ in Theorem 4.

Remark 5: With the aid of Theorem 2 in Section 3 we can generalize Theorem 4 to solutions of the equation

$$\operatorname{div} F_p\Big(.,\nabla u\Big) = nH\Big(.,u\Big) \tag{4.16}$$

with a Lagrangian $F(x, p)$ growing linearly in p. The existence of a number $\rho(\Omega) > 0$ that guarantees the *local solvability* of (3.16) on balls $B \subset\subset \Omega$ with $r(B) \le \rho(\Omega)$ follows from the work of Giaquinta-Modica-Souček [7] under suitable conditions on F and H.
The first to obtain removability results for this type of equations were L. Simon [13] for the case $H = H(x)$ and G. Anzellotti [1] for $H = H(x, z)$. Their results are more complete then ours as they only require $\mathcal{H}^{n-1}(K) = 0$, while we have to assume that K is an admissible singular subset of Ω in the sense of Definition 1. Moreover, L. Simon as well as J.C.C. Nitsche only need the assumption $K \subset \Omega$ (and not $K \subset\subset \Omega$) together with $\mathcal{H}^{n-1}(K) = 0$.

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