# DIVERGENCE, EXOTIC CONVERGENCE AND SELF-BUMPING IN QUASI-FUCHSIAN SPACES

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Abstract. In this paper, we study the topology of the boundaries of quasi-Fuchsian spaces. We first show for a given convergent sequence of quasi-Fuchsian groups, how we can know the end invariant of the limit group from the information on the behaviour of conformal structures at infinity of the groups. This result gives rise to a sufficient condition for divergence of quasi-Fuchsian groups, which generalises Ito's result in the once-punctured torus case to higher genera. We further show that quasi-Fuchsian groups can approach b-groups not along Bers slices only when there are isolated parabolic loci. This makes it possible to give a necessary condition for points on the boundaries of quasi-Fuchsian spaces to be self-bumping points.

## 1. Introduction

In the theory of Kleinian groups, after the major problems like Marden's tameness conjecture and the ending lamination conjecture were solved, the attention is now focused on studying the topological structure of deformation spaces. Although we know, by the resolution of the Bers-Sullivan-Thurston density conjecture ([12], [8], [32] , [22], [23], [34], [30]), that every finitely generated Kleinian group is an algebraic limit of quasi-conformal deformations of a (minimally parabolic) geometrically finite group, the structure of deformation spaces as topological spaces is far from completely understood. For instance, as was observed by work of Anderson-Canary ([2]) and McMullen ([28]), even in the case of Kleinian groups isomorphic to closed surface groups, the deformation spaces are fairly complicated, and in particular are not manifolds since they have singularities caused by the phenomenon called "bumping". Actually, this kind of phenomenon also makes the deformation space not locally connected as was shown by Bromberg [13] and generalised by Magid [24].

The interior of a deformation space is known to be a union of quasi-conformal deformation spaces of minimally parabolic Kleinian groups, which is well understood by work of Ahlfors, Bers, Kra, Maskit, Marden, and Sullivan. In particular, their theory implies that there is a parametrisation of the quasi-conformal deformation space by Teichmüller space of the boundary at infinity of the quotient hyperbolic 3-manifold. Therefore, to understand the global structure of a deformation space, what we need to know is how the boundary is attached to the quasi-conformal deformation space. More concretely, we need to determine, for a sequence of quasi-conformal deformations given as a sequence in the Teichmüller space using this parametrisation, first whether it converges or not, and if it converges what is the

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limit of the sequence. Also, we need to know in what case sequences of quasi-conformal deformations can approach the same group from different directions as in the example of Anderson-Canary and McMullen. This paper tries to answer such problems for the case of Kleinian surface groups.

The interior of a deformation space of Kleinian surface group $AH(S)$ is the quasi-Fuchsian space $QF(S)$. The parametrisation in this case is the Ahlfors-Bers map $qf$ from $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ to $QF(S)$ giving the Ahlfors-Bers parametrisation. The celebrated work of Bers on compatifying the Teichmüller space ([4]) shows that in $AH(S)$, the subspace of $QF(S)$ in the form $\mathcal{T}(S) \times \{\text{pt.}\}$ or $\{\text{pt.}\} \times \mathcal{T}(\bar{S})$ is relatively compact. Considering a quite different direction in $QF(S)$ from that of Bers, Thurston proved that a sequence $\{(m_i, n_i)\} \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$ converges if $\{m_i\}$ converges to a projective lamination $[\lambda]$ and $\{n_i\}$ converges to $[\mu]$ both in the Thurston compactification of the Teichmüller space such that every component of $S \setminus (\lambda \cup \mu)$ is simply connected ([39]).

In contrast to these results on convergence, we showed in ([33]) that if $\{m_i\}$ and $\{n_i\}$ converge to arational laminations with the same support, then $\{qf(m_i, n_i)\}$ always diverges. Since the Teichmüller space of $S$ is properly embedded in $AH(S)$ as a diagonal set of $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$, this may look very natural. However, there is an example by Anderson-Canary in [2] which shows that for a given hyperbolic structure $m_0$, if we consider $(\tau^i(m_0), \tau^{2i}(m_0)) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$, where $\tau$ denotes the deformation of the metric induced by the Dehn twist around a simple closed curve $\gamma$ on $S$, then its image $qf(\tau^i(m_0), \tau^{2i}(m_0))$ in $AH(S)$ converges. These two results show us the situation is quite different depending on the types of the limit projective laminations.

In the case when $\mathcal{T}(S)$ has dimension 2, i.e., if $S$ is either a once-punctured torus or a four-time punctured sphere, a measured lamination is either arational or a weighted simple closed curve, which means that there is nothing between these two situations above. In this case, Ito has given a complete criterion for convergence/divergence ([20]). In the general case when $\mathcal{T}(S)$ has dimension more than 2 however, there is a big room between these two extremes.

Therefore quite naturally, we should ask ourselves what would happen in the cases in between. One of our main theorems in this paper (Theorem 3) is an answer to this question. Consider sequences $\{m_i\}$ in $\mathcal{T}(S)$ and $\{n_i\}$ in $\mathcal{T}(\bar{S})$ converging to $[\mu^+]$ and $[\mu^-]$ such that their supports $|\mu^+|$ and $|\mu^-|$ share a component which is not a simple closed curve. We shall prove that $\{qf(m_i, n_i)\}$ diverges in $AH(S)$ in this setting. More generally, we shall show that if $\mu^+$ and $\mu^-$ have components $\mu_0^+, \mu_0^-$ whose minimal supporting surfaces share a boundary component, then $qf(m_i, n_i)$ diverges.

This theorem is derived from another of our main results Theorem 2, which asserts that if $\{qf(m_i, n_i)\}$ converges, then we can determine the ending laminations of the limit group by considering the shortest pants decomposition of $m_i$ and $n_i$ and their Hausdorff limits. It also implies Theorem 1 stating that if we consider the limit of $\{m_i\}$ and $\{n_i\}$ in the Thurston compactification of the Teichmüller space, then every non-simple-closed-curve component of the limit of $\{m_i\}$ is an ending lamination of an upper end of the limit, and every non-simple-closed-curve component of the limit of $\{n_i\}$ is that of a lower end. For a simple closed curve contained in the limit of $\{m_i\}$ not contained in that of $\{n_i\}$ is a core curve of the parabolic locus of the limit group. A similar thing holds for a simple closed curve in the limit of $\{n_i\}$ not contained that of $\{m_i\}$. These combined with Theorem 5 below can be regarded as a partial answer to the problem of determining limit groups of sequence of quasi-Fuchsian groups given in terms of the parametrisation by the Teichmüller spaces.

In the case when the laminations $|\mu^+|$ and $|\mu^-|$ share only simple closed curves, the convergence or divergence of the sequence depends on whether there is a shared component which is contained in the minimal supporting surfaces of other components of $|\mu^+|$ or $|\mu^-|$. Theorem 4 asserts that if there is such a simple closed curve component, then the sequence diverges. Even in the case when such a component does not exist, $\{qf(m_i, n_i)\}$ can converge only in a special situation which is analogous to an example of Anderson-Canary [2]. Theorem 5 describes the situation where the sequence can converge.

The same example of Anderson-Canary also shows that there is a point in $AH(S)$ where $QF(S)$ bumps itself as explained above. In such a point, the self-bumping is caused by what is called the "exotic convergence". A sequence of quasi-Fuchsian group is said to converge to a b-group exotically when the groups in the sequence are not contained in Bers slices approaching the one containing the limit b-group. The construction of Anderson-Canary gives a sequence converging exotically to a regular b-group. We shall prove such a convergence cannot occur for b-groups which do not have $\mathbb{Z}$-cusps not touching geometrically infinite ends.

As for self-bumping, we conjecture that such a phenomenon cannot happen for geometrically infinite b-groups all of whose $\mathbb{Z}$-cusps touch geometrically infinite ends. What we shall prove is a weaker form of this conjecture: We shall show that if there are two sequences $\{qf(m_i, n_i)\}$ and $\{qf(m'_i, n'_i)\}$ both converging to the same geometrically infinite group all of whose $\mathbb{Z}$-cusps touch geometrically infinite ends, then for any small neighbourhood $U$ of the quasi-conformal deformation space of the limit group, if we take large $i$, then $qf(m_i, n_i)$ and $qf(m'_i, n'_i)$ are connected by an arc in $U$. In particular this shows that under the same condition on $\mathbb{Z}$-cusps, if the limit group is either quasi-conformally rigid or a b-group whose upper conformal structure at infinity is rigid, it cannot be a self-bumping point. More generally, even when there is a cusp not touching a geometrically infinite end, the same argument shows that a Bers slice cannot bump itself at a b-group whose upper conformal structure at infinity is rigid. This latter result has been obtained independently by Brock-Bromberg-Canary-Minsky [9] by a different approach.

Finally, in Theorems 9 and 10, we shall generalise the results for quasi-Fuchsian groups in Theorems 2 and 3 to general surface Kleinian groups.

## 2. Preliminaries

2.1. **Generalities.** Kleinian groups are discrete subgroups of $\mathrm{PSL}_2\mathbb{C}$. In this paper we always assume Kleinian groups to be torsion free. When we talk about deformation spaces, we only consider finitely generated Kleinian groups. However, we also need to consider infinitely generated Kleinian groups which will appear as geometric limits. We refer the reader to Marden [25] for a general references for the theory of Kleinian groups.

Let $S$ be an orientable hyperbolic surface of finite area. In this paper, we focus on Kleinian groups which are isomorphic to $\pi_1(S)$ in such a way parabolic elements correspond to punctures of $S$. We define a deformation space $AH(S)$ to be the quotient space of

$$R(S) = \{(G, \phi) \mid \phi : \pi_1(S) \to \mathrm{PSL}_2\mathbb{C} \text{ is a faithful discrete representation preserving the parabaolicity with } \phi(\pi_1(S)) = G\}$$

by conjugacy in $\mathrm{PSL}_2\mathbb{C}$. The space $R(S)$ has a topology coming from the representation space and we endow $AH(S)$ with its quotient topology. We denote an element of $AH(S)$ also by $(G, \phi)$ for some representative of the equivalence class. We call $\phi$ a marking of the Kleinian group $G$.

The set of faithful discrete representations of $\pi_1(S)$ into $\mathrm{PSL}_2\mathbb{R}$ modulo conjugacy constitutes the Teichmüller space of $S$, which we denote by $\mathcal{T}(S)$. Then, $\mathcal{T}(S)$ is naturally contained in $AH(S)$. More generally, the space of quasi-Fuchsian groups $QF(S)$ lies in $AH(S)$. A quasi-Fuchsian group is a Kleinian group whose domain of discontinuity is a disjoint union of two simply connected components. By the theory of Ahlfors-Bers, $QF(S)$ is parametrised by a homeomorphism $qf : \mathcal{T}(S) \times \mathcal{T}(\bar{S}) \to QF(S)$. Here both $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$ are the Teichmüller space of $S$, but the latter one is identified with $\mathcal{T}(S)$ by an orientation reversing auto-homeomorphism of $S$. For $(m, n) \in \mathcal{T}(S) \times \mathcal{T}(\bar{S})$, its image $qf(m, n)$ is obtained by starting from a Fuchsian group and solving a Beltrami equation so that the conformal structure on the quotient of the lower Jordan domain is $m$ and that on the quotient of the upper Jordan domain is $n$. We call $m$ and $n$ the lower and upper *conformal structures at infinity* of $qf(m, n)$ respectively. The set of Fuchsian groups corresponds to a slice of the form $\{qf(m, m)\}$. By the theory of Ahlfors-Bers combined with Sullivan's rigidity [37], we know that $QF(S)$ is the interior of the entire deformation space $AH(S)$. On the other hand, the Bers-Sullivan-Thurston density conjecture, which was solved by Bromberg [7], Brock-Bromberg [8] in this setting, or is obtained as a corollary of the ending lamination conjecture [10] combined with [31], $AH(S)$ is the closure of $QF(S)$.

By Margulis' lemma, there is a positive constant $\epsilon_0$ such that for any hyperbolic 3-manifold, its $\epsilon_0$-thin part is a disjoint union of cusp neighbourhoods and tubular neighbourhoods of short closed geodesics, which are called Margulis tubes. For a hyperbolic 3-manifold $M$, we denote by $M_0$ the complement of its cusp neighbourhoods.

Bonahon showed in [6] that for any $(G, \phi) \in AH(S)$, the hyperbolic 3-manifold $\mathbb{H}^3/G$ is homeomorphic to $S \times (0, 1)$. We denote by $\Phi$ a homeomorphism from $S \times (0, 1)$ to $\mathbb{H}^3/G$ inducing $\phi$ between the fundamental groups. In general, for an element in $AH(S)$, we denote a homeomorphism from $S \times (0, 1)$ to the quotient hyperbolic 3-manifold by the letter in the upper case corresponding to a Greek letter denoting the marking. Bonahon also proved the every end of $(\mathbb{H}^3/G)_0$ is either geometrically finite or simply degenerate. Here an end is called *geometrically finite* if it has a neighbourhood which is disjoint from any closed geodesic, and *simply degenerate* if it has a neighbourhood of the form $\Sigma \times (0, \infty)$ for an incompressible subsurface $\Sigma$ of $S$ and there is a sequence of simple closed curves $\{c_n\}$ on $\Sigma$ which are homotopic to closed geodesics $c_n^*$ going to the end. For a simply degenerate end, the *ending lamination* is defined to be the support of the projective lamination to which $[c_n]$ converges in the projective lamination space $\mathcal{PML}(\Sigma)$. We shall explain what are laminations and the projective lamination space below.

