SPECTRAL FLOW IS THE INTEGRAL OF ONE FORMS ON THE BANACH MANIFOLD OF SELF ADJOINT FREDHOLM OPERATORS

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Abstract. One may trace the idea that spectral flow should be given as the integral of a one form back to the 1974 Vancouver ICM address of I.M. Singer. Our main theorem gives analytic formulae for the spectral flow along a norm differentiable path of self-adjoint bounded Breuer-Fredholm operators in a semi-finite von Neumann algebra. These formulae have a geometric interpretation which derives from the proof. Namely we define a family of Banach submanifolds of all bounded self-adjoint Breuer-Fredholm operators and on each submanifold define global one forms whose integral on a norm differentiable path contained in the submanifold calculates the spectral flow along this path. We emphasise that our methods do not give a single globally defined one form on the self adjoint Breuer-Fredholms whose integral along all paths is spectral flow rather, as the choice of the plural 'forms' in the title suggests, we need a family of such one forms in order to confirm Singer's idea. The original context for this result concerned paths of unbounded self-adjoint Fredholm operators. We therefore prove analogous formulae for spectral flow in the unbounded case as well. The proof is a synthesis of key contributions by previous authors, whom we acknowledge in detail in the introduction, combined with an additional important recent advance in the differential calculus of functions of non-commuting operators.

1. INTRODUCTION

The notion of spectral flow has been a useful tool in geometry ever since its invention by Lusztig and its application by Atiyah-Patodi-Singer [1, 2]. Until about a decade ago spectral flow was considered primarily in topological terms as an intersection number and there seemed to be no analytic viewpoint. This was despite the observation of I.M. Singer [39] that the eta invariant is the integral of a one form and so, from the variation of eta formula in [1, 2], by implication one is led to ask whether spectral flow is expressible as the integral of a one-form. In this paper we provide an answer to this question.

There has been a succession of contributions leading to our resolution of Singer's question. We mention initial progress on an analytic approach to spectral flow in [24, 26–28]. Then in [33, 34] an analytic definition of spectral flow was given. This definition applied equally to type II von Neumann algebras where operators with continuous spectrum may arise (an issue initially raised in [31, 32]). A key step in synthesising these developments was taken in [10, 11] which exploit an essential contribution of Getzler [26] to produce spectral flow formulae as integrals of one-forms on affine subspaces of the Banach manifold of self adjoint Fredholms. Noncommutative geometry plays a key role in all three of these papers in that theta and finitely summable spectral triples are utilised. The significant new ingredient in [10, 11] was the introduction of general analytic methods which demonstrated that these analytic spectral flow formulae apply equally to the standard type I situation envisaged by Singer and also to spectral flow along paths of self adjoint Breuer-Fredholm operators in a type II_∞ von Neumann algebra. This development was partly motivated by ideas of Mathai on L^2 spectral

invariants for manifolds whose fundamental group has a non-type I regular representation, see [30]. For the benefit of readers unfamiliar with the terminology above we will summarise the relevant definitions in later Sections. Readers interested in more details should see the review in [6], while readers unfamiliar with the Breuer-Fredholm theory may consult [9, 13].

A decisive further development occurred in [5] (see also [4]). The functional calculus methods of [10, 11] simply do not generalise sufficiently to answer Singer's question. A more sophisticated functional calculus is needed and this was provided partly in [5] where it is explained how double operator integrals (DOI) give a differential calculus for functions of operators. It is the key technical tool we exploit in the present paper and in order to make the discussion more self contained we develop, in Section 5, the relevant parts of this DOI technique. A second innovation, which occurred in [42], was a new way to handle paths of unbounded self adjoint Fredholm operators. In [42] a spectral flow formula for paths that lie in an affine space of relatively bounded perturbations of a fixed unbounded self adjoint Fredholm operator was proved. This inspired the present work whose principal aim is to give a very general answer to Singer's question in the case of the Banach manifold of bounded self adjoint Fredholm operators and then to deduce a generalisation of the unbounded results of [42] from our bounded formula. We emphasise that the methods are sufficiently strong to answer Singer's question in a general semifinite von Neumann algebra.

To illustrate our ideas we now summarise a special case of our results. Suppose $\mathcal{M}$ is a semi-finite von Neumann algebra with a normal semifinite faithful (n.s.f.) trace τ which will be fixed throughout. We take the τ-Calkin algebra to be the quotient of $\mathcal{M}$ by the norm closed ideal generated by the $\tau-$finite projections. An operator is τ-Fredholm (and hence Breuer-Fredholm) if it is invertible in the $\tau-$Calkin algebra. Suppose that $t \mapsto F_t \in \mathcal{M}$ is a piecewise C^1-path of self adjoint τ-Fredholm operators such that $\|F_t\| \leqslant 1$ and the spectrum of the image of F_t in the $\tau-$Calkin algebra is $\{\pm 1\}$. If the endpoints of this path, F_0 and F_1, are unitarily equivalent, then the spectral flow, $sf(F_t)$, may be computed by the following analytic formula

$$sf(F_t) = \int_0^1 \tau\left(\dot{F}_t\, h(F_t)\right)\, dt, \tag{1}$$

where h is a positive sufficiently smooth function on $[-1, 1]$. The choice of h is dictated by the requirement that the RHS of (1) is well defined, namely that

$$\int_0^1 \left\|\dot{F}_t\, h(F_t)\right\|_1\, dt < +\infty, \tag{2}$$

where $\|\cdot\|_1$ is the trace norm on $\mathcal{M}$.

In some special cases, formula (1) has been proved by different methods in [5, 10, 11, 42] under various restrictions on the path $\{F_t\}$. A feature of the methods we employ in this paper is that we are able to remove the assumption of [5, 10, 11], that Fredholm paths $\{F_t\}$ must lie in the affine space of τ-compact perturbations of a fixed Fredholm operator F_0. This affine space is contractible so that the spectral flow of any loop in the space is zero and hence these affine space formulae do not directly reveal the rich topology of the space of Breuer-Fredholm operators. In our approach this topology is seen by integrating along non-contractible loops in the space of self adjoint Breuer-Fredholm operators. Note also that each of these affine spaces lies entirely within one of the submanifolds described in the abstract and one may recover the 'global' formulae of the affine space case studied in [5, 10, 11] by an approximation argument (although we give details only in the unbounded case). We emphasise that the consequences of the spectral flow formulae in [10,

11] for affine spaces are quite profound: they imply for example the local index formula in noncommutative geometry in semifinite spectral triples [12, 18].

In the present paper we shall show that a modification of the approach of [5] allows us to prove (1) under only the requirement (2). The formula (1) is a special case of a much more general formula which we prove in Section 3 of this paper. In Section 3 we give expressions for spectral flow along norm differentiable paths in the Banach manifold of bounded self adjoint τ-Fredholms in a semifinite von Neumann algebra. The assumptions under which these formulae hold are the minimal ones: there are no unnecessary side conditions. As Singer's question was originally phrased in the case of spectral flow along paths of unbounded self adjoint operators, we deduce, in Section 4, unbounded formulae from our bounded one. Namely we prove that, for a pair D_0, D_1 of unbounded self adjoint τ-Fredholms, spectral flow along any path joining them that is smooth in the graph norm of D_0 is the integral of a one form defined on the affine space of D_0-graph norm bounded self adjoint perturbations of D_0. This is a strengthening of all previous results (in particular, those in [5, 10, 11, 42]).

We now summarise the geometric meaning of (1). In Section 3 we give more details. The space $\mathcal{F}_*^{\pm 1}$ of all self adjoint τ-Fredholms with norm less than or equal to one and whose images in the Calkin algebra have spectrum ± 1 plays a special role in the theory as we will see later. (In the Appendix we show that the well known lemma of [3] that the space $\mathcal{F}_*^{\pm 1}$ is a deformation retract of the space of self adjoint Fredholms with both positive and negative essential spectrum still holds in a semifinite von Neumann algebra.) As $\mathcal{F}_*^{\pm 1}$ itself does not appear to be a manifold we need to take care in interpreting our construction. We start with an auxiliary bigger class namely all self-adjoint τ-Fredholm operators with no 'essential' spectrum in the interval $[-\delta, \delta]$ for some $\delta > 0$ (more precisely it is the inverse image of the open set of self adjoint invertibles in the τ-Calkin algebra bounded away from zero by δ). This class, denoted $\mathcal{F}_\delta$, is an open subset of the self-adjoint part of the algebra $\mathcal{M}$ and therefore, clearly, is a Banach manifold. Any norm continuous path $t \to F_t$, $t \in [0, 1]$ in the self adjoint τ-Fredholm operators lies in a $\mathcal{F}_\delta$ for some $\delta > 0$. Then the integrand of (1) is a one form in the following sense. The integrand comes from the functional θ_F defined on the tangent space to the manifold of self adjoint τ-Fredholms at $F \in \mathcal{F}_\delta$ by $\theta_F(X) = \tau(Xh(F))$ for a suitably chosen C^1 function h with support in $[-\delta, \delta]$. We will see that this functional gives an exact one form on sufficiently small convex neighbourhoods of F. (Note that this geometric viewpoint can be traced back to [26]). Thus (1) is to be interpreted as the integral of a one form on a path in $\mathcal{F}_\delta$. Note however that, in our approach, there is no global one form on the space of all τ-Fredholm operators that calculates spectral flow. It is necessary to vary δ and hence the function h depending on the path in question. Our general formula (which we do not state in this introduction as it requires much more notation than (1)) applies when the endpoints are not unitarily equivalent.

To recover the affine space formulae of [10, 11] we need to work in $\mathcal{F}_*^{\pm 1}$. To do this in our approach requires an approximation argument and as a result the geometric interpretation of the resulting formulae as integrals of one forms is lost. Thus for the affine space situation the exact one form property of the integrand has to be reproved ab initio (and we do not do this here cf. [10, 11]).

Also, for the unbounded case in Section 4, we remark that an unbounded self adjoint operator D is τ-Fredholm if the operator $F_D = D(1 + D^2)^{-1/2}$ is a τ-Fredholm operator in $\mathcal{M}$. The issue of the differentiability of the map $D \mapsto F_D$ as a function on the unbounded τ-Fredholms has proved in the past to be the principal obstacle to proving spectral flow formulae for the unbounded case using

formulae for the bounded case (see for example the discussion in [42]). One of the novelties of our approach in this paper is a very satisfactory resolution of this differentiability question described in Section 6. This is used in Section 4 to obtain a straightforward proof of the unbounded formula. As in [26] the motivation for our approach comes from questions in noncommutative geometry. In the next Section we will explain one relationship of our results to the latter formalism. This enables one to understand spectral flow as a pairing of K-homology with K-theory.

The remainder of the paper is organised as follows. We prove the most general formula for spectral flow along paths of bounded self adjoint τ-Fredholm operators in Section 3. In Section 4 we deduce from the bounded formula a corresponding formula for paths of unbounded self adjoint τ-Fredholms. We present the proofs in as direct a fashion as possible deferring technical issues on double operator integrals to Section 5 and background on graph norm bounded paths of unbounded operators to Section 6. We present a reasonably detailed discussion in Sections 5 and 6 to make this paper more self contained and independent of previous papers on these two topics.

2. PERTURBATIONS OF SPECTRAL TRIPLES

To explain how the calculation of spectral flow presented in the following Sections fits into the overall picture in noncommutative geometry [17] we describe some preliminary results in this Section.

A semifinite spectral triple consists of an unbounded self adjoint operator D on a Hilbert space $\mathcal{H}$, a unital $*$-subalgebra $\mathcal{A}$ of a semifinite von Neumann algebra $\mathcal{M}$ (with faithful normal semifinite trace τ) acting on $\mathcal{H}$ such that the commutator $[D, a]$ extends to a bounded linear operator on $\mathcal{H}$ for all $a \in \mathcal{A}$ and with D having τ-compact resolvent in $\mathcal{M}$.

In previous work spectral flow between operators in the affine space of bounded self adjoint perturbations of D was studied in the context of spectral triples and a formula for spectral flow proved that provides a first step in the resolution of Singer's question. In [10] it is observed that if A is bounded then

$$F_D - F_{D+A} := D(1 + D^2)^{-1/2} - (D + A)(1 + (D + A)^2)^{-1/2}$$

is τ-compact. This observation is crucial to the method of proof of the spectral flow formulae in [10, 11], namely, one deduces the unbounded formula from a formula for spectral flow in the affine space of τ-compact perturbations of a fixed bounded τ-Fredholm operator F. In [10] it was observed that if A is an unbounded self adjoint operator affiliated to $\mathcal{M}$ that is bounded in the graph norm of a fixed unbounded self adjoint operator D (that is, $\mathrm{dom}(D) \subseteq \mathrm{dom}(A)$ and $\|Av\| \leqslant c(\|v\| + \|Dv\|)$ for some $c > 0$ and all $v \in \mathrm{Dom}(A)$) affiliated to $\mathcal{M}$ then $F_D - F_{D+A}$ is bounded but is not τ-compact in $\mathcal{M}$. Thus to prove a spectral flow formula for spectral flow between D and a graph norm bounded perturbation using the strategy of [10] requires us to prove a formula for spectral flow for general paths of bounded self adjoint Fredholm operators.

The noncommutative geometry framework in the bounded case is that of semifinite pre-Fredholm modules (a special case of Kasparov modules [17]). For our purposes we will only need the following definition. With $\mathcal{A}$ and $\mathcal{M}$ as above a semifinite pre-Fredholm module is given by a self adjoint operator F in $\mathcal{M}$ such that $1 - F^2$ is τ-compact and $[F, a]$ is τ-compact for all $a \in \mathcal{A}$. (If $F^2 = 1$ then the prefix 'pre' is dropped.)

Lemma 1. (i) *Any semifinite spectral triple for $\mathcal{A}$ defines a semifinite pre-Fredholm module where we choose F to be $\mathsf{F_D} = D(1+D^2)^{-1/2}$.*

(ii) *If A is a self adjoint unbounded operator such that the D-graph norm of A is less than 1 and $[A, \mathfrak{a}]$ is bounded for all $\mathfrak{a} \in \mathcal{A}$ then $D+A$ also defines a spectral triple for $\mathcal{A}$. The semifinite pre-Fredholm module for $\mathcal{A}$ given by F_{D+A} is homotopic to that given by $\mathsf{F_D}$ with the homotopy defined by the path $\{\mathsf{F}_{D+tA}, t \in [0,1]\}$.*

Proof. (i) It is sufficient to observe that, since D has a τ-compact resolvent, the operator $1 - \mathsf{F}_D^2$ is τ-compact. Observe also that if $[D, \mathfrak{a}]$ is bounded, then $[\mathsf{F_D}, \mathfrak{a}]$ is τ-compact (see [37, Theorem 11]).

(ii) To see that $D + A$ defines a spectral triple for $\mathcal{A}$, we have to note that the operator $(1 + (D + A)^2)^{-1}$ is τ-compact (see [10, Appendix B, Lemma 7]). In order to define a homotopy, the path $\{\mathsf{F}_{D+tA}, \ t \in [0,1]\}$, should be continuous with respect to the operator norm. This follows from the fact (see Section 6 and the identity (23) in particular) that there is a uniformly bounded family of continuous linear operators $\{T_t\}_{0 \leqslant t \leqslant 1}$ on $\mathcal{M}$ such that

$$\mathsf{F}_{D+tA} - \mathsf{F_D} = t\, T_t (A(1 + D^2)^{-\frac{1}{2}}).$$

The lemma is proved. □

Phillips definition of spectral flow [33, 34], which is extended and explained in some detail in [6], depends on a simple observation. Let χ be the characteristic function of $[0, \infty)$. Let $\mathcal{N}$ be a semifinite von Neumann algebra with semifinite, faithful, normal trace, τ. Let $\{F_t\}$ be any norm continuous path in the bounded self adjoint τ-Fredholms in $\mathcal{N}$ (indexed by some interval $[a, b]$). If we let π be the projection onto the Calkin algebra then one may show that $\pi(\chi(F_t)) = \chi(\pi(F_t))$. As the spectra of $\pi(F_t)$ are bounded away from 0, this latter path is continuous. By compactness we can choose a partition $a = t_0 < t_1 < \cdots < t_k = b$ so that for each $i = 1, 2, \cdots, k$

$$\|\pi(\chi(F_t)) - \pi(\chi(F_s))\| < \frac{1}{2} \quad \text{for all } t, s \text{ in } [t_{i-1}, t_i].$$

Letting $P_i = \chi(F_{t_i})$ for $i = 0, 1, \cdots, k$ then by the previous inequality (see [6]) $P_{i-1}P_i : P_iH \to P_{i-1}H$ is Fredholm. Then we define the *spectral flow of the path* $\{F_t\}$ to be the number:

$$sf(\{F_t\}) = \sum_{i=1}^{k} \mathrm{ind}(P_{i-1}P_i)$$

which is independent of the choice of partition of $[a, b]$. This analytic point of view recovers the intersection number approach to spectral flow when the operators in question have discrete spectrum.

