

HOLOMORPHIC STRUCTURES ON THE QUANTUM PROJECTIVE LINE

MASOUD KHALKHALI, GIOVANNI LANDI AND WALTER D. VAN SUIJLEKOM

ABSTRACT. We show that much of the structure of the 2-sphere as a complex curve survives the q -deformation and has natural generalizations to the quantum 2-sphere – which, with additional structures, we identify with the quantum projective line. Notably among these is the identification of a quantum homogeneous coordinate ring with the coordinate ring of the quantum plane. In parallel with the fact that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information for complex structures on the surface, we formulate a notion of twisted positivity for twisted Hochschild and cyclic cocycles and exhibit an explicit twisted positive Hochschild cocycle for the complex structure on the sphere.

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1. INTRODUCTION

Despite much progress in noncommutative geometry in the past 30 years, noncommutative complex geometry is not developed that much yet. To the best of our knowledge the paper [8] is the first outlining a possible approach to the idea of a complex structure in noncommutative geometry, based on the notion of positive Hochschild cocycle on an involutive algebra. Other contributions include [13] where noncommutative complex structures motivated by supersymmetric quantum field theory were introduced, and [23] where a detailed study of holomorphic structure on noncommutative tori and holomorphic vector bundles on them is carried out. In [7, Section VI.2] Connes shows explicitly that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a holomorphic structure on the surface. Although the corresponding problem of characterizing holomorphic structures on n -dimensional manifolds via positive Hochschild cocycles is still open, nevertheless this result suggests regarding positive Hochschild cocycles as a foundation for holomorphic noncommutative structures. Indeed, this fits very well in the case of noncommutative tori, as complex structures defined in [23] can be shown to define a positive Hochschild 2-cocycle on the noncommutative torus [10, 11].

In the present paper we study a natural complex structure on the Podleś quantum 2-sphere – which, with additional structure, we identify with the quantum projective line $\mathbb{C}P_q^1$ – that resembles in many aspects the analogous structure on the classical Riemann sphere. We shall concentrate on both algebraic and analytic aspects. While at the algebraic level the complex structure we are using on the quantum projective line was already present in [17], we move from this to the analytic level of holomorphic functions and sections. Indeed, it is well known that there are finitely generated projective modules over the quantum sphere that correspond to the canonical line bundles on the Riemann sphere in the $q \rightarrow 1$ limit. In this paper we study a holomorphic structure on these projective modules and give explicit bases for the space of corresponding holomorphic sections. Since these projective modules are in fact bimodules we can define, in terms of their tensor products, a quantum homogeneous coordinate ring for $\mathbb{C}P_q^1$. We are able to identify this ring with the coordinate ring of the quantum plane.

In §2 we define the notion of a *complex structure* on an involutive algebra as a natural and minimal algebraic requirement on structures that ought to be present in any holomorphic structure on a noncommutative space; we also give several examples, some of which already present in the literature. We then define holomorphic structures on modules and bimodules and indicate, in special cases, a tensor product for bimodules. In §3 we look at the quantum projective line $\mathbb{C}P_q^1$ and its holomorphic structure defined via a differential calculus. This differential calculus, induced from the canonical left covariant differential calculus on the quantum “group” $SU_q(2)$, is the unique left covariant one on $\mathbb{C}P_q^1$. In §4 we compute explicit bases for the space of holomorphic sections of the canonical line bundles \mathcal{L}_n on $\mathbb{C}P_q^1$ and we notice that they follow a pattern similar to the classical commutative

case. This allows us to compute the quantum homogeneous coordinate ring of $\mathbb{C}P_q^1$, and to show that it coincides with the coordinate ring of the quantum plane. In §5 we look for a possible positive Hochschild cocycle on the quantum sphere. Given that there are no non-trivial 2-dimensional cyclic cocycles on the quantum 2-sphere we formulate a notion of twisted positivity for twisted Hochschild and cyclic cocycles and show that a natural twisted Hochschild cocycle is positive. The twist here is induced on $\mathbb{C}P_q^1$ by the modular automorphism of the quantum $SU_q(2)$.

2. HOLOMORPHIC STRUCTURES IN NONCOMMUTATIVE GEOMETRY

We start with a general setup for complex and holomorphic structures in noncommutative geometry. In the next sections, we will apply this to the Podleś sphere seen as a quantum Riemann sphere. We try to introduce a scheme that applies also to other noncommutative holomorphic structures already present in the literature. In particular, we have in mind the holomorphic structure on the noncommutative torus that was introduced in [9] and further explored in [23]. Also, we require compatibility with the definitions in [13] of noncommutative complex and Kähler manifolds that originated from supersymmetric quantum theory.

2.1. Noncommutative complex structures.

Suppose \mathcal{A} is an algebra over \mathbb{C} equipped with a *differential $*$ -calculus* $(\Omega^\bullet(\mathcal{A}), d)$. Recall that this is a graded differential $*$ -algebra $\Omega^\bullet(\mathcal{A}) = \bigoplus_{p \geq 0} \Omega^p(\mathcal{A})$ with $\Omega^0(\mathcal{A}) = \mathcal{A}$, thus $\Omega^\bullet(\mathcal{A})$ has the structure of an \mathcal{A} -bimodule. The differential $d : \Omega^\bullet(\mathcal{A}) \rightarrow \Omega^{\bullet+1}(\mathcal{A})$ satisfying a graded Leibniz rule, $d(\alpha\beta) = (d\alpha)\beta + (-1)^{\text{dg}(\alpha)}\alpha(d\beta)$ and $d^2 = 0$. Also the differential commutes with the $*$ -structure: $d(a^*) = d(a)^*$.

Definition 2.1. *A complex structure on \mathcal{A} for the differential calculus $(\Omega^\bullet(\mathcal{A}), d)$ is a bigraded differential $*$ -algebra $\Omega^{(\bullet, \bullet)}(\mathcal{A})$ with two differentials $\partial : \Omega^{(p, q)}(\mathcal{A}) \rightarrow \Omega^{(p+1, q)}(\mathcal{A})$ and $\bar{\partial} : \Omega^{(p, q)}(\mathcal{A}) \rightarrow \Omega^{(p, q+1)}(\mathcal{A})$ ($p, q \geq 0$) such that the following hold:*

$$\Omega^n(\mathcal{A}) = \bigoplus_{p+q=n} \Omega^{(p, q)}(\mathcal{A}); \quad \partial(a)^* = \bar{\partial}(a^*); \quad \text{and} \quad d = \partial + \bar{\partial}.$$

Also, the involution $*$ maps $\Omega^{(p, q)}(\mathcal{A})$ to $\Omega^{(q, p)}(\mathcal{A})$.

In the following, we will also abbreviate $(\mathcal{A}, \bar{\partial})$ for a complex structure on \mathcal{A} . Note that the complex $(\Omega^{(0, \bullet)}(\mathcal{A}), \bar{\partial})$ forms a differential calculus as well.

Remark 2.2. For the purpose of the present paper the above definition suffices. In general one may need to add some kind of integrability condition. We postpone this to future work. These lines of research are also being pursued in [4].

Definition 2.3. *Let $(\mathcal{A}, \bar{\partial})$ be an algebra with a complex structure. The algebra of holomorphic elements in \mathcal{A} is defined as*

$$\mathcal{O}(\mathcal{A}) := \ker \{ \bar{\partial} : \mathcal{A} \rightarrow \Omega^{(0, 1)}(\mathcal{A}) \}.$$

In the following, we will also loosely speak about $\mathcal{O}(\mathcal{A})$ as holomorphic functions. Note that by the Leibniz rule this is indeed an algebra over \mathbb{C} .

Example 2.4. The motivating example is the de Rham complex (with complex coefficients) on a complex manifold. It is of course a complex structure in the above sense for the complexified de Rham differential calculus.

Example 2.5. Let L be a *real* Lie algebra with a *complex structure*:

$$L^{\mathbb{C}} =: L_0 \oplus \overline{L_0}$$

Given $L \rightarrow \text{Der}(\mathcal{A}, \mathcal{A})$, an action of L by $*$ -derivations on an involutive algebra \mathcal{A} , then the Chevalley–Eilenberg complex

$$\Omega^{\bullet} \mathcal{A} := \text{Hom}_{\mathbb{C}}(\Lambda^{\bullet} L^{\mathbb{C}}, \mathcal{A}),$$

for the Lie algebra cohomology of $L^{\mathbb{C}}$ with coefficients in \mathcal{A} , is a differential calculus for \mathcal{A} . A complex structure on \mathcal{A} for this differential calculus is defined by setting

$$\Omega^{(p,q)} \mathcal{A} := \text{Hom}_{\mathbb{C}}(\Lambda^p L_0 \otimes \Lambda^q \overline{L_0}, \mathcal{A})$$

as the space of (p, q) -forms.

Example 2.6. The noncommutative torus \mathcal{A}_{θ} is defined as the involutive algebra generated by two unitaries U_1, U_2 satisfying $U_1 U_2 = e^{2\pi i \theta} U_2 U_1$, for a fixed $\theta \in \mathbb{R}$. The two basic derivations on this torus are given by $\delta_j(U_k) = 2\pi i \delta_{jk} U_k$, for $j, k = 1, 2$, and generate an action of the abelian Lie algebra \mathbb{R}^2 on \mathcal{A}_{θ} . Any $\tau \in \mathbb{C} \setminus \mathbb{R}$ defines a complex structure on the Lie algebra \mathbb{R}^2 :

$$\mathbb{R}^2 \otimes \mathbb{C} = L_0 \oplus \overline{L_0}$$

where $L_0 := e_1 + \tau e_2$ with (e_1, e_2) the standard basis of \mathbb{R}^2 . The construction of Example 2.5 then yields a complex structure on the noncommutative torus which was already present in [9]. The only holomorphic functions are the constants.

Now, the conformal class of a general constant metric in two dimensions is parametrized by a complex number $\tau \in \mathbb{C}$, $\text{Im } \tau > 0$. Up to a conformal factor, the metric is given by

$$g = (g_{ij}) = \begin{pmatrix} 1 & \text{Re } \tau \\ \text{Re } \tau & |\tau|^2 \end{pmatrix}. \quad (2.1)$$

The complex structure on \mathcal{A}_{θ} is then given by

$$\partial_{(\tau)} = \frac{1}{(\tau - \overline{\tau})} (-\overline{\tau} \delta_1 + \delta_2), \quad \overline{\partial}_{(\tau)} = \frac{1}{(\tau - \overline{\tau})} (\tau \delta_1 - \delta_2). \quad (2.2)$$

The cyclic 2-cocycle which ‘integrates’ 2-forms on the noncommutative torus is

$$\Psi(a_0, a_1, a_2) = \frac{i}{2\pi} \text{tr}_{\theta} (a_0 (\delta_1 a_1 \delta_2 a_2 - \delta_2 a_1 \delta_1 a_2)), \quad (2.3)$$

where tr_{θ} indicates the unique invariant normalized faithful trace on \mathcal{A}_{θ} . Its normalization ensures that for any hermitian projection $p \in \mathcal{A}_{\theta}$, the quantity $\Psi(p, p, p)$ is an integer: it

is indeed the index of a Fredholm operator. By working with the metric (2.1), the positive Hochschild cocycle Φ associated with the cyclic one (2.3) is given by

$$\Phi(a_0, a_1, a_2) = \frac{2}{\pi} \operatorname{tr}_\theta (a_0 \partial_{(\tau)} a_1 \bar{\partial}_{(\tau)} a_2). \quad (2.4)$$

The cocycle (2.4) seen as the conformal class of a general constant metric on the torus is in [7, Section VI.2]. We return to this later on in §5.

Example 2.7. The hermitian spectral data introduced in [13] in a noncommutative geometry approach to supersymmetric field theories, is an example of a noncommutative complex structure. We refer in particular to Corollary 2.34 there for more details.

