THE GRUNSKY OPERATOR, AHLFORS' QUESTION AND GEOMETRY OF UNIVERSAL TEICHMÜLLER SPACE

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Abstract. We prove somewhat modified Grinshpan conjecture on the norm of the Grunsky operator generated by univalent functions in the disk. The result implies fundamental consequences for the Ahlfors problem concerning the quasiconformal extension of holomorphic maps and for geometric, plurisubharmonic and pluripotential features of the universal Teichmüller space.

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1. Introduction and main results

1.1. **The Grunsky operator.** In 1939, Grunsky established the necessary and sufficient conditions for univalence of holomorphic functions on finitely connected domains on the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ in terms of an infinite system of coefficient inequalities. The method of Grunsky inequalities was generalized in certain directions and extended to bordered Riemann surfaces with a finite number of boundary components (see, e.g., [Gr], [Le], [M], [Po], [SS]).

In particular, the Grunsky theorem for the disk states that a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood U_0 of $z = \infty$ with a given germ at infinity is extended to a univalent holomorphic function on the disk

$$\Delta^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : \ |z| > 1\}$$

if and only if its **Grunsky coefficients** α_{mn}, determined by the expansion

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in U_0^2, \tag{1.1}$$

with the principal branch of logarithmic function, satisfy the inequalities

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le 1. \tag{1.2}$$

Here the sequences $\mathbf{x} = (x_n)$ run over the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\| = \left(\sum\limits_1^{\infty} |x_n|^2 \right)^{1/2}$. Then the double series (1.1) is convergent on $(\Delta^*)^2$.

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The quantity

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| : \mathbf{x} = (x_n) \in S(l^2) \Big\} \tag{1.3}$$

is called the **Grunsky norm** of f.

Assume that $f(z) \neq 0$ on Δ^*. Then the corresponding functions $F_f(\zeta) = 1/f(1/\zeta)$ are holomorphic and univalent on the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. One can define their Grunsky coefficients similarly to (1.1), getting the equality $\varkappa(F_f) = \varkappa(f)$.

For the functions with quasiconformal extensions, we have instead of (1.2) a stronger inequality

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le k = k(f), \tag{1.4}$$

where $k(f) < 1$ is the **Teichmüller norm** of f, i.e., the minimal dilatation of quasiconformal extensions of f to $\widehat{\mathbb{C}}$ (see [Ku1]). By abuse of notation, we use for extensions the same symbol f.

Due to [KK2], [Kr6], the set of f with $\varkappa(f) < k(f)$ is dense in Σ, and moreover, the **Schwarzian derivatives** of these functions

$$S_f(z) = \Big(\frac{f''}{f'}\Big)' - \frac{1}{2}\Big(\frac{f''}{f'}\Big)^2, \quad z \in \Delta^*,$$

form an open and dense in the universal Teichmüller space $\mathbf{T}$ (see the definitions in Section 2.1). On the other hand, the functions with $\varkappa(f) = k(f)$ play a crucial role in applications of Grunsky inequalities to the Teichmüller space theory.

Note also that by a theorem of Pommerenke and Zhuravlev, any $f \in \Sigma$, with $\varkappa(f) \le k < 1$, has k_1-quasiconformal extensions to $\widehat{\mathbb{C}}$ with $k_1 = k_1(k) \ge k$ (see [Po]; [KK1, pp. 82-84), [Zh]).

The Grunsky (matrix) operator $\mathcal{G}(f) = (\sqrt{mn}\ \alpha_{mn})_{m,n=1}^{\infty}$ has been investigated from different points of views. For each $f \in \Sigma^0$, it acts as a linear operator $l^2 \to l^2$ contracting the norms of elements $\mathbf{x} \in l^2$. The norm of this operator can be evaluated using a stronger form of the inequality (1.2)

$$\sum_{m=1}^{\infty} \Big| \sum_{n=1}^{N} \sqrt{mn}\ \alpha_{mn} x_n \Big| \le \Big| \sum_{n=1}^{N} |x_n|^2, \quad N = 1, 2, \dots ,$$

and this norm equals $\varkappa(f)$.

1.2. **Classes of functions.** We denote by $\Sigma(k)$ and $S(k)$ the subclasses of Σ and S containing the functions having k'-quasiconformal extensions to Δ and Δ^*, respectively, with $k' \le k$, and let

$$S^0 = \bigcup_k S(k), \quad \Sigma^0 = \bigcup_k \Sigma(k).$$

The original normalization of the functions $f \in \Sigma^0$ includes only two conditions $f(\infty) = \infty,\ f'(\infty) = 1$ (respectively, $F(0) = 0,\ F'(0) = 1$ for $F \in S^0$). Thus the Schwarzian equation $S_w(z) = \varphi(z)$ for f or F and the Beltrami equation $\overline{\partial} w = \mu \partial w$ for quasiconformal extensions determine their solutions up to linear transformations, which for $f \in \Sigma$ are the translations $w \mapsto w + b_0$ and for $F \in S$ assume the form $w \mapsto w/(1 - \alpha w)$, where the admissible values of α are determined by $a_2 = -b_0$. This forces one to deal with some subclasses of $\Sigma(k)$ and $S(k)$ consisting of the maps with a complete normalization (fixing the admissible values b_0 and a_2). The appropriate collections are, for example,

$$\Sigma_k(0) = \{f \in \Sigma(k) : f(0) = 0\}, \quad S_k(\infty) = \{F \in S(k) : F(\infty) = \infty\}.$$

Note that the additional normalization conditions concern quasiconformal extensions of the initial conformal maps of the disks Δ^* and Δ, respectively, thus reflect on the minimal dilatations. Some other classes will be presented later.

1.3. **The Grinshpan conjecture.** Using another equivalent definition of norm of the Grunsky operator, A. Grinshpan has established in [G1] that for any function

$$f(z) = z + b_0 + b_1 z^{-1} + \dots \in \Sigma^0 \tag{1.5}$$

and any integer $p \ge 2$, its p-root transform

$$\mathcal{R}_p: \ f(z) \mapsto f_p(z) := f(z^p)^{1/p} = z + \frac{b_0}{p} z^{-p+1} + \dots, \quad p \ge 2, \tag{1.6}$$

does not decrease the Grunsky norm, i.e.,

$$\varkappa(f) \le \varkappa_p(f) := \varkappa(f_p), \tag{1.7}$$

and this inequality is sharp on each subset $\{f \in \Sigma: \ \varkappa(f) = k\}$.

Note that every image $f_p = \mathcal{R}_p f$ is p-symmetric with respect to rotation around the origin, i.e.,

$$f_p(e^{2n\pi i/p} z) = e^{2n\pi i/p} f_p(z), \quad n = 0, 1, \dots, p-1; \quad p \ge 2,$$

for any $z \in \Delta^*$, and is connected with its original $f \in \Sigma$ by the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{C}}_p & \xrightarrow{\mathcal{R}_p f} & \widetilde{\mathbb{C}}_p \\ {\scriptstyle \pi_p}\downarrow & & \downarrow{\scriptstyle \pi_p} \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

where $\widetilde{\mathbb{C}}_p$ denotes the p-sheeted sphere $\widehat{\mathbb{C}}$ branched over 0 and ∞, and the projection $\pi_p(z) = z^p$.

The covering maps $\mathcal{R}_p f$ of f in $\bigcup_k \Sigma_k(0)$ and $\bigcup_k S_k(\infty)$ commute with projection π_p on the whole sphere $\widehat{\mathbb{C}}$, while for $f \in \Sigma^0$, not preserving the origin, only on the punctured sphere $\mathbb{C} \setminus \{0\}$.

The sequence $\varkappa_p(f), \ p = 2, 3, \dots$, is not necessarily nondecreasing. For example, any of the maps

$$f_m(z) = z\big(1 + t z^{-m}\big)^{2/m}: \ \Delta^* \to \widehat{\mathbb{C}}^*, \quad m = 2, 3, \dots, \quad |t| < 1,$$

has $|t|$-quasiconformal extension across the unit circle $S^1 = \partial\Delta$ given by

$$f_m(z) = z\big[1 + t(\overline{z}/z)^{m/2}\big]^{2/m}$$

whose Beltrami coefficient in Δ equals

$$\mu_{f_m}(z) = t\Big(\frac{\overline{z}}{z}\Big)^{m/2-1} = t\Big(\frac{|z|}{z}\Big)^{m-2}.$$

For any odd $m \ge 3$, we have the strict inequality $\varkappa(f) < |t| = k(f)$ (see below Proposition 2.4), while the Beltrami coefficient, for example, of $f_2(z) = f(z^2)^{1/2}$ equals $\mu(z^2)|z|^4/z^4$, hence $\varkappa(f_2) = k(f_2) = |t|$, and similarly for all even m (cf. below Proposition 2.4).

The Grunsky coefficients of f_p will be denoted by $\alpha_{mn}^{(p)}(f)$. Let

$$\mu_p(z) = \mathcal{R}_p^* \mu(z) = \mu(z^p)\overline{z}^{p-1}/z^{p-1}, \quad p = 2, 3, \dots .$$

In his survey paper [G2], Grinshpan posed a deep conjecture, which in our terms can be formulated as follows.

Conjecture. *For every $f \in \Sigma_k(0)$, the upper limit $\limsup_{p\to\infty} \varkappa_p(f)$ must be equal to k (i.e., to the smallest possible values of k', for which $f \in \Sigma_{k'}(0)$).*

In fact, this conjecture was stated in [G2] in an equivalent form for the functions from $S_k(\infty)$.

We call the quantity

$$\widehat{\varkappa}(f) := \limsup_{p\to\infty} \varkappa_p(f) \tag{1.8}$$

the **limit Grunsky norm** of a function f.

The following **example** given by Kühnau shows that transformed functions $\mathcal{R}_p(f|\Delta^*)$ can possess the extremal quasiconformal extensions to the plane with greater dilatations; in other words, the Teichmüller norm can increase under the p-root transforms. The extremal extension of

$$f(z) = z + k^2 z^{-1} + 2k \quad (0 < k < 1)$$

onto the disk Δ (i.e., with minimal dilatation) is the affine map $\widetilde{f}(z) = z + k^2\overline{z} + 2k$ having constant Beltrami coefficient $\mu(z) = k^2$; thus $\varkappa(f) = k(f) = k^2$. In contrast, both Grunsky and Teichmüller norms of the image of this function under the square root transform

$$f_2(z) = f(z^2)^{1/2} = z + kz^{-1} + \dots$$

are equal to k (at least for small k), and the same holds for all roots f_{2p}. Thus

$$\limsup_{p\to\infty} \varkappa_p(f) = k = \limsup_{p\to\infty} \varkappa_{2p}(f) > k^2.$$

In a similar way, one can construct the functions $f \in \Sigma$ with $k(f) < 1$ having $k(f_2) = \limsup \varkappa_{2p}(f)$ arbitrary close to 1. We see that independence of the free coefficient b_0 on the Schwarzian and Beltrami equations defining the functions $f \in \Sigma$ and their quasiconformal extensions can cause an essential change of the Grunsky norm under the root transforms.