For $(G, \phi) \in AH(S)$, we choose an embedding $f : S \to \mathbb{H}^3/G$ inducing $\phi$ between the fundamental groups. Such an embedding is unique up to ambient isotopy in

$\mathbb{H}^3/G$. An end of $\mathbb{H}^3/G$ is called *upper* when it lies above $f(S)$ and *lower* when it lies below $f(S)$. Let $C$ be a compact core of $(\mathbb{H}^3/G)_0$ intersecting each component of $\mathrm{Fr}(\mathbb{H}^3/G)_0$ by a core annulus. Let $P$ denote $C \cap \mathrm{Fr}(\mathbb{H}^3/G)_0$. Each component of $P$ is called a *parabolic locus* of $\mathbb{H}^3/G$ or $C$. Parabolic loci corresponding to punctures of $S$ are called *boundary parabolic loci*. For non-boundary parabolic loci, those contained in $S \times \{1\}$ are called upper and those contained in $S \times \{0\}$ lower. The ends of $(\mathbb{H}^3/G)_0$ correspond one-to-one to the components of $\mathrm{Fr}C$. Since $C$ is homeomorphic to $S \times [0,1]$, each upper end faces a subsurface of $S \times \{1\}$, which is a component of $S \times \{1\} \setminus P$. A non-boundary parabolic locus $p$ in $P$ is called *isolated* when the components of $S \times \{0,1\} \setminus P$ adjacent to $p$ (there are one or two such components) face geometrically finite ends, in other words, no component of $S \times \{0,1\} \setminus P$ facing a simply degenerate end touches $p$ at its frontier.

2.2. **Laminations.** A geodesic lamination on a hyperbolic surface $S$ is a closed subset of $S$ consisting of disjoint geodesics, which are called leaves. For a geodesic lamination $\lambda$, each component of $S \setminus \lambda$ is called a complementary region of $\lambda$. We say that a geodesic lamination is *arational* when every complementary region is simply connected. A subset $l$ of a geodesic lamination is called a *minimal component* when for each leaf $\ell$ of $l$, its closure $\bar{\ell}$ coincides with $l$. Any geodesic lamination is decomposed into finitely many minimal components and finitely many non-compact isolated leaves.

A measured lamination is a (possibly empty) geodesic lamination endowed with a transverse measure which is invariant with respect to a homotopy along leaves. When we consider a measured lamination, we always assume that the support of the transverse measure is the entire lamination. The measured lamination space $\mathcal{ML}(S)$ is the set of measured laminations on $S$ endowed with the weak topology. Thurston proved that $\mathcal{ML}(S)$ is homeomorphic to $\mathbb{R}^{6g-6+2p}$, where $g$ is the genus and $p$ is the number of punctures. A weighted disjoint union of closed geodesics can be regarded as a measured lamination. It was shown by Thurston that the set of weighted disjoint unions of closed geodesics is dense in $\mathcal{ML}(S)$.

The projective lamination space $\mathcal{PML}(S)$ is the space obtained by taking a quotient of $\mathcal{ML}(S) \setminus \{\emptyset\}$ identifying scalar multiples. Thurston constructed a natural compactification of the Teichmüller space whose boundary is $\mathcal{PML}(S)$ in such a way that the mapping class group acts continuously on the compactification.

We need to consider one more space, the unmeasured lamination space. This space, denoted by $\mathcal{UML}(S)$, is defined to be the quotient space of $\mathcal{ML}(S)$, where two laminations with the same support are identified. An element in $\mathcal{UML}(S)$ is called an unmeasured lamination.

For a minimal geodesic lamination $\lambda$ on $S$, its *minimal supporting surface* is defined to be a subsurface of $S$ with geodesic boundary containing $\lambda$ which is minimal among all such surfaces. When $\lambda$ is a closed geodesic, we define its minimal supporting surface to be $\lambda$ itself. It is obvious the minimal supporting surface of $\lambda$ is uniquely determined.

2.3. **Algebraic convergence and geometric convergence.** When $\{(G_i, \phi_i)\}$ converges to $(\Gamma, \psi) \in AH(S)$, we say that the sequence converges *algebraically* to $(\Gamma, \psi)$. We can choose representatives $(G_i, \phi_i)$ so that $\phi_i$ converges to $\psi$ as representations. As a convention, when we say that $\{(G_i, \phi_i)\}$ converges to $(\Gamma, \psi)$, we always take representatives so that $\{\phi_i\}$ converges to $\psi$.

We need to consider another kind of convergence, the geometric convergence. A sequence of Kleinian groups $\{G_i\}$ is said to converge to a Kleinian group $G_\infty$ *geometrically* if (1) for any convergent sequence $\{\gamma_{i_j} \in G_{i_j}\}$, its limit lies in $G_\infty$, and (2) any element $\gamma \in G_\infty$ is a limit of some $\{g_i \in G_i\}$. When $(G_i, \phi_i)$ converges to $(\Gamma, \psi)$ algebraically, the geometric limit $G_\infty$ contains $\Gamma$ as a subgroup.

When $\{G_i\}$ converges to $G_\infty$ geometrically, if we take a basepoint $x$ in $\mathbb{H}^3$ and its projections $x_i \in \mathbb{H}^3/G_i$ and $x_\infty \in \mathbb{H}^3/G_\infty$, then $(\mathbb{H}^3/G_i, x_i)$ converges to $(\mathbb{H}^3/G_\infty, x_\infty)$ with respect to the pointed Gromov-Hausdorff topology: that is, there exists a $(K_i, r_i)$-approximate isometry $B_{r_i}(\mathbb{H}^3/G_i, x_i) \to B_{K_i r_i}(\mathbb{H}^3/G_\infty, x_\infty)$ with $K_i \to 1$ and $r_i \to \infty$.

2.4. **Bers slices and b-groups.** We fix a point $m_0 \in \mathcal{T}(S)$ and consider a subspace $qf(\{m_0\} \times \mathcal{T}(\bar{S}))$ in $QF(S)$. This space is called the *Bers slice* over $m_0$. Kleinian groups lying on its frontier are called *b-groups* with lower conformal structure $m_0$.

Anderson-Canary [2] constructed a sequence of quasi-Fuchsian groups converging to a b-group whereas its coordinates in $QF(S)$ do not approach a Bers slice.

**Definition 2.1.** We say a sequence of quasi-Fuchsian groups $\{(G_i, \phi_i) = qf(m_i, n_i)\}$ converges *exotically* to a b-group $(\Gamma, \psi)$ if $\{(G_i, \phi_i)\}$ converges to $(\Gamma, \psi)$ algebraically and both $\{m_i\}$ and $\{n_i\}$ go out from any compact set in the Teichmüller space.

The existence of exotic convergence is related to the singularities of $AH(S)$ at its boundary. In fact, McMullen showed in [28] that $AH(S)$ has a singular point at infinity where $QF(S)$ bumps itself. For a Kleinian surface group $(\Gamma, \psi) \in AH(S) \setminus QF(S)$, we say that $QF(S)$ *bumps itself* at $(\Gamma, \psi)$, when there is a neighbourhood $V$ of $(\Gamma, \psi)$ such that for any smaller neighbourhood $U \subset V$, its intersection with the quasi-Fuchsian space, $U \cap QF(S)$ is disconnected. McMullen showed the existence of points where $QF(S)$ bumps itself, making use of the construction of Anderson-Canary.

We say that a Bers slice $\mathcal{B}(m_0) = qf(\{m_0\} \times \mathcal{T}(\bar{S}))$ bumps itself at $(\Gamma, \psi)$ when there is a neighbourhood $V$ of $(\Gamma, \psi)$ such that for any smaller neighbourhood $U \subset V$, its intersection with the Bers slice, $U \cap \mathcal{B}(m_0)$ is disconnected. Up to now, it is not known if a Bers slice can bump itself or not.

2.5. **Curve complexes and hierarchies.** When we talk about curve complexes, we usually regard surfaces as either closed (i.e. compact without boundary) or open without boundary. For a surface $S$ of genus $g$ and $p$ punctures, we define $\xi(S)$ to be $3g + p$. We shall define curve complexes for orientable surfaces $S$ with $\xi(S) \geq 4$ or $\xi(S) = 2$. The *curve complex* $\mathcal{CC}(S)$ of $S$ with $\xi(S) \geq 5$ is defined as follows. (This notion was first introduced by Harvey [19].) The vertices of $\mathcal{CC}(S)$, whose set we denote by $\mathcal{CC}_0(S)$, are the homotopy classes of essential simple closed curves on $S$. A set of $n+1$ vertices $\{v_0, \ldots, v_n\}$ spans an $n$-simplex if and only if they can be represented as pairwise disjoint simple closed curves on $S$.

In the case when $\xi(S) = 4$, the curve complex is a 1-dimensional simplicial complex. The vertices are the homotopy classes of essential simple closed curves as in the case when $\xi(S) \geq 5$. Two vertices $v_0, v_1$ are connected by an edge if their intersection number is 1 when $S$ is a once-punctured torus, and 2 when if $S$ is a four-times-punctured sphere. In the case when $\xi(S) = 2$, we consider a compactification of $S$ to an annulus. The curve complex is a 1-dimensional and the vertices are the homotopy classes (relative to the endpoints) of essential arcs with both endpoints on the boundary. Two curves are connected by an edge if they can be made disjoint in their interiors.

A *tight sequence* in $\mathcal{CC}(S)$ with $\xi(S) \geq 5$ is a sequence of simplices $\{s_0, \dots, s_n\}$ with the first one and the last one being vertices such that for any vertices $v_i \in s_i$ and $v_j \in s_j$, we have $d_{\mathcal{CC}(S)}(v_i, v_j) = |j - i|$, and $s_{j+1}$ is homotopic to the union of essential boundary components of a regular neighbourhood of $s_j \cup s_{j+2}$. We also consider an infinite tight sequence such as $\{s_0, \dots\}$ or $\{\dots, s_0\}$ or $\{\dots, s_0, \dots\}$. In the case of surface with $\xi(S) = 2, 4$, we consider a sequence of vertices and ignore the second condition.

For an essential simple closed curve $c$ of $S$, we consider the covering $A_c$ of $S$ associated to the image of $\pi_1(c)$, which is an open annulus. If we fix a hyperbolic metric on $S$, we can compactify the hyperbolic annulus $A_c$ to an annulus $\bar{A}_c$ by regarding $\pi_1(c)$ as acting on $\mathbb{H}^2$ and considering the quotient of $\mathbb{H}^2 \cup \Omega_{\pi_1(c)}$ by $\pi_1(c)$. We call an essential simple arc with endpoints on $\partial \bar{A}_c$ a transversal of $c$. A simple closed curve $d$ intersecting $c$ essentially induces $i(c, d)$ transversals on $c$. Any two transversals induced from $d$ are within distance 1 in $\mathcal{CC}(A_c)$.

For a non-annular domain $\Sigma$ in $S$, we define $\pi_\Sigma : \mathcal{CC}_0(S) \to \mathcal{CC}(\Sigma) \cup \{\emptyset\}$ to be a map sending $c \in \mathcal{CC}_0(S)$ to simple closed curves obtained by connecting the endpoints of each component of $c \cap \Sigma$ by arcs on $\mathrm{Fr}\Sigma$ in a consistent way if $c$ intersects $\Sigma$ essentially. Otherwise we define $\pi_\Sigma(c)$ to be $\emptyset$. When $\Sigma$ is an annulus we define $\pi_\Sigma(c)$ to be the lift of $c$ to $\bar{\Sigma}$ if $c$ intersects $\Sigma$ essentially. (See §2.3 of Masur-Minsky [27] for details.)

A *marking* $\mu$ on a surface $S$ consists of a simplex in $\mathcal{CC}(S)$ and transversals on some of its vertices (at most one for each). The vertices of the simplex are called the base curves of $\mu$, and their union is denoted by base$(\mu)$. A marking $\mu$ is said to be *clean* if every component $c$ of base$(\mu)$ has a transversal and it is induced by a simple closed curve with intersection number 1 when $c$ is non-separating and with intersection number 2 if $c$ is separating, which is disjoint from the other components of base$(\mu)$. A clean marking $\mu'$ is said to be compatible with a marking $\mu$ when base$(\mu) =$ base$(\mu')$, and every transversal of a component $c$ in $\mu$ is within distance 2 from the transversal of $c$ in $\mu'$ as vertices in the curve complex of an annulus with core curve $c$. A marking is called *complete* if its base curves constitute a pants decomposition of $S$.

To deal with the case of geometrically infinite groups, we need a notion of generalised markings. A *generalised marking* consists of an unmeasured lamination on $S$ and transversals on some of its components which are simple closed curves. Also for a generalised marking $\mu$, we denote the lamination by base$(\mu)$. We say that a generalised marking *complete*, if its base lamination is maximal, i.e. it is not a proper sublamination of another unmeasured lamination. From now on, we always assume *markings and generalised markings to be complete.*

A finite *tight geodesic* on a surface $\Sigma$ is a triple $(g, I(g), T(g))$, where $g$ is a tight sequence and $I(g)$ and $T(g)$ are generalised markings on $\Sigma$ which have at least one simple closed curve component, such that the first vertex is a simple closed curve component of the base$(I(g))$ and the last vertex is that of base$(T(g))$. The surface $\Sigma$ is called the support of $g$ and we write $\Sigma = D(g)$. An infinite tight geodesic is defined similarly just letting $T(g)$ be an arational unmeasured lamination to which $s_i$ converges as $i \to \infty$ when $g$ is in the form of $\{s_0, \dots\}$. Similarly, by letting $I(g)$ be an arational unmeasured lamination to which $s_i$ converges as $i \to -\infty$ when $g$ is in the form of $\{\dots, s_0\}$, and by letting both $I(g)$ and $T(g)$ be arational unmeasured laminations which are limits of $s_i$ as $i \to -\infty$ and as $i \to \infty$ respectively when $g$ is in the form of $\{\dots, s_0, \dots\}$. Refer to the following subsection form more explanations on the boundary of $\mathcal{CC}(\Sigma)$.