The spectral flow for a D-graph norm continuous path $\{D_t : \ t \in [0, 1]\}$ of unbounded self adjoint τ-Fredholms joining $D = D_0$ to D_1 affiliated to $\mathcal{M}$ is defined as the spectral flow along the corresponding path F_{D_t} of bounded τ-Fredholms. When $u \in \mathcal{A}$ the spectral flow along the path

$$D_t := (1 - t)D + tuDu^* = D + tu[D, u^*]$$

defines a pairing between the K-homology class defined by the semifinite spectral triple $(\mathcal{H}, D, \mathcal{A})$ and the class of u in $K_1(\mathcal{A})$. The preceding lemma gives a condition on the perturbation A under which the spectral triple defined by $D + A$ gives the same pairing with $K_1(\mathcal{A})$ as does the spectral triple defined by D.

Notice that for an unbounded self adjoint operator D with $(1 + D^2)^{-1}$ being τ-compact the map $D \mapsto \mathsf{F_D}$ has range in the space of self adjoint bounded τ-Fredholms of norm less than or equal to one and such that the essential spectrum is contained in ± 1. That is $1 - \mathsf{F}_D^2 = (1 + D^2)^{-1}$ is τ-compact, explaining in part

the distinguished role played by this retract of the manifold of all bounded τ-Fredholm operators in the subsequent exposition.

3. SPECTRAL FLOW FORMULA, BOUNDED CASE.

Let $\mathcal{M}$ be a von Neumann algebra and let τ be a n.s.f. trace on $\mathcal{M}$. $\|\cdot\|$ stands for the operator norm on $\mathcal{M}$. Let $L^1(\mathcal{M})$ be the predual of $\mathcal{M}$ equipped with the trace norm $\|\cdot\|_1$. Recall that an operator $F \in \mathcal{M}$ is called τ-Fredholm if and only if

   (i) the projections N_F and N_{F*} are τ-finite;
   (ii) there is a τ-finite projection $p \in \mathcal{M}$, such that $\mathrm{Ran}(1-p) \subseteq \mathrm{Ran}(F)$.

Here N_F is the projection onto the $\mathrm{Ker}(F)$ and $\mathrm{Ran}(F)$ is the range of the operator F.

Let $\mathcal{K}$ be the two-sided ideal of all τ-compact operators of $\mathcal{M}$. The quotient space $\mathcal{M}/\mathcal{K}$ is a C^*-algebra. Let π be the homomorphism

$$\pi : \mathcal{M} \mapsto \mathcal{M}/\mathcal{K}.$$

Recall the following characterization of the τ-Fredholm operators due to M. Breuer (see [9, Theorem 1]): *An operator F is τ-Fredholm if and only if the image $\pi(F)$ is invertible.* We set $\delta_F = \left\|\pi(F)^{-1}\right\|^{-1}$. Note that the mapping $F \mapsto \delta_F$ is continuous on the Banach manifold of τ-Fredholm operators.

We shall denote the set of all self adjoint τ-Fredholm operators $F \in \mathcal{M}$ by $\mathcal{F}_*$. We also shall denote the subset of $\mathcal{F}_*$ with $\|F\| \leqslant 1$ and $\delta_F = 1$ by $\mathcal{F}_*^{\pm 1}$.

The characterization of M. Breuer implies that if F is a self adjoint τ-Fredholm operator, then, for every $0 \leqslant \delta < \delta_F$, the spectral projection $E^F(-\delta, \delta)$ is τ-finite i.e.,

$$\tau\left(E^F(-\delta, \delta)\right) < +\infty, \;\; 0 \leqslant \delta < \delta_F.$$

Indeed, fix $0 \leqslant \delta < \delta_F$. Consider the operator $F_\delta = F - F\,E^F(-\delta, \delta)$. We then have that

$$\left\|\pi(F) - \pi(F_\delta)\right\| \leqslant \|F - F_\delta\| \leqslant \delta < \delta_F = \left\|\pi(F)^{-1}\right\|^{-1}.$$

Consequently, the operator $\pi(F_\delta)$ is invertible and therefore F_δ is τ-Fredholm. This means that there is a τ-finite projection p such that $1 - p \subseteq \mathrm{Ran}(F_\delta)$. The latter implies that $E^F(-\delta, \delta) \subseteq p$ which means that the projection $E^F(-\delta, \delta)$ is τ-finite. Furthermore,

Lemma 2.   (i) *For every $F \in \mathcal{F}_*$ and every bounded Borel function g supported on the interval $[-\delta_F, \delta_F]$ such that $\lim_{x \to \pm\delta_F} g(x) = g(\pm\delta_F) = 0$, the operator $g(F)$ is τ-compact. In particular, if $F \in \mathcal{F}_*^{\pm 1}$, then $1 - F^2$ and $F - B$ are τ-compact, where $B = 2\chi_{[0,+\infty)}(F) - 1$.*

   (ii) *For every $F_0 \in \mathcal{F}_*$ and every $0 < \delta < \delta_{F_0}$, there is a neighbourhood N of F_0 such that the mapping $F \mapsto E^F(-\delta, \delta)$ is trace norm bounded on the self adjoint part of N.*

Proof. (i) To see that the operator $g(F)$ is τ-compact, it is sufficient to show that, for every $\epsilon > 0$, there is a τ-compact operator K_ϵ such that $\|g(F) - K_\epsilon\| < \epsilon$.

Fix $\epsilon > 0$. Let x_ϵ be the point $0 < x_\epsilon < \delta_F$ such that

$$|g(x)| < \epsilon, \;\; \text{for every } x_\epsilon \leqslant |x| \leqslant \delta_F.$$

We set $K_\epsilon = g(F)\chi_{[-x_\epsilon, x_\epsilon]}(F)$. Since the function g is bounded and the projection $\chi_{[-x_\epsilon, x_\epsilon]}(F)$ is τ-finite, the operator K_ϵ is τ-compact. On the other hand, by the choice of x_ϵ, we see that

$$\|g(F) - K_\epsilon\| \leqslant \sup_{x_\epsilon \leqslant |x| \leqslant \delta_F} |g(x)| < \epsilon.$$

The proof is finished.

(ii) Assume first that $F_0 \in \mathcal{F}_*^{\pm 1}$. In this special case, the proof employs the argument of [5, Lemma 1.26.(ii)]. Let $F = F_0 + A$ where A is self adjoint and $\|A\| \leqslant \frac{1}{6}(1 - \delta^2)$. Clearly,

$$1 - F^2 = 1 - F_0^2 + B$$

where $B = F_0 A + A F_0 - A^2$ and $\|B\| \leqslant \frac{1}{2}(1 - \delta^2)$. Let $\mu_t(X)$ be a decreasing rearrangement of the operator $X \in \mathcal{M}$ (see [25]). Note that

$$\chi_{(-\delta, \delta)}(x) \leqslant \frac{1 - x^2}{1 - \delta^2}, \quad |x| \leqslant 1.$$

This observation together with [25, Lemma 2.5] implies that

$$\mu_t\left(E^F(-\delta, \delta)\right) \leqslant \frac{1}{1 - \delta^2} \mu_t\left(1 - F^2\right)$$

$$\leqslant \frac{1}{1 - \delta^2}\left[\mu_{\frac{t}{2}}\left(1 - F_0^2\right) + \mu_{\frac{t}{2}}(B)\right]$$

$$\leqslant \frac{1}{1 - \delta^2}\mu_{\frac{t}{2}}\left(1 - F_0^2\right) + \frac{1}{2}.$$

Since the operator $1 - F_0^2$ is τ-compact, the function $t \mapsto \mu_t(1 - F_0^2)$ is decreasing to 0 at $+\infty$. Thus, we see that the functions $t \mapsto \mu_t\left(E^F(-\delta, \delta)\right)$ are uniformly majorized across

$$F \in N = \left\{F_0 + A, \ \|A\| \leqslant \frac{1}{6}(1 - \delta^2)\right\}$$

by a single decreasing function with value $\frac{1}{2}$ at $+\infty$. On the other hand, we know that

$$\mu_t\left(E^F(-\delta, \delta)\right) = \chi_{[0, \tau(E^F(-\delta, \delta))]}.$$

Consequently, the value

$$\tau\left(E^F(-\delta, \delta)\right)$$

is uniformly bounded across N.

Let now $F_0 \in \mathcal{F}_*$. Let θ be a C^2-function such that (i) θ' is nonnegative and supported on the interval $[-\delta_{F_0}, \delta_{F_0}]$; (ii) $\theta(\pm\infty) = \pm 1$. Clearly, $\theta(F_0) \in \mathcal{F}_*^{\pm 1}$. Moreover, the mapping $F \mapsto \theta(F)$ is operator norm continuous (see Remark 18). Consequently, the claim of the lemma for the general $F_0 \in \mathcal{F}_*$ holds with the preimage $\theta^{-1}(N)$ of the ball N constructed with respect to the operator $\theta(F_0) \in \mathcal{F}_*^{\pm 1}$ above. $\qquad\square$

If $t \mapsto F_t \in \mathcal{F}_*$ is a continuous path of self adjoint τ-Fredholm operators, then $sf(F_t)$ stands for the spectral flow as defined in [6,34]. We shall prove the following analytic formula for spectral flow, which extends that of [42, Theorem 6.4] and [5, Theorem 3.18].

Theorem 3. *Let* $F_t : [0, 1] \mapsto \mathcal{F}_*$ *be a piecewise* C^1-*path of self adjoint* τ-*Fredholm operators. If* h *is a positive* C^2-*function supported on* $[-\delta, \delta]$, *where* $\delta = \min_{0 \leqslant t \leqslant 1} \delta_{F_t}$, *such that*

(i) $\int_{-\delta}^{+\delta} h(x)\, dx = 1$;

(ii) $\int_0^1 \left\|\dot{F}_t\, h(F_t)\right\|_1 dt < +\infty$;

(iii) $H(F_1) - H(F_0) + \frac{1}{2}B_0 - \frac{1}{2}B_1 \in L^1(\mathcal{M})$, *where* $H(x)$ *is an antiderivative of* $h(x)$ *such that* $H(\pm\delta) = \pm\frac{1}{2}$ *and* B_j *is the phase of* F_j, *i.e.,* $B_j = 2\chi_{[0,+\infty)}(F_j) - 1$, $j = 0, 1$;

then

$$sf(F_t) = \int_0^1 \tau\left(\dot{F}_t\, h(F_t)\right) dt + \tau\left(H(F_1) - H(F_0) + \frac{1}{2}B_0 - \frac{1}{2}B_1\right). \tag{3}$$

Remark 4. (i) Observe that every positive C^1-function h, which is supported on a proper subinterval of $[-\delta, \delta]$ (where δ is defined in Theorem 3 and such that

$$\int_{-\delta}^{\delta} h(x)\, dx = 1,$$

satisfies the conditions of Theorem 3. Indeed, (i) is trivial; (ii) follows from Lemma 2.(ii) which implies that the function $t \in [0,1] \mapsto \|h(F_t)\|_1$ is bounded; and (iii) follows from Lemma 2.(ii) again and the observation that the function

$$\chi_{(0,+\infty)}(x) - \frac{1}{2} - H(x)$$

is bounded and supported on a proper subinterval of $[-\delta, \delta]$.

(ii) In previous papers the case where we work in a subset of the τ-Fredholms consisting of operators F satisfying the condition $(1 - F^2)^{n/2}$ is trace class [10] or $e^{-|1-F^2|^{-1}}$ is trace class [11] (the n-summable or theta summable cases respectively) were studied. Thus in the setting of Theorem 3 we would choose h to be given on $[-1, 1]$ by either $h(x) = (1-x^2)^{n/2}$ or $h(x) = e^{-|1-x^2|^{-1}}$ and to be zero on the complement of $[-1, 1]$. Notice that these two functions do not satisfy the assumptions of the theorem if we allow operators with essential spectrum ± 1. This minor difficulty is handled by an approximation argument which we describe in the proofs below.

Let $t \in [0,1] \mapsto F_t$ be a loop of self adjoint τ-Fredholm operators, $F_0 = F_1$. It is shown in [34, Remark 2.4] that if the loop F_t lies within sufficiently small neighbourhood in $\mathcal{M}$, then the spectral flow along this loop is 0. One of the steps in proving the analytic formula of Theorem 3 is to show that the integral (3) is also 0 for such loops. This is precisely the part where the proof of the spectral formula in [42] exploits the assumption that the mapping $t \mapsto 1 - F_t^2$ is C^1 with respect to the trace norm. We shall first see that, with some modification, the proof of [5, Proposition 3.5] allows us to avoid the latter restriction (see Theorem 5 below). This modification is based on results from [20, 21, 36].

Let $F_0 \in \mathcal{M}$ be a τ-Fredholm operator and let $N(F_0)$ be the neighbourhood given by

$$N(F_0) = \{F \in \mathcal{M}, \ \|\pi(F - F_0)\| < \delta_{F_0}\}.$$

Clearly, applying M. Breuer's result, every $F \in N(F_0)$ is τ-Fredholm. Observe also that $N(F_0)$ is convex.

Theorem 5. *Let* $t \in [0,1] \mapsto F_t \in \mathcal{F}_*$ *be a piecewise* C^1-*loop* ($F_0 = F_1$) *of* τ-*Fredholm operators such that* $F_t \in N$, $t \in [0,1]$, *where* N *is an open convex subset of* $\mathcal{M}$ *such that the norm closure* $\bar{N}$ *is a subset of* $N(F_0)$. *If* h *is a positive* C^2-*function such that*

(i) $\operatorname{supp} h \subseteq [-\delta, \delta]$ *where* $\delta = \min_{F \in N} \delta_F > 0$;

(ii) $\int_0^1 \left\| \dot{F}_t\, h(F_t) \right\|_1 dt < +\infty$,

then

$$\int_0^1 \tau\left(\dot{F}_t\, h(F_t) \right)\, dt = 0.$$

Proof. Observe that, since the space $L^1([0,1], L^1(\mathcal{M}))$ (= the space of all Bochner integrable $L^1(\mathcal{M})$-valued functions on $[0,1]$) is separable, for every function h such that $\operatorname{supp} h \subseteq [-\delta, \delta]$ there is a sequence[1] of positive C^2-functions $(h_n)_{n=1}^{\infty}$ such that

$$\operatorname{supp} h_n \subseteq (-\delta, \delta) \quad \text{and} \quad \lim_{n \to \infty} \int_0^1 \left\| \dot{F}_t h_n(F_t) - \dot{F}_t h(F_t) \right\|_1 dt = 0.$$

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[1]To this end, it is sufficient to construct a sequence of functions h_n such that the difference $h - h_n$ is uniformly bounded and the support of the difference $h - h_n$ vanishes as $n \to \infty$ and then refer to,

Consequently,

$$\int_0^1 \tau\left(\dot{F}_t\, h_n(F_t)\right)\, dt = 0,\ \forall n \geqslant 1 \implies \int_0^1 \tau\left(\dot{F}_t\, h(F_t)\right)\, dt = 0.$$

Thus, without loss of generality, we may assume that $\operatorname{supp} h \subseteq (-\delta, \delta)$.