1.4. **A modified conjecture.** Keeping in view the applications, we consider here certain modification of this conjecture, which allows us to extend it to the whole class Σ^0 (i.e., drop the condition $f(0) = 0$). In order to ensure nonincreasing the Teichmüller norm under transforms $\mathcal{R}_p$ (which provides, in particular, the bound $\kappa(f_p) \le k(f)$ for all p), we consider for each $f \in \Sigma$ its **power part**

$$f_*(z) = f(z) - b_0 = z + b_1 z^{-1} + \dots \tag{1.9}$$

having the same Grunsky and Teichmüller norms as f. To apply these parts, we must be sure that that for any f_* the image $f_*(\Delta^*)$ is located outside from the origin $w = 0$ and hence, the inversion $1/f_*(1/z)$ belongs to S.

Let $F(z) \in S$. Take its Schwarzian $S_F(z) = q(z)$ and consider the independent solutions w_1, w_2 to the linear differential equation

$$w'' + \frac{q(z)}{2} w = 0 \quad \text{in} \quad \Delta$$

satisfying

$$w_1(0) = 0,\ w_1'(0) = 1; \quad w_2(0) = 1,\ w_2'(0) = 0.$$

The ratio

$$F_0(z) = \frac{w_1(z)}{w_2(z)} = z + c_3 z^3 + \dots. \quad (c_2 = 0), \quad |z| < 1,$$

has the same Schwarzian $q(z)$. This function is univalent for all $|z| < 1$ (together with the original F), and one has to show that F_0 is holomorphic on the whole unit disk.

Note that $F_0(z*) = \infty$ when $w_2(z*) = 0$ (or w_2 has at the point $z*$ zero of a higher order than w_1). By the well-known properties of solutions to the linear differential equations, w_1 and w_2 are connected by

$$w_1 w_2' - w_2 w_1' \equiv \text{const} = 1$$

(with the Wronskian in the left-hand side), which implies (cf. [Go, Ch. 3])

$$F_0'(z) = 1/w_2^2,$$

and

$$q(z) = -2\sqrt{F_0'}\,\frac{d^2\left(1/\sqrt{F_0'}\right)}{dz^2} = -2\frac{w_2''}{w_2},$$

where d stands for the differentiation. It follows from the last equality that $w_2(z)$ does not vanish for all $|z| < 1$ (since $q(z)$ is there holomorphic). Thus $F_0(z) \neq \infty$, and letting

$$f_*(z) = \frac{1}{F_0(1/z)} = z + b_1 z^{-1} + \dots ,$$

one obtains a function from Σ of the form (1.9).

Vice versa, any admissible power part f_* is obtained in such way, and it follows from above that the admissible values of $b_0 = -a_2$ for $f \in \Sigma^0$ are those which range over the closed complementary domain $\widehat{\mathbb{C}} \setminus f_*(\Delta^*)$.

We can now formulate our modified version of Grinshpan's conjecture.

Conjecture. *For every $f \in \Sigma^0$, the upper limit $\limsup\limits_{p\to\infty} \varkappa_p(f_*)$ is equal to the Teichmüller norm $k(f)$.*

It will be shown that in such form the conjecture is naturally related to the universal Teichmüller space $\mathbf{T}$, while its original version for functions of $\Sigma_k(0)$ can be connected with Teichmüller space of the punctured disk.

1.5. **Main theorem.** The purpose of this paper is to prove the modified version of Grinshpan's conjecture and apply the result to solving certain well-known problems. In fact, we prove somewhat more:

Theorem 1.1. *For every function $f \in \Sigma^0$, we have the equalities*

$$\widehat{\varkappa}(f_*) = \limsup_{p\to\infty} \varkappa_p(f_*) = \sup_p \varkappa_p(f_*) = k(f). \tag{1.10}$$

and

$$\widehat{\varkappa}(f) = \max\{k(f), k(f_2)\}. \tag{1.11}$$

First we observe some **remarks** concerning this theorem.

1. To have monotone nondecreasing of Grunsky norms, we pass to a suitable subsequence of roots (1.6). For example, one can take

$$p = 2^m, \quad m = 2, 3, \dots ,$$

because for any m, the inequality (1.7) implies

$$\varkappa_{2^m}(f_*) \leq \varkappa_{2^{m+1}}(f_*). \tag{1.12}$$

It suffices to establish that

$$\lim_{m\to\infty} \varkappa_{2^m}(f_*) = k(f). \tag{1.13}$$

The equality (1.11) is easily obtained from (1.10) using the fact that for any $f \in \Sigma$ its first squaring f_2 is of the form (1.8).

2. The proof of Theorem 1.1 relies on the geometry of the universal Teichmüller space and involves an implicit rapidly increasing subsequence $\{p_m\}$. Iv view of monotonicity, it suffices to establish (1.13) for a subsequence $\{2^{m_j}\}$, $j \to \infty$.

The extremal quasiconformal extensions of the the functions f with sufficiently regular boundary values on $S^1 = \partial\Delta^*$ are of Teichmüller type being defined by holomorphic quadratic differentials on Δ.

The extensions preserving the origin 0 lead to quadratic differentials with simple pole at 0. The corresponding generalization of Theorem 1.1, proving the original Grinshpan conjecture, in fact follows the same line (with a needed modification of arguments) and will be given somewhere.

3. The use of the power parts f_* means that we distinguish in Σ^0 a subclass of functions without free terms, on which the transforms $\mathcal{R}_p$ act with preserving the Teichmüller norm. There are other subsets in Σ with this property, for example, the set of all f whose extremal extensions to $\widehat{\mathbb{C}}$ fix 0 and do not there a singularity. The arguments from the proof of Theorem 1.1 are extended to such maps straightforwardly.

4. Note also that $\mathcal{R}_p$ is a special case of renormalizations applied in complex dynamics. It turned out that Grinshpan's conjecture is deeply connected with the theory of renormalizations and measurable foliations of dynamical systems, which forces some restrictions to admissible subsequences $\{p_m\}$.

1.6. **Some applications.** Theorem 1.1 has many important applications.

1. A question of Ahlfors. In 1963, Ahlfors stated the following question which gave rise to various investigations of quasiconformal extendibility of conformal maps.

Question. *Let f be a conformal map of the disk (or half-plane) onto a domain with quasiconformal boundary (quasicircle). How can this map be characterized?*

He conjectured that the characterization should be in analytic properties of the invariant (logarithmic derivative) f''/f' (see [Ah]). Many results were established on quasiconformal extensions of holomorphic maps in terms of f''/f' and other invariants (see, e.g., the survey [Kr5] and the references there).

Theorem 1.1 solves the problem completely in terms of the limit Grunsky norm of the power part f_* of f.

Theorem 1.2. *A $\widehat{\mathbb{C}}$-holomorphic map f of the disk Δ^* with the expansion*

$$f(z) = z + \text{const} + O(1/z)$$

near the infinity is an embedding and has k-quasiconformal extensions across the unit circle S^1 onto the whole sphere if and only if $k \geq \widehat{\varkappa}(f_*)$, where f_* is the power part of f. The bound $k = \widehat{\varkappa}(f_*)$ equals the minimal dilatation of extensions of f.

The curve $L = f(S^1)$ is a $\widehat{\varkappa}(f_*)$-quasicircle (i.e., the image of S^1 under $\widehat{\varkappa}(f_*)$-quasiconformal maps of $\widehat{\mathbb{C}}$), and its reflection coefficient $q_L = \widehat{\varkappa}(f_*)$.

We see that f provides a quantitative characterization of its quasiconformal extensions to the plane and of the boundary curve $f(S^1)$.

Note that all known criteria for quasiconformal extendibility of holomorphic functions with prescribed bound for dilatations established earlier imply either sufficient or necessary conditions but not both simultaneously.

2. Applications to geometry of universal Teichmüller space. Theorem 1.1 reveals the fundamental geometric and potential features of the universal Teichmüller space $\mathbf{T}$ giving a direct proof of certain basic theorems.

But first observe that each Grunsky norm $\varkappa_p$, as a function of the Schwarzian derivatives $S_f = S_{f_*}$ is plurisubharmonic on $\mathbf{T}$ and can be approximated by holomorphic maps $h_p : \mathbf{T} \to \Delta$ (such maps will be constructed below explicitly). Therefore, the equality (1.13) implies

$$\lim_{p\to\infty} h_{2^p}(S_f) = k(f_*). \tag{1.14}$$

The power parts f_* of the functions $f \in \Sigma^0$ naturally appear in the Teichmüller space theory as the ratios η_2/η_1 of independent solutions of the linear differential equations

$$2\eta'' + \varphi\eta = 0 \quad \text{on} \quad \Delta^*$$

with φ running over $\mathbf{T}$, normalized by

$$\eta_1(z) = \frac{1}{z} + \frac{c_2}{z^2} + \dots , \quad \eta_2(z) = 1 + \frac{d_1}{z} + \dots . \tag{1.15}$$

Denote the Carathéodory, Kobayashi and Teichmüller metrics of $\mathbf{T}$ by $c_{\mathbf{T}}$, $d_{\mathbf{T}}$, $\tau_{\mathbf{T}}$ and their infinitesimal Finsler forms on the tangent bundle $\mathcal{T}(\mathbf{T})$ of $\mathbf{T}$ by $\mathcal{C}_{\mathbf{T}}(\varphi, v), \mathcal{K}_{\mathbf{T}}(\varphi, v), F_{\mathbf{T}}(\varphi, v)$, respectively. It is elementary that

$$c_{\mathbf{T}}(\cdot, \cdot) \le d_{\mathbf{T}}(\cdot, \cdot) \le \tau_{\mathbf{T}}(\cdot, \cdot), \tag{1.16}$$

and similarly for the infinitesimal metrics.