We call an open subsurface of $S$ whose frontier is non-contractible and non-peripheral, a domain. Let $\Sigma$ be a domain of $S$. For a simplex $s$ in $\mathcal{CC}(\Sigma)$, its component domains are defined to be the components of $\Sigma \setminus s$ and annuli whose core curves are components of $s$. We consider only one annulus for each component of $s$. Let $g$ be a tight geodesic in $\mathcal{CC}(\Sigma)$, and suppose that $s$ is a simplex on $g$. Let $\Sigma'$ be a component domain of $s$. (Such a domain is also said to be a component domain of $g$.) Then, following Masur-Minsky [27], we define $T(\Sigma', g)$ to be $\mathrm{succ}(s)|\Sigma'$ if $s$ is not the last vertex of $g$, and to be $T(g)|\Sigma'$ if $s$ is. Similarly, we define $I(\Sigma', g)$ to be $\mathrm{prec}(s)|\Sigma'$ if $s$ is not the first vertex of $g$, and to be $I(g)|\Sigma'$ if $s$ is. We write $\Sigma' \overset{d}{\searrow} g$ or $\Sigma' \overset{d}{\searrow} (g, s)$, if $T(\Sigma', g)$ is non-empty, and $g \overset{d}{\swarrow} \Sigma'$ or $(g, s) \overset{d}{\swarrow} \Sigma'$ if $I(\Sigma', g)$ is non-empty. If a geodesic $k$ is a tight geodesic supported on $\Sigma'$ with $(g, s) \overset{d}{\swarrow} \Sigma'$ and $I(k) = I(\Sigma', g)$, then we write $g \overset{d}{\swarrow} k$ or $(g, s) \overset{d}{\swarrow} k$, and say that $k$ is *directly backward subordinate* to $g$ at $s$. Similarly, if $\Sigma' \overset{d}{\searrow} (g, s)$ and $T(k) = T(\Sigma', g)$, we write $k \overset{d}{\searrow} g$ or $k \overset{d}{\searrow} (g, s)$, and say that $k$ is *directly forward subordinate* to $g$ at $s$.

A *hierarchy* $h$ on $S$, which was introduced by Masur-Minsky [27], is a family of tight geodesics supported on domains of $S$ having the following properties.

(1) There is a unique geodesic $g_h$ supported on $S$.

(2) For any $g \in h$ other than $g_h$, there are geodesics $b, f \in h$ with $b \overset{d}{\swarrow} g \overset{d}{\searrow} f$.

(3) For any $b, f \in h$ and a component domain $\Sigma$ of $b, f$ with $b \overset{d}{\swarrow} \Sigma \overset{d}{\searrow} g$ ($b$ and $f$ may coincide), there is a unique geodesic $k$ supported on $\Sigma$ with $b \overset{d}{\swarrow} k \overset{d}{\searrow} f$.

A hierarchy $h$ is said to be *complete* if every component domain of geodesics in $h$ supports a geodesic in $h$, and 4-*complete* if every non-annular component domain of geodesics in $h$ supports a geodesic in $h$.

We write $g \searrow (f, v)$ if there is a sequence of geodesics in $h$ such that $g = f_0 \overset{d}{\searrow} f_1 \overset{d}{\searrow} \dots \overset{d}{\searrow} (f_n, v) = (f, v)$, and say that $g$ is forward subordinate to $f$. Similarly, we write $(b, u) \swarrow g$ if there is a sequence in $h$ such that $(b, u) = (b_m, u) \overset{d}{\swarrow} \dots \overset{d}{\swarrow} b_1 \overset{d}{\swarrow} b_0 = g$, and say that $g$ is backward subordinate to $b$. We use symbol $\underset{=}{\searrow}$ to mean either $\searrow$ or $=$ and $\underset{=}{\swarrow}$ to mean either $\swarrow$ or $=$.

In §9, we shall use the notions of slices and resolutions of hierarchies invented by Masur-Minsky [27]. We shall review them briefly here. Let $h$ be a complete or 4-complete hierarchy. A *slice* $\sigma$ of $h$ is a set of pairs $(g, v)$, where $g$ is a geodesic in $h$ and $v$ is a simplex on $g$ satisfying the following conditions. (Masur and Minsky call $\sigma$ satisfying the first three conditions a slice, and call it a complete slice if it also satisfies the fourth condition.)

(1) A geodesic can appear at most in one pair of $\sigma$.

(2) There is a pair whose first entry is the main geodesic of $h$.
(3) For each pair $(g, v)$ in $\sigma$ such that $g$ is not the main geodesic, $D(g)$ is a component domain of a simplex $v'$ for some $(g', v') \in \sigma$.
(4) For each component domain $D$ of $v$ for $(g, v) \in \sigma$ with $\xi(D) \neq 3$ if $h$ is complete and $\xi(D) > 3$ if $h$ is 4-complete, there is a pair $(g', v') \in \sigma$ with $D(g') = D$.

Masur and Minsky introduced orders $\prec_p$ between pairs of geodesics and simplices in $h$ and $\prec_s$ between slices. For two pairs $(g, v)$ and $(g', v')$ of a 4-complete hierarchy, we write $(g, v) \prec_p (g', v')$ if either $g = g'$ and $v'$ comes after $v$, or there is a geodesic $g''$ with $(g, v) \underset{=}{\searrow} (g'', w)$ and $(g'', w') \underset{=}{\swarrow} (g', v')$ such that $w'$ is a simplex coming after $w$. For two distinct slices $\sigma$ and $\tau$, we write $\sigma \prec_s \tau$ if for any $(g, v) \in \sigma$, either $(g, v) \in \tau$ or there is $(g', v') \in \tau$ with $(g, v) \prec_p (g', v')$.

A *resolution* $\tau = \{\sigma_i\}$ of a 4-complete hierarchy $h$ is an ordered sequence of slices of $h$ such that $\sigma_{i+1}$ is obtained from $\sigma_i$ by an elementary forward move. An elementary forward move is a change of pairs in $\sigma_i$: We advance $(g, v) \in \sigma_i$ to $(g, \mathrm{succ}(v))$ under the condition that for every pair $(g', v')$ supported on a component domain of $v$ into which $\mathrm{succ}(v)$ is projected to an essential curve, the simplex $v'$ is the last vertex, and after removing all such $(g', v')$ we add pairs $(g'', v'')$ such that $g''$ is supported on a component domain of $\mathrm{succ}(v)$ into which $v$ is projected to an essential curve and $v''$ is the first vertex of $g''$.

A model manifold for a Kleinian surface group is constructed in Minsky [29] as follows. Let $G$ be a Kleinian group with $M = \mathbb{H}^3/G$. From an end invariant $G$, we shall construct a hierarchy of tight geodesics, $h_G$. A clean marking is called shortest with respect to a conformal structure if its base curve is a shortest pants decomposition and transversal are chosen to be shortest among all of those having the same base curves, where we consider the lengths to be the hyperbolic ones. In the case when $G$ is quasi-Fuchsian, we construct a hierarchy by defining the initial and terminal markings to be the shortest clean markings with respect to the upper and lower conformal structures at infinity. When $M$ has a totally degenerate end without accidental parabolics, we define the initial or terminal generalised marking to be its ending lamination. In general, we consider the union of ending laminations of $M_0$, parabolic loci for upper or lower ends, and shortest markings on the remaining geometrically finite upper or lower ends, and let them be terminal or initial generalised markings.

Then, we construct a resolution $\{\tau_i(G)\}$ of $h_G$. In the resolution, we look at each step $\tau_i(G) \to \tau_{i+1}(G)$ that advances a vertex on a 4-geodesic, from $w_i$ to $w_{i+1}$. For such a step, we provide a block, which is topologically homeomorphic to $\Sigma \times I$, where $\Sigma$ is either a sphere with four holes or a torus with one hole. The block has two ditches, one on the top and the other on the bottom, corresponding to the two vertices $w_i$ and $w_{i+1}$. The top and bottom boundary of a block consists of pairs of sphere with three holes. The model manifold is constructed by piling up such blocks by pasting a top component of one block to a bottom component of another, according to the information given by the resolution $\{\tau_i(G)\}$, attach *boundary blocks* to the top and the bottom of the piled up blocks if there are geometrically finite ends of $M_0$, which have special forms and are constructed according to conformal structures at infinity of $G$, and then finally fill in Margulis tubes. In this paper, we let a boundary block have a form $\Sigma \times [s, t)$ or $\Sigma \times (t, s]$ for some subsurface $\Sigma$ of $S$, and do not put extra-annuli as in Minsky's definition. This is because we are constructing a model manifold of the *non-cuspidal part*, not of the entire manifold. By the same reason, in contrast to Minsky's original construction, we do not fill in cusp neighbourhoods to make it compatible with model manifolds for geometric limits developed in [35]. Each slice in $\{\tau_i(G)\}$ corresponds to a *split level surface* in the model manifold which is a disjoint union of horizontal surfaces in blocks which are spheres with three holes. Taking split level surfaces to pleated surfaces, a homotopy equivalent map from the model manifold to $M_0$ is constructed. This can be modified to a uniform bi-Lipschitz map which is called *a model map* to $M_0$. See Minsky [29] and Brock-Canary-Minsky [10] for more details.

2.6. **The boundaries at infinity of curve complexes.** It was proved by Masur-Minsky [26] that $\mathcal{CC}(S)$ is a Gromov hyperbolic space with respect to the path metric defined by setting every edge to have the unit length. For a Gromov hyperbolic space, its boundary at infinity can be defined as a topological space. (Refer for instance to Coornaert-Delzant-Papadopoulos [16].) Klarreich in [21] showed that the boundary at infinity of $\mathcal{CC}(S)$ is the space of ending laminations: that is, the space of arational unmeasured laminations with topology induced from $\mathcal{UML}(S)$. This space is denoted by $\mathcal{EL}(S)$.

We shall show the following lemma, which is an easy consequence of the definition of the topology of $\mathcal{CC}(\Sigma) \cup \mathcal{EL}(\Sigma)$.

**Lemma 2.2.** *Let $\{g_i\}$ be a sequence of geodesics in $\mathcal{CC}(\Sigma)$ converging to a geodesic ray $g_\infty$ uniformly on every compact set. Then the last vertex of $g_i$ converges to the endpoint at infinity of $g_\infty$ with respect to the topology of $\mathcal{CC}(\Sigma) \cup \mathcal{EL}(\Sigma)$.*

*Proof.* Let $\lambda$ be a measured lamination whose support is the endpoint at infinity of $g_\infty$. We can assume that all the $g_i$ have the same initial vertex, which we denote by $v$. Let $w_i$ be the last vertex of $g_i$. Since the length of $g_i$ goes to infinity, the distance between $v$ and $w_i$ goes to infinity. On the other hand, since $\{g_i\}$ converges to $g_\infty$ on every compact set, there is a number $n_{i,j}$ going to $\infty$ such that the first $n_{i,j}$ simplices of $g_i$ and $g_j$ are the same. Since $(w_i|w_j)_v \geq n_{i,j}$, we see that $(w_i|w_j)_v$ goes to $\infty$ as $i, j \to \infty$. Therefore, $\{w_i\}$ converges to some ending lamination after passing to a subsequence. By the definition of the topology on $\mathcal{CC}(\Sigma) \cup \mathcal{EL}(\Sigma)$, there is a measured lamination $\mu$ and positive real numbers $r_i$ such that $\{r_i w_i\}$ converges to $\mu$.

We need to show that $|\mu| = |\lambda|$. Suppose not. Since $\{g_i\}$ converges to $g_\infty$ uniformly on every compact set, we can take a simplex $v_i \in g_\infty$ which is also contained in $g_i$ tending to $|\lambda|$ in $\mathcal{CC}(\Sigma) \cup \mathcal{EL}(\Sigma)$. Since $|\lambda|$ and $|\mu|$ are distinct points on the boundary at infinity, we have $\limsup_{i\to\infty}(v_i|w_i)_v < \infty$. This contradicts the facts that both $v_i$ and $w_i$ lie on the same geodesic $g_i$ and that both $d(v, v_i)$ and $d(v, w_i)$ go to $\infty$. □

## 3. The main results

In this section, we shall state our main theorems.

3.1. **End invariants of limit groups.** We shall first state a theorem showing that for a limit of quasi-Fuchsian groups, the limit laminations of upper conformal structures at infinity appear as ending laminations of upper ends whereas the limit of lower ones appear as ending laminations of lower ends

**Theorem 1.** *Let $\{(m_i, n_i)\}$ be a sequence in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ such that $qf(m_i, n_i)$ converges to $(\Gamma, \psi)$ in $AH(S)$. Let $[\mu^+]$ and $[\mu^-]$ be projective laminations which are limits of $\{m_i\}$ and $\{n_i\}$ in the Thurston compactification of the Teichmüller space. Then every component of $|\mu^+|$ that is not a simple closed curve is the ending lamination of an upper end of $(\mathbb{H}^3/\Gamma)_0$ whereas every component of $|\mu^-|$ that is not a simple closed curve is the ending lamination of a lower end.*

*Moreover every simple closed curve in $|\mu^+|$ that is not contained in $|\mu^-|$ is a core curve of an upper parabolic locus. Similarly every simple closed curve in $|\mu^-|$ that is not contained in $|\mu^+|$ is a core curve of a lower parabolic locus.*

Simple closed curves contained both $|\mu^+|$ and $|\mu^-|$ will be dealt with in Theorem 5.

This theorem will be obtained by combining the following theorem with a simple lemma regarding the Thurston compactification of Teichmüller space.

**Theorem 2.** *In the same setting as in Theorem 1, let $c_{m_i}$ and $c_{n_i}$ be shortest pants decompositions of $(S, m_i)$ and $(S, n_i)$ respectively. Let $\nu^-, \nu^+$ be the Hausdorff limits of $\{c_{m_i}\}$ and $\{c_{n_i}\}$ respectively. Then the minimal components of $\nu^+$ that are not simple closed curves coincide with the ending laminations of upper ends of $(\mathbb{H}^3/\Gamma)_0$. Moreover, no upper parabolic locus can intersect a minimal component of $\nu^+$ transversely. Similarly the minimal components of $\nu^-$ that are not simple closed curves coincide with the ending laminations of lower ends of $(\mathbb{H}/\Gamma)_0$, and no lower parabolic locus can intersect a minimal component of $\nu^-$ transversely.*

3.2. **Divergence theorems.** We shall next state our theorems on divergence of quasi-Fuchsian groups. In the following theorems, we shall give sufficient conditions for sequences to diverge .