Let $\operatorname{supp} h \subseteq (-\delta, \delta)$ and let g be a positive function such that $\operatorname{supp} g \subseteq (-\delta, \delta)$, $g^{\frac{1}{2}} \in C^2$ and $g(x) = 1$ for every x in some neighbourhood of $\operatorname{supp} h$. Let us show that the mapping $t \in [0,1] \mapsto g(F_t)$ is a continuous function with respect to the trace norm. Indeed, by the representation

$$g(F_t) - g(F_s) = g^{\frac{1}{2}}(F_t)\left(g^{\frac{1}{2}}(F_t) - g^{\frac{1}{2}}(F_s)\right) + \left(g^{\frac{1}{2}}(F_t) - g^{\frac{1}{2}}(F_s)\right) g^{\frac{1}{2}}(F_s),$$

is it clear that this mapping is continuous in the trace norm provided the function $t \mapsto g^{\frac{1}{2}}(F_t)$ is continuous in the operator norm and bounded in the trace norm. For the former, note that $t \mapsto F_t$ is operator norm continuous and $g^{\frac{1}{2}} \in C^2$ and therefore the path $t \mapsto g^{\frac{1}{2}}(F_t)$ is also operator norm continuous (see Remark 18). For the latter, observe that (i) $0 \leqslant g(x) \leqslant \chi(x)$, for some indicator function χ of a proper subinterval of $(-\delta, \delta)$; (ii) consequently, according to Lemma 2.(ii), the function $t \mapsto g^{\frac{1}{2}}(F_t)$ is bounded with respect to the trace norm in a small neighbourhood of every point $t \in [0,1]$; (iii) finally, due to compactness, this function is also globally bounded. Observe also that, since h is C^2, the mapping $t \mapsto h(F_t)$ is operator norm continuous (see Remark 18). Furthermore, since $h(F_t) = h(F_t)g(F_t)$, the mapping $t \mapsto h(F_t)$ is also continuous with respect to the trace norm. We shall single out the argument presented above as the following lemma.

Lemma 6. *For every $F_0 \in \mathcal{F}_*$ and every C^2-function h supported on a proper subinterval of $[-\delta_{F_0}, \delta_{F_0}]$, there is a neighbourhood N of F_0 such that the mapping $F \in N \mapsto h(F)$ is trace norm continuous on the self adjoint part of N.*

From this point on, the proof of Theorem 5 follows that of [5, Proposition 3.5]. We shall show that there is a function $\theta : N \mapsto \mathbb{C}$ such that

$$d\theta_F(X) = \tau(Xh(F)),\ \ X \in \mathcal{M}. \tag{4}$$

In other words, we shall show that the one form $X \to \tau(Xh(F))$ is exact. This will finish the proof.

Fix the element $F \in N(F_0)$. For the rest of the proof, $r \in [0,1] \mapsto F_r \in N$ stands for the straight line path connecting F_0 and F (i.e., $F_r = (1-r)F_0 + rF$). We introduce the function $\theta : N \mapsto \mathbb{C}$ as follows

$$\theta(F) = \int_0^1 \tau\left(\dot{F}_r\, h(F_r)\right)\, dr.$$

Let $d\theta_F(X)$, $X \in \mathcal{M}$ be the differential form of θ, i.e.,

$$d\theta_F(X) := \lim_{s \to 0} \frac{1}{s}\left(\theta(F + sX) - \theta(F)\right).$$

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e.g. [16, Proposition 2.1]. Observe also that one can also achieve that

$$\int_{\mathbb{R}} h_n(x)\, dx = 1,\ \ \forall n \geqslant 1.$$

Now we prove (4). Fix $X \in \mathcal{M}$. Following the definition of the function θ, a simple computation yields

$$\frac{1}{s} \left(\theta(F + sX) - \theta(F) \right) = \int_0^1 \tau \left(X \, h(F_r + srX) \right) \, dr$$
$$+ \int_0^1 \tau \left(\dot{F}_r \, \frac{1}{s} \left(h(F_r + srX) - h(F_r) \right) \right) \, dr, \qquad (5)$$

where $r \in [0, 1] \mapsto F_r$ is the straight line path connecting F_0 and F.

Let us consider the algebra $\mathcal{N} = L^\infty[0, 1] \bar{\otimes} \mathcal{M}$ equipped with the trace $\tau_1 = \int_0^1 dr \otimes \tau$ (see [41]). The mapping $\bar{F} : r \mapsto F_r$ is a τ_1-Fredholm operator in $\mathcal{N}$ with $\delta_{\bar{F}} \geqslant \delta$ and the mapping $\bar{X} : r \mapsto rX$ is an element of $\mathcal{N}$. Applying Lemma 6, we see that the mapping $s \mapsto h(\bar{F} + s\bar{X})$ is continuous in $L^1(\mathcal{N})$ in some neighbourhood of 0. Consequently, letting $s \to 0$, yields that the first term in (5) approaches

$$\int_0^1 \tau \left(X \, h(F_r) \right) \, dr.$$

For the second term of (5), we shall show that

$$\lim_{s \to 0} \int_0^1 \tau \left(\dot{F}_r \, \frac{1}{s} \left(h(F_r + srX) - h(F_r) \right) \right) \, dr = \int_0^1 r \, \tau \left(X \, \frac{d}{dr} \left[h(F_r) \right] \right) \, dr. \qquad (6)$$

If (6) is proved, then, letting $s \to 0$, we see from (5) that

$$d\theta_F(X) = \tau \left(X \left(\int_0^1 h(F_r) + r \frac{d}{dr} \left[h(F_r) \right] \, dr \right) \right).$$

Thus, to finish the proof of (4), we have to show that

$$\int_0^1 h(F_r) + r \frac{d}{dr} \left[h(F_r) \right] \, dr = h(F).$$

This readily follows if we integrate the second term by parts. Namely, integrating by parts, we have

$$\int_0^1 r \frac{d}{dr} \left[h(F_r) \right] \, dr = h(F_1) - \int_0^1 h(F_r) \, dr.$$

Next we prove (6). The proof of (6) heavily relies on the theory of Double Operator Integrals (DOIs) developed in [20, 21, 36] recently. We will describe in Section 5 sufficient background on DOIs for the reader to appreciate their role in this Section. Let again $\mathcal{N} = L^\infty[0, 1] \bar{\otimes} \mathcal{M}$ be a tensor product von Neumann algebra with the trace $\tau_1 = \int_0^1 dr \otimes \tau$. It is proved in Lemma 19 below that there are families[2] of linear operators $\{T_s\}$, $\{T_s'\}$ and $\{T_s''\}$ uniformly bounded on $\mathcal{N}$ and on $L^1(\mathcal{N})$ such that

- (a) $T_s = T_s' + T_s''$;
- (b) $T_s'(Y) = g(F_r + srX) \, T_s'(Y)$, $Y \in \mathcal{N}$;
- (c) $T_s''(Y) = T_s''(Y) \, g(F_r)$, $Y \in \mathcal{N}$;
- (d) $h(F_r + srX) - h(F_r) = T_s(srX)$;
- (e) $\frac{d}{dr} \left[h(F_r) \right] = T_0(\dot{F}_r)$;
- (f) $\tau(T_0(Y)Z) = \tau(Y T_0(Z))$, $Y, Z \in \mathcal{N}$;
- (g) $\lim_{s \to 0} \left\| T_s'(Y) - T_0'(Y) \right\|_1 = \lim_{s \to 0} \left\| T_s''(Y) - T_0''(Y) \right\|_1 = 0$, $Y \in L^1(\mathcal{N})$.

---

[2]The property (e) follows from the corresponding statement of Lemma 19 if one takes the group of $*$-automorphisms given by translations in $L^\infty[0, 1] \bar{\otimes} \mathcal{M}$, i.e.

$$\gamma_t(F_r) = F_{r+t}, \quad F_r \in \mathcal{N}, \ r, t \in [0, 1]$$

where the group $[0, 1]$ is equipped with summation modulo 1

Observe first that (a), (b) and (c) together with the fact that the mapping $s \mapsto g(\bar{F} + s\bar{X})$ is trace norm continuous, readily implies that $T_s, T_s', T_s'' \in B(\mathcal{N}, L^1(\mathcal{N}))$. In particular, the fact that the operator $T_0 \in B(\mathcal{N}, L^1(\mathcal{N}))$ and (e) guarantee that the mapping $r \mapsto \frac{d}{dr}[h(F_r)]$ is trace norm continuous and the right hand side of (6) is well-defined. Furthermore,

$$\frac{1}{s}(h(F_r + srX) - h(F_r)) \overset{(d)}{=} T_s(rX) =$$

$$\text{(a), (b) and (c)} \quad = g(F_r + srX)\, T_s'(rX) + T_s''(rX)\, g(F_r)$$

$$= (g(F_r + srX) - g(F_r))\, T_s'(rX) + g(F_r)\, T_s'(rX) + T_s''(rX)\, g(F_r).$$

Letting $s \to 0$ yields

$$\lim_{s \to 0} \frac{1}{s}(h(F_r + srX) - h(F_r)) \overset{(g)}{=} g(F_r)\, T_0'(rX) + T_0''(rX)\, g(F_r)$$

$$\text{(b) and (c)} \quad = T_0'(rX) + T_0''(rX) \overset{(a)}{=} T_0(rX),$$

where the limit converges in $L^1(\mathcal{N})$. Finally,

$$\lim_{s \to 0} \tau_1\left(\dot{F}_r \frac{1}{s}(h(F_r + srX) - h(F_r))\right)$$

$$= \tau_1\left(\dot{F}_r\, T_0(rX)\right) \overset{(f)}{=} \tau_1\left(X T_0(r\dot{F}_r)\right)$$

$$\overset{(e)}{=} \tau_1\left(X\, r\, \frac{d}{dr}[h(F_r)]\right).$$

Thus, (6) is proved. $\qquad\square$

Let us now proceed with the proof of Theorem 3.

Proof of Theorem 3. Note that similarly to the proof of Theorem 5, we may assume that $\operatorname{supp} h \subseteq (-\delta, \delta)$.

Let us show that without loss of generality, we may assume that the path $t \mapsto F_t$ is contained in $\mathcal{F}_*^{\pm 1}$. Indeed, consider a C^2-function θ such that $\operatorname{supp} \theta' \subseteq (-\delta, \delta)$ and $\theta(\pm\delta) = \pm 1$. Let us introduce the path $t \mapsto G_t = \theta(F_t)$ and the function $k(x) = K'(x)$, where K is such that $H(x) = K(\theta(x))$. Observe that $G_t \in \mathcal{F}_*^{\pm 1}$ and $\delta_{G_t} = 1$.

Let us verify that the path $t \mapsto G_t$ and the function $k(x)$ satisfies the assumptions of Theorem 3. (i) is clear from

$$1 = \int_{-\delta}^{\delta} h(x)\, dx = \int_{-\delta}^{\delta} k(\theta(x))\, \theta'(x)\, dx = \int_{-1}^{1} k(y)\, dy.$$

For (ii), note that, (a) since $\operatorname{supp} k$ is a proper subinterval of $(-1, 1)$, the function $t \mapsto k(G_t)$ is bounded with respect to the trace norm $\|\cdot\|_1$; (b) according to Remark 18, the path $t \mapsto G_t$ is C^1 with respect to the operator norm. Consequently, the quantity in the assumption (ii) is finite. (iii) is clear, since

$$K(G_j) = H(F_j) \quad \text{and} \quad \chi_{[0, +\infty)}(G_j) = \chi_{[0, +\infty)}(F_j), \quad j = 0, 1.$$

Thus, we see that the path $t \mapsto G_t$ and the function $k(x)$ satisfies the assumption of Theorem 3 and $G_t \in \mathcal{F}_*^{\pm 1}$.

On the other hand, we also have that

$$sf(F_t) = sf(G_t) \quad \text{and} \quad \tau\left(\dot{F}_t\, h(F_t)\right) = \tau\left(\dot{G}_t\, k(G_t)\right), \quad 0 \leqslant t \leqslant 1.$$

The former is clear from the definition of the spectral flow. For the latter, observe that there is a uniformly bounded family of continuous linear operators $\{T_t\}_{0 \leqslant t \leqslant 1}$

on $\mathcal{M}$ such that $\dot{G}_t = T_t(\dot{F}_t)$ (see Lemma 19.(e)) such that

$$\tau\left(\dot{G}_t k(G_t)\right) = \tau\left(T_t(\dot{F}_t)k(\theta(F_t))\right) = \tau\left(\dot{F}_t\theta'(F_t)k(\theta(F_t))\right) = \tau\left(\dot{F}_t h(F_t)\right),$$

where the second identity is due to Lemma 20[3].

Therefore, for the rest of the proof, we assume that the path $t \mapsto F_t$ is taken from $\mathcal{F}_*^{\pm 1}$ and $\operatorname{supp} h \subseteq (-1, 1)$.

For every $t \in [0, 1]$, let N_t be an open convex set given by

$$N_t = \{F \in \mathcal{M}, \ \|\pi(F - F_t)\| < \epsilon_t\} \subseteq N(F_t),$$

for some $0 < \epsilon_t < 1$ such that $\operatorname{supp} h \subseteq (-\delta_t, \delta_t)$, where $\delta_t = \min_{F \in N_t} \delta_F$ (one can find such N_t since the mapping $F \mapsto \delta_F$ is continuous with respect to the semi-norm $\|\pi(\cdot)\|$). The preimages of the family $\{N_t\}_{0 \leqslant t \leqslant 1}$ under the mapping $t \mapsto F_t$ produce an open covering of $[0, 1]$. Consequently, due to compactness, we can *finitely* partition the segment $[0, 1]$ by some points

$$0 = t_0 < t_1 < \ldots < t_n = 1$$

such that every segment $t \in [t_{k-1}, t_k] \mapsto F_t$ of the path $t \mapsto F_t$ lies within the open convex set $N_k = N_{t_k} \subseteq N(F_k)$ and $\operatorname{supp} h \subseteq (-\delta_k, \delta_k)$, where $\delta_k = \min_{F \in N_k} \delta_F$. Observe also that identity (3) (which we are proving) is additive with respect to partitioning of the path $t \mapsto F_t$. Thus, we need only to prove this identity for each segment $[t_{k-1}, t_k]$. Hence, from now on, we shall assume that the path $t \in [0, 1] \mapsto F_t$ lies entirely within the convex open set

$$N = \{F \in \mathcal{M}, \ \|\pi(F - F_0)\| < \epsilon\} \subseteq N(F)$$

for some $0 < \epsilon < 1$ and that $\operatorname{supp} h \subseteq (-\delta, \delta)$, where $\delta = \min_{F \in N} \delta_F$.