2.2. Holomorphic connections on modules.

We will now lift the complex structures described in the previous section to noncommutative vector bundles. This requires the introduction of connections on (left or right) \mathcal{A} -modules. We will work with left module structures, although this choice is completely irrelevant. Recall that a *connection* on a left \mathcal{A} -module \mathcal{E} for the differential calculus $(\Omega^\bullet(\mathcal{A}), d)$ is a linear map $\nabla : \mathcal{E} \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ satisfying the (left) Leibniz rule:

$$\nabla(a\eta) = a\nabla(\eta) + da \otimes_{\mathcal{A}} \eta, \quad \text{for } a \in \mathcal{A}, \eta \in \mathcal{E}.$$

We can extend the connection to a map $\nabla : \Omega^p(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \Omega^{p+1}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ via the graded (left) Leibniz rule:

$$\nabla(\omega\rho) = (-1)^p \omega \nabla(\rho) + (d\omega)\rho, \quad \text{for } \omega \in \Omega^p(\mathcal{A}), \rho \in \Omega^p(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}. \quad (2.5)$$

The *curvature* of the connection is defined as $F(\nabla) = \nabla^2$; one shows that it is left \mathcal{A} -linear, i.e. it is an element in $\operatorname{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E})$.

Definition 2.8. Let $(\mathcal{A}, \bar{\partial})$ be an algebra with a complex structure. A holomorphic structure on a left \mathcal{A} -module \mathcal{E} with respect to $(\mathcal{A}, \bar{\partial})$ is a flat $\bar{\partial}$ -connection, i.e. a linear map $\bar{\nabla} : \mathcal{E} \rightarrow \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ satisfying

$$\bar{\nabla}(a\eta) = a\bar{\nabla}(\eta) + \bar{\partial}a \otimes_{\mathcal{A}} \eta, \quad \text{for } a \in \mathcal{A}, \eta \in \mathcal{E}. \quad (2.6)$$

and such that $F(\bar{\nabla}) = \bar{\nabla}^2 = 0$.

If in addition \mathcal{E} is a finitely generated projective \mathcal{A} -module, we shall call the pair $(\mathcal{E}, \bar{\nabla})$ a holomorphic vector bundle.

The last part of the definition is motivated by the classical case: a vector bundle on a complex manifold is holomorphic if and only if it admits a flat $\bar{\partial}$ -connection. Indeed, there is a one-to-one correspondence between holomorphic structures on vector bundles over a complex manifold and (equivalence classes of) flat $\bar{\partial}$ -connections. The equivalence is with respect to gauge transformation, a concept which can be generalized to the noncommutative setup as well.

Definition 2.9. Two holomorphic structures $\bar{\nabla}_1$ and $\bar{\nabla}_2$ on an \mathcal{A} -module \mathcal{E} are gauge equivalent if there exists an invertible element $g \in \operatorname{End}_{\mathcal{A}}(\mathcal{E})$ such that $\bar{\nabla}_2 = g^{-1} \circ \bar{\nabla}_1 \circ g$.

More generally, it follows from the Leibniz rule (2.6) that the difference of any two connections is \mathcal{A} -linear, i.e. for $\bar{\nabla}_1$ and $\bar{\nabla}_2$ holomorphic structures on \mathcal{E} , it holds that

$$\nabla_1 - \nabla_2 \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}) \quad (2.7)$$

In particular, for finitely generated projective \mathcal{A} -modules a $\bar{\nabla}$ -connection is given by an element $A \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \Omega^{(0,1)} \otimes_{\mathcal{A}} \mathcal{E})$. Indeed, such a module \mathcal{E} is a direct summand of a free \mathcal{A} -module and so inherits a connection ∇_0 from the trivial connection $\bar{\partial}$ on the free module (acting diagonally, after identifying the free module with N copies of \mathcal{A}). By Eq. (2.7), any other connection on \mathcal{E} is then given by $\bar{\nabla}_0 + A$ with A as said. We call A the *connection (0,1)-form*.

Since $\bar{\nabla}$ is a flat connection, there is the following complex of vector spaces

$$0 \rightarrow \mathcal{E} \xrightarrow{\bar{\nabla}} \Omega^{(0,1)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\bar{\nabla}} \Omega^{(0,2)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \xrightarrow{\bar{\nabla}} \dots \quad (2.8)$$

where $\bar{\nabla}$ is extended to $\Omega^{(0,q)}(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ by a Leibniz rule similar to Eq. (2.6).

Definition 2.10. *The j -th cohomology group of the complex (2.8) is denoted by $H^j(\mathcal{E}, \bar{\nabla})$. In particular the zeroth cohomology group $H^0(\mathcal{E}, \bar{\nabla})$ is called the space of holomorphic sections of \mathcal{E} .*

As a consequence of the Leibniz rule (2.6), the space $H^0(\mathcal{E}, \bar{\nabla})$ is a left $\mathcal{O}(\mathcal{A})$ -module.

2.3. Holomorphic structures on bimodules and their tensor products.

In the previous section we focused on connections on modules carrying a left algebra action. Now, we would like to define tensor products of connections on bimodules, and for that we need some compatibility with the right action of the algebra. The following definition was proposed in [20] for linear connections and [12] for the general case. See also [3, Section II.2].

Definition 2.11. *A bimodule connection on an \mathcal{A} -bimodule \mathcal{E} for the calculus $(\Omega^\bullet(\mathcal{A}), d)$ is given by a connection $\nabla : \mathcal{E} \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$ (for the left module structure) for which there is a bimodule isomorphism*

$$\sigma(\nabla) : \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E},$$

such that the following twisted right Leibniz rule holds

$$\nabla(\eta a) = \nabla(\eta)a + \sigma(\nabla)(\eta \otimes da), \quad \text{for } \eta \in \mathcal{E}, a \in \mathcal{A}. \quad (2.9)$$

In particular, this definition applies to the differential calculus $(\Omega^{(0,\bullet)}(\mathcal{A}), \bar{d})$ thus giving a notion of *holomorphic structure on bimodules*.

Next, suppose we are given two \mathcal{A} -bimodules $\mathcal{E}_1, \mathcal{E}_2$ with two bimodule connections ∇_1, ∇_2 , respectively. Denote the corresponding bimodule isomorphisms by σ_1 and σ_2 . The following result establishes their *tensor product connection*.

Proposition 2.12. *The map $\nabla : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \mapsto \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$ defined by*

$$\nabla := \nabla_1 \otimes 1 + (\sigma_1 \otimes \text{id})(1 \otimes \nabla_2)$$

yields a σ -compatible connection on the \mathcal{A} -bimodule $\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$, with bimodule isomorphism $\sigma : \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2 \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \rightarrow \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$. defined as the composition $(\sigma_1 \otimes 1) \circ (1 \otimes \sigma_2)$.

Proof. Clearly, this map satisfies the left Leibniz rule, since ∇_1 does. Well-definedness of the map on $\mathcal{E}_1 \otimes_{\mathcal{A}} \mathcal{E}_2$ follows by an application of the twisted Leibniz rule (2.9) for ∇_1 and of the left Leibniz rule for ∇_2 . Also, σ -compatibility follows from the twisted Leibniz rule (2.9) and the fact that the σ_i are bimodule maps. \square

It would be desirable that the flatness condition on holomorphic structures survives upon taking the tensor product. Unfortunately, this is not the case for the above tensor product, as shown by the following example.

Example 2.13. Let $\mathcal{E} = \mathcal{A} \otimes V$ and $\mathcal{F} = \mathcal{A} \otimes W$ be two free \mathcal{A} -bimodules, with V and W vector spaces. Since a connection on a free module is determined by its action on basis vectors e_i of V and f_j of W , respectively, we define connections $\nabla_{\mathcal{E}}$ and $\nabla_{\mathcal{F}}$ on \mathcal{E} and \mathcal{F} by two matrices of one-forms A_{ik} and B_{jl} respectively. The tensor product connection is then determined by

$$\nabla(e_i \otimes f_j) = \sum_k A_{ik} \otimes e_k \otimes f_j + \sum_l B_{jl} \otimes e_i \otimes f_l.$$

One then computes that its curvature is

$$\begin{aligned} \nabla^2(e_i \otimes f_j) &= \sum_k F(\nabla_{\mathcal{E}})_{ik} \otimes (e_k \otimes f_j) + \sum_l F(\nabla_{\mathcal{F}})_{jl} \otimes e_i \otimes f_l \\ &\quad - \sum_{kl} (A_{ik}B_{jl} + B_{jl}A_{ik}) \otimes (e_k \otimes f_l) \end{aligned}$$

with $F(\nabla_{\mathcal{E}})_{ik} = dA_{ik} + \sum_r A_{ir}A_{rk}$ and similarly for the curvature of $\nabla_{\mathcal{F}}$. Thus, the curvature of the tensor product connection is not the sum of the curvatures of the two connections, unless of course the differential calculus is (graded) commutative. Indeed, in the latter case, the term $A_{ik}B_{jl} + B_{jl}A_{ik}$ vanishes.

Possibly, a modification of the tensor product could overcome the problem. For the complex *one-dimensional* case of interest for this paper, we can ignore the problem since then flatness of any holomorphic connection is automatic.

3. THE HOPF BUNDLE ON THE QUANTUM PROJECTIVE LINE

The most natural way to define the quantum projective line $\mathbb{C}P_q^1$ is as a quotient of the sphere S_q^3 for an action of $U(1)$. It is the standard Podleś sphere S_q^2 with additional structure and the construction we need is the well known quantum principal $U(1)$ -bundle over the standard Podleś sphere S_q^2 and whose total space is the manifold of the quantum group $SU_q(2)$. This bundle is an example of a quantum homogeneous space [5]. In the

following, without loss of generality we will assume that $0 < q < 1$. We shall also use the ‘ q -number’

$$[s] = [s]_q := \frac{q^{-s} - q^s}{q^{-1} - q}, \quad (3.1)$$

defined for $q \neq 1$ and any $s \in \mathbb{R}$.

3.1. The algebras of S_q^3 and $\mathbb{C}P_q^1$.

The manifold of S_q^3 is identified with the manifold of the quantum group $SU_q(2)$. Its coordinate algebra $\mathcal{A}(SU_q(2))$ is the $*$ -algebra generated by elements a and c , with relations,

$$\begin{aligned} ac &= qca, & ac^* &= qc^*a, & cc^* &= c^*c, \\ a^*a + c^*c &= aa^* + q^2cc^* = 1. \end{aligned} \quad (3.2)$$

These are equivalent to requiring that the ‘defining’ matrix

$$U = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix},$$

is unitary: $UU^* = U^*U = 1$. The Hopf algebra structure for $\mathcal{A}(SU_q(2))$ is given by coproduct, antipode and counit:

$$\Delta U = U \otimes U, \quad S(U) = U, \quad \epsilon(U) = 1.$$

The quantum universal enveloping algebra $\mathcal{U}_q(\mathfrak{su}(2))$ is the Hopf $*$ -algebra generated as an algebra by four elements K, K^{-1}, E, F with $KK^{-1} = 1$ and subject to relations:

$$K^\pm E = q^\pm EK^\pm, \quad K^\pm F = q^\mp FK^\pm, \quad [E, F] = \frac{K^2 - K^{-2}}{q - q^{-1}}. \quad (3.3)$$

The $*$ -structure is simply

$$K^* = K, \quad E^* = F, \quad F^* = E,$$

and the Hopf algebra structure is provided by coproduct Δ , antipode S , counit ϵ :

$$\Delta(K^\pm) = K^\pm \otimes K^\pm, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F,$$

$$S(K) = K^{-1}, \quad S(E) = -qE, \quad S(F) = -q^{-1}F,$$

$$\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0.$$

There is a bilinear pairing between $\mathcal{U}_q(\mathfrak{su}(2))$ and $\mathcal{A}(SU_q(2))$, given on generators by

$$\begin{aligned} \langle K, a \rangle &= q^{-1/2}, & \langle K^{-1}, a \rangle &= q^{1/2}, & \langle K, a^* \rangle &= q^{1/2}, & \langle K^{-1}, a^* \rangle &= q^{-1/2}, \\ \langle E, c \rangle &= 1, & \langle F, c^* \rangle &= -q^{-1}, \end{aligned}$$

and all other couples of generators pairing to 0. One regards $\mathcal{U}_q(\mathfrak{su}(2))$ as a subspace of the linear dual of $\mathcal{A}(SU_q(2))$ via this pairing. Then there are [26] canonical left and right $\mathcal{U}_q(\mathfrak{su}(2))$ -module algebra structures on $\mathcal{A}(SU_q(2))$ such that

$$\langle g, h \triangleright x \rangle := \langle gh, x \rangle, \quad \langle g, x \triangleleft h \rangle := \langle hg, x \rangle, \quad \forall g, h \in \mathcal{U}_q(\mathfrak{su}(2)), \quad x \in \mathcal{A}(SU_q(2)).$$