In the following theorem we assume (without loss of generality) that the functions $f \in \Sigma^0$ defining the universal Teichmüller space $\mathbf{T}$ are chosen as the ratios of (1.15), thus satisfy $f = f_*$. The relations (1.14) and (1.16) imply

Theorem 1.3. *(a) For every point $\varphi = S_f \in \mathbf{T}$, its distance from the origin in each of the Carathéodory, Kobayashi and Teichmüller metrics (hence in any conractible invariant metric on $\mathbf{T}$) equals $\tanh^{-1} \widehat{\varkappa}(f_*)$, i.e.,*

$$c_{\mathbf{T}}(\mathbf{0}, S_f) = d_{\mathbf{T}}(\mathbf{0}, S_f) = \tau_{\mathbf{T}}(\mathbf{0}, S_f) = \tanh^{-1} \widehat{\varkappa}(f). \tag{1.17}$$

(b) For every tangent vector $v = \phi_{\mathbf{T}}(\mathbf{0})\mu$ at the origin of $\mathbf{T}$, the infinitesimal Carathéodory, Kobayashi and Teichmüller metrics are given by

$$\mathcal{C}_{\mathbf{T}}(\mathbf{0}, v) = \mathcal{K}_{\mathbf{T}}(\mathbf{0}, v) = F_{\mathbf{T}}(\mathbf{0}, v) = \limsup_{p \to \infty} \sup_{\mathbf{x} \in S(l^2)} |\langle \mathcal{R}_p^* \mu, \psi_{\mathbf{x}} \rangle_\Delta|, \tag{1.18}$$

where $\phi_{\mathbf{T}}$ is the defining projection for $\mathbf{T}$, $\mathcal{R}_p^* \mu = \mu(z^p)|z|^{2p-2}/z^{2p-2}$,

$$\psi_{\mathbf{x}}(z) = \omega_{\mathbf{x}}(z)^2 = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m x_n z^{m+n-2} \in A_1^2, \tag{1.19}$$

with $\mathbf{x} = (x_n) \in S(l^2)$, and $\|\psi_{p,\mathbf{x}}\|_{A_1} = 1$.

Since the universal Teichmüller space $\mathbf{T}$ is a homogeneous complex domain with respect to the action of its modular group, Theorem 1.3 also implies the equality of invariant distances between any of two points in $\mathbf{T}$ and the corresponding equality for infinitesimal metrics.

Corollary 1.4. *The Carathéodory metric $c_{\mathbf{T}}$ of the universal Teichmüller space $\mathbf{T}$ coincides with its Teichmüller metric $\tau_{\mathbf{T}}$, and, consequently, all contractible (that is, non-increasing under holomorphic maps) invariant distances on $\mathbf{T}$ coincide. In particular, for any pair of points φ_1, φ_2 in $\mathbf{T}$, we have the equality*

$$c_{\mathbf{T}}(\varphi_1, \varphi_2) = \tau_{\mathbf{T}}(\varphi_1, \varphi_2) = d_{\mathbf{T}}(\varphi_1, \varphi_2) = \inf d_\Delta\big(h^{-1}(\varphi_1), h^{-1}(\varphi_2)\big), \tag{1.20}$$

where d_Δ denotes the hyperbolic Poincaré metric on the unit disk of Gaussian curvature -4 and the infimum is taken over all $h \in \mathrm{Hol}(\Delta, \mathbf{T})$.

Similarly, for the corresponding infinitesimal forms $\mathcal{C}_{\mathbf{T}}(\varphi, v), F_{\mathbf{T}}(\varphi, v), \mathcal{K}_{\mathbf{T}}(\varphi, v)$ of these metrics (determined on the tangent bundle $\mathcal{T}(\mathbf{T})$ of $\mathbf{T}$), we have the equality

$$\mathcal{C}_{\mathbf{T}}(\varphi, v) = \mathcal{K}_{\mathbf{T}}(\varphi, v) = F_{\mathbf{T}}(\varphi, v), \tag{1.21}$$

and all these metrics have holomorphic curvature -4.

The assertions of this corollary were established by another method in [Kr4]. The second equality in (1.20) is a special case of the Gardiner-Royden theorem on the coincidence of the Kobayashi and Teichmüller metrics on all Teichmüller spaces (see [GL], [Ro1], [EKK]). Its infinitesimal form (the second equality in (1.21)) was establsihed in [EM].

Corollary 1.5. *(a) For every point $\varphi = S_f \in \mathbf{T}$, the pluricomplex Green function $g_{\mathbf{T}}(\mathbf{0}, S_f)$ with pole at the origin of $\mathbf{T}$ is given by*

$$g_{\mathbf{T}}(\mathbf{0}, S_f) = \log \widehat{\varkappa}(f). \tag{1.22}$$

(b) For any pair (φ, ψ) of points in $\mathbf{T}$, we have

$$g_{\mathbf{T}}(\varphi, \psi) = \log \tanh d_{\mathbf{T}}(\varphi, \psi) = \log \tanh c_{\mathbf{T}}(\varphi, \psi) = \log k(\varphi, \psi), \tag{1.23}$$

where $k(\varphi, \psi)$ denotes the extremal dilatation of quasiconformal maps determining the Teichmüller distance between φ and ψ.

Recall that the **pluricomplex Green function** $g_D(x, y)$ of a domain D in a complex Banach space X with pole y is defined by

$$g_D(x, y) = \sup u_y(x) \quad (x, y \in D)$$

followed by the upper semicontinuous regularization

$$v^*(x) = \limsup_{x' \to x} v(x').$$

The supremum here is taken over all plurisubharmonic functions $u_y(x) : \ D \to [-\infty, 0)$ such that

$$u_y(x) = \log \|x - y\|_X + O(1)$$

in a neighborhood of the pole y. Here $\| \cdot \|_X$ denotes the norm on X, and the remainder term $O(1)$ is bounded from above. The Green function $g_D(x, y)$ is a maximal plurisubharmonic function on $D \setminus \{y\}$ (unless it is identically $-\infty$). The first equality in (1.23) is in fact a special case of the general equality

$$g_D(x, y) = \log \tanh d_D(x, y),$$

which holds for any hyperbolic Banach domain whose Kobayashi metric d_D is logarithmically plurisubharmonic (cf [Di], [Kl], [Kr3]).

3. Fredholm eigenvalues. The Fredholm eigenvalues ρ_n of a smooth closed Jordan curve $L \subset \widehat{\mathbb{C}}$ are the eigenvalues of its double-layer potential, i.e., of the integral equation

$$u(z) + \frac{\rho}{\pi} \int\limits_L u(\zeta) \frac{\partial}{\partial n_\zeta} \log \frac{1}{|\zeta - z|} ds_\zeta = h(z), \tag{1.24}$$

where n_ζ is the outer normal and ds_ζ is the length element at $\zeta \in L$. This equation appears in many applications.

These values are intrinsically connected with the Grunsky coefficients of the corresponding conformal maps. This is qualitatively expressed by the Kühnau-Schiffer theorem on reciprocity of $\varkappa(f)$ to the least positive Fredholm eigenvalue ρ_L. It is defined for any oriented closed Jordan curve $L \subset \widehat{\mathbb{C}}$ by

$$\frac{1}{\rho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)}, \tag{1.25}$$

where G and G^* are, respectively, the interior and exterior of L; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$ (cf. [Ku3], [Sc], [Sch]).

This value ρ_L naturally arises in various subjects. One of the reasons is that by applying to the equation (1.24) the standard approximation method, the speed of approximation is equal to $O(1/\rho_L)$.

For these eigenvalues, Theorem 1.1 implies

Theorem 1.6. *For any quasicircle $L \subset \widehat{\mathbb{C}}$, its reflection coefficient q_L is represented by the appropriate Fredholm eigenvalues:*

$$q_L = \limsup_{m \to \infty} \frac{1}{\rho_{\mathcal{R}_{2m}(L)}}.$$

Also the Teichmüller norms of the (appropriately renormalized) Riemann mapping functions of the interior and exterior domains of L both are equal to the same limit.

This shows that, for example, in the case of $C^{1+\delta}$-smooth curves L ($\delta > 0$), the smallest positive solutions to the equation (1.24) for appropriate covers of L determine completely the quantitative characteristics of L and quasiconformal extensions of its Riemann mapping functions.

The Riemann mapping functions assume the canonical normalizations of maps obtained by applying appropriate Möbius transformations. This does not reflect on the Teichmüller norm.

Another application of Theorem 1.1 is presented in the last section.

2. Preliminaries

We briefly present here certain results underlying the proof of the theorems formulated above. These results hold in a more general setting but this will not be used here.

2.1. **Universal Teichmüller space and its invariant metrics.** The universal Teichmüller space $\mathbf{T}$ is the space of quasisymmetric homeomorphisms of the unit circle $S^1 = \partial\Delta$ factorized by Möbius maps. The canonical complex Banach structure on $\mathbf{T}$ is defined by factorization of the ball of the **Beltrami coefficients** (or complex dilatations)

$$\mathrm{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}) : \ \mu|\Delta^* = 0, \ \|\mu\| < 1\}, \tag{2.1}$$

letting $\mu_1, \mu_2 \in \mathrm{Belt}(\Delta)_1$ be equivalent if the corresponding quasiconformal maps w^{μ_1}, w^{μ_2} (solutions to the Beltrami equation $\partial_{\overline{z}} w = \mu \partial_z w$ with $\mu = \mu_1, \mu_2$) coincide on the unit circle $S^1 = \partial\Delta^*$ (hence, on $\overline{\Delta^*}$). The equivalence classes $[w^\mu]$ are in one-to-one correspondence with the Schwarzian derivatives S_{w^μ}, which range over a bounded domain in the complex Banach space $\mathbf{B}$ of hyperbolically bounded holomorphic functions on Δ^* with the norm

$$\|\varphi\|_{\mathbf{B}} = \sup_{\Delta^*} (|z|^2 - 1)^2 |\varphi(z)|.$$

The derivatives $S_{w^\mu}(z)$ with $\mu \in \mathrm{Belt}(\Delta)_1$ range over a bounded domain in the space $\mathbf{B} = \mathbf{B}(\Delta^*)$. This domain models the universal Teichmüller space $\mathbf{T}$, and the defining projection

$$\phi_{\mathbf{T}}(\mu) = S_{w^\mu} : \ \mathrm{Belt}(\Delta)_1 \to \mathbf{T}$$

is holomorphic. The above definition of $\mathbf{T}$ requires a complete normalization of maps w^μ, which uniquely define the values of w^μ on Δ^* by their Schwarzian derivatives. For example, one can take

$$w^\mu(z) = z + O(1/z) \quad \text{as} \ \ z \to \infty.$$

The intrinsic **Teichmüller metric** of this space is defined by

$$\tau_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi_{\mathbf{T}}(\nu)) = \frac{1}{2} \inf\big\{\log K\big(w^{\mu_*} \circ \big(w^{\nu_*}\big)^{-1}\big) : \ \mu_* \in \phi_{\mathbf{T}}(\mu), \nu_* \in \phi_{\mathbf{T}}(\nu)\big\}; \tag{2.2}$$

it is generated by the Finsler structure

$$F_{\mathbf{T}}(\phi_{\mathbf{T}}(\mu), \phi'_{\mathbf{T}}(\mu)\nu) = \inf\{\|\nu_*/(1 - |\mu|^2)^{-1}\|_\infty : \phi'_{\mathbf{T}}(\mu)\nu_* = \phi'_{\mathbf{T}}(\mu)\nu\} \tag{2.3}$$

on the tangent bundle $\mathcal{T}(\mathbf{T}) = \mathbf{T} \times \mathbf{B}$ of $\mathbf{T}$ (here $\mu \in \mathrm{Belt}(\Delta)_1$ and $\nu, \nu_* \in L_\infty(\mathbb{C})$). This structure is locally Lipschitz (cf. [EE]).