Let $B_j = 2\chi_{[0,+\infty)}(F_j) - 1, j = 0, 1$ be two involutions and let $t \in [0, 1] \mapsto B_t$ be the straight line path connecting B_0 and B_1 (i.e., $B_t = (1 - t)B_0 + tB_1$). Since $F_j \in \mathcal{F}_*^{\pm 1}$, by Lemma 2.(i), the difference $F_j - B_j$ is τ-compact and therefore $\pi(F_j - B_j) = 0$. The latter implies that the loop

$$
\begin{array}{ccc}
F_0 & \longrightarrow & F_1 \\
\uparrow & & \downarrow \\
B_0 & \longleftarrow & B_1
\end{array}
$$

lies within the set N, where the segment $B_0 \to F_0$ and $F_1 \to B_1$ are the straight line paths. Applying Theorem 5 for this loop implies that

$$\int_0^1 \tau\left(\dot{B}_t h(B_t)\right) \, dt = \int_0^1 \tau\left(\dot{F}_t h(F_t)\right) \, dt + \gamma_1 - \gamma_0,$$

where γ_j are the integrals along the straight line paths connecting F_j and B_j, i.e.,

$$\gamma_j = \int_0^1 \tau\left((F_j - B_j) \, h\left((1 - t)F_j + tB_j\right)\right) \, dt, \ j = 0, 1.$$

Let us show that

$$H(F_j) - \frac{1}{2}B_j = \int_0^1 (F_j - B_j) \, h\left((1 - t)F_j + t \, B_j\right) \, dt, \ j = 0, 1 \qquad (7)$$

Observe that every operator in (7) is a function of F_j. Moreover, the operators on both sides of this identity are supported on the projection $E_j = \chi_{\operatorname{supp} h}(F_j)$. Indeed,

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[3]applied to the function

$$\phi(\lambda, \mu) = k^{\frac{1}{2}}(\theta(\lambda)) \frac{\theta(\lambda) - \theta(\mu)}{\lambda - \mu} k^{\frac{1}{2}}(\theta(\mu)).$$

on the right hand side, the support is determined by the function h, and, on the left hand side, observe that the function

$$H - \frac{1}{2}\left(\chi_{[0,+\infty)} - \chi_{(-\infty,0)}\right)$$

vanishes outside of the support $\operatorname{supp} h$, which clearly implies that $H(F_j) - \frac{1}{2}B_j$ is supported on E_j. Thus, we may consider the identity on the algebra generated by the operator $E_j F_j$. Since the projection E_j is τ-finite, the latter algebra is $*$-isomorphic to a subalgebra $L^\infty(\mathbb{R}, d\sigma)$, where $\sigma(\Delta) = \tau\left(\chi_\Delta(F_j) E_j\right)$, $\Delta \subseteq \mathbb{R}$.

In the setting of the algebra $L^\infty(\mathbb{R}, d\sigma)$, identity (7) holds a.e. due to the Newton-Leibniz theorem and the integral converges with respect to the ultra-weak topology. This, in particular, implies that $H(F_j) - \frac{1}{2}B_j \in L^1(\mathbb{R}, d\sigma) \subseteq L^1(\mathcal{M})$. Taking trace τ from the latter identity gives

$$\gamma_j = \tau\left(H(F_j) - \frac{1}{2}B_j\right).$$

Observing that from the definition of the spectral flow, it follows that $\operatorname{sf}(F_t) = \operatorname{sf}(B_t)$, it is clear that to finish the proof we need only to show now that

$$\operatorname{sf}(B_t) = \int_0^1 \tau\left(\dot{B}_t\, h(B_t)\right)\, dt. \tag{8}$$

The argument establishing (8) is similar to [42, Proposition 4.3]. Observe, that the path B_t consists of invertible operators excepting the point $B_{\frac{1}{2}}$. Observe also that the operator $B_{\frac{1}{2}}^2$ commutes with the every B_t, $t \in [0,1]$. Let δ_1 be such that $0 < \delta_1 < \delta$ and $\operatorname{supp} h \subseteq [-\delta_1, \delta_1]$. Since $B_{\frac{1}{2}} \in N$, the projection $E = \chi_{[0,\delta_1^2]}\left(B_{\frac{1}{2}}^2\right)$ is τ-finite. Moreover, the projection E commutes with every B_t, $t \in [0,1]$. Let us decompose the path $t \mapsto B_t$ into the direct sum of two paths

$$t \mapsto EB_t E \quad \text{and} \quad t \mapsto (1-E)\, B_t\, (1-E). \tag{9}$$

Observe now that the second path in the latter decomposition consists of invertible operators (in the algebra $(1-E)\mathcal{M}(1-E)$) and therefore the spectral flow vanishes on this path (see [34, Remark 2.3]). On the other hand, due to the choice of δ_1 and the projection E, the spectrum of $(1-E)B_{\frac{1}{2}}(1-E)$ lies outside of the interval $[-\delta_1, \delta_1]$. Furthermore, it is easy to see the inequality $B_t^2 \geqslant B_{\frac{1}{2}}^2$, which means that the statement about the spectrum of $B_{\frac{1}{2}}$ above is equally valid for every $(1-E)\, B_t\, (1-E)$. Thus, we see that

$$h\left((1-E)B_t(1-E)\right) = 0, \quad t \in [0,1]$$

and therefore the integral in (8) also vanishes on the second path in the decomposition (9).

Identity (8) is additive with respect to direct sums. Consequently, we need to prove (8) only for the path $t \mapsto EB_t E$. Regarding the latter path as a path in a finite algebra $E\mathcal{M}E$, the identity follows from [6, § 5.1]. □

It now follows from the arguments in this Section that we have also proved the following result.

Corollary 7. *Let $\mathcal{F}_\delta$ be the set of self adjoint τ-Fredholm operators in N whose essential spectrum does not intersect the interval $[-\delta, \delta]$. This is an open submanifold of the Banach manifold of all self adjoint τ-Fredholm operators. For h as in Theorem 3 the one form θ on $\mathcal{F}_\delta$, given by defining for each $F \in \mathcal{F}_\delta$ the functional θ_F on the tangent space to $\mathcal{F}_\delta$ at F by $X \to \tau(Xh(F))$, is closed. Spectral flow along any piecewise C^1 path in $\mathcal{F}_\delta$ may be interpreted as being obtained by integrating this one form.*

4. SPECTRAL FLOW FORMULA, UNBOUNDED CASE.

We shall now discuss analytic formulae for paths of unbounded self adjoint linear operators. Across this Section, $t \in [0,1] \mapsto D_t$ stands for a path of unbounded self adjoint linear operators affiliated[4] with $\mathcal{M}$. In order to to be able to compute the spectral flow of this path we assume that the path $t \in [0,1] \mapsto F_t = \vartheta(D_t)$ is a continuous path of τ-Fredholm operators, where

$$\vartheta(x) = \frac{x}{(1+x^2)^{\frac{1}{2}}}. \tag{10}$$

In this case, by definition, we set $sf(D_t) = sf(F_t)$. Furthermore, to be able to consider analytic formulae for the spectral flow of the path D_t, we shall also impose a smoothness assumption onto D_t. Namely, the following definition is in order.

Definition 8.   (i) *A path* $t \in [0,1] \mapsto D_t$ *is called* Γ-*differentiable at the point* $t = t_0$ *if and only if there is a bounded linear operator* G *such that*

$$\lim_{t \to t_0} \left\| \frac{D_t - D_{t_0}}{t}(1 + D_{t_0}^2)^{-\frac{1}{2}} - G \right\| = 0.$$

In this case, we set $\dot{D}_{t_0} = G\,(1 + D_{t_0}^2)^{\frac{1}{2}}$. *The operator* $\dot{D}_t$ *is a symmetric linear operator with the domain* $\mathrm{dom}(D_t)$ *(see Lemma 25 below).*

   (ii) *If the mapping* $t \mapsto \dot{D}_t(1 + D_t^2)^{-\frac{1}{2}}$ *is defined and continuous with respect to the operator norm, then we call the path* $t \mapsto D_t$ *continuously* Γ-*differentiable or* C_Γ^1-*path*[5].

The main analytic spectral flow formula in the unbounded case is given by the following theorem.

Theorem 9. *Let* $t \in [0,1] \mapsto D_t$ *be a piecewise* C_Γ^1-*path of linear operators and* $\vartheta(D_t) \in \mathcal{F}_*^{\pm 1}$. *If* $g : \mathbb{R} \mapsto \mathbb{R}$ *is a positive* C^2-*function such that*

   (i) $\int_{-\infty}^{+\infty} g(x)\,dx = 1$;
   (ii) $\int_0^1 \left\| \dot{D}_t g(D_t) \right\|_1 dt < +\infty$;
   (iii) $G(F_1) - \frac{1}{2}B_1 - G(F_0) + \frac{1}{2}B_0 \in L^1(\mathcal{M})$, *where* B_j *are the phases of* D_j, $j = 0,1$, *i.e.,* $B_j = 2\chi_{[0,+\infty)}(D_j) - 1$, *and* G *is the antiderivative of* g *such that* $G(\pm\infty) = \pm\frac{1}{2}$;

then

$$sf(D_t) = \int_0^1 \tau\left(\dot{D}_t\, g(D_t) \right) dt + \tau\left(G(D_1) - \frac{1}{2}B_1 - G(D_0) + \frac{1}{2}B_0 \right).$$

Since the spectral flow for a path of unbounded linear operators is defined by the spectral flow of the corresponding path of bounded operators (via the mapping $D \mapsto \vartheta(D)$), the proof of the theorem above is based on a reduction to the "bounded" spectral flow formula given by Theorem 3. In this reduction the main part is the question whether the path $t \mapsto \vartheta(D_t)$ is C^1 in the operator norm provided the path $t \mapsto D_t$ is C_Γ^1. The latter question has been left open in [42] (see p. 21). We shall resolve this problem in Theorem 22 below.

Proof of Theorem 9. As in the proof of Theorem 3, we may assume that the function g is compactly supported (see footnote 1 on p. 9). Let h be a function such

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[4]Recall that a linear operator $D : \mathrm{dom}(D) \mapsto \mathcal{H}$ is called *affiliated with* a von Neumann algebra $\mathcal{M}$ if and only if $u(\mathrm{dom}(D)) \subseteq \mathrm{dom}(D)$ and $u^*Du = D$ for every $u \in \mathcal{M}'$.

[5]It may be shown that the class of all C_Γ^1-paths is the class of all paths which are continuously differentiable with respect to the graph norm of some fixed operator on this path (see [42]). We, however, will not use this connection below

that

$$g(x) = \frac{h(\vartheta(x))}{(1+x^2)^{\frac{3}{2}}}$$

and let $F_t = \vartheta(D_t)$. The function h is a C^2-function supported on a proper subinterval of $[-1,1]$ and $F_t \in \mathcal{F}_*^{\pm 1}$ by assumption.

Let us verify that the path $t \in [0,1] \mapsto F_t \in \mathcal{F}_*^{\pm 1}$ and the function h satisfies the hypothesis of Theorem 3.

(i) Due to Theorem 22, the mapping $t \mapsto F_t$ is piecewise C^1.

(ii) Observe that

$$\int_{-1}^{1} h(\vartheta)\, d\vartheta = \int_{-\infty}^{+\infty} h(\vartheta(x))\, \vartheta'(x)\, dx = \int_{-\infty}^{+\infty} g(x)\, dx = 1.$$

(iii) Since g is compactly supported, the function h is supported on a proper subinterval of $[-1,1]$ and therefore the mapping $t \mapsto h(F_t)$ is continuous in the trace norm (see Lemma 6). In particular,

$$\int_{0}^{1} \left\| \dot{F}_t h(F_t) \right\|_1 dt < +\infty.$$

Applying Theorem 3, we readily obtain that

$$\mathrm{sf}(D_t) = \int_{0}^{1} \tau\left(\dot{F}_t\, h(F_t) \right) dt + \tau\left(H(F_1) - H(F_0) - \frac{1}{2}B_1 + \frac{1}{2}B_0 \right).$$

Note that if $H(x)$ is the antiderivative of $h(x)$ such that $H(\pm 1) = \pm\frac{1}{2}$, then $H(\vartheta(x)) = G(x)$. Consequently,

$$H(F_j) = G(D_j), \quad j = 0, 1$$

and

$$\tau\left(\dot{F}_t\, h(F_t) \right) = \tau\left(\dot{D}_t \vartheta'(D_t)\, h(\vartheta(D_t)) \right) = \tau\left(\dot{D}_t\, g(D_t) \right), \quad t \in [0,1].$$

The theorem is proved. $\qquad\square$

Before we consider applications of Theorem 9, let us note that applying the argument of the proof above to Theorem 5, we obtain the answer to Singer's question in the form framed for elliptic operators on compact manifolds.

Theorem 10. *If D is a self-adjoint linear operator with τ-compact resolvent and g is a C^2-function, then the one form $\tau(Vg(D))$ is exact on the affine space of all D-bounded perturbations V such that $Vg(D) \in L^1(\mathcal{M})$. In other words, if $t \in [0,1] \mapsto D_t$ is a C_Γ^1-loop ($D_0 = D_1$) of unbounded self-adjoint linear operators with τ-compact resolvent such that*

$$\int_{0}^{1} \left\| \dot{D}_t g(D_t) \right\|_1 dt$$

is finite, then

$$\int_{0}^{1} \tau\left(\dot{D}_t\, g(D_t) \right) dt = 0.$$

By choosing specific functions g, Theorem 9 above allows a number of important corollaries. We shall state only two of them in Theorems 11 and 12 below. These theorems extend [42, Propositions 6.7 and 6.9] and earlier results of [10, 11].

Theorem 11. *If $t \in [0,1] \mapsto D_t$ is a piecewise C_Γ^1-path of unbounded linear operators such that D_t is θ-summable[6], $t \in [0,1]$ and*

$$\int_{0}^{1} \left\| \dot{D}_t e^{-\epsilon D_t^2} \right\|_1 dt < +\infty, \quad \epsilon > 0,$$

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[6] A self adjoint operator D is called θ-summable if and only if $e^{-\epsilon D^2} \in L^1(\mathcal{M})$ for every $\epsilon > 0$.

then

$$sf(D_0, D_1) = \sqrt{\frac{\epsilon}{\pi}} \int_0^1 \tau \left(\dot{D}_t \, e^{-\epsilon D_t^2} \right) \, dt$$

$$+ \tau \left(G(D_1) - \frac{1}{2} B_1 - G(D_0) + \frac{1}{2} B_0 \right), \quad \epsilon > 0,$$

where

$$G(x) = \sqrt{\frac{\epsilon}{\pi}} \int_{-\infty}^x e^{-\epsilon t^2} \, dt - \frac{1}{2}.$$

Proof. The proof is specialization of Theorem 9 to the case $g(x) = \sqrt{\frac{\epsilon}{\pi}} e^{-\epsilon x^2}$, provided we have checked that

$$\vartheta(D_t) \in \mathcal{F}_*^{\pm 1}, \; t \in [0, 1] \quad \text{and} \quad G(D_j) - \frac{1}{2} B_j \in L^1(\mathcal{M}), \; j = 0, 1. \tag{11}$$

For the first statement in (11), observe that

$$e^{-\epsilon n^2} \chi_{[-n,n]}(x) \leqslant e^{-\epsilon x^2}, \quad x \in \mathbb{R}, \; n \geqslant 1,$$

which means that every projection $\chi_{[-n,n]}(D_t)$ is τ-finite. Furthermore, note also that, under the mapping $x \mapsto \vartheta(x)$, compactly supported indicator functions are mapped onto indicators of proper subintervals of $[-1, 1]$. Thus, we see that if $F_t = \vartheta(D_t)$, then every projection $\chi(F_t)$ is τ-finite where χ is an indicator of a proper subinterval of $[-1, 1]$. Consequently, $F_t \in \mathcal{F}_*^{\pm 1}$.