They are given by $h \triangleright x := \langle (\text{id} \otimes h), \Delta x \rangle$ and $x \triangleleft h := \langle (h \otimes \text{id}), \Delta x \rangle$, or equivalently,

$$h \triangleright x := x_{(1)} \langle h, x_{(2)} \rangle, \quad x \triangleleft h := \langle h, x_{(1)} \rangle x_{(2)},$$

in the Sweedler notation, $\Delta(x) = x_{(1)} \otimes x_{(2)}$, for the coproduct. These right and left actions are mutually commuting:

$$(h \triangleright a) \triangleleft g = (a_{(1)} \langle h, a_{(2)} \rangle) \triangleleft g = \langle g, a_{(1)} \rangle a_{(2)} \langle h, a_{(3)} \rangle = h \triangleright (\langle g, a_{(1)} \rangle a_{(2)}) = h \triangleright (a \triangleleft g),$$

and since the pairing satisfies

$$\langle (Sh)^*, x \rangle = \overline{\langle h, x^* \rangle}, \quad \forall h \in \mathcal{U}_q(\mathfrak{su}(2)), \quad x \in \mathcal{A}(\text{SU}_q(2)),$$

the $*$ -structure is compatible with both actions:

$$h \triangleright x^* = ((Sh)^* \triangleright x)^*, \quad x^* \triangleleft h = (x \triangleleft (Sh)^*)^*, \quad \forall h \in \mathcal{U}_q(\mathfrak{su}(2)), \quad x \in \mathcal{A}(\text{SU}_q(2)).$$

We list here the left action on powers of generators. For $s = 0, 1, \dots$, one finds:

$$\begin{aligned} K^\pm \triangleright a^s &= q^{\mp \frac{s}{2}} a^s, & K^\pm \triangleright a^{*s} &= q^{\pm \frac{s}{2}} a^{*s}, & K^\pm \triangleright c^s &= q^{\mp \frac{s}{2}} c^s, & K^\pm \triangleright c^{*s} &= q^{\pm \frac{s}{2}} c^{*s}; \\ F \triangleright a^s &= 0, & F \triangleright a^{*s} &= q^{(1-s)/2} [s] c a^{*s-1}, & F \triangleright c^s &= 0, & F \triangleright c^{*s} &= -q^{-(1+s)/2} [s] a c^{*s-1}; \\ E \triangleright a^s &= -q^{(3-s)/2} [s] a^{s-1} c^*, & E \triangleright a^{*s} &= 0, & E \triangleright c^s &= q^{(1-s)/2} [s] c^{s-1} a^*, & E \triangleright c^{*s} &= 0. \end{aligned} \quad (3.4)$$

The principal bundle structure comes from an additional (right) action of the group $\text{U}(1)$ on $\text{SU}_q(2)$, given via a map $\alpha : \text{U}(1) \rightarrow \text{Aut}(\mathcal{A}(\text{SU}_q(2)))$, explicit on generators by

$$\alpha_u \left(\begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \right) = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \text{for } u \in \text{U}(1). \quad (3.5)$$

The invariant elements for this action form a subalgebra of $\mathcal{A}(\text{SU}_q(2))$ which is by definition the coordinate algebra $\mathcal{A}(\text{S}_q^2)$ of the standard Podleś sphere S_q^2 of [21]. The inclusion $\mathcal{A}(\text{S}_q^2) \hookrightarrow \mathcal{A}(\text{SU}_q(2))$ is a quantum principal bundle [5] with classical structure group $\text{U}(1)$. Moreover, the sphere S_q^2 (or the projective line \mathbb{CP}_q^1 then) is a quantum homogeneous space of $\text{SU}_q(2)$ and the (left) coaction of $\mathcal{A}(\text{SU}_q(2))$ onto itself restricts to a left coaction $\Delta_L : \mathcal{A}(\text{S}_q^2) \rightarrow \mathcal{A}(\text{SU}_q(2)) \otimes \mathcal{A}(\text{S}_q^2)$; dually, it survives a right action of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{A}(\text{S}_q^2)$ as twisted derivations.

As a set of generators for $\mathcal{A}(\text{S}_q^2)$ we may take

$$B_- := ac^*, \quad B_+ := ca^*, \quad B_0 := cc^*, \quad (3.6)$$

for which one finds relations:

$$\begin{aligned} B_- B_0 &= q^2 B_0 B_-, \\ B_- B_+ &= q^2 B_0 (1 - q^2 B_0), & B_+ B_- &= B_0 (1 - B_0), \end{aligned}$$

and $*$ -structure: $(B_0)^* = B_0$ and $(B_\pm)^* = B_\mp$.

Later on in §3.3 we shall describe a natural complex structure on the quantum 2-sphere S_q^2 for the unique 2-dimensional covariant calculus on it. This will transform the sphere S_q^2 into a quantum Riemannian sphere or quantum projective line \mathbb{CP}_q^1 . Having this in mind, with a slight abuse of ‘language’ from now on we will speak of \mathbb{CP}_q^1 rather than S_q^2 .

3.2. The line bundles over \mathbb{CP}_q^1 .

The left action of the group-like element K on the algebra $\mathcal{A}(\mathrm{SU}_q(2))$ allows one [18, Eq. (1.10)] to give a vector space decomposition $\mathcal{A}(\mathrm{SU}_q(2)) = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n$, where,

$$\mathcal{L}_n := \{x \in \mathcal{A}(\mathrm{SU}_q(2)) : K \triangleright x = q^{n/2} x\}. \quad (3.7)$$

These could equivalently be defined in terms of the $U(1)$ -action. Indeed, if H denotes the infinitesimal generator of the action α , the group-like element K can be written as $K = q^{-H}$. In particular $\mathcal{L}_0 = \mathcal{A}(\mathbb{CP}_q^1)$. Also, $\mathcal{L}_n^* = \mathcal{L}_{-n}$ and $\mathcal{L}_n \mathcal{L}_m = \mathcal{L}_{n+m}$. Each \mathcal{L}_n is clearly a bimodule over $\mathcal{A}(\mathbb{CP}_q^1)$. It was shown in [25, Prop. 6.4] that each \mathcal{L}_n is isomorphic to a projective left $\mathcal{A}(\mathbb{CP}_q^1)$ -module of rank 1 and winding number $-n$. We have indeed the following proposition.

Proposition 3.1.

- (1) *Each \mathcal{L}_n is the bimodule of equivariant elements associated with the irreducible weight $-n$ representation of $U(1)$.*
- (2) *The natural map $\mathcal{L}_n \otimes \mathcal{L}_m \rightarrow \mathcal{L}_{n+m}$ defined by multiplication induces an isomorphism of $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules*

$$\mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_m \simeq \mathcal{L}_{n+m}.$$

Proof. These follow by using the representation theory of $U(1)$ as well as relations

$$a \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} c - qc \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} a = 0, \quad a \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} c^* - qc^* \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} a = 0, \quad c \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} c^* - c^* \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} c,$$

and so on, that can be easily established. \square

Using the explicit action of K in the first line of (3.4), a generating set for the \mathcal{L}_n 's as $\mathcal{A}(\mathbb{CP}_q^1)$ -modules is found to be given by elements

$$\begin{aligned} & \{c^{*\mu} a^{*n-\mu}, \mu = 0, \dots, n\} && \text{for } n \geq 0, \\ & \{a^\mu c^{|\mu|-n}, \mu = 0, \dots, |n|\} && \text{for } n \leq 0, \end{aligned} \quad (3.8)$$

and any $\phi \in \mathcal{L}_n$ is written as

$$\phi_f = \begin{cases} \sum_{\mu=0}^n f_\mu c^{*\mu} a^{*n-\mu} = \sum_{\mu=0}^n c^{*\mu} a^{*n-\mu} \tilde{f}_\mu & \text{for } n \geq 0, \\ \sum_{\mu=0}^{|n|} f_\mu a^\mu c^{|\mu|-n} = \sum_{\mu=0}^{|n|} a^\mu c^{|\mu|-n} \tilde{f}_\mu & \text{for } n \leq 0, \end{cases}$$

with f_μ and \tilde{f}_μ generic elements in $\mathcal{A}(\mathbb{CP}_q^1)$. It is worth stressing that the elements in (3.8) are not independent over $\mathcal{A}(\mathbb{CP}_q^1)$ since the \mathcal{L}_n are not free modules.

From the definition of the \mathcal{L}_n 's in (3.7) and the relations (3.3) of $\mathcal{U}_q(\mathrm{su}(2))$ one gets that,

$$E \triangleright \mathcal{L}_n \subset \mathcal{L}_{n+2}, \quad F \triangleright \mathcal{L}_n \subset \mathcal{L}_{n-2}. \quad (3.9)$$

On the other hand, commutativity of the left and right actions of $\mathcal{U}_q(\mathrm{su}(2))$ yields that

$$\mathcal{L}_n \triangleleft h \subset \mathcal{L}_n, \quad \forall h \in \mathcal{U}_q(\mathrm{su}(2)).$$

A PBW-basis for $\mathcal{A}(\mathrm{SU}_q(2))$ is given by monomials $\{a^m c^k c^{*l}\}$ for $k, l = 0, 1, \dots$, and $m \in \mathbb{Z}$ with the convention that a^{-m} is a short-hand notation for a^{*m} ($m > 0$). Also, a similar basis for \mathcal{L}_n is given by the monomials $a^{l-k} c^k c^{*l+n}$; indeed, from (3.4) it follows that $K \triangleright (a^m c^k c^{*l}) = q^{(-m-k+l)/2} a^m c^k c^{*l}$; then the requirement that $-m - k + l = n$ is met by renaming $l \rightarrow l + n$ forcing in turn $m = l - k$. In particular, the monomials $a^{l-k} c^k c^{*l}$ are the only K -invariant elements thus providing a PBW-basis for $\mathcal{L}_0 = \mathcal{A}(\mathbb{CP}_q^1)$.

3.3. The calculi on the principal bundle.

The principal $U(1)$ -bundle $\mathcal{A}(\mathbb{CP}_q^1) \hookrightarrow \mathcal{A}(\mathrm{SU}_q(2))$ is endowed [5, 6] with compatible nonuniversal calculi obtained from the 3-dimensional left-covariant calculus [26] on $\mathrm{SU}_q(2)$ we describe first. We then give the unique left covariant 2-dimensional calculus [22] on the projective line \mathbb{CP}_q^1 obtained by restriction.

The differential calculus we take on the quantum group $\mathrm{SU}_q(2)$ is the three dimensional left-covariant one already developed in [26]. Its quantum tangent space $\mathcal{X}(\mathrm{SU}_q(2))$ is generated by the three elements:

$$X_z = \frac{1 - K^4}{1 - q^{-2}} = (X_z)^*, \quad X_- = q^{-1/2} F K, \quad X_+ = q^{1/2} E K = (X_-)^*, \quad (3.10)$$

whose coproducts are easily found:

$$\Delta X_z = 1 \otimes X_z + X_z \otimes K^4, \quad \Delta X_{\pm} = 1 \otimes X_{\pm} + X_{\pm} \otimes K^2.$$

From these one also infers that

$$S(X_z) = -X_z K^{-4}, \quad S(X_{\pm}) = -X_{\pm} K^{-2}.$$

The dual space of 1-forms $\Omega^1(\mathrm{SU}_q(2))$ has a basis

$$\omega_z = a^* da + c^* dc, \quad \omega_- = c^* da^* - qa^* dc^*, \quad \omega_+ = adc - qcda, \quad (3.11)$$

of left-invariant forms, that is

$$\Delta_L^{(1)}(\omega_s) = 1 \otimes \omega_s, \quad s = z, \pm, \quad (3.12)$$

with $\Delta_L^{(1)}$ the (left) coaction of $\mathcal{A}(\mathrm{SU}_q(2))$ onto itself extended to forms. The differential $d : \mathcal{A}(\mathrm{SU}_q(2)) \rightarrow \Omega^1(\mathrm{SU}_q(2))$ is then given by

$$df = (X_+ \triangleright f) \omega_+ + (X_- \triangleright f) \omega_- + (X_z \triangleright f) \omega_z, \quad (3.13)$$

for any $f \in \mathcal{A}(\mathrm{SU}_q(2))$. By taking the conjugate of (3.11) the requirement of a differential $*$ -calculus, $df^* = (df)^*$ (see §2.1), yields $\omega_-^* = -\omega_+$ and $\omega_z^* = -\omega_z$. The bimodule structure is:

$$\begin{aligned} \omega_z a &= q^{-2} a \omega_z, & \omega_z a^* &= q^2 a^* \omega_z, & \omega_{\pm} a &= q^{-1} a \omega_{\pm}, & \omega_{\pm} a^* &= q a^* \omega_{\pm} \\ \omega_z c &= q^{-2} c \omega_z, & \omega_z c^* &= q^2 c^* \omega_z, & \omega_{\pm} c &= q^{-1} c \omega_{\pm}, & \omega_{\pm} c^* &= q c^* \omega_{\pm}, \end{aligned} \quad (3.14)$$

Higher dimensional forms can be defined in a natural way by requiring compatibility for commutation relations and that $d^2 = 0$. One has:

$$d\omega_z = -\omega_- \wedge \omega_+, \quad d\omega_+ = q^2(1 + q^2)\omega_z \wedge \omega_+, \quad d\omega_- = -(1 + q^{-2})\omega_z \wedge \omega_-, \quad (3.15)$$

together with commutation relations:

$$\omega_{\pm} \wedge \omega_{\pm} = \omega_z \wedge \omega_z = 0, \quad \omega_- \wedge \omega_+ + q^{-2} \omega_+ \wedge \omega_- = 0, \quad \omega_z \wedge \omega_{\mp} + q^{\pm 4} \omega_{\mp} \wedge \omega_z = 0. \quad (3.16)$$

Finally, there is a unique top form $\omega_- \wedge \omega_+ \wedge \omega_z$.