The space $\mathbf{T}$ as a complex Banach manifold also has invariant metrics (with respect to its biholomorphic automorphisms); the largest and the smallest invariant metrics are the Kobayashi and the Carathéodory metrics, respectively. Namely, the **Kobayashi metric** $d_{\mathbf{T}}$ on $\mathbf{T}$ is the largest pseudometric d on $\mathbf{T}$ which does not get increased by holomorphic maps $h: \Delta \to \mathbf{T}$ so that for any two points $\varphi_1, \varphi_2 \in \mathbf{T}$, we have

$$d_{\mathbf{T}}(\varphi_1, \varphi_2) \le \inf\{d_\Delta(0, t): h(0) = \varphi_1, h(t) = \varphi_2\},$$

where d_Δ is the hyperbolic metric on Δ with the differential form

$$ds = \lambda_{\text{hyp}}(z)|dz| := |dz|/(1 - |z|^2). \tag{2.4}$$

The **Carathéodory distance** between φ_1 and φ_2 in $\mathbf{T}$ is

$$c_{\mathbf{T}}(\varphi_1, \varphi_2) = \sup d_\Delta(h(\varphi_1), h(\varphi_2)),$$

where the supremum is taken over all holomorphic maps $h: \mathbf{T} \to \Delta$.

The corresponding differential (infinitesimal) forms of the Kobayashi and Carathéodory metrics are defined for the points $(\varphi, v) \in \mathcal{T}(\mathbf{T})$, respectively, by

$$\mathcal{K}_{\mathbf{T}}(\varphi, v) = \inf\{1/r: r > 0, h \in \text{Hol}(\Delta_r, \mathbf{T}), h(0) = \varphi, h'(0) = v\},$$
$$\mathcal{C}_{\mathbf{T}}(\varphi, v) = \sup\{|df(\varphi)v|: f \in \text{Hol}(\mathbf{T}, \Delta), f(\varphi) = 0\},$$

where $\text{Hol}(X, Y)$ denotes the collection of holomorphic maps of a complex manifold X into Y and Δ_r is the disk $\{|z| < r\}$. For the properties of invariant metrics we refer, for example, to [Di], [Ko].

The following strengthened version of the Gardiner-Royden theorem was established in [Kr3] for universal Teichmüller space.

Proposition 2.1. [Kr2] *The differential Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\varphi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$ of the universal Teichmüller space $\mathbf{T}$ is logarithmically plurisubharmonic in $\varphi \in \mathbf{T}$, equals the canonical Finsler structure $F_{\mathbf{T}}(\varphi, v)$ on $\mathcal{T}(\mathbf{T})$ generating the Teichmüller metric of $\mathbf{T}$ and has constant holomorphic sectional curvature $\kappa_{\mathcal{K}}(\varphi, v) = -4$ on $\mathcal{T}(\mathbf{T})$.*

This implies that the Teichmüller metric $\tau_{\mathbf{T}}(\varphi, \psi)$ is logarithmically plurisubharmonic in each of its variables, which is an underlying fact for many results of geometric complex analysis. This proposition also follows from Theorem 1.1.

Recall that the sectional **holomorphic curvature** of an upper semicontinuous Finsler metric on a complex Banach manifold X is defined as the supremum of the Gaussian curvatures

$$\kappa_\lambda(t) = -\frac{\mathbf{\Delta} \log \lambda(t)}{\lambda(t)^2} \tag{2.5}$$

over an appropriate collection of holomorphic maps from the disk into X for a given tangent direction at the image. Here $\mathbf{\Delta}$ means the generalized Laplacian

$$\mathbf{\Delta}\lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \Big\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta}) d\theta - \lambda(t) \Big\}$$

(provided that $-\infty \le \lambda(t) < \infty$). Similar to C^2 functions, for which $\mathbf{\Delta}$ coincides with the usual Laplacian $4\bar{\partial}\partial$, one obtains that λ is subharmonic on a domain Ω if and only if $\mathbf{\Delta}\lambda(t) \ge 0$; hence, at the points t_0 of local maxima of λ with $\lambda(t_0) > -\infty$, we have $\mathbf{\Delta}\lambda(t_0) \le 0$.

The holomorphic curvature of the Kobayashi metric $\mathcal{K}(x, v)$ of any complete hyperbolic manifold X satisfies $\kappa_{\mathcal{K}_X} \ge -4$ at all points (x, v) of the tangent bundle $\mathcal{T}(X)$ of X, and for the Carathéodory metric $\mathcal{C}_X$ we have $\kappa_{\mathcal{C}}(x, v) \le -4$. For details and general properties of invariant metrics, we refer to [Di], [Ko] (see also [AP], [Kr3]).

2.2. **Frame maps and Strebel points.** Let $f_0 := f^{\mu_0} \in \Sigma^0$ be an extremal representative of its equivalence class $[F_0]$ with dilatation

$$k(F_0) = \|\mu_0\|_\infty = \inf\{k(f^\mu) : f^\mu|S^1 = f_0|S^1\} = k,$$

and assume that there exists in this class a quasiconformal map f_1 whose Beltrami coefficient μ_{f_1} satisfies the strong inequality $\operatorname{ess\,sup}_{A_r} |\mu_{f_1}(z)| < k$ in some annulus $A_r := \{z : r < |z| < 1\}$. Then f_1 is called a **frame map** for the class $[f_0]$ and the corresponding point of the space $\mathbf{T}$ is called a **Strebel point**.

The following two results are fundamental in the theory of extremal quasiconformal maps and Teichmüller spaces.

Proposition 2.2. [St] *If a class $[f]$ has a frame map, then the extremal map f_0 in this class is unique and either conformal or a Teichmüller map with Beltrami coefficient of the form $\mu_0 = k|\psi_0|/\psi_0$ on Δ (and equal to zero on Δ^*), defined by an integrable holomorphic function (quadratic differential) ψ on Δ and a constant $k \in (0,1)$.*

This holds, for example, when the curves $f(S^1)$ are asymptotically conformal; this case includes all smooth curves.

Proposition 2.3. [GL] *The set of Strebel points is open and dense in $\mathbf{T}$.*

2.3. **Grunsky coefficients revised.** Each Grunsky coefficient $\alpha_{mn}^{(p)}(f) = \alpha_{mn}(f_p)$ is represented as a polynomial of a finite number of the initial coefficients $b_1, b_2, \dots, b_s$ of

$$f(z) = z + b_0 + b_1 z^{-1} + \dots.$$

Thus the coefficients $\alpha_{mn}^{(p)}$ considered as the functions of the Schwarzian derivatives S_f are holomorphic on the space $\mathbf{T}$ and generate for each $\mathbf{x} = (x_n) \in S(l^2)$ the holomorphic maps

$$h_{p,\mathbf{x}}(\varphi) = \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}^{(p)}(\varphi) x_m x_n : \ \mathbf{T} \to \Delta. \tag{2.6}$$

This is an underlying fact in applications of the Grunsky operators $\mathcal{G} = (\alpha_{mn}^{(p)})$ to geometry of the universal Teichmüller space and to geometric analysis.

The holomorphy of functions (2.6) is a consequence of the mentioned holomorphy of $\alpha_{mn}^{(p)}$ and of the well-known inequality (cf. [Po, p. 23]) : for any $1 \le j \le M,\ 1 \le l \le N$,

$$\Big| \sum_{m=j}^{M} \sum_{n=l}^{N} \sqrt{mn}\ \alpha_{mn}^{(p)} x_m x_n \Big| \le \sum_{m=j}^{M} |x_m|^2 \sum_{n=l}^{N} |x_n|^2.$$

As was shown in [Kr5], these coefficients give rise to a noninvariant Finsler structure $F_\varkappa(\varphi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$, which is dominated by the canonical Finsler structure of this space and generates the indicated metric. It naturally relates to quasiconformal extensions of conformal maps of quasidisks and reflections across their boundaries. This allows one to construct on holomorphic disks in $\mathbf{T}$ the complex Finsler metrics of generalized Gaussian curvatures at most -4 which can be compared with the basic Kobayashi metric.

Define for $\mu \in L_\infty(\Delta)$ and $\psi \in L_1(\Delta)$ the pairing

$$\langle \mu, \psi \rangle_\Delta = \iint_\Delta \mu(z)\psi(z) dxdy \quad (z = x + iy \in \Delta), \tag{2.7}$$

and assign to each Beltrami coefficient $\mu \in \mathrm{Belt}(\Delta)$ the corresponding coefficient μ^* with norm 1 by

$$\mu^*(z) = \mu(z)/\|\mu\|_\infty.$$

Due to the well-known criterion for extremality (the Hamilton-Krushkal-Reich-Strebel theorem), a Beltrami coefficient $\mu_0 \in \mathrm{Belt}(\Delta)_1$ is extremal if an only if

$$\|\mu_0\|_\infty = \sup_{\|\psi\|_{A_1}=1} |\langle \mu_0, \psi \rangle_\Delta|$$

(see, e.g., [EKK], [GL]). In contrast, the Grunsky norm is intrinsically connected with holomorphic functions from A_1^2, i.e., with abelian differentials.

The following two propositions underly the proof of Theorem 1.1.

Proposition 2.4. [Kr2], [Kr5]. *The equality $\varkappa(f) = k(f)$ holds if and only if the function f is the restriction to Δ^* of a quasiconformal self-map w^{μ_0} of $\widehat{\mathbb{C}}$ with Beltrami coefficient μ_0 satisfying the condition*

$$\sup |\langle \mu_0, \psi \rangle_\Delta| = \|\mu_0\|_\infty, \tag{2.8}$$

where the supremum is taken over holomorphic functions $\psi \in A_1^2(\Delta)$ with $\|\varphi\|_{A_1(\Delta)} = 1$.