For the second statement in (11), let us consider the function

$$f(x) = G(x) - \chi_{[0, +\infty)}(x) + \frac{1}{2}. \tag{12}$$

Clearly,

$$G(D_j) - \frac{1}{2} B_j = f(D_j).$$

Thus, the required assertion follows from the estimate

$$|f(x)| \leqslant e^{-\frac{\epsilon}{2} x^2} \sqrt{\frac{\epsilon}{\pi}} \int_{-\infty}^{-|x|} e^{-\frac{\epsilon}{2} t^2} \, dt$$

and the fact that D_t is θ-summable. $\qquad\square$

This last result should be compared with Corollary 8.11 of [11]

Theorem 12. *If $t \in [0, 1] \mapsto D_t$ is a piecewise C_r^1-path such that D_t is p-summable[7] for some $1 \leqslant p < \infty$ and*

$$\int_0^1 \left\| (1 + D_t^2)^{-\frac{p}{2}} \right\|_1 \, dt < +\infty,$$

then

$$sf(D_0, D_1) = \frac{1}{c_p} \int_0^1 \tau \left(\dot{D}_t \, (1 + D_t^2)^{-\frac{p}{2} - \frac{1}{2}} \right) \, dt$$

$$+ \tau \left(G(D_1) - \frac{1}{2} B_1 - G(D_0) + \frac{1}{2} B_0 \right),$$

where

$$c_p = \int_{-\infty}^{+\infty} (1 + x^2)^{-\frac{p}{2} - \frac{1}{2}} \, dx, \quad G(x) = \frac{1}{c_p} \int_{-\infty}^x (1 + t^2)^{-\frac{p}{2} - \frac{1}{2}} \, dt - \frac{1}{2}.$$

---

[7]A self adjoint operator D is called p-summable, for some $1 \leqslant p < \infty$ if and only if $(1 + D^2)^{-\frac{p}{2}} \in L^1(\mathcal{M})$

Proof. The proof consists of specialization of Theorem 9 (and justifying (11)) for the case $g(x) = \frac{1}{c_p}(1 + x^2)^{-\frac{p}{2} - \frac{1}{2}}$.

By the assumption, the mapping $t \mapsto \dot{D}_t (1 + D_t^2)^{-\frac{1}{2}}$ is operator norm continuous, hence

$$\int_0^1 \left\| \dot{D}_t \, (1 + D_t^2)^{-\frac{p}{2} - \frac{1}{2}} \right\|_1 \, dt < +\infty.$$

Now, for the first statement in (11) as in the proof of Theorem 11 it is sufficient to note the estimate

$$(1 + n^2)^{-\frac{p}{2} - \frac{1}{2}} \chi_{[-n,n]}(x) \leqslant (1 + x^2)^{-\frac{p}{2} - \frac{1}{2}}, \quad x \in \mathbb{R}, \ n \geqslant 1.$$

Consequently, every projection $\chi_{[-n,n]}(D_t)$ is τ-finite and the argument is repeated verbatim.

For the second statement in (11), we shall estimate the function $f(x)$ given in (12) as follows

$$|f(x)| = \frac{1}{c_p} \int_{-\infty}^{-|x|} (1 + t^2)^{-\frac{p}{2} - \frac{1}{2}} \, dt \leqslant \frac{1}{c_p} \int_{-\infty}^{-|x|} |t|^{-p-1} \, dt$$

$$= \frac{1}{p \, c_p} \, |x|^{-p} \leqslant \frac{2^{\frac{p}{2}}}{p \, c_p} \, (1 + x^2)^{-\frac{p}{2}}, \quad |x| \geqslant 1.$$

$\square$

A customary assumption in non-commutative geometry (see [5, 10, 11]) is that the path $t \in [0, 1] \mapsto D_t$ is C^1 with respect to the operator norm (which is a stronger assumption than the C_Γ^1 assumption). Under this assumption, the statement of Theorem 11 remains exactly the same (except the symbol $\dot{D}_t$ now stands for the ordinary Gâteaux derivative). On the other hand, when $t \mapsto D_t$ is a C^1-path in the operator norm, Theorem 12 changes to Theorem 13 below. In the latter theorem, we no longer need the additional resolvent factor under the trace in the spectral flow formula to guarantee summability. The observations above, regarding piecewise C^1-paths $t \mapsto D_t$, cover the spectral flow formulae proved in [5, 10]. Observe also that, in the latter case, the p-summability assumption is no longer sufficient to guarantee that the end points satisfy the boundary assumptions of Theorem 9 and therefore we have to require this explicitly in Theorem 13 below.

Theorem 13. *If $t \in [0, 1] \mapsto D_t$ is a piecewise C^1-path (with respect to the operator norm) such that D_t is p-summable for some $1 \leqslant p < \infty$,*

$$\int_0^1 \left\| (1 + D_t^2)^{-\frac{p}{2}} \right\|_1 \, dt < +\infty$$

and

$$G(D_1) - \frac{1}{2}B_1 - G(D_0) + \frac{1}{2}B_0 \in L^1(\mathcal{M}),$$

then

$$sf(D_0, D_1) = \frac{1}{c_p} \int_0^1 \tau \left(\dot{D}_t \, (1 + D_t^2)^{-\frac{p}{2}} \right) \, dt$$

$$+ \tau \left(G(D_1) - \frac{1}{2}B_1 - G(D_0) + \frac{1}{2}B_0 \right),$$

where

$$c_p = \int_{-\infty}^{+\infty} (1 + x^2)^{-\frac{p}{2}} \, dx, \quad G(x) = \frac{1}{c_p} \int_{-\infty}^{x} (1 + t^2)^{-\frac{p}{2}} \, dt - \frac{1}{2}.$$

Compare this result with Theorem 9.3 of [11].

5. DOUBLE OPERATOR INTEGRALS.

In this section, we shall briefly outline the theory of double operator integrals (DOI) , developed recently in [19–21, 36, 37]. This theory unifies several different approaches of harmonic analysis to smoothness properties of operator functions. In the present section, we shall mostly present the results (proved somewhere else) needed to complete the proof of Theorem 5 (see properties (a)–(g)) and those needed in Section 6.

The theory of double operator integrals is a method of giving an integral representation of the difference $f(A) - f(B)$ where f is a bounded Borel function and A and B are self adjoint. In the case when A, B are $n \times n$ matrices with spectral representation $A = \sum_{j=1}^{n} \lambda_j E_j$, $B = \sum_{j=1}^{n} \mu_k F_k$ (here E_j and F_k denote spectral projections) this integral representation is obtained from the following elementary computation

$$f(A) - f(B) = \sum_{j,k=1}^{n} (f(\lambda_j) - f(\mu_k)) E_j F_k = \sum_{j,k=1}^{n} \frac{f(\lambda_j) - f(\mu_k)}{\lambda_j - \mu_k} E_j(A - B) F_k.$$

In other words, we have just represented the difference $f(A) - f(B)$ as the Stieltjes double operator integral $\int \int \frac{f(\lambda) - f(\mu)}{\lambda - \mu} E_j(A - B) F_k$. Notice that we are making use of the bimodule property of the $n \times n$ matrices. An exposition of an early version of DOI which may assist the reader may be found in [38].

It is precisely the generalisation of this perturbation formula to infinite dimensional analogues that constitutes the essence of the double operator integration theory initiated by Daletskii and Krein and developed by Birman and Solomyak for type I factors, and further extended to semifinite von Neumann algebras in [17,18,16] and [31,32,34] to which we refer for additional historical information and references.

Let $\mathcal{M}$ be a semi-finite von Neumann algebra and let τ be a n.s.f. trace. The symbol $\mathcal{E}$ stands for a non-commutative fully[8] symmetric ideal associated with the couple $(\mathcal{M}, \tau)$ (see [15, 22]). In particular, $\mathcal{L}^p$, $1 \leqslant p \leqslant \infty$ stands for the non-commutative L^p-Schatten ideal. Furthermore, the symbol $\mathcal{E}^\times$ stands for the Köthe dual $\mathcal{E}^\times$ of a symmetric ideal $\mathcal{E}$ (see, [23]). In particular, if $\mathcal{E} = \mathcal{L}^p$, $1 \leqslant p \leqslant \infty$, then $\mathcal{E}^\times = \mathcal{L}^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.

We shall let D_0, D_1 denote self adjoint unbounded operators affiliated with $\mathcal{M}$. Let dE_λ^0, dE_μ^1 be the corresponding spectral measures. Recall that for every $K_1, K_2 \in \mathcal{L}^2$

$$\tau(K_1 \, dE_\lambda^0 \, K_2 \, dE_\mu^1), \ \ \lambda, \mu \in \mathbb{R}$$

is a σ-additive complex-valued measure on the plane $\mathbb{R}^2$ with the total variation bounded by $\|K_1\|_2 \|K_2\|_2$, see [21, Remark 3.1].

Let $\phi = \phi(\lambda, \mu)$ be a bounded Borel function on $\mathbb{R}^2$. We call the function $\phi \, dE^0 \otimes dE^1$-*integrable* in the symmetric ideal $\mathcal{E}$, if and only if there is a linear operator $T_\phi = T_\phi(D_0, D_1) \in B(\mathcal{E})$ such that

$$\tau(K_1 T_\phi(K_2)) = \int_{\mathbb{R}^2} \phi(\lambda, \mu) \, \tau(K_1 \, dE_\lambda^0 \, K_2 \, dE_\mu^1), \tag{13}$$

for every

$$K_1 \in \mathcal{L}^2 \cap \mathcal{E}^\times \ \text{ and } \ K_2 \in \mathcal{L}^2 \cap \mathcal{E}.$$

---

[8]We shall omit the word "fully" in the sequel.

If the operator $T_\phi(D_0, D_1)$ exists, then it is unique, [21, Definition 2.9]. The latter definition is in fact a special case of [21, Definition 2.9]. See also [21, Proposition 2.12] and the discussion there on pages 81–82. The operator T_ϕ is called *the Double Operator Integral*.

We shall write $\phi \in \Phi(\mathcal{E})$ if and only if the function ϕ is $dE^0 \otimes dE^1$-integrable in the symmetric ideal $\mathcal{E}$ for any measures dE^0 and dE^1.

Theorem 14 ([20, 21]). *Let* D_0, D_1 *be unbounded self adjoint operators affiliated to* $\mathcal{M}$. *The mapping*

$$\phi \mapsto T_\phi = T_\phi(D_0, D_1) \in B(\mathcal{E}), \quad \phi \in \Phi(\mathcal{E})$$

satisfies $T_{\phi^*} = T_\phi^*$ *and* $T_{\phi\psi} = T_\phi T_\psi$. *Moreover, if* $\alpha, \beta : \mathbb{R} \mapsto \mathbb{C}$ *are bounded Borel functions and if* $\phi(\lambda, \mu) = \alpha(\lambda)$ *(resp.* $\phi(\lambda, \mu) = \beta(\mu)$), $\lambda, \mu \in \mathbb{R}$, *then*

$$T_\phi(K) = \alpha(D_0)\, K \quad (\text{resp.}\ T_\phi(K) = K\, \beta(D_1)), \quad K \in \mathcal{E}.$$

The latter result allows the construction of a sufficiently large class of functions in $\Phi(\mathcal{E})$. Indeed, let us consider the class $\mathfrak{A}_0$ which consists of all bounded Borel functions $\phi(\lambda, \mu)$, $\lambda, \mu \in \mathbb{R}$ admitting the representation

$$\phi(\lambda, \mu) = \int_S \alpha_s(\lambda)\, \beta_s(\mu)\, d\nu(s) \tag{14}$$

such that

$$\int_S \|\alpha_s\|_\infty \|\beta_s\|_\infty\, d\nu(s) < \infty,$$

where $(S, d\nu)$ is a measure space, $\alpha_s, \beta_s : \mathbb{R} \mapsto \mathbb{C}$ are bounded Borel functions, for every $s \in S$ and $\|\cdot\|_\infty$ is the operator norm. The space $\mathfrak{A}_0$ is endowed with the norm

$$\|\phi\|_{\mathfrak{A}_0} := \inf \int_S \|\alpha_s\|_\infty \|\beta_s\|_\infty\, d\nu(s),$$

where the minimum runs over all possible representations (14). The space $\mathfrak{A}_0$ together with the norm $\|\cdot\|_{\mathfrak{A}_0}$ is a Banach algebra, see [20] for details. The subspace of $\mathfrak{A}_0$ of all functions ϕ admitting representation (14) with continuous functions α_s and β_s is denoted by $\mathfrak{C}_0$. The following result is a straightforward corollary of Theorem 14.

Corollary 15 ([20, Proposition 4.7]). *Every* $\phi \in \mathfrak{A}_0$ *is* $dE^0 \otimes dE^1$-*integrable in the symmetric ideal* $\mathcal{E}$ *for any measures* dE^0, dE^1, *i.e.* $\mathfrak{A}_0 \subseteq \Phi(\mathcal{E})$. *Moreover, if* $T_\phi = T_\phi(D_0, D_1)$, *for some self adjoint operators* D_0, D_1, *affiliated with* $\mathcal{M}$, *then*

$$\|T_\phi\|_{B(\mathcal{E})} \leqslant \|\phi\|_{\mathfrak{A}_0},$$

for every $\phi \in \mathfrak{A}_0$.

The major benefit delivered by the double operator integral theory is the observation that, if D is a self adjoint linear operator affiliated with $\mathcal{M}$ and A is a self adjoint perturbation from $\mathcal{E}$, then the perturbation of the operator function $f(D)$ (where $f : \mathbb{R} \mapsto \mathbb{C}$) is given by a double operator integral. Namely,

$$f(D + A) - f(D) = T_{\psi_f}(A),$$

where $T_{\psi_f} := T_{\psi_f}(D + A, D) \in B(\mathcal{E})$ and

$$\psi_f(\lambda, \mu) = \frac{f(\lambda) - f(\mu)}{\lambda - \mu}, \ \lambda \neq \mu, \ \psi(\lambda, \lambda) = f'(\lambda). \tag{15}$$

The identity above is proved in Theorem 17 below. The proof is based on the following lemma which is a slight generalization of [21, Lemma 7.1]. The proof of the lemma is a repetition of that of [21, Lemma 7.1] and therefore is omitted.

Lemma 16. *Let* D_j *be self adjoint linear operators affiliated with* $\mathcal{M}$ *with corresponding spectral measures* $E_n^j = E^{D_j}[-n, n]$, $j = 0, 1$, $n = 1, 2, \ldots$. *If*

$$\phi(\lambda_1, \lambda_0) = \frac{\beta_1(\lambda_1)\,\beta_0(\lambda_0)}{\alpha_1(\lambda_1)\,\alpha_0(\lambda_0)}\,\psi_f(\lambda_1, \lambda_0) \in \Phi(\mathcal{E}), \tag{16}$$

where f *is a Borel function and* α_j, β_j *are bounded Borel functions, then, for every* $K \in \mathcal{E}$,

$$E_n^1\beta_1(D_1)\,[f(D_1)\,K - K\,f(D_0)]\,\beta_0(D_0)\,E_n^0$$
$$= T_\phi\left(E_n^1\alpha_1(D_1)\,[D_1\,K - K\,D_0]\,\alpha_0(D_0)\,E_n^0\right).$$

Theorem 17. *Let* f *be a Borel function and let* α_j *and* β_j, $j = 0, 1$ *be bounded Borel functions. Let* D_j, $j = 0, 1$ *be self adjoint linear operators affiliated with* $\mathcal{M}$ *and let* $A \in \mathcal{M}$ *be such that*

$$B = \alpha_1(D_1)\,[D_1\,A - A\,D_0]\,\alpha_0(D_0) \in \mathcal{M}.$$

If $\phi \in \Phi(\mathcal{M})$, *where the function* ϕ *is given in (16), then*

$$C = \beta_1(D_1)\,[f(D_1)\,A - A\,f(D_0)]\,\beta_0(D_0) \in \mathcal{M}$$

and

$$C = T_\phi(B).$$

Proof. To be able to define the operator C, first we have to ensure that

$$A\beta_0(D_0)[\mathrm{dom}(D_0)] \subseteq \mathrm{dom}(f(D_1)\,\beta_1(D_1)). \tag{17}$$

To this end, we shall consider the bilinear form

$$q(\xi, \eta) = \langle A\beta_0(D_0)\,\xi, \bar{\beta}_1(D_1)\,\bar{f}(D_1)\,\eta\rangle$$
$$- \langle \beta_1(D_1)\,A\beta_0(D_0)\,f(D_0)\,\xi, \eta\rangle, \quad \xi \in \mathrm{dom}(D_0), \eta \in \mathrm{dom}(D_1).$$