We summarize the above results in the following proposition.

Proposition 3.2. *For the 3-dimensional left covariant differential calculus on $SU_q(2)$ the bimodules of forms are all (left, say) trivial $\mathcal{A}(SU_q(2))$ -modules given explicitly as follows:*

$$\begin{aligned} \Omega^0(SU_q(2)) &= \mathcal{A}(SU_q(2)), \\ \Omega^1(SU_q(2)) &= \mathcal{A}(SU_q(2)) \langle \omega_-, \omega_+, \omega_z \rangle, \\ \Omega^2(SU_q(2)) &= \mathcal{A}(SU_q(2)) \langle \omega_- \wedge \omega_+, \omega_- \wedge \omega_z, \omega_+ \wedge \omega_z \rangle, \\ \Omega^3(SU_q(2)) &= \mathcal{A}(SU_q(2)) \omega_- \wedge \omega_+ \wedge \omega_z; \end{aligned}$$

the exterior differential and commutation relations are obtained from (3.15) and (3.16), whereas the bimodule structure is obtained from (3.14).

The restriction of the above 3 dimensional calculus to the projective line $\mathbb{C}P_q^1$ yields the unique left covariant 2-dimensional calculus on the latter [17]. This unique calculus was realized in [24] via a Dirac operator.

The calculus on $\mathbb{C}P_q^1$ is broken into a holomorphic and anti-holomorphic part in a natural way. The module of 1-forms $\Omega^1(\mathbb{C}P_q^1)$ is shown to be isomorphic to the direct sum $\mathcal{L}_{-2} \oplus \mathcal{L}_2$, that is the line bundles with winding number ± 2 . Since the element K acts as the identity on $\mathcal{A}(\mathbb{C}P_q^1)$, the differential (3.13) becomes, when restricted to the latter,

$$df = (X_- \triangleright f) \omega_- + (X_+ \triangleright f) \omega_+ \quad \text{for } f \in \mathcal{A}(\mathbb{C}P_q^1).$$

These lead one to break the exterior differential into a holomorphic and an anti-holomorphic part, $d = \bar{\partial} + \partial$, with:

$$\bar{\partial}f = (X_- \triangleright f) \omega_-, \quad \partial f = (X_+ \triangleright f) \omega_+, \quad \text{for } f \in \mathcal{A}(\mathbb{C}P_q^1). \quad (3.17)$$

Lemma 3.3. *The two differentials ∂ and $\bar{\partial}$ satisfy*

$$(\partial f)^* = \bar{\partial} f^*.$$

Proof. This follows by direct computation:

$$(\partial f)^* = -\omega_-(X_+ \triangleright f)^* = -\omega_-(S(X_+)^* \triangleright f^*) = \omega_-(K^{-2} X_- \triangleright f^*) = (X_- \triangleright f^*) \omega_- = \bar{\partial} f^*,$$

using the compatibility of the $*$ -structure with the left action. \square

The decomposition of the calculus shows that

$$\Omega^1(\mathbb{C}P_q^1) = \Omega^{(0,1)}(\mathbb{C}P_q^1) \oplus \Omega^{(1,0)}(\mathbb{C}P_q^1)$$

where $\Omega^{(0,1)}(\mathbb{C}P_q^1) \simeq \mathcal{L}_{-2} \omega_-$ is the $\mathcal{A}(\mathbb{C}P_q^1)$ -bimodule generated by:

$$\{\bar{\partial}B_-, \bar{\partial}B_0, \bar{\partial}B_+\} = \{a^2, ca, c^2\} \omega_- = q^2 \omega_- \{a^2, ca, c^2\}, \quad (3.18)$$

and $\Omega^{(1,0)}(\mathbb{CP}_q^1) \simeq \mathcal{L}_{+2}\omega_+$ is the $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodule generated by:

$$\{\partial B_+, \partial B_0, \partial B_-\} = \{a^{*2}, c^*a^*, c^{*2}\}\omega_+ = q^{-2}\omega_+\{a^{*2}, c^*a^*, c^{*2}\}. \quad (3.19)$$

That these two modules of forms are not free is also expressed by the existence of relations among the differential:

$$\partial B_0 - q^{-2}B_-\partial B_+ + q^2B_+\partial B_- = 0, \quad \bar{\partial}B_0 - B_+\bar{\partial}B_- + q^{-4}B_-\bar{\partial}B_+ = 0.$$

The 2-dimensional calculus on \mathbb{CP}_q^1 has a unique top 2-form ω with $\omega f = f\omega$, for all $f \in \mathcal{A}(\mathbb{CP}_q^1)$, and $\Omega^2(\mathbb{CP}_q^1)$ is the free $\mathcal{A}(\mathbb{CP}_q^1)$ -module generated by ω , that is

$$\Omega^2(\mathbb{CP}_q^1) = \omega\mathcal{A}(\mathbb{CP}_q^1) = \mathcal{A}(\mathbb{CP}_q^1)\omega.$$

Now, both ω_{\pm} commutes with elements of $\mathcal{A}(\mathbb{CP}_q^1)$ and so does $\omega_- \wedge \omega_+$, which is taken as the natural generator $\omega = \omega_- \wedge \omega_+$ of $\Omega^2(\mathbb{CP}_q^1)$. The exterior derivative of any 1-form $\alpha = x\omega_- + y\omega_+ \in \mathcal{L}_{-2}\omega_- \oplus \mathcal{L}_{+2}\omega_+$ is given by

$$d\alpha = d(x\omega_- + y\omega_+) = \partial x \wedge \omega_- + \bar{\partial}y \wedge \omega_+ = (X_{-}\triangleright y - q^2X_{+}\triangleright x)\omega_- \wedge \omega_+. \quad (3.20)$$

We summarize the above results in the following proposition.

Proposition 3.4. *The 2-dimensional differential calculus on the projective line \mathbb{CP}_q^1 is:*

$$\Omega^{\bullet}(\mathbb{CP}_q^1) = \mathcal{A}(\mathbb{CP}_q^1) \oplus (\mathcal{L}_{-2}\omega_- \oplus \mathcal{L}_{+2}\omega_+) \oplus \mathcal{A}(\mathbb{CP}_q^1)\omega_- \wedge \omega_+,$$

Moreover, the splitting $\Omega^1(\mathbb{CP}_q^1) = \Omega^{(1,0)}(\mathbb{CP}_q^1) \oplus \Omega^{(0,1)}(\mathbb{CP}_q^1)$ together with the two maps ∂ and $\bar{\partial}$ given in (3.17) constitute a complex structure (in the sense of Definition 2.1) for the differential calculus.

3.4. The holomorphic connection.

The next ingredient is a connection on the quantum principal bundle with respect to the left covariant calculus $\Omega(\mathbb{CP}_q^1)$. The connection will in turn determine a covariant derivative on any $\mathcal{A}(\mathbb{CP}_q^1)$ -module \mathcal{E} , in particular on the modules of sections of line bundles \mathcal{L}_n given in (3.7).

The most natural way to define a connection on a quantum principal bundle (with given calculi) is by splitting the 1-forms on the total space into horizontal and vertical ones [5, 6]. Since horizontal 1-forms are given in the structure of the principal bundle, one needs a projection on forms whose range is the subspace of vertical ones. The projection is required to be covariant with respect to the right coaction of the structure Hopf algebra.

For the principal bundle over the quantum projective line \mathbb{CP}_q^1 that we are considering, a principal connection is a covariant left module projection $\Pi : \Omega^1(\mathrm{SU}_q(2)) \rightarrow \Omega_{\mathrm{ver}}^1(\mathrm{SU}_q(2))$. That is $\Pi^2 = \Pi$ and $\Pi(x\omega) = x\Pi(\omega)$, for $\omega \in \Omega^1(\mathrm{SU}_q(2))$ and $x \in \mathcal{A}(\mathrm{SU}_q(2))$, and $\alpha \circ \Pi = \Pi \circ \alpha$, with a natural extension of the $U(1)$ -action α to 1-forms. Equivalently it is a covariant splitting $\Omega^1(\mathrm{SU}_q(2)) = \Omega_{\mathrm{ver}}^1(\mathrm{SU}_q(2)) \oplus \Omega_{\mathrm{hor}}^1(\mathrm{SU}_q(2))$. It is not difficult to realize that with the left covariant 3 dimensional calculus on $\mathcal{A}(\mathrm{SU}_q(2))$, a basis for $\Omega_{\mathrm{hor}}^1(\mathrm{SU}_q(2))$ is given by $\{\omega_-, \omega_+\}$. Furthermore:

$$\alpha_u(\omega_z) = \omega_z, \quad \alpha_u(\omega_-) = \omega_- u^{*2}, \quad \alpha_u(\omega_+) = \omega_+ u^2, \quad \text{for } u \in U(1).$$

Thus, a natural choice of a connection [5, 17] is to define ω_z to be vertical:

$$\Pi(\omega_z) := \omega_z, \quad \Pi(\omega_{\pm}) := 0.$$

With a connection, one has a covariant derivative for any $\mathcal{A}(\mathbb{CP}_q^1)$ -module \mathcal{E} :

$$\nabla : \mathcal{E} \rightarrow \Omega^1(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{E}, \quad \nabla := (\text{id} - \Pi_z) \circ d,$$

and one readily shows the Leibniz rule property: $\nabla(f\phi) = f\nabla(\phi) + (df) \otimes \phi$, for all $\phi \in \mathcal{E}$ and $f \in \mathcal{A}(\mathbb{CP}_q^1)$. We shall concentrate on \mathcal{E} being the line bundles \mathcal{L}_n given in (3.7). Then, with the left covariant 2-dimensional calculus on $\mathcal{A}(\mathbb{CP}_q^1)$ (coming from the left covariant 3-dimensional calculus on $\mathcal{A}(\text{SU}_q(2))$ as explained before) we have

$$\nabla\phi = (X_{+\triangleright}\phi)\omega_+ + (X_{-\triangleright}\phi)\omega_- = q^{-n-2}\omega_+(X_{+\triangleright}\phi) + q^{-n+2}\omega_-(X_{-\triangleright}\phi), \quad (3.21)$$

since $X_{\pm\triangleright}\phi \in \mathcal{L}_{n\pm 2}$. Using Proposition 3.1 we see that

$$\nabla\phi \in \omega_+\mathcal{L}_{n+2} \oplus \omega_-\mathcal{L}_{n-2} \simeq \Omega^1(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{E},$$

as required.

The curvature of the connection ∇ is the $\mathcal{A}(\mathbb{CP}_q^1)$ -linear map

$$\nabla^2 : \mathcal{E} \rightarrow \Omega^2(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{E}.$$

On $\phi \in \mathcal{L}_n$ one finds

$$\nabla^2\phi = -q^{-2n-2}\omega_+ \wedge \omega_-(X_z\triangleright\phi),$$

with X_z the (vertical) vector field in (3.10), or

$$\nabla^2 = -q^{-n-1}[n]\omega_+ \wedge \omega_- \quad (3.22)$$

as an element in $\text{Hom}_{\mathcal{A}(\mathbb{CP}_q^1)}(\mathcal{L}_n, \Omega(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_n)$.