**Theorem 3.** *Let $\{(m_i, n_i)\}$ be a sequence in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ satisfying the following conditions.*

*(1) $\{m_i\}$ converges to a projective lamination $[\mu^-] \in \mathcal{PML}(S)$ whereas $\{n_i\}$ converges to $[\mu^+] \in \mathcal{PML}(S)$ in the Thurston compactification of the Teichmüller space.*
*(2) There are components $\mu_0^+$ of $\mu^-$ and $\mu_0^-$ of $\mu^+$ which are not simple closed curves and have the minimal supporting surfaces sharing at least one boundary component.*

*Then the sequence $\{qf(m_i, n_i)\} \subset QF(S)$ diverges in $AH(S)$.*

Theorem 3 follows rather easily from Theorem 1. If we use Theorem 2 instead of Theorem 1, we get the following.

**Theorem 4.** *Let $\{m_i\}$ and $\{n_i\}$ be sequences in $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$ without convergent subsequences, and let $c_{m_i}$ and $c_{n_i}$ be shortest pants decomposition of the hyperbolic surfaces $(S, m_i)$ and $(S, n_i)$ respectively. Suppose that $c_{m_i}$ and $c_{n_i}$ converge $\mu^-$ and $\mu^+$ in the Hausdorff topology respectively. Suppose that there are minimal components $\mu_0^-$ of $\mu^-$ and $\mu_0^+$ of $\mu^+$ which are not simple closed curves and have minimal supporting surfaces sharing at least one boundary component. Then $\{qf(m_i, n_i)\} \subset QF(S)$ diverges in $AH(S)$.*

In the setting of these theorems, the case when $\mu_0^-$ and $\mu_0^+$ have the same support is most interesting. In fact, if they do not, it is much easier to prove the theorems just by using the continuity of length function in hyperbolic manifolds and the fact that ending lamination for a simply degenerate end is uniquely determined. Also the assumption that $\mu_0^-$ and $\mu_0^+$ are not simple closed curves is essential. In the case when $\mu^-$ and $\mu^+$ share simple closed curves, the construction of Anderson-Canary [2] gives an example of convergent sequence. Still, we can show the following theorem.

**Theorem 5.** *Let $\mu^-$ and $\mu^+$ be two measured laminations on $S$ such that the components shared by $|\mu^-|$ and $|\mu^+|$ are all simple closed curves, which we denote by $c_1, \ldots, c_r$. Suppose that at last one of $c_1, \ldots, c_r$ lies on the boundary of the minimal supporting surface of a component of either $\mu^-$ or $\mu^+$. Then for every $\{m_i\}$ converging to $[\mu^-]$ and $\{n_i\}$ converging to $[\mu^+]$ in the Thurston compactification of the Teichmüller space, the sequence $\{qf(m_i, n_i)\} \subset QF(S)$ diverges in $AH(S)$.*

In the case when none of $c_1, \ldots, c_r$ lies on the boundary of minimal supporting surface of a component of $\mu^-$ or $\mu^+$, we need to take into accounts the weights on $c_1, \ldots, c_r$, as was done in Ito [20] in the case of once-punctured torus groups.

**Theorem 6.** *Consider, as in Theorem 4, sequences $\{m_i\}$ and $\{n_i\}$ converging to $[\mu^-]$ and $[\mu^+]$ respectively whose supports share only simple closed curves $c_1, \ldots, c_r$. Suppose that none of $c_1, \ldots, c_r$ lies on the boundary of the minimal supporting surface of a component of $\mu^-$ or $\mu^+$. Then $\{qf(m_i, n_i)\}$ converges after taking a subsequence only if the following conditions are satisfied.*

*(1) For each $c_j$ among $c_1, \ldots, c_r$, neither $\mathrm{length}_{m_i}(c_j)$ nor $\mathrm{length}_{n_i}(c_j)$ goes to 0.*

*(2) There are sequence of integers $\{p_i^1\}, \ldots, \{p_i^r\}, \{q_i^1\}, \ldots, \{q_i^r\}$ and constants $a, b \in \mathbb{Z}$ such that the following hold after passing to a subsequence:*

*(a) If $|\mu^-| \setminus \cup_{j=1}^r c_j$ is non-empty, then $(\tau_{c_1}^{p_i^1} \circ \cdots \circ \tau_{c_r}^{p_i^r})_*(m_i)$ converges to $[\mu^- \setminus \cup_{j=1}^r w_j c_j]$ in the Thurston compactification, where $w_j$ is the transverse measure on $c_j$ in $\mu^-$. Otherwise $(\tau_{c_1}^{p_i^1} \circ \cdots \circ \tau_{c_r}^{p_i^r})_*(m_i)$ either stays in a compact set of the Teichmüller space or converges to a projective lamination $[\nu^-]$ which contains none of $c_1, \ldots, c_r$ as leaves.*

*(b) Similarly, if $|\mu^+| \setminus \cup_{j=1}^r c_j$ is non-empty, $(\tau_{c_1}^{q_i^1} \circ \cdots \circ \tau_{c_r}^{q_i^r})_*(n_i)$ converges to $[\mu^+ \setminus \cup_{j=1}^r v_j c_j]$ in the Thurston compactification, where $v_j$ is the transverse measure on $c_j$ in $\mu^+$. Otherwise $(\tau_{c_1}^{q_i^1} \circ \cdots \circ \tau_{c_r}^{q_i^r})_*(n_i)$ either stays in a compact set of the Teichmüller space or converges in the Thurston compactification to a projective lamination $[\nu^+]$ which contains none of $c_1, \ldots, c_r$ as leaves.*

*(c) There exists $a_j \in \mathbb{Z}$ $(j = 1, \ldots, r)$ such that $(a_j + 1)p_i^j = a_j q_i^j$ for every $j = 1, \ldots, r$ and large $i$.*

*In this situation, $c_j$ is a core curve of an upper parabolic locus of the algebraic limit if $a_j \leq 0$, and that of a lower parabolic locus otherwise.*

*Conversely, let $a_j \in \mathbb{Z}\,(j = 1, \ldots, r)$ be any number, and $\mu^-, \mu^+$ measured laminations with the following conditions.*

*(1*) The laminations $\mu^-$ and $\mu^+$ do not have non-simple-closed-curve minimal components $\mu_0^-$ and $\mu_0^+$ whose minimal supporting surfaces sharing a boundary component.*

*(2*) In the case when the minimal supporting surfaces of $\mu^-$ and $\mu^+$ are the entire surface $S$, the supports $|\mu^-|$ and $|\mu^+|$ do not coincide.*

*(3*) For (possibly empty) simple closed curves $c_1, \ldots, c_r$ shared by $|\mu^-|$ and $|\mu^+|$, none of $c_1, \ldots, c_r$ lies on the boundary of the minimal supporting surface of a component of $\mu^-$ or $\mu^+$.*

*Then, there is a sequence of $\{(m_i, n_i)\}$ in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ with algebraically convergent $\{qf(m_i, n_i)\}$ such that $\{m_i\}$ converges to $[\mu^-]$ and $\{n_i\}$ converges to $[\mu^+]$ in the Thurston compactification, and the two conditions (1) and (2) above are satisfied.*

The latter half of this theorem shows that *as sufficient conditions for divergence expressed in term of the limits of the conformal structures in the Thurston compactification, Theorems 3 and 4 together with the main theorem of* [33] *are best possible.*

3.3. **Non-existence of exotic convergence.** What Theorem 4 claims is related to the fact that such a sequence cannot converge exotically to a b-group. The condition that none of $c_1, \ldots, c_r$ lies on the boundary of the supporting surface of a component of $\mu^-$ or $\mu^+$ is essential for the exotic convergence. In fact, we can prove the following. (Recall that a non-boundary parabolic locus is said to be isolated if it does not touch a simply degenerate end.)

**Theorem 7.** *Let $G$ be a b-group without isolated parabolic locus. Then there is no sequence of quasi-Fuchsian groups exotically converging to $G$.*

3.4. **Self-bumping.** As the results of McMullen [28], Bromberg [7] and Magid [24] suggest, the singularities of $AH(S)$ which are found thus far are all related to the construction of Anderson-Canary. The following results show that convergence to geometrically infinite groups in $AH(S)$ without isolated parabolic loci is quite different from the situation for regular b-groups where $QF(S)$ bumps itself.

**Theorem 8.** *Let $\Gamma$ be a geometrically infinite group with isomorphism $\psi : \pi_1(S) \to \Gamma$ in $AH(S)$. Suppose that $\Gamma$ does not have an isolated parabolic locus. Let $\{(m_i, n_i)\}$ and $\{(m_i', n_i')\}$ be two sequences in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ such that both $\{qf(m_i, n_i)\}$ and $\{qf(m_i', n_i')\}$ converge to $(\Gamma, \psi)$. Then for any neighbourhood $U$ of the quasi-conformal deformation space $QH(\Gamma, \psi)$ of $(\Gamma, \psi)$ in $AH(S)$, we can take $i_0$ so that if $i > i_0$, then there is an arc $\alpha_i$ in $U \cap QF(S)$ connecting $qf(m_i, n_i)$ with $qf(m_i', n_i')$. In the case when $\Gamma$ is a b-group whose lower conformal structure at infinity is $m_0$, we can take $\alpha_i$ satisfying further the following condition. For any neighbourhood $V$ of $m_0$, we can take $i_0$ so that for any $i > i_0$, the arc $\alpha_i$ is also contained in $qf(V \times \mathcal{T}(\bar{S}))$.*

We shall then get a corollary as follows.

**Corollary 9.** *In the setting of Theorem 7, suppose furthermore that each component of $\Omega_\Gamma/\Gamma$ that is not homeomorphic to $S$ is a thrice-punctured sphere. Then $QF(S)$ does not bump itself at $(\Gamma, \psi)$, and in particular $AH(S)$ is locally connected at $(\Gamma, \psi)$.*

We can generalise this corollary by dropping the assumption that there are no isolated parabolic loci. The same result has been obtained by substantially different methods in Brock-Bromberg-Canary-Minsky [9]; see also Canary [15]. Also, a related result has been obtained by Anderson-Lecuire [3].

**Corollary 10.** *Let $\Gamma$ be a group on the boundary of $QF(S)$ with isomorphism $\psi : \pi_1(S) \to \Gamma$. This time we allow $\Gamma$ to have isolated parabolic loci.*

*(1) If every component of $\Omega_\Gamma/\Gamma$ is a thrice-punctured sphere, then $QF(S)$ does not bump itself at $(\Gamma, \psi)$.*
*(2) If $\Gamma$ is a b-group and every component of $\Omega_\Gamma/\Gamma$ corresponding to upper ends of $(\mathbb{H}^3/\Gamma)_0$ is a thrice-punctured sphere, then the Bers slice containing $(\Gamma, \psi)$ on the boundary does not bump itself.*

3.5. **General Kleinian surface groups.** We can generalise Theorems 2 and 3 to sequences of Kleinian surface groups which may not be quasi-Fuchsian.

Let $G$ be a Kleinian surface group with marking $\phi : \pi_1(S) \to G$, and set $M = \mathbb{H}^3/G$. The marking $\phi$ determines a homeomorphism $h : S \times \mathbb{R} \to M$. Let $P_+$ be the upper parabolic loci on $S$. We consider all the upper ends of the non-cuspidal part $M_0$. For a geometrically finite end, we consider its minimal pants decomposition, and for a simply degenerate end, we consider its ending lamination. Take the union of all these curves and laminations together with the core curves of $P_+$, which we denote by $e_+$. In the same way, we define $e_-$ for the lower ends. We call $e_+$ and $e_-$ the upper and lower *generalised shortest pants decompositions* respectively.

We now state a generalisation of Theorem 2

**Theorem 11.** *Let $\{(G_i, \phi_i)\}$ be a sequence of Kleinian surface groups which have upper and lower generalised shortest pants decompositions $e(i)_+$ and $e(i)_-$. Suppose that $\{(G_i, \phi_i)\}$ converges algebraically to $(\Gamma, \psi)$. Consider the Hausdorff limit $e(\infty)_+$ of $\{e(i)_+\}$ and $e(\infty)_-$ of $\{e(i)_-\}$ after passing to a subsequence. Then every minimal component of $e(\infty)_+$ (resp. $e(\infty)_-$) that is not a simple closed curve is the ending lamination of an upper end (resp. lower end) of $(\mathbb{H}^3/\Gamma)_0$. Conversely any ending lamination of an upper (resp. lower) end of $(\mathbb{H}^3/\Gamma)_0$ is a minimal component of $e(\infty)_+$ (resp. $e(\infty)_-$). Moreover, no upper (resp. lower) parabolic locus can intersect a minimal component of $e(\infty)_+$ (resp. $e(\infty)_-$) transversely.*

Next we shall state a generalisation of Theorem 3.

**Theorem 12.** *Let $\{(G_i, \phi_i)\}$ be a sequence of Kleinian surface groups which have upper and lower generalised shortest pants decompositions $e(i)_+$ and $e(i)_-$. Let $e(\infty)_+$ and $e(\infty)_-$ be the Hausdorff limits of $\{e_+(i)\}$ and $\{e_-(i)\}$ respectively. If there are minimal components $\lambda$ of $e(\infty)_-$ and $\mu$ of $e(\infty)_+$ which are not simple closed curves and whose minimal supporting surfaces share a boundary component, then $\{(G_i, \phi_i)\}$ diverges in $AH(S)$.*

## 4. Models of geometric limits

4.1. **Brick decompositions of geometric limits.** In this section, we shall review the results in Ohshika-Soma [35] and show some facts derived from them, which are essential in our discussion.