Consider the spectral projections $E_n^j = E^{D_j}[-n, n]$. Let

$$B_n = E_n^1\,\alpha_1(D_1)\,[D_1\,A - A\,D_0]\,\alpha_0(D_0)\,E_n^0$$

and

$$C_n = E_n^1\,\beta_1(D_1)\,[f(D_1)\,A - A\,f(D_0)]\,\beta_0(D_0)\,E_n^0.$$

We obtain from Lemma 16 that

$$C_n = T_\phi(B_n).$$

Since the set of operators $\{B_n\}_{n \geqslant 1}$ is uniformly bounded and $\phi \in \Phi(\mathcal{M})$, this implies that the operators C_n are also uniformly bounded. Furthermore, we see that

$$\lim_{n \to \infty} \langle C_n\xi, \eta\rangle = q(\xi, \eta), \quad \xi \in \mathrm{dom}(D_1), \eta \in \mathrm{dom}(D_0).$$

Consequently, the form $q(\xi, \eta)$ is bounded and therefore we have (17). Thus, the operator C is properly defined and bounded. Moreover, since $\mathrm{dom}(D_j)$, $j = 0, 1$ are dense in $\mathcal{H}$, we also have that

$$wo - \lim_{n \to \infty} C_n = C.$$

Observe that we also have that

$$wo - \lim_{n \to \infty} B_n = B.$$

Since the operator T_ϕ is continuous with respect to the weak operator topology (see [36, Lemma 2.4 and the proof of Proposition 2.6]), we finally obtain that

$$T_\phi(B) = C.$$

$\square$

Remark 18. It is clearly seen from Theorem 17 that, if $\psi_f \in \Phi(\mathcal{M})$, then the function f maps (uniformly) operator norm continuous paths into themselves. On the other hand, we know from [37, Theorem 4] that, for every function $f : \mathbb{R} \mapsto \mathbb{C}$ such that

$$\|f\|_{\Lambda_\theta} + \|f'\|_{\Lambda_\epsilon} < +\infty, \ 0 \leqslant \theta < 1, \ 0 < \epsilon \leqslant 1,$$

we have $\psi_f \in \mathfrak{C}_0 \subseteq \Phi(\mathcal{M})$. Here Λ_θ is the semi-norm on functions on $\mathbb{R}$ given by

$$\|f\|_{\Lambda_\theta} = \sup_{x_1, x_2} \frac{|f(x_1) - f(x_2)|}{|x_1 - x_2|}.$$

Thus, in particular, every C^2-function maps operator norm (respectively, trace norm) continuous paths into operator norm (respectively, trace norm) continuous paths.

Finally, we complete the proof of Theorems 5 and 3 by establishing the following lemmas.

Lemma 19. *Let* g, h *be compactly supported* C^2-*functions on* $\mathbb{R}$ *such that* $g(x) = 1$ *for every* x *from some neighbourhood of* supp h *and let* $r \mapsto \gamma_r$ *be a weakly continuous group of* τ-*invariant* $\ast$-*isomorphisms on* $\mathcal{M}$ *with the generator* $\delta : \mathrm{dom}(\delta) \mapsto \mathcal{M}, \mathrm{dom}(\delta) \subseteq \mathcal{M}$. *If* $F, X \in \mathcal{M}$ *are self adjoint, then there are families of linear operators* $\{T_s\}$, $\{T'_s\}$ *and* $\{T''_s\}$ *uniformly bounded on* $\mathcal{M}$ *and on* $L^1(\mathcal{M})$ *such that*

(a) $T_s = T'_s + T''_s$;
(b) $T'_s(Y) = g(F + srX) T'_s(Y), Y \in \mathcal{M}$;
(c) $T''_s(Y) = T''_s(Y) g(F), Y \in \mathcal{M}$;
(d) $h(F + sX) - h(F) = T_s(sX)$;
(e) *if* $F \in \mathrm{dom}(\delta)$ *and* $\lim_{r \to 0} \|\gamma_r(F) - F\| = 0$, *then* $h(F) \in \mathrm{dom}(\delta)$ *and* $\delta(h(F)) = T_0(\delta(F))$;
(f) $\tau(T_0(Y)Z) = \tau(YT_0(Z)), Y, Z \in \mathcal{M}$;
(g) $\lim_{s \to 0} \|T'_s(Y) - T'_0(Y)\|_1 = \lim_{s \to 0} \|T''_s(Y) - T''_0(Y)\|_1 = 0, Y \in L^1(\mathcal{M})$.

Proof. We set $T_s = T_{\psi_h}(F + sX, F)$. It follows from [20, Corollary 7.6] (see also [37, Theorem 4]) that $\psi_h \in \mathfrak{C}_0$. Consequently, we readily see that (d) follows from [21, Corollary 7.2] (or Theorem 17); and (f) — from [36, Lemma 2.4].

Let g_1 be a compactly supported C^2-function such that $g_1(x) = 1$ when $x \in$ supp h and $g(x) = 1$ when $x \in \mathrm{supp}\, g_1$. We set

$$\psi_1(\lambda, \mu) = g_1(\lambda) \psi_h(\lambda, \mu) \ \text{ and } \ \psi_2 = \psi_h - \psi_1.$$

We also set $T'_s = T_{\psi_1}(F + sX, F)$ and $T''_s = T_{\psi_2}(F + sX, F)$. We instantly have (a).

Note that $\psi_1, \psi_2 \in \mathfrak{C}_0$. Consequently, (g) follows from [20, Lemma 5.14].

We readily have from the construction that

$$\psi_1(\lambda, \mu) = g(\lambda) \psi_1(\lambda, \mu). \tag{18}$$

Furthermore, it may be observed that we also have

$$\psi_2(\lambda, \mu) = \psi_2(\lambda, \mu) g(\mu). \tag{19}$$

Indeed, since the function h is compactly supported, the function ψ_h is supported in the cross $S_1 \cup S_2 \subseteq \mathbb{R} \times \mathbb{R}$, where the strips $S_j, j = 1, 2$ are given by

$$S_1 = \{(\lambda, \mu) : \ \lambda \in \mathrm{supp}\, h\}, \ S_2 \{(\lambda, \mu) : \ \mu \in \mathrm{supp}\, h\}.$$

By construction the function ψ_1 coincides with ψ_h on the strip S_1, i.e.

$$\psi_1(\lambda, \mu) = \psi_h(\lambda, \mu), \ (\lambda, \mu) \in S_1.$$

Consequently, the function $\psi_2 = \psi_h - \psi_1$ is supported within S_2 which justifies (19).

Clearly, (b) and (c) follows from Theorem 14 and (18) and (19).

Let us show (e) (we refer the reader to [19, 35] for a more complete study of the connection between double operator integrals and domains of derivations). By definition

$$F \in \mathrm{dom}(\delta) \iff \lim_{t \to 0} \tau\left(\frac{\gamma_t(F) - F}{t} Y\right) = \tau\left(\delta(F) Y\right), \quad Y \in L^1(\mathcal{M}). \tag{20}$$

From [21, Corollary 7.2], we obtain that

$$h(\gamma_t(F)) - h(F) = S_t(\gamma_t(F) - F),$$

where $S_t = T_{\psi_h}(\gamma_t(F), F)$. Observe also that the family $\{S_t\}$ is different from $\{T_s\}$. However, $S_0 = T_0$. Now,

$$\tau\left(Y \left[\frac{\gamma_t(h(F)) - h(F)}{t} - S_0(\delta(F))\right]\right)$$
$$= \tau\left(Y S_0 \left[\frac{\gamma_t(F) - F}{t} - \delta(F)\right]\right) + \tau\left(Y \left[(S_t - S_0) \frac{\gamma_t(F) - F}{t}\right]\right)$$
$$= \tau\left(S_0^*(Y) \left[\frac{\gamma_t(F) - F}{t} - \delta(F)\right]\right) + \tau\left((S_t^* - S_0^*)(Y) \frac{\gamma_t(F) - F}{t}\right).$$

Letting $t \to 0$, we see that the first term vanishes due to the fact that S_0^* is bounded on $L^1(\mathcal{M})$ (see [36, Lemma 2.4]) and $F \in \mathrm{dom}(\delta)$. Noting that the dual[9] family $\{S_t^*\}$ is a family of double operator integrals bounded on $L^1(\mathcal{M})$ (see [36, Lemma 2.4]) and the family

$$\left\{\frac{\gamma_t(F) - F}{t}\right\}$$

is uniformly bounded with respect to the operator norm (see (20)), the second term vanishes due to [20, Lemma 5.14]. Thus, according to (20) $h(F) \in \mathrm{dom}(\delta)$ and $\delta(h(F)) = T_0(\delta(F))$. $\qquad\square$

Lemma 20. *Let* $\phi \in \mathfrak{C}_0$ *such that* $\phi(\lambda, \mu) = \phi(\mu, \lambda)$, $\lambda, \mu \in \mathbb{R}$ *and such that* ϕ *is supported in a square* $I \times I$, *where* I *is an interval. Let* D *be a linear self adjoint operator affiliated with* $\mathcal{M}$ *and let* $T = T_\phi(D, D)$ *be a double operator integral (see [20]) which is a bounded linear operator on* $\mathcal{M}$. *If* $E = \chi_I(D)$ *is* τ-*finite, then, for every* $V \in \mathcal{M}$,

$$\tau\left(T(V)\right) = \tau\left(f(D) V\right), \quad \text{where } f(\lambda) = \phi(\lambda, \lambda). \tag{21}$$

Proof. Observe first that

$$T(1) = f(D). \tag{22}$$

Indeed, if $\phi \in \mathfrak{C}_0$, then there is a measure space (S, ν) and continuous functions α_s and β_s, $s \in S$, such that

$$\phi(\lambda, \mu) = \int_S \alpha_s(\lambda) \beta_s(\mu) \, d\nu(s)$$

and

$$T(A) = \int_S \alpha_s(D) A \beta_s(D) \, d\nu(s), \quad A \in \mathcal{M},$$

where

$$\int_S \|\alpha_s\|_\infty \|\beta_s\|_\infty \, d\nu(s) < +\infty.$$

Consequently, (22) follows from

$$T(1) = \int_S \alpha_s(D) \, 1 \, \beta_s(D) \, d\nu(s) = f(D).$$

---

[9]Here, we consider dual operator S_t^* restricted on $L^1(\mathcal{M}) \subseteq \mathcal{M}^*$, see details in [36].

Since the operator T is self-dual (see [36, Lemma 2.4]), we obtain that

$$\tau\left(f(D)\,V\right) = \tau\left(f(D)\,EV\right) = \tau\left(T(1)\,EV\right) = \tau\left(T(EV)\right) = \tau\left(T(V)\right),$$

where the last identity follows from Theorem 14. □

6. PATHS OF SELF ADJOINT LINEAR OPERATORS SMOOTH IN GRAPH NORM.

As we observed in Section 4, analytic spectral flow formulae for paths of unbounded self adjoint linear operators are deduced from corresponding formulae for paths of bounded Fredholm operators via the mapping $D \mapsto \vartheta(D)$, where the function ϑ is given in (10). Consequently, the question of smoothness properties of this mapping become of significant importance. This question has been studied deeply in [10, 11, 14, 37, 40, 42].

The main result of the present Section is that the function $x \mapsto \vartheta(x)$ maps C_Γ^1-paths onto C^1-paths with respect to the operator norm (see Theorem 22). This answers the question asked in [42, p. 21]. The proof is based on the following observation, which is a development of the technique presented in [37]. For every pair of self adjoint operators D_j, $j = 0, 1$, such that $D_1 - D_0 \in B(\mathcal{H})$ there is a linear[10] operator T on $B(\mathcal{H})$ such that

$$F_1 - F_0 = T\left[(D_1 - D_0)\,(1 + D_0^2)^{-\frac{1}{2}}\right], \tag{23}$$

where the operators F_j are given[11] by $F_j = \vartheta(D_j)$, $j = 0, 1$. In the present section, we shall further develop the above construction under the weaker assumption that $\mathrm{dom}(D_0) \subseteq \mathrm{dom}(D_1)$ and the operator

$$(D_1 - D_0)\,(1 + D_0^2)^{-\frac{1}{2}}$$

is bounded which is equivalent to the operator $D_1 - D_0$ being bounded with respect to the graph norm of the operator D_0 (see Lemma 23 below).

Now let us state the two major results of the present Section.

Theorem 21. *If $\{D_t\}$ is a collection of self adjoint operators Γ-differentiable (see Definition 8) at the point $t = 0$, then the collection $\{F_t\}$, $F_t = \vartheta(D_t)$ is differentiable with respect to the operator norm at the point $t = 0$.*

Theorem 22. *If $t \in [0, 1] \mapsto D_t$ is a C_Γ^1-path (see Definition 8) of self adjoint linear operators, then $t \in [0, 1] \mapsto \vartheta(D_t)$ is a C^1-path with respect to the operator norm, where the function ϑ is given by (10).*

6.1. **D-bounded operators.** Let $\mathcal{H}$ be Hilbert space and let $D : \mathrm{dom}(D) \mapsto \mathcal{H}$ be a linear operator with the domain $\mathrm{dom}(D) \subseteq \mathcal{H}$. A linear operator $A : \mathrm{dom}(A) \mapsto \mathcal{H}$ ($\mathrm{dom}(A) \subseteq \mathcal{H}$) is called D-bounded if and only if $\mathrm{dom}(D) \subseteq \mathrm{dom}(A)$ and there is a constant $c > 0$ such that

$$\|A(\xi)\|_{\mathcal{H}} \leqslant c \left(\|\xi\|_{\mathcal{H}}^2 + \|D(\xi)\|_{\mathcal{H}}^2\right)^{\frac{1}{2}}, \quad \xi \in \mathrm{dom}(D). \tag{24}$$

We let $\|A\|_D$ be the smallest possible constant $c > 0$ such that (24) holds.

---

[10]In the present section, we shall consider only the operator norm. Hence, we do not need an abstract von Neumann algebra. Instead, $B(\mathcal{H})$ will suffice.

[11]When the operator $D_1 - D_0$ is bounded and the resolvent of D_0 is $\mathcal{E}$-summable (i.e., $\left(1 + D_0^2\right)^{-\frac{1}{2}} \in \mathcal{E}$), then, since the operator T is bounded on the space $\mathcal{E}$, the identity (23) yields that

$$\|F_1 - F_0\|_{\mathcal{E}} \leqslant c \, \|D_1 - D_0\| \, \left\|(1 + D_0^2)^{-\frac{1}{2}}\right\|_{\mathcal{E}}.$$

The latter is proved in [37, Theorem 17] for an arbitrary symmetric ideal $\mathcal{E}$.