In §4.2 we shall study at length holomorphic vector bundles on \mathbb{CP}_q^1 coming from the natural splitting of the connection ∇ defined in (3.21) into a holomorphic and anti-holomorphic part: $\nabla = \nabla^\partial + \nabla^{\bar{\partial}}$, with

$$\nabla^\partial\phi = q^{-n-2}\omega_+(X_{+\triangleright}\phi), \quad \nabla^{\bar{\partial}}\phi = q^{-n+2}\omega_-(X_{-\triangleright}\phi), \quad (3.23)$$

for which there are corresponding Leibniz rules: $\nabla^\partial(f\phi) = f\nabla^\partial(\phi) + (\partial f) \otimes \phi$ and $\nabla^{\bar{\partial}}(f\phi) = f\nabla^{\bar{\partial}}(\phi) + (\bar{\partial}f) \otimes \phi$, for all $\phi \in \mathcal{E}$ and $f \in \mathcal{A}(\mathbb{CP}_q^1)$. The operator $\nabla^{\bar{\partial}}$ clearly satisfies the conditions of a holomorphic structure as given in Definition 2.8. In particular, these connections are automatically flat as it is evident from the curvature in (3.22) being a $(1, 1)$ -form. Thus we have the following:

Definition 3.5. *The operator $\nabla^{\bar{\partial}}$ in (3.23) will be called the standard holomorphic structure on \mathcal{L}_n ; we denote $\bar{\nabla}_{(n)} = \nabla^{\bar{\partial}}$ or even simply $\bar{\nabla}$ when there is no room for confusion.*

It is natural to expect that modulo gauge equivalence as given in Definition 2.9 the holomorphic structure on \mathcal{L}_n is unique; as of now we are unable to prove the uniqueness.

3.5. Tensor products.

We next study the tensor product of two such noncommutative line bundles with connections. Similar to [16, Lemma 18] we have the following

Lemma 3.6. *For any integer n there is a ‘twisted flip’ isomorphism*

$$\Phi_{(n)} : \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \Omega^{(0,1)}(\mathbb{CP}_q^1) \xrightarrow{\sim} \Omega^{(0,1)}(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_n$$

as $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules. There is a analogous map for $\Omega^{(1,0)}(\mathbb{CP}_q^1)$ replacing $\Omega^{(0,1)}(\mathbb{CP}_q^1)$.

Proof. First, note that due to relations (3.14) not only $\Omega^{(0,1)}(\mathbb{CP}_q^1) \simeq \mathcal{L}_{-2}\omega_-$ but also $\Omega^{(0,1)}(\mathbb{CP}_q^1) \simeq \omega_- \mathcal{L}_{-2}$ as $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules (this was indeed given in (3.18)). It follows, using Proposition 3.1, that we have the following diagram of $\mathcal{A}(\mathbb{CP}_q^1)$ -bimodules:

$$\begin{array}{ccc} \mathcal{L}_n \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \Omega^{(0,1)}(\mathbb{CP}_q^1) & \dashrightarrow & \Omega^{(0,1)}(\mathbb{CP}_q^1) \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_n \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{L}_{n-2}\omega_- & \xrightarrow{\sim} & \omega_- \mathcal{L}_{n-2} \end{array}$$

where for the bottom row we have used again the commutation relations (3.14) that hold inside $\Omega^1(\mathrm{SU}_q(2))$. The dashed arrow is the desired bimodule isomorphism $\Phi_{(n)}$. \square

Given the twisted flip, the following propositions can be easily verified.

Proposition 3.7. *The holomorphic structure $\bar{\nabla}$ on \mathcal{L}_n is a bimodule connection with $\sigma(\bar{\nabla}) = \Phi_{(n)}$, i.e. it satisfies the twisted right Leibniz rule*

$$\bar{\nabla}(\eta f) = \bar{\nabla}(\eta) f + \Phi_{(n)}(\eta \otimes \bar{\partial} f), \quad \text{for } \eta \in \mathcal{L}_n, \quad f \in \mathcal{A}(\mathbb{CP}_q^1).$$

Proposition 3.8. *Let $(\mathcal{L}_{n_i}, \bar{\nabla}_{n_i})$, $i = 1, 2$, be two holomorphic bundles with standard holomorphic structure. Then the tensor product connection $\bar{\nabla}_{n_1} \otimes 1 + (\Phi_{(n_1)} \otimes 1)(1 \otimes \bar{\nabla}_{n_2})$ coincides with the standard holomorphic structure on $\mathcal{L}_{n_1} \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_{n_2}$ when identified with $\mathcal{L}_{n_1+n_2}$.*

4. HOLOMORPHIC STRUCTURES ON THE QUANTUM PROJECTIVE LINE

We will study more closely the holomorphic structure on the quantum projective line given in terms of the ∂ and $\bar{\partial}$ -operator in (3.17) and of the ∇^∂ and $\nabla^{\bar{\partial}}$ -connection in (3.23). We start with holomorphic functions before moving to holomorphic sections.

4.1. Holomorphic functions.

Recall the Definition 2.3 above of holomorphic (polynomial) functions. On \mathbb{CP}_q^1 , these are elements in the kernel of the operator $\bar{\partial} : \mathcal{A}(\mathbb{CP}_q^1) \rightarrow \Omega^{(0,1)}(\mathbb{CP}_q^1)$.

Proposition 4.1. *There are no non-trivial holomorphic polynomial functions on \mathbb{CP}_q^1 .*

Proof. From the definition of $\bar{\partial}$ in (3.17) we see that $f \in \mathcal{A}(\mathbb{CP}_q^1)$ is holomorphic if and only if it is in the kernel of $F \in \mathrm{U}_q(\mathrm{su}(2))$. At the end of §3.2 we mentioned that the

monomials $a^{l-k}c^k c^{*l}$ are a PBW-basis for $\mathcal{A}(\mathbb{CP}_q^1)$. Thus, any element $f \in \mathcal{A}(\mathbb{CP}_q^1)$ is written as

$$f = \sum_{k,l \geq 0} f_{kl} a^{l-k} c^k c^{*l},$$

with only finite many of non-zero coefficients f_{kl} . Let us determine the coefficients f_{kl} for elements $f \in \mathcal{A}(\mathbb{CP}_q^1)$ in the kernel of F . We start by expressing $F \triangleright (a^{l-k} c^k c^{*l})$ and distinguish the two cases $l - k \geq 0$ and $l - k < 0$. In the first case, we compute

$$F \triangleright (a^{l-k} c^k c^{*l}) = K^{-1} \triangleright (a^{l-k} c^k) F \triangleright c^{*l} = -q^{-\frac{1}{2}-k} [l] a^{l-k+1} c^k c^{*l-1}, \quad (4.1)$$

using the twisted Leibniz rule dictated by the coproduct in $U_q(su(2))$, the explicit expressions (3.4) for the action and finally the commutation relations (3.2) in $\mathcal{A}(SU_q(2))$.

In the second case, when $k > l$, we write $a^{l-k} c^k c^{*l} = a^{*k-l} c^k c^{*l}$ and, using the same rules as above, compute

$$\begin{aligned} F \triangleright (a^{*k-l} c^k c^{*l}) &= (F \triangleright a^{*k-l}) K \triangleright (c^k c^{*l}) + (K^{-1} \triangleright a^{*k-l} c^k) F \triangleright (c^{*l}) \\ &= q^{l-k+\frac{1}{2}} [k-l] c a^{*k-l-1} c^k c^{*l} - q^{-\frac{1}{2}} [l] a^{*k-l} c^k a c^{*l-1} \\ &= q^{-\frac{1}{2}} [k-l] a^{*k-l-1} c^{k+1} c^{*l} - q^{-\frac{1}{2}-k} [l] a^{*k-l} a c^k c^{*l-1}. \end{aligned}$$

Then, we use the spherical relation $a^* a + c^* c = 1$ to write

$$\begin{aligned} F \triangleright (a^{*k-l} c^k c^{*l}) &= q^{-\frac{1}{2}} \left(([k-l] + q^{-k} [l]) a^{*k-l-1} c^{k+1} c^{*l} - q^{-k} [l] a^{*k-l-1} c^k c^{*l-1} \right) \\ &= q^{-\frac{1}{2}} \left(q^{-l} [k] a^{*k-l-1} c^{k+1} c^{*l} - q^{-k} [l] a^{*k-l-1} c^k c^{*l-1} \right). \end{aligned} \quad (4.2)$$

We combine the above expressions (4.1) and (4.2) to obtain for the action of F on f :

$$\begin{aligned} F \triangleright f &= q^{-\frac{1}{2}} \sum_{l \geq k \geq 0} -f_{kl} q^{-k} [l] a^{l-k+1} c^k c^{*l-1} \\ &\quad + q^{-\frac{1}{2}} \sum_{0 \leq l < k} \left(f_{kl} q^{-l} [k] a^{*k-l-1} c^{k+1} c^{*l} - f_{kl} q^{-k} [l] a^{*k-l-1} c^k c^{*l-1} \right). \end{aligned} \quad (4.3)$$

In the first sum only strictly positive powers of a appear, whereas the remaining terms involve positive or zero powers of a^* and the two sums are therefore independent. Imposing $F \triangleright f$ thus means that $f_{kl} = 0$ for $l \geq k$ unless $l = k = 0$ (which are the constants) while for $l < k$ we obtain the relation

$$\sum_{0 \leq l < k} f_{kl} q^{-l} [k] a^{*k-l-1} c^{k+1} c^{*l} - f_{kl} q^{-k} [l] a^{*k-l-1} c^k c^{*l-1} = 0. \quad (4.4)$$

Since the sum starts with $l = 0, k = 1$ for which the second summand vanishes, we can shift l and k by one in that second summand to obtain

$$\sum_{0 \leq l < k} \left\{ f_{kl} q^{-l} [k] - \sum_{0 \leq l < k} f_{k+1, l+1} q^{-k-1} [l+1] \right\} a^{*k-l-1} c^{k+1} = 0.$$

In other words, the vanishing of $F \triangleright f$ implies the following recursion relations between the coefficients f_{kl} with $0 \leq l < k$:

$$f_{kl}q^{-l}[k] = f_{k+1,l+1}q^{-k-1}[l+1],$$

the solutions of which are given by

$$f_{kl} = \frac{[k-1][k-2]\cdots[k-l]q^k q^{k-1}\cdots q^{k-l+1}}{[l]!q^{l-1}q^{l-2}\cdots q^0} f_{k-l,0} = q^{(k-l+1)l} \begin{bmatrix} k-1 \\ l \end{bmatrix}_q \tilde{f}_{k-l}. \quad (4.5)$$

We have denoted $\tilde{f}_{k-l} = f_{k-l,0}$, the choice of which is the only remaining freedom. Clearly, the coefficients f_{kl} in Eq. (4.5) do not define a polynomial function. The only polynomial solution to $F \triangleright f$ is given by $f \in \mathbb{C}$, which corresponds to $k = l = 0$. \square

Nevertheless, the expression (4.5) indicates that in some ‘reasonable’ closure of $\mathcal{A}(\mathbb{CP}_q^1)$ there might be non-trivial elements in the kernel of $\bar{\partial}$. We will see soon that, in quite a subtle way, this is not the case.

Lemma 4.2. *For any $N > 0$, the formal series $f_{(N)}$ defined by*

$$f_{(N)} = (a^{*N}c^N) \sum_{l=0}^{\infty} q^{(N+1)l} \begin{bmatrix} N+l-1 \\ l \end{bmatrix}_q c^l c^{*l} \quad (4.6)$$

has coefficients of exponential decay which satisfy the recursion relations (4.5).