Throughout this section, we assume that we have a sequence $\{(G_i, \phi_i)\}$ in $AH(S)$ converging to $(\Gamma, \psi)$, and that $\{G_i\}$ converges geometrically to $G_\infty$, which contains $\Gamma$ as a subgroup. We do not assume that $G_i$ is quasi-Fuchsian, to make our argument work also for the proofs of Theorems 9 and 10. Recall that $M_\infty = \mathbb{H}^3/G_\infty$ is a Gromov-Hausdorff limit of $M_i = \mathbb{H}^3/G_i$ with basepoints which are projections of some fixed point in $\mathbb{H}^3$. We denote $\mathbb{H}^3/\Gamma$ by $M'$. Let $p : M' \to M_\infty$ denote the covering associated to the inclusion of $\Gamma$ to $G_\infty$. Let $\rho_i : B_{r_i}(M_i, y_i) \to B_{K_i r_i}(M_\infty, y_\infty)$ denote an approximate isometry corresponding to the Gromov convergence of $(M_i, y_i)$ to $(M_\infty, y_\infty)$.

In [35], we introduced the notion of brick manifolds. A brick manifold is a 3-manifold constructed from "bricks" which are defined as follows. We note that a brick is an entity different from a block introduced by Minsky which we explained in Preliminaries. Still, as we shall see, in our settings each brick is decomposed into blocks.

**Definition 4.1.** A *brick* is a product interval bundle of the form $\Sigma \times J$, where $\Sigma$ is an incompressible subsurface of $S$ and $J$ is a closed or half-open interval in $[0,1]$. For a brick $B = \Sigma \times J$, the *lower front*, denoted by $\partial_- B$, is defined to be $\Sigma \times \inf J$, and the *upper front*, denoted by $\partial_+ B$, is defined to be $\Sigma \times \sup J$. When $J$ is an half-open interval, one of them may not really exist, but is regarded as an end. In such a case, a front is called an *ideal front*. A brick naturally admits two foliations: one is a horizontal foliation whose leaves are horizontal surfaces $\Sigma \times \{t\}$, the other is a vertical foliation whose leaves are vertical lines $\{p\} \times J$.

A brick manifold is a manifold consisting of countably many bricks, whose boundary consists of tori and open annuli. Two bricks can intersect only at their fronts in such a way that an essential subsurface is the upper front of one brick is pasted to an essential surface in the lower front of the other brick. (We say that a subsurface is essential if none of its boundary components are null-homotopic or peripheral.)

The following is one of the main theorems of [35] which is fitted into our present situation.

**Theorem 4.2** (Ohshika-Soma [35]). *Let $\{(G_i, \phi_i)\}$ be an algebraically convergent sequence in $AH(S)$, and $M_\infty$ a geometric limit of $M_i = \mathbb{H}^3/G_i$ with basepoint at $y_i$. Then, there are a model manifold $\mathbf{M}$ of $(M_\infty)_0$, which has a structure of brick manifold, and a model map $f : \mathbf{M} \to (M_\infty)_0$ which is a $K$-bi-Lipschitz homeomorphism for a constant $K$ depending only on $\chi(S)$. The model manifold $\mathbf{M}$ has the following properties.*

*(1) $\mathbf{M}$ is embedded in $S \times [0,1]$ preserving the vertical and horizontal foliations of the bricks.*
*(2) There is no essential properly embedded annulus in $\mathbf{M}$.*
*(3) An end contained in a brick is either geometrically finite or simply degenerate. The model map takes geometrically finite ends geometrically finite ends of $(M_\infty)_0$, simply degenerate ends to a simply degenerate ends of $(M_\infty)_0$.*
*(4) Every geometrically finite end of $\mathbf{M}$ corresponds to an incompressible subsurface of either $S \times \{0\}$ or $S \times \{1\}$.*
*(5) An end not contained in a brick is neither geometrically finite nor simply degenerate. For such an end, there is no open annulus tending to the end which is not properly homotopic into a boundary component. We call such an end* wild.
*(6) $\mathbf{M}$ has a brick of the form $S \times J$, where $J$ is an interval containing $1/2$, and $f_\#(\pi_1(S \times \{t\}))$ with $t \in J$ carries the image of $\pi_1(M')$ in $\pi_1(M_\infty)$.*

Regard $\mathbf{M}$ as a subset of $S \times [0,1]$ embedded preserving the horizontal and vertical foliations. Then a brick of $B$ has a form $\Sigma \times J$ with respect to the parametrisation of $S \times [0,1]$. We denote $\sup J$ by $\sup B$ and $\inf J$ by $\inf B$. Note that $\sup B$ is the level of the horizontal leaf on which the upper front of $B$ lies and $\inf B$ that on which the lower front of $B$ lies. Each end of $\mathbf{M}$, even if it is wild, corresponds to $\Sigma \times \{t\}$ for some essential subsurface $\Sigma$ of $S$. By the condition (ii), every geometrically finite end is contained in either $S \times \{0\}$ or $S \times \{1\}$. We call those contained in $S \times \{0\}$

*lower geometrically finite ends* and those in $S \times \{1\}$ *upper geometrically finite ends*. By moving the embedding vertically if necessary, we can assume that $S \times \{0\}$ and $S \times \{1\}$ consists of union of ends and annuli homotopic to the closure of annulus boundary components in $S \times [0,1] \setminus \mathrm{Int}\mathbf{M}$.

Sometimes it is convenient to consider the complement of $\mathbf{M}$ in $S \times [0,1]$. Let $B$ be a component of $S \times [0,1] \setminus \mathbf{M}$. Then $\mathrm{Fr}B \cap B$ consists of (countably many) horizontal surfaces each of which corresponds to an end of $\mathbf{M}$ which is either simply degenerate or wild. On the other hand, $\mathrm{Fr}B \setminus B$ consists of annuli and tori which are boundary components of $\mathbf{M}$.

We can associate to each geometrically finite end of $\mathbf{M}$ a marked conformal structure at infinity of the corresponding geometrically finite end of $(M_\infty)_0$. Similarly, we can associate to each simply degenerate end of $\mathbf{M}$ the ending lamination of the corresponding simply degenerate end of $(M_\infty)_0$. We call these conformal structures and ending laminations *labels*, and call a brick manifold with labels a *labelled brick manifold*. We showed in [35] that a labelled brick manifold is decomposed into blocks in the sense of Minsky [29] and tubes. The complement of the tubes is denoted by $\mathbf{M}[0]$. We put a metric of a Margulis tube into each tube so that the flat metric induced on the boundary coincides with that induced from the metric on $\mathbf{M}[0]$ determined by blocks.

Each Margulis tube $V = A \times [s,t]$ has a coefficient $\omega_{\mathbf{M}}(V)$ which is defined as follows. The boundary of the tube $\partial V$ has a flat metric induced from the metric of $\mathbf{M}[0]$ determined by blocks. We can give a marking (longitude-meridian system) $(\alpha, \beta)$ to $\partial V$ by defining $\alpha$ to be a horizontal curve and $\beta$ to be $\partial(a \times [s,t])$ for some essential arc $a$ connecting the two boundary components of $A$. The metric and the marking determine a point in the Teichmüller space of a torus identified with $\{z \in \mathbb{C} \mid \Im z > 0\}$, which we define to be $\omega_{\mathbf{M}}(V)$. We define $\mathbf{M}[k]$ to be the complement of the tubes whose $\omega_{\mathbf{M}}$ have absolute value greater than or equal to $k$.

Since $G_i$ is a Kleinian surface group, it also has a bi-Lipschitz model constructed by Minsky [29] and proved to be bi-Lipschitz in Brock-Canary-Minsky [10]. As was explained in Preliminaries, we ignore cusp neighbourhoods in the model manifolds of Minsky to make them models for the non-cuspidal parts. Let $\mathbf{M}_i$ be a model manifold for $(M_i)_0 = (\mathbb{H}^3/G_i)_0$ in the sense of Minsky with a bi-Lipschitz model map $f_i : \mathbf{M}_i \to (M_i)_0$. Minsky's construction is based on complete hierarchies of tight geodesics which are determined by the end invariants of $M_i$, as we briefly explained in Preliminaries. The model manifold has decomposition into blocks and Margulis tubes, which corresponds to a resolution of a complete hierarchy. When we talk about model manifolds $\mathbf{M}_i$, *we always assume the existence of complete hierarchies $h_i$ beforehand*, and that the manifolds are decomposed into blocks and Margulis tubes using resolutions. The metric of model manifolds are defined as the union of metrics on internal blocks and metrics determined by conformal structures at infinity on boundary blocks. We should note that as was shown in [29], the decomposition of $\mathbf{M}_i$ into blocks and the metric on $\mathbf{M}_i$ depend only on $h_i$ and end invariants, and are independent of choices of resolutions.

In the proof of Theorem A in [35] (§5.2), the following was also shown.

**Proposition 4.3.** *Let $x_i \in \mathbf{M}_i$ be a point in $\mathbf{M}_i$ such that $f_i(x_i)$ is within uniformly bounded distance from the basepoint $y_i$ of $M_i$. Then $(\mathbf{M}_i[k], x_i)$ converges to $\mathbf{M}[k]$ for any $k \in [0, \infty)$ after passing to a subsequence. The model manifolds $\mathbf{M}$ and $\mathbf{M}_i$ have structures of labelled brick manifolds admitting a block decomposition with the* *following condition. Let $\rho_i^{\mathbf{M}}$ be an approximate isometry between $\mathbf{M}_i[k]$ and $\mathbf{M}[k]$ corresponding to the geometric convergence. Then the following hold.*

1. *For any compact set $K$ in $\mathbf{M}[k]$, the restriction $\rho_i \circ f_i \circ (\rho_i^{\mathbf{M}})^{-1}|K$ converges to $f|K$ uniformly as $i \to \infty$.*
2. *For any block $b$ of $\mathbf{M}[k]$, its pull-back $(\rho_i^{\mathbf{M}})^{-1}(b)$ is a block in $\mathbf{M}_i[k]$ for large $i$.*
3. *For any brick $B$ of $\mathbf{M}[k]$, its pull-back $(\rho_i^{\mathbf{M}})^{-1}(B)$ is contained in a brick of $\mathbf{M}_i[k]$ for large $i$.*
4. *We can arrange $\rho_i^{\mathbf{M}}$ so that it preserves the horizontal foliations.*

By defining a brick to be a maximal union of parallel horizontal leaves, we can define a brick decomposition of $\mathbf{M}_i[k]$. (Such a brick decomposition is called the *standard brick decomposition*.) A brick in $\mathbf{M}_i[k]$ in the condition (3) means that in this brick decomposition of $\mathbf{M}_i[k]$.

4.2. **Algebraic limits in the models.** Since the algebraic limit $\Gamma$ of $G_i$ is contained in the geometric limit $G_\infty$, there is an inclusion of $\pi_1(S)$ in $\pi_1(\mathbf{M})$ corresponding to the inclusion of $\Gamma$ into $G_\infty$. We realise this inclusion by a $\pi_1$-injective immersion $g : S \to \mathbf{M}$ so that $(f \circ g)_\# \pi_1(S)$ is equal to the image of $\pi_1(M')$ in $\pi_1(M_\infty)$ under the covering projection. We call such $g$ an *algebraic locus*.

**Lemma 4.4.** *The immersion $g$ can be homotoped to a map $g'$ as follows.*

1. *The surface $S$ is decomposed into essential subsurfaces $\Sigma_1, \ldots, \Sigma_m$, none of which is an annulus, and annuli $A_1, \ldots, A_n$.*
2. *The restriction of $g'$ to $\Sigma_j$ is a horizontal embedding into $\Sigma_j \times \{t_j\}$.*
3. *Each annulus $g'(A_j)$ is composed of $2n-1$ horizontal annuli and $2n$ vertical annuli for some $n \in \mathbb{N}$ and goes around a torus boundary of $\mathbf{M}$ $n$-times. See Figure 1.*

*Proof.* By the last condition in Theorem 4.2, we see that $g$ is $\pi_1$-injective in $S \times [0,1]$. Since every $\pi_1$-injective map from $S$ to $S \times [0,1]$ is homotopic to a horizontal surface, $g$ is homotopic to a horizontal surface $S \times \{t\}$ in $S \times [0,1]$.

Since $\mathbf{M}$ is a brick manifold, we can homotope $g$ within $\mathbf{M}$ so that $g(S)$ consists of horizontal leaves in bricks and vertical annuli. By the additivity of Euler characteristics, we see that the sum of the Euler characteristics of the horizontal leaves is equal to $\chi(S)$. We consider the projection of horizontal leaves to $S$. Since $g$ is homotopic to $S \times \{t\}$ in $S \times I$, we see by the invariance of the algebraic intersection number that for each point $x \in S$ the surface $g(S)$ contains $x \times \{s\}$ for some $s \in I$. This implies that the horizontal leaves cannot overlap along a surface with negative Euler characteristic. It follows that only compact regions that $g(S)$ can bound in $S \times I$ are solid tori. If such a solid torus is contained in $\mathbf{M}$, we can eliminate it by a homotopy. Therefore the only remaining possibility is that each solid torus contains components of $\partial\mathbf{M}$. By the second condition of Theorem 4.2, there is only one boundary component contained in each solid torus. Thus the only possible situation is as in our statement. □

We call a map $g' : S \to \mathbf{M}$ as in Lemma 4.4 *a standard algebraic immersion*. Recall that there is a homotopy equivalence $\Phi_i : S \to M_i$ realising $\phi_i$. By composing the inverse of the model map, we have a homotopy equivalence from $S$ to $\mathbf{M}_i$, which *we denote by* $\Phi_i^{\mathbf{M}}$.