Observe that if an operator A is D_0-bounded and $\|A\|_{D_0} < 1$, then the operator A is also D-bounded, where $D = A + D_0$. Indeed, suppose that $0 < c < 1$ is the constant such that (24) holds. It then follows that

$$\|A(\xi)\|_{\mathcal{H}} \leqslant c \left(\|\xi\|_{\mathcal{H}} + \|D_0(\xi)\|_{\mathcal{H}}\right) \leqslant c \left(\|\xi\|_{\mathcal{H}} + \|D(\xi)\|_{\mathcal{H}} + \|A(\xi)\|_{\mathcal{H}}\right).$$

This implies that

$$\|A(\xi)\|_{\mathcal{H}} \leqslant \frac{c}{1-c} \left(\|\xi\|_{\mathcal{H}} + \|D(\xi)\|_{\mathcal{H}}\right) \leqslant \frac{c\sqrt{2}}{1-c} \left(\|\xi\|_{\mathcal{H}}^2 + \|D(\xi)\|_{\mathcal{H}}^2\right)^{\frac{1}{2}}.$$

In other words,

$$\|A\|_D \leqslant \frac{\sqrt{2}\,\|A\|_{D_0}}{1 - \|A\|_{D_0}}. \tag{25}$$

Furthermore, if $D_j : \operatorname{dom}(D_j) \mapsto \mathcal{H}$, $j = 0,1$ and $A : \operatorname{dom}(A) \mapsto \mathcal{H}$ are linear operators and $\|D_1 - D_0\|_{D_0} < 1$, then A is D_0-bounded if and only if A is D_1-bounded. Indeed, firstly since $D_1 - D_0$ is D_0-bounded, we see that $\operatorname{dom}(D_0) = \operatorname{dom}(D_1)$. Secondly, according to (25), $D_1 - D_0$ is also D_1-bounded. Finally, if c is the D_0-norm of A and c' is the D_1-norm of $D_1 - D_0$, then

$$\begin{aligned}
\|A(\xi)\|_{\mathcal{H}} &\leqslant c \left(\|\xi\|_{\mathcal{H}} + \|D_0(\xi)\|_{\mathcal{H}}\right) \\
&\leqslant c \left(\|\xi\|_{\mathcal{H}} + \|D_1(\xi)\|_{\mathcal{H}} + \|(D_1 - D_0)(\xi)\|_{\mathcal{H}}\right) \\
&\leqslant c \left(\|\xi\|_{\mathcal{H}} + \|D_1(\xi)\|_{\mathcal{H}} + c' \left(\|\xi\|_{\mathcal{H}} + \|D_1(\xi)\|_{\mathcal{H}}\right)\right) \\
&\leqslant \sqrt{2}\,c\,(1 + c') \left(\|\xi\|_{\mathcal{H}}^2 + \|D_1(\xi)\|_{\mathcal{H}}^2\right)^{\frac{1}{2}}, \quad \xi \in \operatorname{dom}(D_0).
\end{aligned}$$

Thus, if A is D_0-bounded then A is D_1-bounded. The opposite implication is similar. In other words, we proved that

$$c_1 \|A\|_{D_0} \leqslant \|A\|_{D_1} \leqslant c_2 \|A\|_{D_0}, \tag{26}$$

where c_1 and c_2 are positive constants depending on $\|D_1 - D_0\|_{D_0} < 1$.

Lemma 23. *Let* $D : \operatorname{dom}(D) \mapsto \mathcal{H}$ *be a self adjoint linear operator and let* $A : \operatorname{dom}(A) \mapsto \mathcal{H}$ *be a linear operator such that* $\operatorname{dom}(D) \subseteq \operatorname{dom}(A)$. *The following are equivalent:*

(i) A *is* D-*bounded;*
(ii) $A\,(i + D)^{-1}$ *and* $A\,(-i + D)^{-1}$ *are bounded;*
(iii) $A\,(1 + D^2)^{-\frac{1}{2}}$ *is bounded.*

Proof. From (i) to (ii). If $c > 0$ is a constant such that (24) holds, then, since

$$(\pm i + D)^{-1}(\mathcal{H}) \subseteq \operatorname{dom}(D),$$

we have that

$$\begin{aligned}
\left\|A\,(\pm i + D)^{-1}(\xi)\right\|_{\mathcal{H}} & \\
&\leqslant c \left(\left\|(\pm i + D)^{-1}(\xi)\right\|_{\mathcal{H}}^2 + \left\|D\,(\pm i + D)^{-1}(\xi)\right\|_{\mathcal{H}}^2\right)^{\frac{1}{2}} \\
&\leqslant c\,\sqrt{2}\,\|\xi\|_{\mathcal{H}}, \quad \xi \in \mathcal{H}.
\end{aligned}$$

This means that the operator $A\,(\pm i + D)^{-1}$ is bounded.

From (ii) to (i). Let $c = \left\|A\,(\pm i + D)^{-1}\right\| < +\infty$. We instantly obtain that

$$\begin{aligned}
\|A(\xi)\|_{\mathcal{H}} &= \left\|A\,(\pm i + D)^{-1}(\pm i + D)(\xi)\right\|_{\mathcal{H}} \\
&\leqslant c \left\|(\pm i + D)(\xi)\right\|_{\mathcal{H}} \leqslant c \left(\|\xi\|_{\mathcal{H}} + \|D(\xi)\|_{\mathcal{H}}\right) \\
&\leqslant c\,\sqrt{2} \left(\|\xi\|_{\mathcal{H}}^2 + \|D(\xi)\|_{\mathcal{H}}^2\right)^{\frac{1}{2}}, \quad \xi \in \operatorname{dom}(D),
\end{aligned}$$

which means that A is D-bounded.

The equivalence of (ii) and (iii) follows from the fact that the operators

$$(\pm i + D)(1 + D^2)^{-\frac{1}{2}} \text{ and } (1 + D^2)^{\frac{1}{2}}(\pm i + D)^{-1}$$

are unitary. $\square$

Remark 24. It follows from the proof of Lemma 23 that

$$\frac{1}{\sqrt{2}} \|A\|_D \leqslant \left\| A (1 + D^2)^{-\frac{1}{2}} \right\| \leqslant \sqrt{2} \|A\|_D.$$

Observe that, according to Lemma 23 and Remark 24, a path $t \mapsto D_t$ is Γ-differentiable at the point $t = 0$ (as defined in Section 4) if and only if $\text{dom}(D_0) \subseteq \text{dom}(D_t)$ in some neighbourhood of $t = 0$ and

$$\lim_{t \to 0} \left\| \frac{D_t - D_0}{t} - \dot{D}_0 \right\|_{D_0} = 0,$$

in other words, if and only if the path $t \mapsto D_t$ is differentiable with respect to the graph norm of the operator D_0 at the point $t = 0$. This observation further extends to

6.2. **The subspace** $L(D)$. Recall that an operator $A : \text{dom}(A) \mapsto \mathcal{H}$ is called *symmetric* if and only if

$$\langle A(\xi), \eta \rangle = \langle \xi, A(\eta) \rangle, \quad \xi, \eta \in \text{dom}(A).$$

Let $D : \text{dom}(D) \to \mathcal{H}$ be a self adjoint linear operator and let $L(D)$ be the real linear space consisting of all symmetric $A : \text{dom}(A) \mapsto \mathcal{H}$ where $\text{dom}(D) \subseteq \text{dom}(A)$ and such that[12] $A (1 + D^2)^{-\frac{1}{2}} \in B(\mathcal{H})$.

Next, observe that, since A is symmetric and $(1 + D^2)^{-\frac{1}{2}}$ is self adjoint, we have

$$\langle (1 + D^2)^{-\frac{1}{2}} A(\xi), \eta \rangle = \langle A(\xi), (1 + D^2)^{-\frac{1}{2}}(\eta) \rangle = \langle \xi, A (1 + D^2)^{-\frac{1}{2}}(\eta) \rangle.$$

Consequently, we have the implication

$$A (1 + D^2)^{-\frac{1}{2}} \in B(\mathcal{H}) \implies (1 + D^2)^{-\frac{1}{2}} A \in B(\mathcal{H}). \tag{27}$$

More precisely, if the operator $A (1 + D^2)^{-\frac{1}{2}}$ is bounded, then the operator $(1 + D^2)^{-\frac{1}{2}} A$ is closable and its closure is also bounded.

Lemma 25. $L(D)$ *is a closed subspace of* $B(\mathcal{H})$.

Proof. We prove that the subspace $L(D)$ is closed with respect to the weak operator topology. Let $B_n \in L(D)$, $n \geqslant 1$, and

$$wo - \lim_{n \to \infty} B_n = B \in B(\mathcal{H}).$$

This means that there is a sequence A_n of symmetric operators such that

$$B_n = A_n (1 + D^2)^{-\frac{1}{2}} \text{ and } \text{dom}(D) \subseteq \text{dom}(A_n).$$

We need to show that $B \in L(D)$. Introduce the operator $A : \text{dom}(A) \mapsto \mathcal{H}$ by setting

$$\text{dom}(A) = \text{dom}(D) \text{ and } A = B (1 + D^2)^{\frac{1}{2}}.$$

We clearly have that the operator $A (1 + D^2)^{-\frac{1}{2}}$ is closable and its closure coincides with B i.e.,

$$B = \overline{A (1 + D^2)^{-\frac{1}{2}}}.$$

---

[12]Here $A (1 + D^2)^{-\frac{1}{2}} \in B(\mathcal{H})$ means that the operator $A (1 + D^2)^{-\frac{1}{2}}$ is closable and the closure belongs to $B(\mathcal{H})$.

Consequently, we have only to verify that the operator A is symmetric. To this end, observe that

$$
\begin{aligned}
\langle A(\xi), \eta \rangle &= \langle B \, (1 + D^2)^{\frac{1}{2}} (\xi), \eta \rangle \\
&= \lim_{n \to \infty} \langle B_n \, (1 + D^2)^{\frac{1}{2}} (\xi), \eta \rangle \\
&= \lim_{n \to \infty} \langle A_n(\xi), \eta \rangle \\
&= \lim_{n \to \infty} \langle \xi, A_n(\eta) \rangle \\
&= \lim_{n \to \infty} \langle \xi, B_n \, (1 + D^2)^{\frac{1}{2}} (\eta) \rangle \\
&= \langle \xi, B \, (1 + D^2)^{\frac{1}{2}} (\eta) \rangle \\
&= \langle \xi, A(\eta) \rangle, \quad \xi, \eta \in \mathrm{dom}(A).
\end{aligned}
$$

Thus, A is symmetric and therefore $B \in \mathcal{L}(D)$. $\square$

For a closely related argument to the following see [29].

Lemma 26. *Let* $D_j : \mathrm{dom}(D_j) \mapsto \mathcal{H}$, $j = 0, 1$ *be a self adjoint linear operator and let* $B \in \mathcal{L}(D_0)$. *If* $D_1 - D_0$ *is* D_0-*bounded and* $\|D_1 - D_0\|_{D_0} < 1$, *then the operator* $B_\theta = (1 + D_1^2)^{-\frac{\theta}{2}} B \, (1 + D_0^2)^{\frac{\theta}{2}}$ *is bounded and*

$$
\|B_\theta\| \leqslant c_0 \, \|B\|,
$$

for some constant $c_0 > 0$ *and every* $0 \leqslant \theta \leqslant 1$.

Proof. Let $A : \mathrm{dom}(A) \mapsto \mathcal{H}$ be a symmetric linear operator such that

$$
B = A \, (1 + D_0^2)^{-\frac{1}{2}} \in B(\mathcal{H}). \tag{28}
$$

In particular,

$$
\mathrm{dom}(D_0) \subseteq \mathrm{dom}(A).
$$

The operator A is D_0-bounded (see Lemma 23). According to (26), the operator A is also D_1-bounded. This further means (using (27) and Lemma 23) that

$$
(1 + D_1^2)^{-\frac{1}{2}} A \in B(\mathcal{H}). \tag{29}
$$

Let $E_n^j = E^{D_j}(-n, n)$ be the spectral projection of the operator D_j, $j = 0, 1$. The operator $E_n^1 A E_n^0$ is bounded since

$$
E_n^1 A E_n^0 = E_n^1 \, B \, (1 + D_0^2)^{\frac{1}{2}} \, E_n^0.
$$

Let

$$
C_{\theta,n} = (1 + D_1^2)^{-\frac{\theta}{2}} E_n^1 B E_n^0 (1 + D_0^2)^{\frac{\theta}{2}}.
$$

We clearly have that

$$
\lim_{n \to \infty} \langle C_{\theta,n}(\xi), \eta \rangle = \langle B_\theta(\xi), \eta \rangle, \quad \xi \in \mathrm{dom}(D_0), \eta \in \mathrm{dom}(D_1).
$$

Thus, it is sufficient to show that the operators $C_{n,\theta}$ are uniformly bounded with respect to n. This follows from (28), (29) and Lemma 27 below. $\square$

Lemma 27. *Let* A, B_j, $j = 0, 1$ *be bounded linear operators. If* B_j, $j = 0, 1$ *are positive, then the operator* $B_1^{1-\theta} A \, B_0^\theta$ *is bounded and*

$$
\left\| B_1^{1-\theta} A \, B_0^\theta \right\| \leqslant \|B_1 A\|^{1-\theta} \, \|A B_0\|^\theta, \quad 0 \leqslant \theta \leqslant 1. \tag{30}
$$

Proof. The lemma is a straightforward application of the three lines lemma (see [7, Lemma 1.1.2]) to the holomorphic function

$$
f_{\xi,\eta}(z) = \|B_1 A\|^{-1+z} \, \|A \, B_0\|^{-z} \, \langle B_1^{1-z} A \, B_0^z(\xi), \eta \rangle, \quad \xi, \eta \in \mathcal{H},
$$

considered in the strip $S = \{z \in \mathbb{C} : 0 < \Re z < 1\}$. $\square$

6.3. **The proof of Theorems 21 and 22.** The proof of Theorems 21 and 22 rests on the properties of the operator $T_\phi(D_1, D_0)$ where the function ϕ is given by

$$\phi(\lambda, \mu) = \frac{\vartheta(\lambda) - \vartheta(\mu)}{\lambda - \mu} (1 + \mu^2)^{\frac{1}{2}}. \tag{31}$$

The difficulty about this operator is the fact that it is not bounded on $B(\mathcal{H})$. Thus, a direct application of the methods of double operator integrals and harmonic analysis is not feasible. Nevertheless, we shall show that this operator, when considered on the subspace $L(D_0)$, is bounded and possesses all the properties needed to prove Theorems 21 and 22.

Let $D_j : \mathrm{dom}(D_j) \mapsto \mathcal{H}, j = 0, 1$ be a self adjoint linear operator such that $D_1 - D_0$ is D_0-bounded and $\|D_1 - D_0\|_{D_0} \leqslant \frac{1}{2}$. In order to introduce the operator $T_\phi = T_\phi(D_1, D_0) : L(D_0) \mapsto B(\mathcal{H})$ (ϕ is given in (31)), let us consider another function

$$\psi(\lambda, \mu) = (1 + \lambda^2)^{\frac{1}{4}} \frac{\vartheta(\lambda) - \vartheta(\mu)}{\lambda - \mu} (1 + \mu^2)^{\frac{1}{4}}. \tag{32}$$

The operator $T_\psi = T_\psi(D_1, D_0)$ is bounded on $B(\mathcal{H})$, (T_ψ is equal to the operator T_θ with $\theta = \frac{1}{2}$ introduced in the proof of [37, Theorem 14]). Observe also that the bound of operator T_ψ does not depend of the operators $D_j, j = 0, 1$.

The fact that the operator T_ψ is bounded on $B(\mathcal{H})$ and Lemma 26 imply that the mapping

$$B \in L(D_0) \mapsto T_\psi((1 + D_1^2)^{-\frac{1}{4}} B (1 + D_0^2)^{\frac{1}{4}}) \in B(\mathcal{H}) \tag{33}$$

is a bounded linear operator $L(D_0) \mapsto B(\mathcal{H})$ whose norm depends only on the quantity $\|D_1 - D_0\|_{D_0}$. Furthermore, it is known (see [36], see also [35] for a more complete and detailed exposition) that the operators T_ϕ and T_ψ are bounded from $\mathcal{L}^2 \mapsto \mathcal{L}^2$, where $\mathcal{L}^2 \subseteq B(\mathcal{H})$ is the Hilbert-Schmidt ideal and the following identity holds

$$T_\phi(B) (1 + D_0^2)^{-\frac{1}{4}} = (1 + D_1^2)^{-\frac{1}{4}} T_\psi(B), \quad B \in \mathcal{L}^2.$$

The latter identity suggests that the mapping (33) is a (unique) bounded extension of the operator T_ϕ from $\mathcal{L}^2 \cap L(D_0)$ to the space $L(D_0)$. Motivated by this observation, we shall write $T_\phi = T_\phi(D_1, D_0)$ for the mapping (33).