Proof. With the estimates $q^{-n-1}[n]^{-1} < 1$ and $q^n[n-1] < q$, we obtain

$$\left| q^{(N+1)l} \begin{bmatrix} N+l-1 \\ l \end{bmatrix}_q \right| < q^l$$

which yields a sequence of exponential decay. The second claim follows directly from the proof of Proposition 4.1, since the coefficients of $f_{(N)}$ coincide with the f_{kl} of Eq. (4.5) upon identifying $N = k - l$. \square

Lemma 4.3. *For any $N, N' \geq 0$ the following relations hold between the functions $f_{(N)}$:*

$$f_{(N)}f_{(N')} = q^{NN'} f_{(N+N')}. \quad (4.7)$$

Proof. We note that $f_{(N)}$ is proportional to a^{*N} , and that (up to scaling) it is the unique solution to (4.5) with this property – for a formal power series expansion f . Thus, since $f_{(N)}f_{(N')}$ is proportional to $a^{*(N+N')}$, it must be a complex multiple of $f_{(N+N')}$. The coefficient is determined to be $q^{NN'}$ by comparing the lowest terms in powers of cc^* (the one for $l = 0$) on both sides. \square

We consider the functions $f_{(N)}$ as elements in the C^* -algebra $C(\mathbb{CP}_q^1) \simeq \mathcal{K} \oplus \mathbb{C}1$ which is the subalgebra of the C^* -algebra $C(\mathrm{SU}_q(2))$ made of $\mathrm{U}(1)$ invariants elements in the latter. These are the universal C^* -algebras in the generators B_{\pm}, B_0 and a, c respectively, with

corresponding relations. That $f_{(N)} \in C(\mathrm{SU}_q(2))$ follows from the fact that the C^* -norm of the generators a and c is less than one by the spherical relation (3.2), so that

$$\|f_{(N)}\| < \frac{1}{1-q}.$$

Indeed, $f_{(N)} \in C(\mathbb{CP}_q^1)$ by its $U(1)$ invariance.

We next have to check if $\bar{\partial}$ vanishes when acting on the elements $f_{(N)}$. This cannot follow directly from Lemma 4.2 since $\bar{\partial}$ acts as an unbounded derivation on $\mathcal{A}(\mathbb{CP}_q^1)$. One needs to consider the domain $\mathrm{Dom}(\bar{\partial})$ of $\bar{\partial}$ in $C(\mathrm{SU}_q(2))$ or, rather, its restriction to $C(\mathbb{CP}_q^1)$, and check whether $f_{(N)} \in \mathrm{Dom}(\bar{\partial}) \cap C(\mathbb{CP}_q^1)$; as we shall see this is not the case. Due to the relation between $\bar{\partial}$ and F , it is equivalent (and easier) to consider the domain of F , defined explicitly by

$$\mathrm{Dom}(F) := \{f \in C(\mathrm{SU}_q(2)) : \|F \triangleright f\| < \infty\}. \quad (4.8)$$

The advantage of F over $\bar{\partial}$ comes from its acting on $\mathcal{A}(\mathrm{SU}_q(2))$ and thus it easily extends (as an unbounded derivation) to the whole C^* -algebra. A natural extension of $\bar{\partial}$ as acting on the infinite sum $f_{(N)}$ is through its action on the corresponding partial sums, i.e. we define $\bar{\partial}(f_{(N)})$ as

$$\bar{\partial}(f_{(N)}) := \lim_{L \rightarrow \infty} \sum_{l=0}^L q^{(N+1)l} \begin{bmatrix} N+l-1 \\ l \end{bmatrix}_q \bar{\partial}(a^{*N} c^N c^l c^{*l}).$$

By Lemma 4.7 and Leibniz rule for $\bar{\partial}$ we can restrict to the case $N = 1$, for which we have $f_{(1)} = (a^*c) \sum_{l=0}^{\infty} q^{2l} c^l c^{*l}$.

Lemma 4.4. *The series whose partial sums are given by*

$$\bar{\partial} \left((a^*c) \sum_{l=0}^L q^{2l} c^l c^{*l} \right)$$

does not converge as $L \rightarrow \infty$. As a consequence, for any $N > 0$ the function $f_{(N)}$ is not in $\mathrm{Dom}(\bar{\partial}) \cap C(\mathbb{CP}_q^1)$.

Proof. Recall the definition in (3.17) of $\bar{\partial}$ in terms of X_- and consequently F via (3.10). Consider the expression:

$$F \triangleright \left((a^*c) \sum_{l=0}^L q^{2l} c^l c^{*l} \right) = c \sum_{l=0}^L q^{2l-\frac{1}{2}} (c^{l+1} c^{*l} - q^{-l-1} [l] c^l c^{*l-1} + q^{-l-1} [l] c^{l+1} c^{*l})$$

obtained by steps similar to the ones for the proof of Proposition 4.1 above. The coefficients of the first and last term combine to give

$$1 + q^{-l-1} [l] = \frac{q - q^{-2l-1}}{q - q^{-1}} = q^{-l} [l+1]$$

We thus obtain

$$\begin{aligned}
F \triangleright \left((a^*c) \sum_{l=0}^L q^{2l} c^l c^{*l} \right) &= c \sum_{l=0}^L q^{l-\frac{1}{2}} [l+1] c^{l+1} c^{*l} - \sum_{l=0}^L q^{l-3/2} [l] c^l c^{*l-1} \\
&= c \sum_{l=0}^L q^{l-\frac{1}{2}} [l+1] c^{l+1} c^{*l} - \sum_{l=0}^{L-1} q^{l-\frac{1}{2}} [l+1] c^{l+1} c^{*l} \\
&= q^{L-\frac{1}{2}} [L+1] c^{L+2} c^{*L}
\end{aligned}$$

where in going to the second line we have shifted l in the second sum by $+1$ (allowed by the vanishing of that term when $l=0$). This series does not converge in the C^* -algebra (in particular does not converge to zero) since

$$\lim_{L \rightarrow \infty} q^{L-\frac{1}{2}} [L+1] = \frac{q^{-1/2}}{1-q^2}.$$

□

We have established that there are no non-trivial ‘analytical’ holomorphic functions on the quantum projective line \mathbb{CP}_q^1 . More precisely,

Theorem 4.5. *There are no non-trivial holomorphic functions in $\text{Dom}(\bar{\partial}) \cap C(\mathbb{CP}_q^1)$.*

Consequently, $\mathcal{O}(\mathbb{CP}_q^1) \simeq \mathbb{C}$ (with a slight abuse of notation), a result which completely parallels the classical case ($q=1$) of holomorphic functions on the Riemann sphere.

4.2. Holomorphic vector bundles.

In this section we shall study holomorphic vector bundles on \mathbb{CP}_q^1 coming from the natural splitting of the connection ∇ defined in (3.21) into a holomorphic and anti-holomorphic part. The anti-holomorphic connection on the modules \mathcal{L}_n is given by (cf. (3.23)):

$$\bar{\nabla}_{(n)} = q^{-n+2} \omega_- (X_{-\triangleright} \phi),$$

that we simply denote by $\bar{\nabla}$ when no confusion arises.

Theorem 4.6. *Let n be a positive integer. Then*

- (1) $H^0(\mathcal{L}_n, \bar{\nabla}) = 0$,
- (2) $H^0(\mathcal{L}_{-n}, \bar{\nabla}) \simeq \mathbb{C}^{n+1}$.

Also, these results continues to hold when the $\mathcal{L}_{\pm n}$ are considered as modules over the C^ -algebra $C(\mathbb{CP}_q^1)$.*

Proof. From the definition of the anti-holomorphic connection in (3.23) we see that $\phi \in \mathcal{L}_n$ is holomorphic, that is to say $\bar{\nabla} \phi = 0$, if and only if it is in the kernel of $F \in \mathcal{U}_q(\mathfrak{su}(2))$. In particular, it should be in the domain $\text{Dom}(F)$ of F , as defined in Eq. (4.8).

Now, we know from the end of §3.2 that any $\phi \in \mathcal{L}_n$ with $n > 0$ can be written in the PBW-basis as

$$\phi = \sum_{l \geq -n, k \geq 0} \phi_{kl} a^{l-k} c^k c^{*l+n}.$$

As before, we will distinguish two cases: $l - k \geq 0$ and $l - k < 0$.

For $l \geq k \geq 0$ we compute:

$$F \triangleright (a^{l-k} c^k c^{*l+n}) = K^{-1} \triangleright (a^{l-k} c^k) F \triangleright c^{*l+n} = -q^{-\frac{1}{2}(n+1)-k} [l+n] a^{l-k+1} c^k c^{*l+n-1}.$$

For $l - k < 0$ we have $k > l \geq -n$ and $k \geq 0$ and we compute:

$$\begin{aligned} F \triangleright (a^{*k-l} c^k c^{*l+n}) &= F \triangleright a^{*k-l} K \triangleright (c^k c^{*l+n}) + K^{-1} \triangleright (a^{*k-l} c^k) F \triangleright c^{*l+n} \\ &= q^{\frac{1}{2}(n+1)-k+l} [k-l] c a^{*k-l-1} c^k c^{*l+n} - q^{-\frac{1}{2}(n+1)} [l+n] a^{*k-l} c^k a c^{*l+n-1} \\ &= q^{\frac{1}{2}(n-1)} [k-l] a^{*k-l-1} c^{k+1} c^{*l+n} - q^{-\frac{1}{2}(n+1)-k} [l+n] a^{*k-l} a c^k c^{*l+n-1}. \end{aligned}$$

We use again the spherical relation $a^* a + c^* c = 1$ in the second term and obtain:

$$\begin{aligned} F \triangleright (a^{*k-l} c^k c^{*l+n}) &= \left(q^{\frac{1}{2}(n-1)} [k-l] + q^{-\frac{1}{2}(n+1)-k} [l+n] \right) a^{*k-l-1} c^{k+1} c^{*l+n} \\ &\quad - q^{-\frac{1}{2}(n+1)-k} [l+n] a^{*k-l-1} c^k c^{*l+n-1} \\ &= q^{-\frac{1}{2}(n+1)} \left(q^{-l} [n+k] a^{*k-l-1} c^{k+1} c^{*l+n} - q^{-k} [l+n] a^{*k-l-1} c^k c^{*l+n-1} \right). \end{aligned}$$

Finally, these combine to give for a holomorphic section of \mathcal{L}_n , for $n > 0$, the condition:

$$\begin{aligned} F \triangleright \phi &= -q^{-\frac{1}{2}(n+1)} \sum_{l \geq k \geq 0} \phi_{kl} q^{-k} [l+n] a^{l-k-1} c^k c^{*l+n-1} \\ &\quad + q^{-\frac{1}{2}(n+1)} \sum_{\substack{k > l \geq -n \\ k \geq 0}} \phi_{kl} \left(q^{-l} [n+k] a^{*k-l-1} c^{k+1} c^{*l+n} - q^{-k} [l+n] a^{*k-l-1} c^k c^{*l+n-1} \right) = 0. \end{aligned}$$

The first sum must vanish independently of the other sum since it contains the only terms with powers of a . Moreover, since $l+n \neq 0$ for any $l \geq 0$, it also follows that each coefficient $\phi_{kl} = 0$ for any $l \geq k \geq 0$.

For the terms in the second sum, we distinguish two cases: either $0 < k-l < n$ or $k-l \geq n$; both of these should vanish independently. For the first case, we write

$$\begin{aligned} \sum_{\substack{0 < k-l < n \\ k \geq 0, l \geq -n}} \left(\phi_{kl} q^{-l} [n+k] - \phi_{k+1, l+1} q^{-k-1} [l+n+1] \right) a^{*k-l-1} c^{k+1} c^{*l+n} \\ + \sum_{-n < l < 0} \phi_{0l} [l+n] a^{*-l-1} c^{*l+n-1} = 0 \end{aligned}$$

where we have shifted both indices k, l by 1 to obtain the second term of the first summand, while the last sum is the remaining $k=0$ term to correct for this shift. As a matter of fact, the last sum is independent of the other, so that $\phi_{0l} = 0$ for $-n < l < 0$. In addition, just as in the proof of Proposition 4.1 one can show that the first sum vanishes for $0 < k-l < n$ if and only if $\phi_{kl} \propto \phi_{0, l-k}$. But we have just noted that these last coefficients vanish.

Finally, we are left with the case $k-l \geq n$. Now, the above expression for $F \triangleright \phi = 0$ reads

$$\sum_{\substack{k-l \geq n \\ k \geq 0, l \geq -n}} \left(\phi_{kl} q^{-l} [n+k] - \phi_{k+1, l+1} q^{-k-1} [l+n+1] \right) a^{*k-l-1} c^{k+1} c^{*l+n} = 0.$$

Again we have shifted the l and k indices by 1, with no need for a correction term here. This clearly has no polynomial solutions. An explicit *formal* solution ϕ_{kl} is obtained with the same procedure of the proof of Proposition 4.1, laddering now ϕ_{kl} down to $\phi_{k-l-n,-n}$. By denoting $\phi_{(N)}$ the formal solution corresponding to a fixed $k-l = N \geq n$, it follows that $f_{(N')}\phi_{(N)} \propto \phi_{(N'+N)}$ for $f_{(N')} \in \mathcal{O}(\mathbb{CP}_q^1)$ (see Eq. (4.7)). It is thus enough to consider the single formal element $\phi_{(n)}$. As can be easily checked $\phi_{(n)}$ is in the C^* -algebra $C(\mathrm{SU}_q(2))$ but not in the domain of F . Thus there are no holomorphic sections in \mathcal{L}_n for n positive.