FIGURE 1. $g'$ going around a torus boundary component

**Lemma 4.5.** *Let $\rho_i^{\mathbf{M}}$ be an approximate isometry between $\mathbf{M}_i[k]$ and $\mathbf{M}[k]$ with basepoints at the thick parts as in Proposition 4.3. For sufficiently large $i$, the immersion $(\rho_i^{\mathbf{M}})^{-1} \circ g'$ is homotopic to $\Phi_i^{\mathbf{M}}$.*

*Proof.* By the definition of $g'$, we see that $f \circ g'$ is homotopic to $\Psi$. Since $\rho_i^{-1} \circ \Psi$ is homotopic to $\Phi_i$ for large $i$, our lemma follows from the condition (1) of Proposition 4.3. □

**Definition 4.6.** Let $e$ be a simply degenerate end of $\mathbf{M}$ contained in a brick $B = \Sigma \times J$. We say that $e$ is *an upper algebraic end* if $J = [s, s')$ and $J \times \{s' - \epsilon\}$ is freely homotopic to $g'(\Sigma)$ which lies in a horizontal part of $g'(S)$ in $\mathbf{M}$ for sufficiently small $\epsilon$. In the same way, we say that $e$ is *a lower algebraic end* if $J = (s, s']$ and $J \times \{s + \epsilon\}$ is freely homotopic to $g'(\Sigma)$ which lies in a horizontal part $g'(S)$ for sufficiently small $\epsilon$.

Similarly, a core curve of an open annulus boundary component or a longitude (i.e. a horizontal curve) of a torus boundary component of $\mathbf{M}$ is said to be an *algebraic parabolic curve* if it is homotopic to a simple closed curve lying on a horizontal part of $g'(S)$. We also call its image of the vertical projection to $S$ an algebraic parabolic curve. A parabolic curve is said to be upper or lower in the same way as simply degenerate ends, but we should note if it lies on a torus boundary component around which $g'(S)$ goes, then the curve is defined to be both upper and lower at the same time. An algebraic parabolic curve is said to be *isolated* if it corresponds to an isolated parabolic locus of $M'$.

**Lemma 4.7.** *Any algebraic parabolic curve lying on a torus boundary component of $\mathbf{M}$ is isolated.*

*Proof.* Let $c$ be an algebraic parabolic curve lying on a torus boundary component of $\mathbf{M}$. Let $P$ be a parabolic locus of a relative core $C$ of $(M')_0$ into which the lift of $f(c)$ to $M'$ is homotopic. Suppose, seeking a contradiction, that $P$ is not isolated. Then, there is a simply degenerate end $e$ touching the $\mathbb{Z}$-cusp corresponding to $P$. By the Thurston's covering theorem (see [38] and Canary [14]), there is a neighbourhood $U$ of $e$ which is projected in to $M_\infty$ homeomorphically by the covering projection. This implies that $\mathbf{M}$ has a simply degenerate brick which touches an open annulus boundary of $\mathbf{M}$ into which $c$ is homotopic. Since no two boundary components of $\mathbf{M}$ have homotopic essential closed curves, this contradicts the assumption that $c$ lies on a torus boundary. □

**Lemma 4.8.** *Algebraic simply degenerate ends of* $\mathbf{M}$ *ends correspond one-to-one to simply degenerate ends of* $M'$ *by mapping them by* $f$ *and lifting them to* $M'$*. The upper (resp. lower) ends correspond to upper (resp. lower) ends of* $M'$*.*

*Proof.* Consider a simply degenerate end corresponding to the upper ideal front of a brick $B = \Sigma \times [s, s')$ of $\mathbf{M}$. By the definition of the model map $f$, for each simply degenerate end of $\mathbf{M}$, there is an infinite sequence of horizontal surfaces $\Sigma \times \{t_j\}$ in $B'$ tending to $\Sigma \times \{s'\}$ which are mapped to a sequence of pleated surfaces $f(\Sigma \times \{t_j\})$ tending to the corresponding simply degenerate end $e$ of $(M_\infty)_0$. Since $\Sigma \times \{t\} \in B'$ is freely homotopic into $g'(S)$, the pleated surfaces $f(\Sigma \times \{t_j\})$ lift to pleated surfaces $\tilde{f}_i$ tending to an end of $M'_0$.

Since the model map $f$ has degree 1, with respect to the orientation of $(M_\infty)_0$, the end $e$ is situated above $f \circ g'(S)$. Lifting this to $M'$, we see that the end to which the $\tilde{f}_i$ tend is an upper end. Similarly, we can show that if the simply degenerate brick $B$ has the form $\Sigma \times (s, s']$, then the corresponding end of $M'_0$ is a lower end.

Conversely, suppose that $e'$ is a simply degenerate end of $M'_0$. By Thurston's covering theorem (see Thurston [38] and Canary [14]), there is a neighbourhood $E$ of $e'$ such that $p|E$ is a proper embedding into $(M_\infty)_0$. Let $\bar{e}$ denote the simply degenerate end of $(M_\infty)_0$ contained in $p(E)$. Then, there is a simply degenerate $\hat{e}$ of $\mathbf{M}$ which is sent to $\bar{e}$ by $f$. Since $e'$ is simply degenerate, there is an essential subsurface $\Sigma$ of $S$ and a sequence of pleated surfaces $h_i : \Sigma \to M'$ taking $\partial\Sigma$ to cusps which tend to $e'$. Their projections $p \circ h_i$ are pleated surfaces tending to $\bar{e}$. This implies that the end $\hat{e}$ is contained in a simply degenerate brick $B_e \cong \Sigma \times J$, where $J$ is a half-open interval. Since $f(\Sigma \times \{t\})$ is homotopic to $p \circ h_i$ and $p \circ h_i$ is homotopic into $f \circ g'(S)$, we see that $\Sigma \times \{t\}$ is freely homotopic to $g'(\Sigma)$. By Lemma 4.4, this is possible only when $\Sigma \times \{t\}$ is homotopic into a horizontal part of $g'(S)$, and we see that the end $\hat{e}$ is algebraic. □

As was shown in [35], except for the geometrically finite ends lying on $S \times \{0, 1\}$, all the tame ends of $\mathbf{M}$ are simply degenerate.

Next we shall see how simply degenerate ends in the model manifold $\mathbf{M}$ are approximated in $\mathbf{M}_i$. Recall that the model manifold $\mathbf{M}_i$ corresponds to a hierarchy $h_i$ of tight geodesics. Recall also that we have a homeomorphism $\hat{\Phi}_i : S \times (0, 1) \to M_i$ inducing $\phi_i$ between the fundamental groups. This determines an embedding $\iota$ of the standard $S \times (0, 1)$ into $S \times [0, 1]$ in which $\mathbf{M}_i$ is embedded. We identify the standard $S \times [0, 1]$ and $S \times [0, 1]$ in which $\mathbf{M}_i$ is embedded so that this $\iota$ becomes an inclusion. In other words, by this identification, the model map $f_i$ is homotopic to $\hat{\Phi}_i$ if restricted to $S \times (0, 1)$. We identify two $S \times [0, 1]$ in which $\mathbf{M}_{i_1}$ and $\mathbf{M}_{i_2}$ are embedded for every pair $i_1, i_2$ using $\hat{\Phi}_{i_1}$ and $\hat{\Phi}_{i_2}$. We also identify the standard $S \times [0, 1]$ with $S \times [0, 1]$ in which $\mathbf{M}$ is embedded so that $S \times J$ in (6) of Theorem 4.2, the map $f_\#$ is the same as $\psi$.

We fix a complete marking $\mu$ on $S$. For a domain $\Sigma$ in $S$, by considering a component of $\mu|\Sigma$, we can define a basepoint in $\mathcal{CC}(\Sigma)$. We call this basepoint the *basepoint determined by marking*.

**Proposition 4.9.** *Let* $B = \Sigma \times I$ *be a simply degenerate brick in* $\mathbf{M}$ *whose end* $e$ *is algebraic. Then there is a geodesic* $\gamma_i$ *contained in the hierarchy* $h_i$ *as follows.*

*(1) The support of* $\gamma_i$ *is* $\Sigma$*.*
*(2) After passing to a subsequence, either all* $\gamma_i$ *are geodesic rays, or they are finite geodesics and the length of* $\gamma_i$ *goes to* $\infty$ *as* $i \to \infty$*.*

(3) *In the case when the $\gamma_i$ are geodesic rays, their endpoints at infinity converge to the ending lamination of $e$ in $\mathcal{EL}(\Sigma)$.*
(4) *Suppose that the $\gamma_i$ are finite geodesics. In the case when the end $e$ is upper, the last vertex of $\gamma_i$ converges to the ending lamination of $e$, whereas the first vertices stay in a bounded distance from a basepoint in $\mathcal{C}(\Sigma)$ determined by marking. In the case when the end is lower, the first vertex of $\gamma_i$ converges to the ending lamination whereas the last vertices stay in a bounded distance from the basepoint determined by marking.*

4.3. **Proof of Proposition 4.8.** Let $\mathcal{T}$ be the union of all boundary components of $\mathbf{M}$ that meet the vertical boundary of $B$. The preimage by $\rho_i^{\mathbf{M}}$ of each component of $\mathcal{T}$ lies on either the boundary of a Margulis tube or a $\mathbb{Z}$-cusp neighbourhood of $\mathbf{M}_i$. For sufficiently large $i$, we consider the union $\mathbf{V}_i$ of all Margulis tubes and cusp neighbourhoods whose intersection with $B_{r_i}(\mathbf{M}_i, x_i)$ are mapped by $\rho_i^{\mathbf{M}}$ into $\mathcal{T}$. Let $\mathbf{B}(i)$ be a maximal subset of the form $\Sigma \times J$ with $J \subset I$ which lies in the component of $B \cap B_{K_i r_i}(\mathbf{M}, x_\infty)$ containing $x_\infty$.

Let $\hat{\mathbf{M}}_i$ be the complement of Int$\mathbf{V}_i$ in $\mathbf{M}_i$. Regarding $\mathbf{M}_i$ as embedded in $S\times[0,1]$, we can decompose $\hat{\mathbf{M}}_i$ into bricks by defining a brick to be a maximal union of parallel horizontal leaves, i.e. by considering a standard brick decomposition. Since every horizontal leaves in $B$ are parallel in $\mathbf{M}$ and $\rho_i^{\mathbf{M}}$ preserves the horizontal foliation, we see that $(\rho_i^{\mathbf{M}})^{-1}(\mathbf{B}(i))$ is contained in a brick $B_i = \Sigma_i \times J_i$ in $\hat{\mathbf{M}}_i$. Since $B$ is algebraic, $\Sigma \times \{t\}$ in $B$ is homotopic to $g'(\Sigma)$ in $\mathbf{M}$. Pulling this back to $\mathbf{M}_i$, we see that $\Sigma_i$ is homotopic to $\Phi_i^{\mathbf{M}}(\Sigma)$. By our way of identifying the $S \times [0,1]$ in which the $\mathbf{M}_i$ lie, we can regard $\Sigma_i$ as $\Sigma$.

We shall now review the notion of tube unions corresponding to $h_i$ introduced in §3.3 of [35].

**Definition 4.10.** Let $g$ be a tight geodesic contained in the hierarchy $h_i$ with $\xi(D(g)) > 4$. Corresponding to $g$, we have a sequence $\mathcal{V}_g$ of disjoint unions of tubes called tube simplices, which is either a finite sequence $\{V_1^g, \dots V_k^g\}$ or infinite sequence of the forms $\{V_1^g, \dots\}$ or $\{\dots, V_{-1}^g\}$ embedded in $D(g) \times J$ in $\mathbf{M}_i$ with the following properties. We use the symbol $\sup V_j^g$ and $\inf V_j^g$, in the same way as for bricks.

(i) $D(g) \times J$ is a brick with respect to the standard brick decomposition of the complement of tubes in the tube unions $\mathcal{V}_{g'}$ for all $g'$ in $h_i$ to which $g$ is subordinate.
(ii) Each tube simplex $V_j^g$ is a union of disjoint solid tori bounded by two horizontal annuli and two vertical annuli. (Therefore $\inf V_j^g$ and $\sup V_j^g$ are well defined.)
(iii) The lower horizontal boundaries of the components of $V_j^g$ lie on the same horizontal leaf, and so do the upper horizontal boundaries.
(iv) The core curves of the tube simplex $V_j^g$ are taken to the $j$-th simplex of $\gamma_i$ by the vertical projection $p : S \times [0,1] \to S$. (We shall say that the core curve is vertically homotopic to the $j$-th simplex in this situation.) In particular, the first and last tube simplices are connected, i.e. solid tori.
(v) We have $\inf V_{j+1}^g = \sup V_j^g$.
(vi) The lower front of the first tube lies on $D(g) \times \inf J$ and the upper front of the last tube lies on $D(g) \times \sup J$. Unless $g$ is the main geodesic of $h_i$, the first tube is attached along its lower front to a component of a tube simplex

corresponding to a simplex $u$ on $g_1$ with $(g_1, u) \overset{d}{\swarrow} g$ and the last tube is attached along its upper front to a component of a tube simplex corresponding to a simplex $w$ on $g_2$ with $g \overset{d}{\searrow} (g_2, w)$.

(vii) The core curves of $\mathcal{V}_{g_1} \cap (D(g) \times \inf J)$ are vertically homotopic to $I(g)$, and those of $\mathcal{V}_{g_2} \cap (D(g) \times \sup J)$ are vertically homotopic to $T(g)$ for $g_1, g_2$ as above.

**Definition 4.11.** In the case when $\xi(D(g)) = 4$, the properties (ii)-(v) should be modified as follows while the conditions (i), (vi) and (vii) remain the same.

(ii*) Each $V^g_j$ is a solid torus bounded by two horizontal annuli and two vertical annuli.

(iii*) The core curve of $V^g_j$ is vertically homotopic to the $j$-th vertex of $g$.

(iv*) We have $\inf V^g_{j+1} > \sup V^g_j$.

(v*) There is no tube $V$ in a tube union $\mathcal{V}_{g'}$ with $g' \in h_i$ such that either $\inf V$ or $\sup V$ lies in the open interval $(\sup V^g_j, \inf V^g_{j+1})$.