In addition, if the equivalence class $[f]$ contains a frame map (is a Strebel point), then the restriction of μ_0 onto the disk Δ must be of the form

$$\mu_0(z) = k|\psi_0(z)|/\psi_0(z) \quad \textit{with} \quad \psi_0 \in A_1^2. \tag{2.9}$$

In a special case when the curve $f(S^1)$ is analytic, the equality (2.9) was obtained by a different method in [Ku3].

The Teichmüller disks admit a stronger assertion:

Proposition 2.5. [Kr6] *If a function $f \in \Sigma^0$ has Teichmüller extension across S^1 with $\mu_f = k|\psi_0|/\psi_0$, then its Grunsky and Teichmüller norms are related by*

$$\varkappa(f) \geq \alpha(f)k(f), \tag{2.10}$$

where the factor $\alpha(f) > 0$ is given by

$$\alpha(f) = \sup_{\psi \in A_1^2, \|\psi\|_{A_1=1}} |\langle |\psi_0|/\psi_0, \psi \rangle_\Delta| \tag{2.11}$$

and cannot be replaced by a larger quantity.

The proof of this important fact relies on certain properties of conformal metrics $ds = \lambda(t)|dt|$ of negative integral curvature bounded from above (see [Ro2], [Kr6]).

Similarly to (2.11), we define for each p-root transform the corresponding quantity

$$\alpha_p(f) = \sup_{\psi \in A_1^2, \|\psi\|_{A_1=1}} |\langle |\psi_{0p}|/\psi_{0p}, \psi_p \rangle_\Delta|, \tag{2.12}$$

where

$$\psi_{0p}(z) = \psi_0(z^p)p^2 z^{2p-2}, \quad \psi_p(z) = \psi(z^p)p^2 z^{2p-2}.$$

2.4. **Action of the Möbius group $\mathbf{PSL(2}, \mathbb{C})/\{\pm 1\}$ on Σ^0.** The following transform preserves the classes Σ^0 and $\Sigma^0(k)$.

For fixed a, $0 < |a| < 1$, consider the conformal automorphism

$$\gamma_a(z) = \frac{\bar{a}}{a} \frac{z + a}{1 + \bar{a}z}$$

of the sphere $\widehat{\mathbb{C}}$; it preserves both disks Δ and Δ^*. Composition with $f \in \Sigma^0$ yields

$$f \circ \gamma_a(z) = f\left(\frac{\frac{1}{z} + \frac{1}{a}}{1 + \frac{1}{\bar{a}z}}\right) = f\left(\frac{1}{a}\right) + \left(1 - \frac{1}{|a|^2}\right) f'\left(\frac{1}{a}\right) \frac{1}{z} + \dots .$$

Thus the transform

$$\begin{aligned}\mathcal{L}_a:\ f(z)\mapsto f_a(z)&=\frac{(1-1/|a|^2)f'(1/a)}{f\circ\gamma_a(z)-f(1/a)}+\frac{1}{2}\left[\left(1-\frac{1}{|a|^2}\right)\frac{f''(1/a)}{f'(1/a)}-\frac{2}{a}\right]+\dots\\&=z+\frac{b_{1,a}}{z}+\dots\end{aligned}\tag{2.13}$$

leaves the set Σ^0 invariant.

Lemma 2.6. *Every $\varkappa_p(f)$ is invariant under the Möbius automorphisms of the unit disk, i.e., for each $\mathcal{L}_a$,*

$$\varkappa_p(f)=\varkappa_p(\mathcal{L}_af).$$

Proof. For each $p\ge 1$, the right-hand side of the corresponding equality (1.25) for $\rho_{f_p(S^1)}=1/\varkappa_p(f)$ is invariant under the Möbius transformations $\gamma\in\mathbf{PSL}(2,\mathbb{C})/\{\pm1\}$. Since by (2.13) the image of S^1 under the map $\mathcal{L}_a(f)$ is obtained from $f(S^1)$ under suitable γ_a, the lemma follows.

3. Proof of Theorem 1.1

The proof will be accomplished in two stages. It does not give explicitly a subsequence $\{p_m\}$ on which the upper limit in (1.10) is attained. To simplify the notations, we can assume that the initial functions f are of the form (1.8) (with $b_0=0$); this does not reflect on the arguments.

Step 1: Strebel points. Let $f\in\Sigma^0$ admit extremal quasiconformal extension onto Δ with Beltrami coefficient

$$\mu_0(z)=k|\psi_0(z)|/\psi_0(z),\quad \psi_0\in A_1.\tag{3.1}$$

The defining quadratic differential ψ_0 is determined up to a positive factor, which can be chosen so that

$$\|\psi_0\|_{A_1}=1.\tag{3.2}$$

In the case when all zeros of ψ_0 in Δ are of even order, the assertion of the theorem trivially follows from Proposition 2.4. Similarly, if ψ_0 has a single zero of odd order at the origin, i.e.,

$$\psi_0(z)=z^{2p'-1}\psi_1(z),\quad \psi_1(z)\ne 0\ \text{ in }\ \Delta,$$

one can apply Proposition 2.4 to the second Grunsky norm $\varkappa_2(f)=\varkappa(f_2)$, getting the desired equality (1.9) with $m=1$. The case when the single zero of ψ_0 is at a point $a\ne 0$ is reduced to the previous one using Lemma 2.6.

Thus one only needs to consider the functions f with extremal differentials (3.1) defined by the quadratic differential ψ_0 having more than one zero of odd order in Δ. For any such function, the assertion of Theorem 1.1 is a consequence of the following proposition.

Proposition 3.1. *For any $\psi_0\in A_1$ having zeros of odd order in the unit disk, the Grunsky norms α_p of the corresponding map $f^{\mu_0}\in\Sigma^0$ defined by (3.1) satisfy*

$$\limsup_{p\to\infty}\ \alpha_p(f^{\mu_0})=1.\tag{3.3}$$

Proof. Let

$$\psi_0(z)=c_nz^n+c_{n+1}z^{n+1}+\dots,\quad n\ge 0,$$

with $c_n>0$ chosen so that (3.2) holds. One can assume that the order n of ψ_0 at the origin is even. Otherwise, we start with the squaring map $f_2(z)=f(z^2)^{1/2}$ whose defining differential $\psi_{0,2}=\mathcal{R}_2^*\psi_0$ satisfies this assumption on the evenness.

We distinguish the first nonzero term $\omega_0(z)=c_nz^n$ and split μ_0^* in two terms

$$\mu_0^*(z)=\frac{|\psi_0(z)|}{\psi_0(z)}=\frac{|z|^n}{z^n}\ \frac{|1+\frac{c_{n+1}}{c_n}z+\dots|}{1+\frac{c_{n+1}}{c_n}z+\dots}=\mu_n^*+\widetilde{\mu},\tag{3.4}$$

where

$$\mu_n^*(z) = \frac{|z|^n}{z^n} = \frac{|\omega_0(z)|}{\omega_0(z)}.$$

The remainder $\widetilde{\mu}$ is of the form

$$\widetilde{\mu}(re^{i\theta}) = \sum_{j=-\infty}^{\infty} C_j(r)e^{i(j-n)\theta}, \tag{3.5}$$

and its Fourier coefficients $C_j(r)$ satisfy

$$C_0(r) \text{ real, and } \overline{C}_j(r) = -C_j(r) \text{ for all } j = 1, 2, \dots .$$

Note also that $\widetilde{\mu}$ as a function of z is real analytic on Δ except for the zeros of ψ_0 in this disk. For small $|z|$, we have a more explicit representation

$$\begin{aligned} \mu_0^*(z) &= \frac{|z|^n}{z^n} \Big[\Big(1 + \frac{\overline{c}_{n+1}}{c_n}\overline{z} + \dots\Big)\Big(1 - \frac{c_{n+1}}{c_n}z - \dots\Big)\Big]^{1/2} \\ &= \frac{|z|^n}{z^n} \Big[1 + \frac{1}{2}\Big(\frac{\overline{c}_{n+1}}{c_n}\overline{z} - \frac{c_{n+1}}{c_n}z - \frac{|c_{n+1}|^2}{c_n^2}|z|^2\Big) + \dots\Big]. \end{aligned}$$

and accordingly,

$$\mu_{0,p}^*(z) := \mathcal{R}_p^*\mu_0^* = \frac{|z|^{p(n+2)-2}}{z^{p(n+2)-2}} \Big[1 + \frac{1}{2}\Big(\frac{\overline{c}_{n+1}}{c_n}\overline{z}^p - \frac{c_{n+1}}{c_n}z^p - \frac{|c_{n+1}|^2}{c_n^2}|z|^{2p}\Big) + \dots\Big].$$

Our aim is to estimate the values $\langle \mathcal{R}_p^*\mu_0^*, \psi\rangle_\Delta$ on A_1^2 for sufficiently large p. First we mention the following simple lemma providing a rough bound for the Taylor coefficients on $A_1(\Delta)$.

Lemma 3.2. *For any function* $\psi(z) = \sum_0^\infty a_n z^n \in A_1(D)$,

$$|a_n| < 2^{n-1}\ \|\psi\|_{A_1}, \quad n = 0, 1, 2, \dots .$$

Proof. Letting $z = x + iy = re^{i\theta}$ with $r \in (1/2, 1)$, one can write

$$a_n = (2\pi i)^{-1} \int\limits_{|z|=r} \psi(z)z^{-n-1}dz = (2\pi)^{-1} \int\limits_0^{2\pi} \psi(re^{i\theta})r^{-n}e^{-in\theta}d\theta,$$

and after multiplying both sides by r and integration along the interval $(1/2, 1)$,

$$\frac{3}{8}a_n = \frac{1}{2\pi} \int\limits_{1/2}^{1} \int\limits_0^{2\pi} \frac{\psi(re^{i\theta})}{r^n e^{in\theta}} r\,dr\,d\theta = \iint\limits_{1/2<|z|<1} \frac{\psi(z)}{z^n}dxdy.$$

Therefore,

$$|a_n| \le \frac{2^n 8}{6\pi} \iint\limits_{1/2<|z|<1} |\psi(z)|dxdy < 2^{n-1}\|\psi\|_{A_1}.$$

The lemma follows.