When $n < 0$ a section $\phi \in \mathcal{L}_n$ in the PBW-basis is written as

$$\phi = \sum_{l \geq 0, k \geq n} \phi_{kl} a^{l-k} c^{k+|n|} c^{*l}.$$

If $l \geq k$ we compute

$$F \triangleright (a^{l-k} c^{k+|n|} c^{*l}) = K^{-1} \triangleright (a^{l-k} c^{k+|n|}) F \triangleright c^{*l} = -q^{-\frac{1}{2}(|n|+1)-k} [l] a^{l-k+1} c^{k+|n|} c^{*l-1},$$

whereas for $l < k$ we find

$$\begin{aligned} F \triangleright (a^{*k-l} c^{k+|n|} c^{*l}) &= F \triangleright a^{*k-l} K \triangleright (c^{k+|n|} c^{*l}) + K^{-1} \triangleright (a^{*k-l} c^{k+|n|}) F \triangleright c^{*l} \\ &= q^{-\frac{1}{2}(|n|-1)-k+l} [k-l] c a^{*k-l-1} c^{k+|n|} c^{*l} - q^{\frac{1}{2}(|n|-1)} [l] a^{*k-l} c^{k+|n|} a c^{*l-1} \\ &= q^{-\frac{1}{2}(|n|+1)} [k-l] a^{*k-l-1} c^{k+|n|+1} c^{*l} - q^{-\frac{1}{2}(|n|+1)-k} [l] a^{*k-l} a c^{k+|n|} c^{*l-1}. \end{aligned}$$

Using the spherical relation $a^*a + c^*c = 1$ in the second term yields

$$\begin{aligned} F \triangleright (a^{*k-l} c^{k+|n|} c^{*l}) &= q^{-\frac{1}{2}(|n|+1)} ([k-l] + q^{-k} [l]) a^{*k-l-1} c^{k+|n|+1} c^{*l} - q^{-k} [l] a^{*k-l-1} c^{k+|n|} c^{*l-1} \\ &= q^{-\frac{1}{2}(|n|+1)} (q^{-l} [k] a^{*k-l-1} c^{k+|n|+1} c^{*l} - q^{-k} [l] a^{*k-l-1} c^{k+|n|} c^{*l-1}). \end{aligned}$$

Thus, for a holomorphic section of \mathcal{L}_n , when $n < 0$, we arrive at the condition:

$$\begin{aligned} F \triangleright \phi &= -q^{-\frac{1}{2}(|n|+1)} \sum_{\substack{k \geq n \\ l \geq 0, l \geq k}} \phi_{kl} q^{-k} [l] a^{l-k+1} c^{k+|n|} c^{*l-1} \\ &\quad + q^{-\frac{1}{2}(|n|+1)} \sum_{\substack{k \geq n \\ l \geq 0, k > l}} (\phi_{kl} q^{-l} [k] a^{*k-l-1} c^{k+|n|+1} c^{*l} - \phi_{kl} q^{-k} [l] a^{*k-l-1} c^{k+|n|} c^{*l-1}) = 0. \end{aligned}$$

Again the first sum must vanish independently of the other sum since it contains the only terms with powers of a . This means that $\phi_{kl} = 0$ for $l \geq k$, unless $l = 0$ for which we get $|n|+1$ independent non vanishing coefficients $\phi_{k,0} = 0$ for $n \leq k \leq 0$. Renaming the label $\mu = -k$ this gives the solutions

$$a^\mu c^{|n|-\mu}, \quad \mu = 0, \dots, |n|,$$

which are precisely the module basis elements for \mathcal{L}_n given in (3.8).

As for the second sum, since $k > l \geq 0$ it never involves a negative integer k , we can factor out the term $c^{|n|}$ to get

$$c^{|n|} \sum_{\substack{k \geq n \\ l \geq 0, k > l}} \phi_{kl} (q^{-l} [k] a^{*k-l-1} c^{k+1} c^{*l} - q^{-k} [l] a^{*k-l-1} c^k c^{*l-1}) = 0.$$

This is exactly the expression of $F \triangleright f$ in Eq. (4.4). As a consequence, there are no solutions to $\bar{\partial}(\phi) = 0$ in the intersection of $\text{Dom}(F)$ with the C^* -algebraic closure of \mathcal{L}_n . \square

We finally have:

Theorem 4.7. *The space $R = \bigoplus_{n \geq 0} H^0(\mathcal{L}_{-n}, \bar{\nabla})$ carries a ring structure and is isomorphic to the quantum plane:*

$$R \simeq \mathbb{C}\langle a, c \rangle / (ac - qca)$$

Proof. The ring structure is induced from the tensor product $\mathcal{L}_{-n} \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} \mathcal{L}_{-m} \simeq \mathcal{L}_{-n-m}$. From the proof of Theorem 4.6, we know that $H^0(\mathcal{L}_{-1}, \bar{\nabla}) = \mathbb{C}a \oplus \mathbb{C}c$. The result follows from the identity $a \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} c - qc \otimes_{\mathcal{A}(\mathbb{CP}_q^1)} a = 0$ which can be easily established. \square

Note that this quantum homogeneous coordinate ring R coincides precisely with the twisted homogeneous coordinate rings of [1, 2] associated to the line bundle $\mathcal{O}(1)$ on \mathbb{CP}_1 and a suitable twist.

5. TWISTED POSITIVE HOCHSCHILD COCYCLE

In [7, Section VI.2] it is shown that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information needed to define a complex structure on the surface. The relevant positive cocycle is in the same Hochschild cohomology class of the cyclic cocycle giving the fundamental class of the manifold. Although the problem of characterizing complex structures on n -dimensional manifolds via positive Hochschild cocycles is still open, nevertheless, Connes' result suggests regarding positive cyclic and Hochschild cocycles as a starting point in defining complex noncommutative structures.

It is well known that there are no non-trivial 2-dimensional cyclic cocycles on the quantum 2-sphere [19]. Thus we shall try and formulate a notion of *twisted positivity* for twisted Hochschild and cyclic cocycles and exhibit an example of it in the case of our complex structure on the quantum 2-sphere.

Recall that, a Hochschild $2n$ -cocycle φ on an $*$ -algebra A is called *positive* [8] if the following pairing defines a positive sesquilinear form on the vector spaces $A^{\otimes(n+1)}$:

$$\langle a_0 \otimes a_1 \otimes \cdots \otimes a_n, b_0 \otimes b_1 \otimes \cdots \otimes b_n \rangle = \varphi(b_0^* a_0, a_1, \cdots, a_n, b_n^*, \cdots, b_1^*).$$

For $n = 0$ one recovers the standard notion of a positive trace on an $*$ -algebra. Given a differential graded $*$ -algebra of noncommutative differential forms $(\Omega A, d)$ on A , a Hochschild $2n$ -cocycle on A defines a sesquilinear pairing on the space $\Omega^n A$ of n -forms (typically these would be middle-degree forms). For $\omega = a_0 da_1 \cdots da_n$ and $\eta = b_0 db_1 \cdots db_n$ one defines

$$\langle \omega, \eta \rangle := \varphi(b_0^* a_0, a_1, \cdots, a_n, b_n^*, \cdots, b_1^*),$$

extended by linearity. One has that $\langle a\omega, \eta \rangle = \langle \omega, a^*\eta \rangle$ for all $a \in A$. Positivity of φ is equivalent to positivity of this sesquilinear form on $\Omega^n A$.

Before we introduce a twisted analogue of the notion of positivity, we need to briefly recall twisted Hochschild and cyclic cohomologies.

Then, let A be an algebra and $\sigma : A \rightarrow A$ be an automorphism of A . For $n \geq 0$, let $C^n(A) = \text{Hom}_{\mathbb{C}}(A^{\otimes(n+1)}, \mathbb{C})$ be the space of $(n+1)$ -linear functionals (the n -cochains) on A . Define the twisted cyclic operator $\lambda_\sigma : C^n(A) \rightarrow C^n(A)$ by

$$(\lambda_\sigma \varphi)(a_0, \dots, a_n) = (-1)^n \varphi(\sigma(a_n), a_0, a_1, \dots, a_{n-1}).$$

Clearly, $(\lambda_\sigma^{n+1} \varphi)(a_0, \dots, a_n) = \varphi(\sigma(a_0), \dots, \sigma(a_n))$. Let

$$C_\sigma^n(A) = \ker((1 - \lambda_\sigma^{n+1}) : C^n(A) \rightarrow C^n(A))$$

denote the space of *twisted Hochschild n -cochains* on A . Define the *twisted Hochschild coboundary* operator $b_\sigma : C_\sigma^n(A) \rightarrow C_\sigma^{n+1}(A)$ and the operator $b'_\sigma : C_\sigma^n(A) \rightarrow C_\sigma^{n+1}(A)$ by

$$\begin{aligned} b_\sigma \varphi(a_0, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}) \\ &\quad + (-1)^{n+1} \varphi(\sigma(a_{n+1}) a_0, a_1, \dots, a_n), \\ b'_\sigma \varphi(a_0, \dots, a_{n+1}) &= \sum_{i=0}^n (-1)^i \varphi(a_0, \dots, a_i a_{i+1}, \dots, a_{n+1}). \end{aligned}$$

One checks that b_σ sends twisted cochains to twisted cochains. The cohomology of the complex $(C_\sigma^*(A), b_\sigma)$ is the *twisted Hochschild cohomology* of A . We also need the twisted cyclic cohomology. An n -cochain $\varphi \in C^n(A)$ is called *twisted cyclic* if $(1 - \lambda_\sigma)\varphi = 0$, or, equivalently

$$\varphi(\sigma(a_n), a_0, \dots, a_{n-1}) = (-1)^n \varphi(a_0, a_1, \dots, a_n),$$

for all a_0, a_1, \dots, a_n elements in A . Denote the space of cyclic n -cochains by $C_{\lambda_\sigma}^n(A)$. Obviously $C_{\lambda_\sigma}^n(A) \subset C_\sigma^n(A)$. The relation $(1 - \lambda_\sigma)b_\sigma = b'_\sigma(1 - \lambda_\sigma)$ shows that the operator b_σ preserves the space of cyclic cochains and we obtain the twisted cyclic complex of the pair (A, σ) , denoted $(C_{\lambda_\sigma}^n(A), b_\sigma)$. The cohomology of the twisted cyclic complex is called the *twisted cyclic cohomology* of A , and denoted $\text{HC}_\sigma^\bullet(A)$.

Definition 5.1. *A twisted Hochschild $2n$ -cocycle φ on an $*$ -algebra A is called twisted positive if the pairing:*

$$\langle a_0 \otimes a_1 \otimes \dots \otimes a_n, b_0 \otimes b_1 \otimes \dots \otimes b_n \rangle := \varphi(\sigma(b_0^*) a_0, a_1, \dots, a_n, b_n^*, \dots, b_1^*).$$

defines a positive sesquilinear form on the vector space $A^{\otimes(n+1)}$.