We call the union of tubes in $\mathcal{V}_g$ the *tube union* corresponding to $g$. We define $\mathcal{V}(h_i)$ to be $\cup_{g \in h_i} \mathcal{V}_g$, and $\mathcal{V}_i$ to be the union of all the tubes in $\mathcal{V}(h_i)$. The complement $\mathcal{V}_i$ in $\mathbf{M}_i$ is the brick manifold $\mathbf{M}_i[0]$. For a tube simplex $V$ in $\mathcal{V}(h_i)$, we denote by $\hat{V}$ the union of $V$ and the tubes to which $V$ is attached and which existed before $V$ is put. By the condition (v*) above for any brick $D(g) \times J$ appearing in the construction of $\mathcal{V}_g$, its real front $F$ intersects $\mathcal{V}_i$ so that every component of $F \setminus \mathcal{V}_i$ is a thrice-punctured sphere.

Now we consider to put tube unions corresponding to geodesics of $h_i$, one by one starting from the main geodesic. We assume that when we put a tube union corresponding to a geodesic $g$, all the tube unions corresponding to the geodesics to which $g$ is subordinate have already been put in $\mathbf{M}$.

We shall show the intersection of the tube unions in $\mathcal{V}(h_i)$ and $B_i$ gives a hierarchy of tight geodesics in $\Sigma$. For that, we shall first show that each tube union intersecting $B_i$ defines a tight geodesic on a subsurface of $\Sigma$. First, we consider the case when a tube union penetrates $B_i$ in the following sense. We say that a tube union $\mathcal{V}_g$ *penetrates* $B_i$ if one of the following conditions is satisfied.

(a) There is a non-empty consecutive sequence $\{V^g_{j_0}, \ldots, V^g_{j_1}\}$ in $\mathcal{V}_g$ such that $\inf B_i \leq \inf V^g_{j_0}$, $\sup V^g_{j_1} \leq \sup B_i$ whereas $\inf \hat{V}^g_{j_0-1} < \inf B_i$ and $\sup \hat{V}^g_{j_1+1} > \sup B_i$ and one of $V^g_{j_0-1}, V^g_{j_0}, \ldots, V^g_{j_1+1}$ intersects $B_i$. (Note that putting ˆ is necessary only when $V_{j_0-1}$ or $V_{j_1+1}$ is the first or last tube of $\mathcal{V}_g$.)

(b) There are consecutive pair of tube simplices $V$ and $V'$ in $\mathcal{V}_g$ both intersecting $B_i$ such that $\inf \hat{V} < \inf B_i \leq \sup V$, $\inf V'' \leq \sup B_i < \sup \hat{V}'$.

(c) There is a tube simplex $V$ in $\mathcal{V}_g$ with $V \cap B_i \neq \emptyset$ and $\inf V < \inf B_i, \sup V > \sup B_i$.

(d) The brick $B_i$ has an upper simply degenerate end. The geodesic $g$ is a ray, and there is a consecutive sequence $\{V^g_{j_0-1}, V^g_{j_0}, \ldots\}$ in $\mathcal{V}_g$ one of which intersects $B_i$ such that $\inf \hat{V}^g_{j_0-1} < \inf B_i$ and $\inf B_i \leq \inf V^g_{j_0}$.

(e) The brick $B_i$ has a lower simply degenerate end. The geodesic $g$ is a ray in the negative direction, and there is a consecutive sequence $\{\ldots, V^g_{j_1}, V^g_{j_1+1}\}$ in $\mathcal{V}_g$ one of which intersects $B_i$ such that $\sup \hat{V}^g_{j_1+1} > \sup B_i$ and $\sup B_i \geq \sup V^g_{j_1}$.

FIGURE 2. Tube union penetrating $B_i$.

The subsequences of $\mathcal{V}_g$ appearing the conditions (a)-(e) are called *penetrating subsequence*. See Figure 2.

**Lemma 4.12.** *Let $\mathcal{V}_g$ is a tube union for $g \in h_i$ which penetrates $B_i$. Then the core curves of the tubes in $\mathcal{V}_g$ are vertically homotopic to simplices in a tight sequence supported on a domain contained in $\Sigma$ with the order preserved. Moreover, in the cases (a), (d) and (e), the domain $D(g)$ is contained in $\Sigma$.*

*Proof.* This is obvious in the cases of (b) and (c). We now consider the case (a). By the condition of the case (a), there are at least three consecutive tube simplices $V^g_j, V^g_{j+1}, V^g_{j+2}$ in $V^g_{j_0-1}, \ldots, V^g_{j_1+1}$. We let $c^g_j, c^g_{j+1}$ and $c^g_{j+2}$ be the simplices vertically homotopic to their core curves. Since $g$ is a tight geodesic, $c^g_j \cup c^g_{j+2}$ fills up $D(g) \setminus c^g_{j+1}$. If $D(g)$ is not a subsurface of either $\Sigma$ or $S \setminus \Sigma$, there is a component $\gamma$ of $\mathrm{Fr}\Sigma$ contained in $D(g)$. On the other hand, the curve $\gamma \subset D(g)$ is disjoint from all of the three simplices $c^g_j, c^g_{j+1}$ and $c^g_{j+2}$ since tubes $V^g_j, V^g_{j+1}$ and $V^g_{j+2}$ are disjoint from $\mathbf{V}_i$. This is a contradiction. Also, since one of $V^g_{j_0-1}, V^g_{j_0}, \ldots, V^g_{j_1+1}$ intersects $B_i$ by assumption, we see that $D(g)$ cannot be contained in $S \setminus \Sigma$. Therefore, $D(g)$ is a subsurface of $\Sigma$, and the core curves of $V^g_{j_0-1}, \ldots, V^g_{j_1+1}$ constitute a tight sequence supported on a domain in $\Sigma$. We also note that this implies that all of $V^g_{j_0-1}, \ldots, V^g_{j_1+1}$ intersect $B_i$ and $V^g_{j_0}, \ldots, V^g_{j_1}$ are contained in $B_i$. We can argue in the same way in the cases (d) and (e), and conclude that $D(g)$ is a domain in $\Sigma$ also in these cases and that the core curves of $V^g_{j_0-1}, \ldots$ or $\ldots, V^g_{j_1+1}$ constitute a tight geodesic ray supported there. $\square$

Suppose that $g$ penetrates $B_i$ and that $D(g)$ is a proper subsurface of $\Sigma$. Let $c$ be a component of $\mathrm{Fr}_\Sigma D(g)$. Since $\mathcal{V}_g$ lies in $D(g) \times J$, we see that there is a tube $V_c$ in $\mathcal{V}_i$ whose core curve is vertically homotopic to $c$ and it penetrates $B_i$ as in the case (c) above. Note that $V_c$ may be a union of tubes corresponding to vertices lying on several different geodesics. The tube $V_c \cap B_i$ must be already put into $\mathbf{M}_i$ when we start to put $\mathcal{V}_g$.

Secondly, we consider tube unions which do not penetrate $B_i$ but stop inside $B_i$. A tube union $\mathcal{V}_g$ is said to *stop inside* $B_i$ when one of the following holds.

(a*) There is a consecutive sequence $\{V^g_{j_0-1}, \ldots, V^g_k\}$ in $\mathcal{V}_g$ ending at the last tube of $\mathcal{V}_g$, one of which intersects $B_i$ and such that $\inf V^g_{j_0-1} < \inf B_i$, $\inf V^g_{j_0} \geq \inf B_i$, whereas $\sup \hat{V}^g_k \leq \sup B_i$. When $k = 2$, we need to assume that both $V^g_{j_0}$ and $V^g_k$ intersect $B_i$.

(b*) There is a consecutive sequence $\{V^g_1, \ldots, V^g_{j_1+1}\}$ of $\mathcal{V}_g$ starting at the first tube of $\mathcal{V}_g$, one of which intersects $B_i$ and such that $\sup V^g_{j_1+1} > \sup B_i$, $\sup V^g_{j_1} \leq \sup B_i$, whereas $\inf \hat{V}^g_1 \geq \inf B_i$. When $k = 2$, we need to assume that both $V^g_1$ and $V^g_2$ intersect $B_i$.

(c*) The last tube $V^g_k$ in $\mathcal{V}_g$ intersects $B_i$ and $\inf V^g_k < \inf B_i$ and $\sup \hat{V}^g_k \leq \sup B_i$.

(d*) The first tube $V^g_1$ of $\mathcal{V}_g$ intersects $B_i$ and $\inf \hat{V}^g_1 \geq \inf B_i$ and $\sup V^g_1 > \sup B_i$.

By the same argument as Lemma 4.11, we can show the following.

**Lemma 4.13.** *Let $\mathcal{V}_g$ is a tube union for $g \in h_i$ which stops inside $B_i$. Then the core curves of the tubes in $\mathcal{V}_g$ are vertically homotopic to a tight sequence supported on a subsurface of $\Sigma$. Moreover, in the cases (a*) and (b*) with $k > 2$, the domain $D(g)$ is contained in $\Sigma$.*

Thirdly, we consider the case when a tube union is totally contained in $B_i$. We say a tube union $\mathcal{V}_g$ is *totally contained* in $B_i$ when one of the following conditions holds.

(a**) For $\mathcal{V}_g = \{V^g_1, \ldots, V^g_k\}$, at least one of the tubes intersects $B_i$ and it holds that $\inf B_i \leq \inf V^g_1$ and $\sup V^g_k \leq \sup B_i$. When $k = 2$, we need to assume that both of the tubes $V^g_1, V^g_2$ intersect $B_i$.

(b**) For an infinite sequence $\mathcal{V}_g = \{V^g_1, \ldots\}$ or $\mathcal{V}_g = \{\ldots V^g_{-1}\}$ at least one of whose entries intersects $B_i$, it holds $\inf B_i \leq \inf V^g_j$ and $\sup V^g_j \leq \sup B_i$ for all $j$.

Again by the same argument as Lemma 4.11, we have the following.

**Lemma 4.14.** *Let $\mathcal{V}_g$ is a tube union for $g \in h_i$ which is totally contained in $B_i$. Then the core curves of the tubes in $\mathcal{V}_g$ are vertically homotopic to a tight sequence supported on a subsurface of $\Sigma$. Moreover except for the case (a**) with $k = 2$, the domain $D(g)$ is contained in $\Sigma$.*

We should note that, by our definitions above, if a tube union has a tube intersecting $B_i$, then it either penetrates or stops inside or is totally contained in $B_i$ except for the case when $k = 2$ and one of the tube simplices stays outside $B_i$. In this latter case, we can ignore the effect of putting the tube union in $B_i$ since it does not affect the construction of hierarchies below. As a convention, when we say that a tube union intersects $B_i$, we always *assume that it either penetrates or stops inside or is totally contained in $B_i$.*

Now, we shall show that the geodesics on $\Sigma$ which $\mathcal{V}_g(g \in h_i)$ gives as above has a structure of a hierarchy. Suppose that $\mathcal{V}_g$ either penetrates or stops inside or is totally contained in $B_i$. Let $\gamma^g$ a tight sequence supported on a subsurface of $\Sigma$ induced by $\mathcal{V}_g$ as shown in Lemmata 4.11-4.13. We define $D(\gamma^g)$ to be $D(g)$ except for the cases (b), (c) for the penetrating sequences and the case when $k \leq 2$ for the sequences either stopping inside or totally contained in $B_i$. In these latter cases, we define $D(\gamma^g)$ to be $D(g) \cap \Sigma$. We define $I(B_i)$ to be the core curves of $\mathcal{V}_i$ intersecting the upper front of $B_i$ if it is not ideal, with empty transversals, and similarly define the terminal marking $T(B_i)$.

**Lemma 4.15.** *If $\mathcal{V}_g$ is the first tube union in the above construction that intersects $B_i$, then $\mathcal{V}_g$ must penetrate $B_i$ and $D(\gamma^g) = \Sigma$. Moreover, in the cases (a) and (d), the tube $V^g_{j_0-1}$ is the first tube of $\mathcal{V}_g$, and in the cases (a) and (e), the tube $V^g_{j_1+1}$ is the last tube of $\mathcal{V}_g$. The sequence $\gamma^g$ becomes a tight geodesic by defining its initial marking to be $I(B_i)$ and its terminal marking to be $T(B_i)$. If one of the fronts of $B_i$ is ideal, then we define the endpoint of $\gamma^g$ at infinity to be the initial or terminal marking. Unless $\mathcal{V}_g$ is the first tube union intersecting $B_i$, the support of $\mathcal{V}_g$ is a proper subsurface of $\Sigma$, and there are geodesics $g_1, g_2$ in $h_i$ such that $\gamma^{g_1} \overset{d}{\swarrow} \gamma^g \overset{d}{\searrow} \gamma^{g_2}$ if we let $I(\gamma^g)$ be $I(D(\gamma^g), \gamma^{g_1})$ and $T(\gamma^g)$ be $T(D(\gamma^g), \gamma^{g_2})$. The family $\{\gamma^g \mid g \in h_i\}$ constitutes a 4-complete hierarchy on $\Sigma$.*

*Proof.* Suppose that $\mathcal{V}_g$ is the first tube union intersecting $B_i$. Let $h_i(g)$ be the subfamily of $h_i$ consisting of the geodesics to which $g$ is subordinate, $\mathcal{V}_{h_i(g)}$ the union of all $\mathcal{V}_{g'}$ with $g' \in h_i(g)$, and $\mathbf{V}_{h_i(g)}$ the union of all the tubes contained in $\mathcal{V}_{h_i(g)}$. Since no tubes in $\mathcal{V}_{h_i(g)}$ can intersect $B_i$, the first and last tube simplices of $\mathcal{V}_g$ that intersect $B_i$ (provided that they exist) must intersect the fronts of $B_i$. This implies that $\mathcal{V}_g$ penetrates $B_i$.