We proceed to the proof of Proposition 3.1 and take a rapidly increasing subsequence $\{p_m\}$, for example,

$$p_m \ge 2^m, \quad m = 1, 2, \dots ,$$

so that in view of (1.10), the norms $\varkappa_{p_m}(f)$ increase. Denote

$$
\begin{aligned}
\mu^*_{n,m}(z) &:= \mathcal{R}^*_{p_m}\mu^*_n = \mu^*_n(z^{p_m})\overline{z}^{p_m-1}/z^{p_m-1}, \quad n = 0,1,2,\dots;\\
\widetilde{\mu}_m &:= \mathcal{R}^*_{p_m}\widetilde{\mu};\\
\psi_{0,m}(z) &:= B_m\mathcal{R}^*_{p_m}\psi_0 = \psi_0(z^{p_m})p_m^2 z^{p_m(n+2)-2};\\
\omega_{n,m}(z) &:= \mathcal{R}^*_{p_m}\omega^*_0 = c_n p_m^2 z^{p_m(n+2)-2};\\
\varphi_{n,m}(z) &= \psi_{0,m} - B_m\mathcal{R}^*_{p_m}(c_n z^n) = (c_{n+1}z^{n+1} + \dots)p_m^2 z^{p_m(n+2)-2}.
\end{aligned}
\tag{3.6}
$$

The renormalizing factor B_m for $\psi_{0,m}$ is again defined from the condition $\|\psi_{0,m}\|_{A_1} = 1$, which yields the equality

$$
\langle \mu^*_{0,m}, \psi_{0,m}\rangle_\Delta = \iint\limits_\Delta \frac{|\psi_{0,m}|}{\psi_{0,m}}\psi_{0,m}\,dxdy = 1. \tag{3.7}
$$

Using the mutual orthogonality of the powers z^j, $j \in \mathbf{Z}$, on the circles $\{|z| = r\}$, one derives from (3.5)-(3.7) the equalities

$$
\begin{aligned}
\langle \mu^*_{0,m}, \psi_{0,m}\rangle_\Delta &= \langle \mu^*_{0,m}, \omega_{n,m}\rangle_\Delta + \langle \widetilde{\mu}^*_m, \omega_{n,m}\rangle_\Delta + \langle \mu^*_{0,m}, \varphi_{n,m}\rangle_\Delta + \langle \widetilde{\mu}^*_m, \varphi_{n,m}\rangle_\Delta\\
&= \langle \mu^*_{0,m}, \omega_{n,m}\rangle_\Delta + \langle \widetilde{\mu}^*_m, \psi_{0,m}\rangle_\Delta.
\end{aligned}
\tag{3.8}
$$

The condition (3.7) implies that the terms in the right-hand side of (3.8) are related by

$$
\langle \mu^*_{0,m}, \omega_{n,m}\rangle_\Delta = 1 - \mathcal{A}_{1,m}, \quad \langle \widetilde{\mu}^*_m, \psi_{0,m}\rangle_\Delta = \mathcal{A}_{1,m}. \tag{3.9}
$$

For sufficiently large m, the zeros of each $\psi_{0,m}$ are located in the annulus $\{1 - r_m < |z| < 1\}$. We choose a sequence of monotone decreasing r_m so that these annuli degenerate to the unit circle. Then the differentials

$$
\psi_{0,m}(rz) = B_m c_n p_m^2 \ r^{p_m(n+2)-2} \ z^{p_m(n+2)-2} + \dots \quad \text{with} \ \ r < r_m
$$

have no zeros in Δ and thus belong to A_1^2.

Using this fact, we construct the differentials

$$
\psi^r_{0,m}(z) := B^r_m p_m^2[c_n z^{p_m(n+2)-2} + rc_{n+1}z^{p_m(n+2)-1} + r^2 c_{n+2}z^{p_m(n+2)} + \dots] \tag{3.10}
$$

with appropriate $r < r'_m$ ($r'_m < r_m$), which also do not vanish in the unit disk. The renormalizing factor B^r_m is added again to have $\|\psi^r_{0,m}\|_{A_1} = 1$. This renormalization ensures that the admissible bound r'_m can be chosen so that it approaches 1 as $m \to \infty$, simultaneously with r_m. Note also that (for a given n)

$$
\psi^r_{0,m}(z) = c(m,r)\omega_{n,m}(z) + \varphi_{n,m}\big(r^{1/[p_m(n+2)-1]}z\big)
$$

with some constant $c(m,r) > 0$.

Replacing $\psi_{0,m}$ by $\psi^r_{0,m}$, one perturbs (decreases) the value of $|\langle \mu^*_{0,m}, \psi_{0,m}\rangle_\Delta|$ on the quantity

$$
O\big((1-r)^{1/[p_m(n+2)-1]}\big).
$$

The above relations result in

$$
\langle \mu^*_{0,m}, \psi^r_{0,m}\rangle_\Delta = 1 - \mathcal{A}_{1,m} + \alpha(r)\mathcal{A}_{1,m}, \tag{3.11}
$$

with

$$
\alpha(r) \asymp r^{1/[p_m(n+2)-1]} \quad \text{as} \ \ r \to 1. \tag{3.12}
$$

This implies that, for a given small $\varepsilon > 0$, one can choose a large number m and $r = r(\varepsilon, m)$ satisfying (3.12) and close to 1 so that the corresponding $\psi^r_{0,m}$ in (3.10) belongs to A_1^2 and in view of (3.11) is estimated from below by

$$
|\langle \mu^*_{0,m}, \psi^r_{0,m}\rangle_\Delta| \geq 1 - \varepsilon.
$$

Therefore, taking a sequence $\varepsilon_j \to 0$, one obtains for suitable p_m that the corresponding quantities $\alpha_{p_m}(f)$, defined by (2.12), approach 1. This implies the desired equality (3.3) for $f \in \Sigma^0$, which admit Teichmüller extremal extensions.

Step 2: Non-Strebel points. Take an arbitrary $f \in \Sigma^0$ whose Schwarzian derivative S_f represents a non-Strebel point of $\mathbf{T}$. By Proposition 2.2, there exists a sequence of Strebel points $\varphi_n \to \varphi = S_f$ in $\mathbf{T}$ which define the functions $f_n \in \Sigma^0$ by the equations $S_{f_n} = \varphi_n$. Letting $f_n(0) = 0$, consider the functions

$$\mathcal{R}_p f_n =: f_{np}(z) = f_n(z)^{1/p}, \quad p = 2, 3, \dots .$$

By Step 1, for each f_n,

$$\widehat{\varkappa}(f_n) = \limsup_{p\to\infty} \varkappa(f_{np}) = k(f_n). \tag{3.13}$$

We shall now consider the Grunsky norms $\varkappa_p$ as functions of the Schwarzian derivatives S_f on the space $\mathbf{T}$. As was mentioned in Section 2, each $\varkappa_p(S_f)$ is continuous and plurisubharmonic on $\mathbf{T}$; thus for every $p = 2, 3, \dots$,

$$\lim_{n\to\infty} \varkappa_p(\varphi_n) = \varkappa_p(\varphi). \tag{3.14}$$

Let us consider also the limit Grunsky norm as a function on the space $\mathbf{T}$. Letting

$$\widehat{\varkappa}(\varphi) = \limsup_{p\to\infty} \varkappa_p(\varphi).$$

Taking its upper semicontinuous regularization

$$\widehat{\varkappa}^*(\varphi) = \limsup_{\varphi_1\to\varphi} \widehat{\varkappa}(\varphi_1), \tag{3.15}$$

one obtains a plurisubharmonic function on $\mathbf{T}$. The regularization (3.15) can decrease $\widehat{\varkappa}$, thus we have

$$\widehat{\varkappa}(S_f) \geq \widehat{\varkappa}^*(S_f). \tag{3.16}$$

As $\|\varphi_n - \varphi\|_{\mathbf{B}} \to 0$, the Teichmüller norm $k(S_f)$, where $S_f = \varphi$, behaves continuously, while the function (3.15) is only upper semicontinuous, and therefore,

$$\lim_{n\to\infty} k(S_{f_n}) = k(S_f), \quad \widehat{\varkappa}^*(S_f) \geq \lim_{n\to\infty} \widehat{\varkappa}^*(S_{f_n}).$$

Combining these relations with (3.13) and (3.16), one derives

$$\widehat{\varkappa}(S_f) \geq \widehat{\varkappa}^*(S_f) \geq k(S_f),$$

completing the proof of Theorem 1.1, because the inequality $\widehat{\varkappa}(S_f) \geq k(S_f)$ can only hold when $\widehat{\varkappa}(f) = k(f)$.

Corollary 3.3 (from the proof). *The limit Grunsky norm*

$$\widehat{\varkappa}(\varphi) = \limsup_{p\to\infty} \varkappa_p(\varphi)$$

is a continuous plurisubharmonic function on the universal Teichmüller space $\mathbf{T}$.

Cf. this corollary with representation (1.19).

4. Proof of Theorem 1.3

For every $p = 1, 2, \dots$, the functions $h_{2^p,\mathbf{x}}(S_f)$ given by (2.4) provide the equality

$$\varkappa_{2^p}(f) = \sup\{|h_{2^p,\mathbf{x}}(S_f)| : \ \mathbf{x} \in S(l^2)\}.$$

We take for a given point $\varphi_0 \in \mathbf{T}$ the corresponding $f_0 \in \Sigma^0$ with $S_{f_0} = \varphi_0$ and select a sequence $\{\mathbf{x}_p = (x_n^{(p)})\} \subset S(l^2)$ so that

$$|\varkappa_{2^p}(f_0) - h_{2^p,\mathbf{x}_p}(S_{f_0})| \le \frac{1}{2^p}, \quad p = 1, 2, \dots \ .$$

Then the equality (1.12) implies

$$c_{\mathbf{T}}(\mathbf{0}, S_{f_0}) \ge \lim_{p\to\infty} |h_{2^p,\mathbf{x}_p}(S_{f_0})| = k(f_0), \tag{4.1}$$

which yields that the Carathéodory and Teichmüller (hence also Kobayashi) distances $c_{\mathbf{T}}(\varphi_0, \mathbf{0})$ and $\tau_{\mathbf{T}}(\varphi_0, \mathbf{0})$ coincide for any point φ_0 of $\mathbf{T}$. Together with (1.14), this gives the equalities (1.15).

To establish the equalities for the infinitesimal distances (1.18), let us consider the corresponding differentials $h'_{2^p,\mathbf{x}_p}(\mathbf{0})v$. Then, using the representation (1.16) and the differential version of the equality (1.12), one obtains similarly to above that the Carathéodory and Teichmüller infinitesimal metrics are related by

$$\mathcal{C}_{\mathbf{T}}(\mathbf{0}, v) \ge \lim_{p\to\infty} |h'_{2^p,\mathbf{x}_p}(\mathbf{0})v| = F_{\mathbf{T}}(\mathbf{0}, v) = \inf\{\|\mu\|_\infty : \ \phi'_{\mathbf{T}}(\mathbf{0})\mu = v\}. \tag{4.2}$$

We now apply the variation of maps $f^\mu \in \Sigma^0$ with small dilatation $\|\mu\|_\infty$. Explicitly, this variation is represented by

$$f^\mu(z) = z - \frac{1}{\pi} \iint\limits_{|\zeta|<1} \frac{\mu(\zeta)}{\zeta - z} \, d\xi d\eta + O(\|\mu\|_\infty^2);$$

its remainder is estimated uniformly on compact sets of $\mathbb{C}$. It implies the following asymptotic equality for the Grunsky coefficients $\alpha_{mn}(f)$ as the functions of Beltrami coefficients

$$\alpha_{mn}(\phi_{\mathbf{T}}(\mu)) = -\pi^{-1} \iint\limits_{\Delta} \mu(z) z^{m+n-2} dxdy + O(\|\mu\|_\infty^2), \quad \|\mu\|_\infty \to 0.$$

The coefficients $\alpha_{mn}^{(p)}(f)$ depend on μ_{2^p} in a similar way.