Let us now go back to the quantum projective line $\mathbb{C}P_q^1$. Let $h : \mathcal{A}(\text{SU}_q(2)) \rightarrow \mathbb{C}$ denote the normalized Haar state of $\text{SU}_q(2)$. It is a positive twisted trace obeying

$$h(xy) = h(\sigma(y)x), \quad \text{for } x, y \in \mathcal{A}(\text{SU}_q(2)),$$

with (modular) automorphism $\sigma : \mathcal{A}(\text{SU}_q(2)) \rightarrow \mathcal{A}(\text{SU}_q(2))$ given by

$$\sigma(x) = K^2 \triangleright x \triangleleft K^2.$$

(cf. [15, Prop. 4.15]). When restricted to $\mathcal{A}(\mathbb{CP}_q^1)$, it induces the automorphism

$$\sigma : \mathbb{CP}_q^1 \rightarrow \mathbb{CP}_q^1, \quad \sigma(x) = x \triangleleft K^2, \quad \text{for } x \in \mathcal{A}(\mathbb{CP}_q^1).$$

The bi-invariance of h on $\mathcal{A}(\mathrm{SU}_q(2))$ reduces to left invariance on $\mathcal{A}(\mathbb{CP}_q^1)$, that is to say:

$$(\mathrm{id} \otimes h) \circ \Delta_L(x) = h(x) 1_{\mathcal{A}(\mathbb{CP}_q^1)}, \quad \text{for } x \in \mathcal{A}(\mathbb{CP}_q^1),$$

where Δ_L is the coaction of $\mathcal{A}(\mathrm{SU}_q(2))$ on $\mathcal{A}(\mathbb{CP}_q^1)$ alluded to at the end of §3.1. Dually, there is invariance for the right action of $\mathcal{U}_q(\mathrm{su}(2))$ on $\mathcal{A}(\mathbb{CP}_q^1)$:

$$h(x \triangleleft v) = \varepsilon(v)h(x), \quad \text{for } x \in \mathcal{A}(\mathbb{CP}_q^1), v \in \mathcal{U}_q(\mathrm{su}(2)).$$

With $\omega_- \wedge \omega_+$ the central generator of $\Omega^2(\mathbb{CP}_q^1)$, h the Haar state on $\mathcal{A}(\mathbb{CP}_q^1)$ and σ its above modular automorphism, the linear functional

$$\int_h : \Omega^2(\mathbb{CP}_q^1) \rightarrow \mathbb{C}, \quad \int_h a \omega_- \wedge \omega_+ := h(a),$$

defines [24] a non-trivial twisted cyclic 2-cocycle τ on $\mathcal{A}(\mathbb{CP}_q^1)$ by

$$\tau(a_0, a_1, a_2) := \frac{1}{2} \int_h a_0 da_1 \wedge da_2.$$

The non-triviality means that there is no twisted cyclic 1-cochain α on $\mathcal{A}(\mathbb{CP}_q^1)$ such that $b_\sigma \alpha = \tau$ and $\lambda_\sigma \alpha = \alpha$. Thus τ is a non-trivial class in $\mathrm{HC}_\sigma^2(\mathbb{CP}_q^1)$.

Proposition 5.2. *The cochain $\varphi \in C^2(\mathcal{A}(\mathbb{CP}_q^1))$ defined by*

$$\varphi(a_0, a_1, a_2) = \int_h a_0 \partial a_1 \bar{\partial} a_2$$

is a twisted Hochschild 2-cocycle on $\mathcal{A}(\mathbb{CP}_q^1)$, that is to say $b_\sigma \varphi = 0$ and $\lambda_\sigma^3 \varphi = \varphi$; it is also positive, with positivity expressed as:

$$\int_h a_0 \partial a_1 (a_0 \partial a_1)^* \geq 0$$

for all $a_0, a_1 \in \mathcal{A}(\mathbb{CP}_q^1)$.

Before giving the proof we prove a preliminary result.

Lemma 5.3. *For any $a_0, a_1, a_2, a_3 \in \mathcal{A}(\mathbb{CP}_q^1)$ it holds that:*

$$\int_h a_0 (\partial a_1 \bar{\partial} a_2) a_3 = \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2.$$

Proof. Write $\partial a_1 \bar{\partial} a_2 = y \omega_- \wedge \omega_+$, for some $y \in \mathcal{A}(\mathbb{CP}_q^1)$. Using the fact that $\omega_- \wedge \omega_+$ commutes with elements in $\mathcal{A}(\mathbb{CP}_q^1)$, we have that

$$\begin{aligned} \int_h a_0 (\partial a_1 \bar{\partial} a_2) a_3 - \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2 &= \int_h a_0 y \omega_- \wedge \omega_+ a_3 - \int_h \sigma(a_3) a_0 y \omega_- \wedge \omega_+ \\ &= \int_h a_0 y a_3 \omega_- \wedge \omega_+ - \int_h \sigma(a_3) a_0 y \omega_- \wedge \omega_+ \\ &= h(a_0 y a_3) - h(\sigma(a_3) a_0 y) = 0 \end{aligned}$$

from the twisted property of the Haar state. □

Proof. of Proposition 5.2.

Using the derivation property of ∂ and $\bar{\partial}$ we have that

$$\begin{aligned} (b_\sigma \varphi)(a_0, a_1, a_2, a_3) &= \int_h a_0 a_1 \partial a_2 \bar{\partial} a_3 - \int_h a_0 \partial(a_1 a_2) \bar{\partial} a_3 \\ &+ \int_h a_0 \partial a_1 \bar{\partial}(a_2 a_3) - \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2 = \int_h a_0 (\partial a_1 \bar{\partial} a_2) a_3 - \int_h \sigma(a_3) a_0 \partial a_1 \bar{\partial} a_2 = 0, \end{aligned}$$

from the previous Lemma.

Next, the cyclic condition follows from invariance of the Haar state and of the calculus. Indeed, from the commutativity of the left and right $\mathcal{U}_q(\mathfrak{su}(2))$ -actions it holds that:

$$\varphi(\sigma(a_0), \sigma(a_1), \sigma(a_2)) = \int_h \sigma(a_0) \partial \sigma(a_1) \bar{\partial} \sigma(a_2) = \int_h \sigma(a_0 \partial a_1 \bar{\partial} a_2);$$

writing $a_0 \partial a_1 \bar{\partial} a_2 = y \omega_- \wedge \omega_+$, for some $y \in \mathcal{A}(\mathbb{C}\mathbb{P}_q^1)$, left $\mathcal{U}_q(\mathfrak{su}(2))$ -invariance of the forms ω_\pm , that is $\omega_\pm \triangleleft K = \omega_\pm$ (part of the dual invariance statement of (3.12)), yields $\sigma(a_0 \partial a_1 \bar{\partial} a_2) = \sigma(y) \omega_- \wedge \omega_+$ and in turn,

$$\begin{aligned} \varphi(\sigma(a_0), \sigma(a_1), \sigma(a_2)) &= \int_h \sigma(y) \omega_- \wedge \omega_+ = h(\sigma(y)) = h(y \triangleleft K^2) = h(y) = \int_h y \omega_- \wedge \omega_+ \\ &= \int_h a_0 \partial a_1 \bar{\partial} a_2 = \varphi(a_0, a_1, a_2). \end{aligned}$$

Finally, for the twisted positivity of φ , the hermitian scalar product on $\Omega^{(1,0)}(\mathbb{C}\mathbb{P}_q^1)$,

$$\langle a_0 \partial a_1, b^0 \partial b^1 \rangle := \varphi(\sigma(b_0^*) a_0, a_1, b_1^*) = \int_h \sigma(b_0^*) a_0 \partial a_1 \bar{\partial} b_1^*,$$

determines a positive sesquilinear form if for all $a_0, a_1 \in A(\mathbb{C}\mathbb{P}_q^1)$ it holds that

$$\int_h \sigma(a_0^*) a_0 \partial a_1 \bar{\partial} a_1^* = \int_h a_0 \partial a_1 (a_0 \partial a_1)^* \geq 0.$$

The first equality follows again from the Lemma. Indeed,

$$\int_h a_0 \partial a_1 (a_0 \partial a_1)^* = \int_h a_0 \partial a_1 (\partial a_1)^* a_0^* = \int_h \sigma(a_0^*) a_0 \partial a_1 \bar{\partial} a_1^*.$$

Then, if $\partial a_1 = y \omega_+$ it follows that $\bar{\partial} a_1^* = (\partial a_1)^* = -\omega_- y^*$; then

$$\begin{aligned} \int_h \sigma(a_0^*) a_0 \partial a_1 \bar{\partial} a_1^* &= - \int_h \sigma(a_0^*) a_0 y \omega_+ \wedge \omega_- y^* = q^2 \int_h \sigma(a_0^*) a_0 y y^* \omega_- \wedge \omega_+ \\ &= q^2 h(\sigma(a_0^*) a_0 y y^*) = q^2 h(a_0 y y^* (a_0)^*) = q^2 h(a_0 y (a_0 y^*)^*) \geq 0, \end{aligned}$$

the positivity being evident. \square

Proposition 5.4. *The twisted Hochschild cocycles τ and φ are cohomologous.*

Proof. Let us define a twisted Hochschild 1-cochain ψ on $\mathcal{A}(\mathbb{C}\mathbb{P}_q^1)$ by

$$\psi(a, b) = \frac{1}{2} \int_h a \partial \bar{\partial}(b),$$

for $a, b \in \mathcal{A}(\mathbb{C}\mathbb{P}_q^1)$. It is a twisted cochain since

$$\begin{aligned} 2\psi(\sigma(a), \sigma(b)) &= \int_h \sigma(a) \partial \bar{\partial}(\sigma(b)) = \int_h \sigma(a) \sigma(\partial \bar{\partial}(b)) \\ &= \int_h \sigma(a \partial \bar{\partial}(b)) = \int_h a \partial \bar{\partial}(b) = 2\psi(a, b), \end{aligned}$$

for the invariance of the integral as seen before. We have

$$\begin{aligned} (b_\sigma \psi)(a_0, a_1, a_2) &= \psi(a_0 a_1, a_2) - \psi(a_0, a_1 a_2) + \psi(\sigma(a_2) a_0, a_1) \\ &= \frac{1}{2} \int_h (a_0 a_1 \partial \bar{\partial}(a_2) - a_0 \partial \bar{\partial}(a_1 a_2) + \sigma(a_2) a_0 \partial \bar{\partial}(a_1)) \\ &= \frac{1}{2} \int_h (a_0 a_1 \partial \bar{\partial}(a_2) - a_0 \partial \bar{\partial}(a_1 a_2) + a_0 (\partial \bar{\partial}(a_1)) a_2). \end{aligned}$$

On the other hand:

$$\frac{1}{2} a_0 da_1 \wedge da_2 = a_0 \partial a_1 \bar{\partial} a_2 + \frac{1}{2} a_0 (-\partial \bar{\partial}(a_1 a_2) + (\partial \bar{\partial} a_1) a_2 + a_1 \partial \bar{\partial} a_2).$$

Comparing these last two relations, we find that

$$\frac{1}{2} \int_h a_0 da_1 \wedge da_2 = \int_h a_0 \partial a_1 \bar{\partial} a_2 + (b_\sigma \psi)(a_0, a_1, a_2)$$

or $\varphi - \tau = b_\sigma \psi$, as stated. \square

It is worth stressing that φ is not a twisted cyclic cocycle, only a Hochschild one. In fact, it is trivial as a twisted Hochschild cocycle since it is known that for the modular automorphism σ , the twisted Hochschild cohomology of the algebra $\mathcal{A}(\mathbb{C}\mathbb{P}_q^1)$ is trivial [14]. To get non-trivial twisted Hochschild cohomology one needs twisting with the inverse modular automorphism.

6. FINAL REMARKS

We have shown that much of the structure of the 2-sphere as a complex curve, or equivalently as a conformal manifold, actually survive the q -deformation and have natural generalizations on the quantum 2-sphere. Chiefly among these is the identification of a quantum homogeneous coordinate ring with the coordinate ring of the quantum plane. Also, in parallel with the fact that positive Hochschild cocycles on the algebra of smooth functions on a compact oriented 2-dimensional manifold encode the information for complex structures on the surface [7, Section VI.2], we have formulated a notion of twisted positivity for twisted Hochschild and cyclic cocycles – given that there are no non-trivial 2-dimensional cyclic cocycles on the quantum 2-sphere – and exhibited an example of a twisted positive Hochschild cocycle in the case of our complex structure on this sphere. Now, additional tools of noncommutative geometry are available there, including the abstract perturbation of conformal structures by Beltrami differentials as explained in [7, Example 8, Section VI.4]. A great challenge is to prove an analogue of the measurable Riemann mapping theorem in the q -deformed case. The formalism of q -groups allows one

to set-up a simple algebraic framework but the real challenge resides in the analysis. An attack of these problems should await a future time.

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MATHEMATICS DEPARTMENT, THE UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, N6A 5B7, CANADA

E-mail address: masoud@uwo.ca

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI TRIESTE, VIA A. VALERIO 12/1, I-34127 TRIESTE, ITALY, AND INFN, SEZIONE DI TRIESTE, TRIESTE, ITALY

E-mail address: landi@univ.trieste.it

INSTITUTE FOR MATHEMATICS, ASTROPHYSICS AND PARTICLE PHYSICS, FACULTY OF SCIENCE, RADOBOD UNIVERSITY NIJMEGEN, TOERNOOIVELD 1, 6525 ED NIJMEGEN, THE NETHERLANDS

E-mail address: waltervs@math.ru.nl