Proof of Theorem 21. Let $T_{\phi,t} = T_\phi(D_t, D_0)$ and let $H = T_{\phi,0}(G)$ where $G = \dot{D}_0 (1 + D_0^2)^{-\frac{1}{2}}$ (observe that the subspace $L(D_0)$ is closed in $B(\mathcal{H})$ and therefore $G \in L(D_0)$). It now follows from Theorem 17 that

$$\frac{F_t - F_0}{t} - H = T_{\phi,t} \left(\frac{D_t - D_0}{t} (1 + D_0^2)^{-\frac{1}{2}} - G \right) + (T_{\phi,t}(G) - T_{\phi,0}(G)).$$

When $t \to 0$, the first term vanishes due to assumptions of the theorem and the fact that the operators $T_{\phi,t}$ are uniformly bounded. To finish the proof of the theorem, we need to justify that

$$\lim_{t \to 0} \|T_{\phi,t}(G) - T_{\phi,0}(G)\| = 0. \tag{34}$$

Letting $T_{\psi,t} = T_\psi(D_t, D_0)$ and

$$C_t = (1 + D_t^2)^{-\frac{1}{4}} G (1 + D_0^2)^{\frac{1}{4}},$$

we infer (via (33)) that

$$T_{\phi,t}(G) - T_{\phi,0}(G) = T_{\psi,t}(C_t) - T_{\psi,0}(C_0) = T_{\psi,t}(C_t - C_0) + (T_{\psi,t}(C_0) - T_{\psi,0}(C_0)).$$

Observing that $T_{\psi,t}$ are uniformly bounded, we see that it is sufficient to show

$$\lim_{t \to 0} \|C_t - C_0\| = 0 \quad \text{and} \quad \lim_{t \to 0} \|T_{\psi,t}(C_0) - T_{\psi,0}(C_0)\| = 0. \tag{35}$$

The first limit in (35) is due to the following identity

$$C_t - C_0 = (1 + D_t^2)^{-\frac{1}{4}} G (1 + D_0^2)^{\frac{1}{4}} - (1 + D_0^2)^{-\frac{1}{4}} G (1 + D_0^2)^{\frac{1}{4}}$$
$$= \left((1 + D_t^2)^{-\frac{1}{4}} (1 + D_0^2)^{\frac{1}{4}} - 1 \right) C_0$$

and the following estimate (see Lemma 28.(i) below)

$$\left\| (1 + D_t^2)^{-\frac{1}{4}} (1 + D_0^2)^{\frac{1}{4}} - 1 \right\| \leqslant c_0 \left\| (D_t - D_0) (1 + D_0^2)^{-\frac{1}{2}} \right\|. \tag{36}$$

For the second limit in (35), observe that the operator $T_{\psi,t}$ is precisely the operator $T_{\psi,t}$ introduced in the proof of [37, Theorem 21]. Thus, the required limit follows from [37, Formula (6.8)] and the estimate (see Lemma 28.(ii) below)

$$\left\| (1 + D_t^2)^{\frac{is}{2}} - (1 + D_0^2)^{\frac{is}{2}} \right\| \leqslant c_0 \left\| (D_t - D_0) (1 + D_0)^{-\frac{1}{2}} \right\|, \quad |s| \leqslant s_0, \tag{37}$$

for some constant $c_0 > 0$ which may depend on s_0. The latter estimate is an improvement of [37, Formula (6.16)]. The proof of the theorem is finished. □

Proof of Theorem 22. The proof is similar to the proof of Theorem 21. Indeed, letting $T_{\phi,t,s} = T_\phi (D_t, D_s)$ (ϕ is given in (31)), we obtain

$$H_t - H_0 = T_{\phi,t,t}(G_t) - T_{\phi,0,0}(G_0) = T_{\phi,t,t}(G_t) - T_{\phi,t,t}(G_0)$$
$$+ T_{\phi,t,t}(G_0) - T_{\phi,t,0}(G_0)$$
$$+ T_{\phi,t,0}(G_0) - T_{\phi,0,0}(G_0).$$

The first term vanishes since the operators $T_{\phi,t,t}$ are uniformly bounded and the assumption of the theorem; the last one does due to (34). To finish the proof, we need to show that

$$\lim_{t \to 0} \| T_{\phi,t,t}(G_0) - T_{\phi,t,0}(G_0) \| = 0.$$

Let $T_{\psi,t,s} = T_\psi(D_t, D_s)$, where the function ψ is given in (32) and let

$$C_{t,s} = (1 + D_t^2)^{-\frac{1}{4}} G_0 (1 + D_s^2)^{\frac{1}{4}} \in B(\mathcal{H}).$$

It then follows from the definition of the operator $T_{\phi,t,s}$ that

$$T_{\phi,t,t}(G_0) - T_{\phi,t,0}(G_0) = T_{\psi,t,t}(C_{t,t}) - T_{\psi,t,0}(C_{t,0})$$
$$= T_{\psi,t,0}(C_{t,0} - C_{0,0}) + T_{\psi,t,t}(C_{t,t} - C_{0,0})$$
$$+ T_{\psi,t,t}(C_{0,0}) - T_{\psi,t,0}(C_{0,0}). \tag{38}$$

The first term vanishes due to the fact that the operators $T_{\psi,t,s}$ are uniformly bounded and (35). For the last term in (38), observe that the operator $T_{\psi,t,s}$ is the operator $\bar{T}_{t,s}$ introduced in the proof of [37, Theorem 21]. Therefore, the last term in (38) vanishes due to [37, Formula (6.11)] and the estimate (37). Thus, to finish the proof of the theorem, we need only justify that

$$\lim_{t \to 0} \| C_{t,t} - C_{0,0} \| = 0. \tag{39}$$

For the latter, observe that

$$C_{t,t} - C_{0,0} = (1 + D_t^2)^{-\frac{1}{4}} G_0 (1 + D_t^2)^{\frac{1}{4}} - (1 + D_0^2)^{-\frac{1}{4}} G_0 (1 + D_0^2)^{\frac{1}{4}}$$
$$= C_{t,t} \left(1 - (1 + D_t^2)^{-\frac{1}{4}} (1 + D_0^2)^{\frac{1}{4}} \right)$$
$$+ \left((1 + D_t^2)^{-\frac{1}{4}} (1 + D_0^2)^{\frac{1}{4}} - 1 \right) C_{0,0}.$$

It is now clear that (39) follows from (36) and the fact that the operators $C_{t,t}$ are uniformly bounded. The proof of the theorem is finished. □

Lemma 28. *Let* $D_j : \mathrm{dom}(D_j) \mapsto \mathcal{H}, j = 0, 1$ *be a linear self adjoint operator. If* $D_1 - D_0$ *is* D_0*-bounded and* $\| D_1 - D_0 \|_{D_0} \leqslant \frac{1}{2}$, *then*

(i) *there is a constant $c_0 > 0$ such that*

$$\left\| (1 + D_1^2)^{-\frac{1}{4}} (1 + D_0^2)^{\frac{1}{4}} - 1 \right\| \leqslant c_0 \left\| (D_1 - D_0)(1 + D_0^2)^{-\frac{1}{2}} \right\|;$$

(ii) *for every $s_0 > 0$, there is a constant $c_1 > 0$ such that*

$$\left\| (1 + D_1^2)^{\frac{is}{2}} - (1 + D_0^2)^{\frac{is}{2}} \right\| \leqslant c_1 \left\| (D_1 - D_0)(1 + D_0^2)^{-\frac{1}{2}} \right\|, \quad |s| \leqslant s_0.$$

Proof. (i) Let us set $G_j = (1 + D_j^2)^{\frac{1}{2}}$, $j = 0, 1$ for brevity. By Lemma 26, we have

$$\left\| G_1^{-\frac{3}{4}} (D_1 - D_0) G_0^{-\frac{1}{4}} \right\| \leqslant c_0 \left\| (D_1 - D_0) G_0^{-1} \right\|.$$

Thus, it is sufficient to show that

$$\left\| G_1^{-\frac{1}{2}} G_0^{\frac{1}{2}} - 1 \right\| \leqslant c_0 \left\| G_1^{-\frac{3}{4}} (D_1 - D_0) G_0^{-\frac{1}{4}} \right\|.$$

Consider the function

$$\eta(\lambda_1, \lambda_0) = \gamma_1^{\frac{3}{4}} \frac{\gamma_1^{-\frac{1}{2}} \gamma_0^{\frac{1}{2}} - 1}{\lambda_1 - \lambda_0} \gamma_0^{\frac{1}{4}}, \quad \gamma_j = (1 + \lambda_j^2)^{\frac{1}{2}}, \; j = 0, 1. \tag{40}$$

Suppose that $\eta \in \Phi(B(\mathcal{H}))$. The required estimate then follows from Theorem 17 which guarantees the identity

$$G_1^{-\frac{1}{2}} G_0^{\frac{1}{2}} - 1 = T_\eta (G_1^{-\frac{3}{4}} (D_1 - D_0) G_0^{-\frac{1}{4}}),$$

where $T_\eta = T_\eta(D_1, D_0)$. Thus, to finish the proof of (i) we need show that $\eta \in \Phi(B(\mathcal{H}))$. This is justified by [37, Lemmas 7 and 9] and the following representation of the function η given in (40)

$$\eta(\lambda_1, \lambda_0) = \frac{\lambda_1}{\gamma_1} f\left(\frac{\gamma_0}{\gamma_1}\right) + \frac{\lambda_0}{\gamma_0} f\left(\frac{\gamma_1}{\gamma_0}\right),$$

where the function f is given by

$$f(t) = (1 + t)^{-1} \left(t^{\frac{1}{4}} + t^{-\frac{1}{4}}\right)^{-1}, \quad t > 0.$$

(ii) We keep the notations of the proof above. Let s_0 be fixed. Referring to Lemma 26 again, we need only show that

$$\left\| G_1^{is} - G_0^{is} \right\| \leqslant c_0 \left\| G_1^{-\frac{1}{2}} (D_1 - D_0) G_0^{-\frac{1}{2}} \right\|.$$

Let

$$\zeta(\lambda_1, \lambda_0) = \gamma_1^{\frac{1}{2}} \frac{\gamma_1^{is} - \gamma_0^{is}}{\lambda_1 - \lambda_0} \gamma_0^{\frac{1}{2}}.$$

If $\zeta \in \Phi(B(\mathcal{H}))$, then we have the identity (see Theorem 17)

$$G_1^{is} - G_0^{is} = T_\zeta \left(G_1^{-\frac{1}{2}} (D_1 - D_0) G_0^{-\frac{1}{2}} \right),$$

where $T_\zeta = T_\zeta(D_1, D_0)$ and (ii) follows. Thus, we need to establish that $\zeta \in \Phi(B(\mathcal{H}))$. To this end, note that the latter function admits the representation

$$\zeta(\lambda_1, \lambda_0) = \gamma_1^{\frac{is}{2}} \gamma_0^{\frac{is}{2}} \left[\frac{\lambda_1}{\gamma_1} f\left(\frac{\gamma_0}{\gamma_1}\right) + \frac{\lambda_0}{\gamma_0} f\left(\frac{\gamma_1}{\gamma_0}\right) \right],$$

where the function f is given by

$$f(t) = \frac{t^{\frac{is}{2}} - t^{-\frac{is}{2}}}{(1 + t)\left(t^{\frac{1}{4}} - t^{-\frac{1}{4}}\right)}.$$

This, together with [37, Lemmas 7 and 9], implies that $\zeta \in \Phi(B(\mathcal{H}))$ (with the norm depending on s). The proof of the lemma is finished. $\qquad\square$

7. APPENDIX

Homotopical equivalence between $\mathcal{F}_*$ and $\mathcal{F}_*^{\pm 1}$. We regard the sets $\mathcal{F}_*$ and $\mathcal{F}_*^{\pm 1}$ as topological spaces endowed with norm topology.

Theorem 29. *The space $\mathcal{F}_*^{\pm 1}$ is a deformation retract of the space $\mathcal{F}_*$ i.e., there is a continuous mapping $r : [0, 1] \times \mathcal{F}_* \mapsto \mathcal{F}_*$ such that*

  (i) $r(0, F) = F$, $F \in \mathcal{F}_*$;
  (ii) $r(1, F) \in \mathcal{F}_*^{\pm 1}$, $F \in \mathcal{F}_*$;
  (iii) $r(1, F) = F$, $F \in \mathcal{F}_*^{\pm 1}$.

Proof. Recall that $\mathcal{K}$ is the two-sided ideal of all τ-compact operators of $\mathcal{M}$ and π is the homomorphism

$$\pi : \mathcal{M} \mapsto \mathcal{M}/\mathcal{K}.$$

Also recall that if F is a self-adjoint τ-Fredholm operator and $\delta_F = \left\|\pi(F)^{-1}\right\|^{-1}$, then, for every $0 \leqslant \delta < \delta_F$, the spectral projection $E^F(-\delta, \delta)$ is τ-finite (see Lemma 2) i.e.,

$$\tau\left(E^F(-\delta, \delta)\right) < +\infty, \quad 0 \leqslant \delta < \delta_F.$$

Consider the intermediate space $\mathcal{F}_*^{\pm 1} \subseteq \mathcal{F}_*' \subseteq \mathcal{F}_*$ of all τ-Fredholm operators F such that the projection $E^F(-\delta, \delta)$ is τ-finite, for every $0 \leqslant \delta < 1$. Clearly, it is sufficient to show that $\mathcal{F}_*'$ is a deformation retract of $\mathcal{F}_*$ and that $\mathcal{F}_*^{\pm 1}$ is a deformation retract of $\mathcal{F}_*'$.

Observing that the function $F \mapsto \left\|\pi(F)^{-1}\right\|$ is continuous, we see that the deformation retract between $\mathcal{F}_*$ and $\mathcal{F}_*'$ is given by the mapping

$$r_1(t, F) = F\left(1 - t + t \left\|\pi(F)^{-1}\right\|\right)^{-1} \in \mathcal{F}_*, \quad 0 \leqslant t \leqslant 1.$$

To construct the deformation retract between $\mathcal{F}_*'$ and $\mathcal{F}_*^{\pm 1}$, let us consider the continuous function $\chi(t)$ which is constantly 1 for $t \geqslant 1$, constantly -1 for $t \leqslant -1$ and linear for $-1 \leqslant t \leqslant 1$. The function χ is given by

$$\chi(t) = \frac{1}{2}|t + 1| - \frac{1}{2}|t - 1|.$$

Observe that the mapping $F \mapsto \chi(F)$ is continuous in the norm topology (see [8, Theorem X.2.1]). Observe also that $\chi(F) \in \mathcal{F}_*^{\pm 1}$, for every $F \in \mathcal{F}_*'$. Indeed, let us fix $F \in \mathcal{F}_*'$ and let $F_1 = \chi(F)$. Clearly, $\|F_1\| \leqslant 1$. To see that $1 - F_1^2$ is τ-compact, consider the points $\delta_n = 1 - \frac{1}{n}$, $n = 1, 2, \ldots$, and the projections

$$E_n = E^F(-\delta_{n+1}, -\delta_n] + E^F[\delta_n, \delta_{n+1}).$$

Observe that, since $F \in \mathcal{F}_*'$, the projection E_n is τ-finite, for every $n = 1, 2, \ldots$. Noting that

$$E^F(-1, 1) = E^{F_1}(-1, 1) \ \text{ and } \ (1 - F_1^2)E^{F_1}\{\pm 1\} = 0,$$

we obtain that

$$1 - F_1^2 = (1 - F_1^2)\, E^F(-1, 1) = (1 - F_1^2) \sum_{n=1}^{\infty} E_n = \sum_{n=1}^{\infty} (1 - F_1^2)\, E_n \leqslant \sum_{n=1}^{\infty} \frac{2}{n}\, E_n.$$

The latter means that the operator $1 - F_1^2$ is τ-compact. Thus, we may define the deformation retract between $\mathcal{F}_*'$ and $\mathcal{F}_*^{\pm 1}$ by setting

$$r_2(t, F) = (1 - t) F + t \chi(F) \in \mathcal{F}_*', \quad 0 \leqslant t \leqslant 1.$$

The theorem is proved. $\qquad\square$

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