Substituting these expressions of $\alpha_{mn}^{(p)}(f)$ into the series (2.6) and calculating the derivatives $h'_{2^p,\mathbf{x}_p}(\mathbf{0})$, one obtains the explicit representation of the tangent vectors $h'_{2^p,\mathbf{x}_p}(\mathbf{0})v$ by the differentials of the functions (1.16). This representation gives, similarly to case of (4.1), that in fact the first inequality in (4.2) only can hold when it is an equality. The theorem is proven.

5. Grinshpan's conjecture and holomorphic homotopy

This section concerns the properties of a dilatation function generated by an univalent function. We answer here some questions and conjectures raised by R. Kühnau in [KK3].

5.1. **Holomorphic homotopy of a univalent function.** The following proposition concerns the dynamical properties of quasiconformal extensions of univalent functions and implies the best bounds for their dilatations.

Define for any $f \in \Sigma$ its complex isotopy

$$f_t(z) = tf(z/t) = z + b_0 t + b_1 t^2 z^{-1} + b_2 t^3 z^{-2} + \dots : \ \Delta^* \times \Delta \to \widehat{\mathbb{C}} \tag{5.1}$$

to the identity map $f_0(z) \equiv z$. Then

$$S_{f_t}(z) = t^{-2} S_f(t^{-1} z),$$

and the map $g_f: \ \Delta \to S_{f_t}$ is holomorphic as a function $\Delta \to \mathbf{B}$. The corresponding **homotopy disk**

$$\Delta(S_f) = g_f(\Delta) = \{S_{f_t}\} \tag{5.2}$$

is holomorphic at noncritical points of the map g_f. These disks foliate the set Σ^0.

Proposition 5.1. *(a) If a function $f(z) = z + b_0 + b_1 z^{-1} + \dots$ belongs to $\Sigma(k)$, then for any $t \in \Delta$ the map $f_t(z) = tf(t^{-1}z)$ belongs to $\Sigma(k|t|^2)$. This bound $\|\mu_{f_t}\|_\infty \le k|t|^2$ for the smallest dilatations of possible quasiconformal extensions f_t^μ of f_t is sharp and occurs only for the maps*

$$f_{b_0,b_1;1}(z) := z + b_0 + b_1 z^{-1} \quad \text{with} \ |b_1| = k,$$

whose homotopy maps

$$f_{b_0,b_1;1} = z + b_0 t + b_1 t^2/z$$

have the affine extensions $\widehat{f}_{b_0,b_1;1}(z) = z + b_0 t + kt^2\overline{z}$ onto Δ.

(b) More generally, if

$$f(z) = z + b_0 + b_m z^{-m} + b_{m+1} z^{-(m+1)} + \dots\ , \quad m \ge 1 \ (b_m \ne 0), \tag{5.3}$$

then $k(f_t) \le k|t|^{m+1}$; this bound is also sharp.

Accordingly, for f of the form (5.3), the expansion of the map g_f assumes the form

$$g_f(t) = g_{m+1} t^{m+1} + g_{m+2} t^{m+2} + \dots\ ,$$

where the first coefficient g_{m+1} does not vanish simultaneously with b_m (here all $g_j \in \mathbf{B}$). The proof of all these assertions can be found in [Kr4]. Note that the map g_f also can vanish at the points $t \ne 0$.

For small $|t|$, there is a sharp asymptotic estimate (cf. [KK3])

$$k(f_t) = \frac{m+1}{2} |b_m| |t|^{m+1} + O(|t|^{m+2}), \quad t \to 0. \tag{5.4}$$

Note also that by Proposition 2.2 each homotopy map f_t has (a unique) Teichmüller extension $f^{k(t)\mu_t^*}$ onto Δ across S^1, where

$$k(t) = k(f_t), \quad \mu_t(z) = |\psi_t(z)|/\psi_t(z), \quad \psi_t \in A_1(\Delta). \tag{5.5}$$

The homotopy (5.1) is a special case of holomorphic motions. Proposition 5.1 implies a stronger and sharper bound for the dilatations of extremal quasiconformal extensions of univalent functions than the estimate that follows from the general theory of holomorphic motions.

Let $f \in \Sigma$. Take its renormalized functions

$$f_p(z) = \mathcal{R}_p f = f(z^p)^{1/p}, \quad p = 2, 3, \dots,$$

and consider their homotopies

$$f_{p,t}(z) = t^{-1} f_p(z), \quad t \in \Delta.$$

By Proposition 5.1 there exists a quasiconformal extension $f_{p,t}^*$ of $f_{p,t}|\Delta^*$ onto $\widehat{\mathbb{C}}$ with $k(f_{p,t})^* \le |t|^{m+1}$, provided that f is of the form (5.3). Generically, such an extension does not preserve the rotational symmetry of maps $f_{p,t}$.

We need quasiconformal extensions $f_{p,t}^{**}$ of $f_{p,t}$ preserving symmetry and thus descending into the homotopy f_t of the original map f. The extremal dilatations of symmetric extensions of any renormalized map $\mathcal{R}_p f$ are the same as for the underlying extensions of f.

Note also that the invariant version of the extended lambda-lemma (Slodkowski's theorem), established in [EKK], implies quasiconformal extensions of holomorphic motions compatible with arbitrary subgroups of $\mathbf{PSL}(2, \mathbb{C})$.

5.2. **Grunsky norm of homotopy functions.** Again consider first the case of Strebel points. Let $f \in \Sigma^0$ be represented in D^* by the series of the form (5.3) and admit Teichmüller extremal extension onto the unit disk Δ whose Beltrami coefficient μ_0 is given by (3.1). Its homotopy maps $f_t(z)$ have, by Propositions 2.2 and 5.1, the extremal extensions with coefficients

$$\mu_t := \mu_{f_t} = k(f_t)|\psi_t|/\psi_t$$

defined by holomorphic $\psi_t \in A_1(\Delta)$, and by (5.4), for small $|t|$,

$$\mu_t = \frac{m+1}{2}|b_m||t|^{m+1}\frac{|\psi_t|}{\psi_t} + O(t^{m+2}),$$

where the remainder is estimated in L_∞-norm.

Consider also the restrictions of holomorphic maps (2.6) to the homotopy disk $\Delta(S_f)$ of this function in the space $\mathbf{T}$. In terms of parameter $t \in \Delta$ these maps assume the form

$$\widetilde{h}_{p,\mathbf{x}}(t) := h_{1,\mathbf{x}} \circ g_f(t) = \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}^{(p)}(f) x_m x_n. \tag{5.6}$$

The following theorem discovers the interesting geometric features of the homotopy maps.

Theorem 5.2. *For every function $f \in \Sigma^0$ of the form (5.3), there exists a number $r_1(f)$, $0 < r_1(f) \le 1$, such that for all $|t| < r_1(f)$ the defining quadratic differential ψ_t for μ_{f_t} satisfies:*

(i) if m is odd, $m = 2m' - 1$, then ψ_t has on the disk Δ only zeros of even order;

(ii) if m is even, $m = 2m'$, then ψ_t has on Δ zeros of odd order.

(iii) The radius $r_1(f)$ in the part (i) is equal to the greatest value of $|t|$ for which the defining differential ψ_t has only zeros of even order in Δ.

Proof. Consider the maps

$$f_m(z) = z\Big(1 + \frac{a_m t^{m+1}}{z^{m+1}}\Big)^{2/(m+1)}, \quad z \in \Delta^* \quad (|t| < 1) \tag{5.7}$$

with $a_m = (m+1)b_m/2$ and their extremal extensions onto Δ with Beltrami coefficients

$$\mu_m(z) = a_m t^{m+1}\Big(\frac{\overline{z}}{z}\Big)^{(m-1)/2} = a_m t^{m+1}\Big(\frac{|z|}{z}\Big)^{m-1}.$$

A simple calculation gives

$$S_{f_t}(z) = S_{f_m}(z) + H(z,t), \tag{5.8}$$

where the remainder is well-defined by the chain rule for the Schwarzian derivatives

$$S_{w_1 \circ w} = S_{w_1} \circ w\ (w')^2 + S_w,$$

and (5.3) implies the asymptotic bound

$$\|H(\cdot, t)\|_{\mathbf{B}} = O(t^{m+2}), \quad t \to 0. \tag{5.9}$$

The case of even m is simple. Then we have for the corresponding maps (5.7) the inequality $\varkappa(f_m) < k(f_m)$ (cf. the examples in Section 1.2 and Proposition 2.4). The equality (5.8) implies that, by continuity of both Teichmüller and Grunsky norms on $\mathbf{T}$, the inequality $\varkappa(f_t) < k(f_t)$ is preserved for all t in a neighborhood of the origin $t = 0$.

In the case of odd m,

$$\varkappa(f^{t^{m+1}\mu_m}) = k(f^{t^{m+1}\mu_m}) \quad \text{for any} \quad |t| < 1.$$

By (5.9), $\tau_{\mathbf{T}}(S_{f_t}, S_{f_m}) = O(t^{m+2})$.

We now apply the canonical local complex parameters and geometry defined by the integrable holomorphic quadratic differentials (cf., e.g. [GL], [Kr1]). Let $w = f(z)$ be an extremal map of the disk Δ into $\mathbb{C}$, $f(0) = 0$, with Beltrami coefficient $\mu_f(z) = k|\psi|/\psi$ defined by $\psi \in A_1(\Delta)$. Assume that ψ is holomorphic on the closed disk $\overline{\Delta}$.

We introduce in a neighborhood of every point $z_0 \in \Delta$ a new local parameter

$$\zeta = \left(\int\limits_{z_0}^{z} \sqrt{\psi} dz + k \int\limits_{z_0}^{z} \sqrt{\overline{\psi}} d\overline{z} \right)^{2/(n+2)},$$

where n is the order of ψ (in fact the order of its zero) at the point z_0. This parameter is uniquely defined up to multiplication at the $(n+2)$th root of 1.

The map $\zeta(z)$ is a local homeomorphism satisfying for $\psi(z) \neq 0$ the same Beltrami equation $\partial_{\overline{z}} w = (k|\psi|/\psi)\partial_z w$ as does $w = f(z)$. Hence, $\zeta = \Phi(w)$ is a holomorphic function on Δ generating the integrable holomorphic quadratic differential

$$\Psi(w) = ((n+2)^2/4)\Phi(w)^n \Phi'(w)$$

on the disk Δ, which connects with ψ by

$$\sqrt{\Psi} dw = \sqrt{\psi} dz + k\sqrt{\overline{\psi}} d\overline{z}$$

and defines the Beltrami coefficient of the inverse map f^{-1}. Note that ψ and Ψ have the equal orders in the corresponding points z and $w = f(z)$.

Then the map f admits the factorization

$$f = \gamma_2^{-1} \circ \gamma \circ \gamma_1, \tag{5.10}$$

whose factors are locally represented by

$$\gamma_1(z) = \left(\int\limits_{z_0}^{z} \sqrt{\psi} dz \right)^{2/(n+2)}, \quad \gamma_2(w) = \left(\int\limits_{w_0}^{w} \sqrt{\Psi} dw \right)^{2/(n+2)};$$

$$\gamma(\zeta) = \left(\frac{\zeta^{(n+2)/2} + k\overline{\zeta}^{(n+2)/2}}{1-k} \right)^{2/(n+2)}$$

(for $n = 0$ the middle factor γ is an affine map).

Applying this to the original map f and to its associated map f_m by (5.7), we obtain that (up to the integration constants) f_m is represented by (5.11) with factors of the form

$$z \mapsto z^{(m+1)/2} \quad \text{and} \quad z \mapsto A(z^{(m+1)/2} + k\overline{z}^{(m+1)/2}), \tag{5.11}$$

while f_t admits two factorizations

$$f_t = \gamma_2^{-1} \circ \widetilde{\gamma}_t \circ \gamma_1, \quad f_t = \gamma_{2,t}^{-1} \circ \gamma \circ \gamma_{1,t},$$

where $\gamma_{1,t}$ and $\gamma_{2,t}$ are constructed similar to above for $\psi = \psi_t$. The maps γ_1 and γ_2 are of the same form $z \mapsto z^{(m+1)/2}$ as for f_m, and $\widetilde{\gamma}_t = \gamma_2 \circ \ f_t \circ \gamma_1^{-1}$; for small $|t|$, it is close to the second map in (5.11) on $\overline{\Delta}$.

We see that, if $|t| \leq t_0$ is sufficiently small, the corresponding factors in representation (5.10) for the maps f_m and f_t are homotopic and have equal topological degrees (hence the same winding numbers about the origin). This implies that the defining differentials ψ_t must have zero of even order $m-1$ at the origin and no other zeros in the disk Δ.

The proof of (iii) relies on Minda's maximum principle [Mi].

Lemma 5.3. [Mi] *If a function $u: D \to [-\infty, +\infty)$ is upper semicontinuous in a domain $D \subset \mathbb{C}$ and its generalized Laplacian satisfies the inequality $\boldsymbol{\Delta} u(t) \geq Ku(t)$ with some positive constant K at any point $t \in D$, where $u(t) > -\infty$, and if*

$$\limsup_{t \to t_*} u(t) \leq 0 \quad \textit{for all } t_* \in \partial D,$$

then either $u(t) < 0$ for all $t \in D$ or else $u(t) = 0$ for all $t \in D$.

Consider the homotopy disk (5.2) of f and, using the composite maps (5.6) for $p = 1$,

$$\widetilde{h}_{1,\mathbf{x}}(t) := h_{1,\mathbf{x}} \circ g_f(t) = \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(f_t) x_m x_n, \tag{5.12}$$

and pull back the hyperbolic metric (2.4) of Δ onto $\Delta(S_f)$, define on this disk and Δ the conformal metric $ds = \lambda_{\widetilde{h}_{1,\mathbf{x}}}(t)|dt|$ with

$$\lambda_{\widetilde{h}_{1,\mathbf{x}}}(t) = \frac{|\widetilde{h}'_{1,\mathbf{x}}(t)||dt|}{1 - |\widetilde{h}_{1,\mathbf{x}}(t)|^2}. \tag{5.13}$$

Its Gaussian curvature equals -4 at noncritical points of the maps (5.12). Note that the map g_f defining the disk $\Delta(S_f)$ has at most a countable set of critical points in Δ, which do not reflect on the next assertion. Consider the upper envelope of metrics (5.13)

$$\lambda_{\varkappa_1}(t) := \sup\{\lambda_{\widetilde{h}_{1,\mathbf{x}}}(t) : \ \mathbf{x} \in S(l^2)\} \quad (t \in \Delta) \tag{5.14}$$

followed by its upper semicontinuous regularization

$$\lambda_{\varkappa_1}(t) = \limsup_{t' \to t} \lambda_{\varkappa_1}(t').$$

Similarly to Lemmas 3.1 and 3.2 in [Kr6], one obtains

Lemma 5.4. *The enveloping metric $\lambda_{\varkappa_1}$ is a logarithmically subharmonic on Δ, and its generalized Gaussian curvature on this disk satisfies $\kappa_{\lambda_{\varkappa_1}} \le -4$.*

The following lemma shows that the Grunsky norm can be reconstructed from its infinitesimal form $\lambda_{\varkappa_1}$ along the geodesic Teichmüller disks.

Lemma 5.5. [Kr6] *On any extremal Teichmüller disk $\Delta(\mu_0) = \{\phi_{\mathbf{T}}(t\mu_0) : \ t \in \Delta\}$, we have the equality*

$$\tanh^{-1}[\varkappa_1(f^{r\mu_0})] = \int_0^r \lambda_{\varkappa_1}(t) dt. \tag{5.15}$$

It follows from Proposition 2.4 and from the part (i) that for $|t| < r_1(f)$ we have the equality

$$\lambda_{\varkappa_1}(t) = \lambda_{\mathcal{K} \circ g_f}(t), \tag{5.16}$$

where $\lambda_{\mathcal{K} \circ g_f}$ denotes the restriction of the infinitesimal Kobayashi metric of $\mathbf{T}$ to the homotopy disk $\Delta(S_f)$ composed with the map $g_f : \ t \to S_{f_t}$. Except for the critical points of g_f, this metric is a continuous and logarithmically plurisubharmonic and has generalized Gaussian curvature -4.

We may now prove the last part of Theorem 5.2. Take a point t_0 with $|t_0| < r_1(f)$ at which the metrics $\lambda_{\varkappa_1}$ and $\lambda_{\mathcal{K} \circ g_f}$ coincide. Letting

$$M = \{\sup \lambda_{\mathcal{K} \circ g_f}(t) : t \in U_0\},$$

in a sufficiently small neighborhood U_0 of t_0, we get that in this neighborhood,

$$\lambda_{\mathcal{K} \circ g_f}(t) + \lambda_{\varkappa_1}(t) \le 2M.$$

Thus the metric

$$u = \log \frac{\lambda_{\varkappa_1}}{\lambda_{\mathcal{K} \circ g_f}} = \log \lambda_{\varkappa_1} - \log \lambda_{\mathcal{K} \circ g_f}$$

satisfies

$$\mathbf{\Delta} u = \mathbf{\Delta} \log \lambda_{\varkappa_1} - \mathbf{\Delta} \log \lambda_{\mathcal{K} \circ g_f} = 4(\lambda_{\varkappa_1}^2 - \lambda_{\mathcal{K} \circ g_f}^2) \ge 8M(\lambda_{\varkappa_1} - \lambda_{\mathcal{K} \circ g_f})$$

(cf. [Mi], [Di]). Since

$$M \log \frac{t - t_0}{s - t_0} \geq t - s \quad \text{for} \quad t_0 < s \leq t < M$$

(with equality only for $t = s$), one derives

$$M \log \frac{\lambda_{\varkappa_1}(t)}{\lambda_{\mathcal{K}\circ g_f}(t)} \geq \lambda_{\varkappa_1}(t) - \lambda_{\mathcal{K}\circ g_f}(t),$$

and therefore, $\boldsymbol{\Delta} u(t) \geq 4M^2 u(t)$. By Lemma 5.2, the metrics $\lambda_{\varkappa_1}$ and $\lambda_{\mathcal{K}\circ g_f}$ must coincide in the neighborhood U_0. Using that these metrics are circularly symmetric (radial), one obtains successively their equality in the disks $\Delta_r = \{|t| < r\}$ with $r < r_1' \leq 1$. Together with Lemma 5.5, this implies that the norms $\varkappa(f_t)$ and $k(f_t)$ are equal on such disks, completing the proof of the theorem.

5.3. **Remarks.** 1) The arguments in the proof of Theorem 5.2 actually give more and allow one to strengthen it as follows:

(a) For each even m, the corresponding differentials ψ_t have in the disk Δ zeros of odd order for all $|t| < 1$.

(b) Assume that m in (5.3) is odd. If for some $t \neq 0$ the zeros of ψ_t are only of even order, then this holds also for all $|t| \leq |t_0|$.

2) In a special case $m = 1$, i.e., for the functions $f(z) = z + b_0 + b_1 z^{-1} + \dots \in \Sigma$ with $b_1 \neq 0$, the assertion (i) of Theorem 5.2 was discovered by Kühnau in [KK2].

5.4. **Application of Theorem 1.1.** Take the homotopies $f_{p,t}(z) = t f_p(t^{-1} z)$, $t \in \Delta$, of the renormalized functions $f_p(z) = \mathcal{R}_p f$, with symmetric quasiconformal extensions onto $\widehat{\mathbb{C}}$. Then, similarly to above, $\varkappa(f_{p,t}) = k(f_{p,t})$, provided that $|t| < r_p(f) = r_1(f)^{1/p}$.

Open question. *For which* $f \in \Sigma^0$ *must* $\limsup\limits_{p\to\infty} r_p(f) = 1$ *?*

Take again the functions of the form (1.9). Replacing $\varkappa(f_r)$ by the limit Grunsky norm $\widehat{\varkappa}(f_r)$, one gets, as a corollary of Theorem 1.1,

Theorem 5.6. *For any* $f(z) = z + b_1 z^{-1} + \dots \in \Sigma^0$ *and any* $r \in [0, 1]$,

$$\widehat{\varkappa}(f_r) = k(f_r).$$

Similarly to (5.15), the limit Grunsky norm $\widehat{\varkappa}(f)$ can be reconstructed on Teichmüller disks from its infinitesimal form, in other words, from the enveloping metric $\lambda_{\widehat{\varkappa}}$.

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