Now suppose that we are in the situation of (a). Recall that by the definition of tube unions, $D(g) \times J$ is a brick with respect to the standard brick decomposition of $\mathbf{M}_i \setminus \mathbf{V}_{h_i(g)}$, and that the first and last tubes of $\mathcal{V}_g$ are attached to tubes in $\mathcal{V}_{h_i(g)}$ along their fronts. In particular $\mathcal{V}_{h_i(g)}$ contains tubes with core curves whose union is vertically homotopic to $\mathrm{Fr}D(g)$. We let $\mathcal{V}(\mathrm{Fr}D(g))$ be the family consisting of these tubes. Since $D(g) \times J$ intersects $B_i$ and $D(g) \subset \Sigma$, every tube of $\mathcal{V}(\mathrm{Fr}D(g))$ either intersects $B_i$ or touches the vertical boundary of $B_i$. Since we assumed that $\mathcal{V}_g$ is the first tube union intersecting $B_i$, there can be no tube in $\mathcal{V}(\mathrm{Fr}D(g))$ intersecting $B_i$. Therefore, every tube in $\mathcal{V}(\mathrm{Fr}D(g))$ touches the vertical boundary of $B_i$, and we have $D(g) = \Sigma$. This also implies that the tubes in $\mathcal{V}(\mathrm{Fr}D(g))$ are contained in $V_{h_i(g)}$, and that the first and last tubes in $\mathcal{V}_g$ intersect $B_i$. Even in the cases (b) and (c), the same argument implies that $D(\gamma^g) = \Sigma$. We can also deal with the case when $\gamma^g$ is infinite in the same way.

Next suppose that $\mathcal{V}_g$ is not the first tube union intersecting $B_i$. Such a tube union exists only when $\xi(\Sigma) > 4$. Then there is $g_0 \in h_i(g)$ such that $\mathcal{V}_{g_0}$ is the first tube union intersecting $B_i$. As was shown above, $\mathcal{V}_{g_0}$ penetrates $B_i$. It follows that every horizontal leaf of $B_i$ must intersect a tube in $\mathcal{V}_{g_0}$. Therefore, there can be no $\Sigma \times J$ in $\mathbf{M}_i \setminus \mathbf{V}(h_i(g))$ touching $B_i$. This shows that the support of $\gamma^g$ must be a proper subsurface of $\Sigma$.

Suppose that $\mathcal{V}_g$ penetrates $B_i$. We shall show that if $B_i$ has a lower real front, then there must be a tube simplex $V$ in $\mathcal{V}_g$ with $\hat{V}$ intersecting it, and that the same holds if $B_i$ has a real upper front. Suppose, seeking a contradiction, that there is no

$V_j \in \mathcal{V}_g$ with $\hat{V}_j$ intersecting the lower real front of $B_i$. Then there is a consecutive tube simplices $V_j$, $V_{j+1}$ such that $\sup V_j < \inf B_i$ and $\inf V_{j+1} > \inf B_i$. This can happen only when $\xi(D(g)) = 4$, for otherwise $\sup V_j = \inf V_{j+1}$. In the case when $\xi(D(g)) = 4$, by the condition (v*) of the definition of tube unions, we see that this cannot happen either. The same argument works also for the upper front. Thus we have shown that there are $\hat{V}_j$ and $\hat{V}_{j'}$ in $\mathcal{V}_g$ intersecting the lower and upper fronts respectively if they are real. Recall that each component of the complement of $\mathcal{V}_i$ in the real fronts of $B_i$ is a thrice-punctured sphere. Therefore, if $\gamma^g$ is finite, then its first and last vertices are contained in $I(B_i)$ and $T(B_i)$ respectively. These show that $\gamma^g$ is both forward and backward subordinate to $\gamma^{g^*}$ at its only one vertex.

Next we consider the case when $g$ stops inside $B_i$. Then for either the first or last tube $V$ of $\mathcal{V}_g$, the tube $\hat{V}$ is contained in $B_i$. We shall only consider the case when $V$ is the last tube. The case when $V$ is the first tube can be dealt with in the same way just turning everything upside down.

Let $W_1, \ldots, W_p$ be tubes in $\mathbf{V}(h_i(g))$ intersecting $\mathrm{Fr}D(g) \times J$, and let $g^*$ be a geodesic in $h_i(g)$ such that $\mathcal{V}_{g^*}$ contains a tube contained in one of $W_1, \ldots, W_p$ which appears or is prolonged within $\Sigma \times [\inf B_i, \sup J]$ latest. Let $v$ be a simplex of $g^*$ giving this tube in $W_1, \ldots, W_p$. The tube union $\mathcal{V}_{g^*}$ was put into $D(g^*) \times J^*$ which is a brick in the standard brick decomposition of $\mathbf{M}_i \setminus \mathbf{V}(h_i(g^*))$. Since $\inf J^* < \sup J$ by our definition of $g^*$, there is a horizontal leaf of $D(g^*) \times J^*$ containing that of $D(g) \times J$. Therefore we see that $D(g^*)$ must contain $D(g)$, and that $D(g)$ is a component domain of $v$. The last tube $V$ of $\mathcal{V}_g$ is attached along its upper front to a tube $V'$ which is a component of $\mathbf{V}(h_i(g))$. Let $X_1, \ldots, X_q$ be tubes in $\mathbf{V}(h_i(g))$ whose core curves lie on $D(g)$ such that $\inf X_1 = \cdots = \inf X_q = \inf V'$. If $v$ is not the last vertex of $g^*$, then $\sup J' > \sup J$, and hence $\mathrm{succ}(v)$ contains curves vertically homotopic to the core curves of $X_1, \ldots X_q$ since $D(g^*) \supset D(g)$. By setting $T(\gamma^g)$ to be $\mathrm{succ}(v)|D(g)$, we have $\gamma^g \overset{d}{\searrow} \gamma^{g^*}$ at $v$ then. If $v$ is the last vertex, then $T(g^*)$ must contain curves vertically homotopic to the core curves of $X_1, \ldots, X_1$ by the condition (vii) in the definition of tube unions. By setting $T(\gamma^g)$ to be $T(g^*)|D(\gamma^g)$, we have $\gamma^g \overset{d}{\searrow} \gamma^{g^*}$ at $v$.

We can show that $\gamma^g = g$ is subordinate to some $\gamma^{g_1}$ and $\gamma^{g_2}$ with $g_1, g_2 \in h_i(g)$ in the same way even when $g$ is totally contained in $B_i$ just by repeating the argument above at both $\sup J$ and $\inf J$.

The 4-completeness of $\{\gamma^g\}$ follows immediately from the completeness of $h_i$. □

Let $h(B_i)$ denote the 4-complete hierarchy $\{\gamma^g\}$ in the above lemma.

Now we return to the proof of Proposition 4.8. We shall next show that the main geodesic of $h(B_i)$ gets longer and longer whereas the lengths of other geodesics which are located near the basepoint have bounded lengths.

**Lemma 4.16.** *The length of the main geodesic $g_{h(B_i)}$ goes to $\infty$ as $i \to \infty$.*

*Proof.* Since $B$ is simply degenerate, the number of tubes in $\mathcal{V}_i \cap B_i$ goes to $\infty$ as $i \to \infty$. By our assumption the end contained in $B$ is algebraic. We can assume, as before, that the end is upper without loss of generality, for the case when it is lower can be dealt with just turing everything upside down.

Consider a resolution $R_i = \{\tau^i_j\}$ of the hierarchy $h_i(B_i)$, where each $\tau^i_j$ denotes a slice in $h_i(B_i)$. Since $B$ is simply degenerate, by our definition of $h_i(B_i)$, the number of slices in $R_i$ goes to $\infty$ as $i \to \infty$. Recall as was shown in Minsky [29], that each slice corresponds to a split level surface in $B_i$. Note that each component of a split level surface is a thrice-punctured sphere whose isometry type is unique. Let $D$ be its diameter. We define $D_S$ to be $(\xi(S)-2)D$, and let $E$ be the diameter of interior blocks, which are all isometric. Take a point $w_\infty$ in $B$ outside the Margulis tubes so that the distance modulo the Margulis tubes between $w_\infty$ and the front of $B$ is greater than the constant $2(E+D_S)$, and let $w_i$ be $(\rho_i^{\mathbf{M}})^{-1}(w_\infty)$, which is contained in $B_i$. Let $b_i$ be a block of $B_i$ containing $w_i$. We say that a slice contains a block $b_i$ if the corresponding split level surface passes through $b_i$. By the definition of resolution, the subset of slices in $R_i$ consisting of those containing $b_i$ is a consecutive sequence. Let $R(b_i)$ denote this subsequence of $R_i$. We shall first show the following.

**Claim 4.17.** The number of slices in $R(b_i)$ is bounded independently of $i$.

*Proof.* Suppose, seeking a contradiction, that the number of slices in $R(b_i)$ is unbounded as $i \to \infty$. Then, there must be a geodesic $g_i \in h_i(B_i)$ such that the number of distinct pairs which appear in slices in $R(b_i)$ and whose first entries are $g_i$ goes to $\infty$ as $i \to \infty$. Since all of the slices in $R(b_i)$ contain $b_i$, the geodesic $g_i$ cannot be the main geodesic of $h_i(B_i)$. (For, by our choice of $w_i$, the tubes touching $b_i$ are disjoint from the fronts of $B_i$.) Let $c_i$ be a frontier component of $D(g_i)$ in $\Sigma$, and let $V_i$ denote the Margulis tube in $\mathbf{M}_i$ whose core curve is vertically homotopic to $c_i$. As was shown in §9 of Minsky [29], we see that $\Im\omega_{\mathbf{M}_i}(V_i)$ goes to $\infty$ as $i \to \infty$. Pick up a slice $\sigma_i$ of $R(b_i)$ containing a pair with its first entry $g_i$, and consider the corresponding split level surface $f_i : \Sigma \to B_i$. The Margulis tubes which $f_i$ touches give rise to a pants decomposition $t_i$ of $\Sigma$. Since there are only finitely many ways to decompose a surface into pairs of pants up to homeomorphisms, by taking a subsequence, we can assume that there is a homeomorphism $k_i : \Sigma \to \Sigma$ such that the pants decomposition of $\Sigma$ induced from $t_i$ by pulling it back by $k_i$ to $\Sigma$ is independent of $i$. We denote by $\mathcal{K}$ the collection of simple closed curves giving this decomposition on $\Sigma$.

We say that a curve $d \in \mathcal{K}$ is *parabogenic* if for a Margulis tube $V_i'$ in $\mathbf{M}_i$ with core curve homotopic to $k_i(d)$, we have $|\omega_{\mathbf{M}_i}(V_i')| \to \infty$ passing to a subsequence, whereas $k_i^{-1} \circ f_i^{-1}(b_i)$ and $d$ can be connected by a path $\alpha$ such that for any curve $e$ in $\mathcal{K}$ that $\alpha$ passes, the Margulis tube $V_i''$ with core curve $k_i(e)$ has the property that $\omega_{\mathbf{M}_i}(V_i'')$ is bounded as $i \to \infty$. A parabogenic curve always exists since $c_i$ as above is a core curve of the Margulis tube $V_i$ with $|\omega_{\mathbf{M}_i}(V_i)| \to \infty$. Choose a parabogenic curve $d$ and a Margulis tube $V_i'$ with core curve vertically homotopic to $k_i(d)$. Note that the diameters of split level surfaces modulo the Margulis tubes are universally bounded, and that $b_i$ and $V_i'$ can be connected by a path passing only through a split level surface and Margulis tubes with bounded $\omega_{\mathbf{M}_i}$. Therefore, the distance between $b_i$ and $V_i'$ is bounded as $i \to \infty$. Since $|\omega_{\mathbf{M}_i}(V_i')| \to \infty$, the tubes $V_i'$ converge geometrically to a cusp neighbourhood $V_\infty'$ intersecting $B$.

Since $V_i'$ is within distance $D_S$ modulo the Margulis tubes from $b_i$, we see that $V_\infty'$ cannot intersect the fronts of $B$ by our choice of $w_\infty$. Moreover, since $d$ is a curve in $\mathcal{K}$, we see that $\rho_i^{\mathbf{M}}(k_i(d))$ cannot be homotopic to a curve on the vertical boundary of $B$. Therefore $V_\infty'$ is a cusp in $B$ whose core is not vertically homotopic into $\mathrm{Fr}\Sigma$. This contradicts the facts that $B$ is a brick in the standard brick decomposition of $\mathbf{M}$ and that consequently $B$ can intersect a cusp neighbourhood only at its vertical boundary and real front. □

Now we return to the proof of Lemma 4.15. Starting from a slice $\tau^i_{j(i)}$ in $R(b_i)$, we proceed in the resolution $R_i$. For any block $b'$ in $B$, we consider the subfamily $R_i(b')$ of $R_i$ consisting slices $\tau^i_j$ with $j \geq j(i)$ such that the split level surface corresponding to $\tau^i_j$ intersects $(\rho^{\mathbf{M}}_i)^{-1}(b')$. By almost the same argument as in the proof of Claim 4.16, we can show the the number of slices contained in $R_i(b')$ is bounded as $i \to \infty$. The only difference of the argument for $b'$ from that for $b_i$ is that $b'$ does not contain the basepoint $w_\infty$. We used the assumption that $b_i$ contains $(\rho^{\mathbf{M}}_i)^{-1}(w_\infty)$ in the proof of Claim 4.16 to show that the cusp neighbourhood $V'_\infty$ does not touch the front of $B$. Since the end in $B$ is upper, the front which we are considering is the lower one. Since we are considering a subsequence of $R_i$ going forward from $j(i)$, the split level surfaces corresponding to slices in $R_i(b')$ are above that corresponding to $\tau^i_{j(i)}$. This implies that a cusp neighbourhood appearing as a limit of Margulis tubes meeting these split level surfaces cannot intersect the lower front of $B$ either. Similarly, we can show that for any Margulis tube or cusp neighbourhood $V$ in $B$, the number of slices $\tau^i_j$ in $R_i$ such that $j \geq j(i)$ and the split level surface corresponding to $\tau^i_j$ intersect $(\rho^{\mathbf{M}}_i)^{-1}(V)$ is bounded as $i \to \infty$.

Recall that the total number of the slices in $R_i$ goes to $\infty$. For the split level surface $h(\tau^i_{j(i)})$ corresponding to $\tau^i_{j(i)}$, its image $\rho^{\mathbf{M}}_i(h(\tau^i_{j(i)}))$ is bounded distance from the lower front of $B$ since it intersects $b$. Therefore there must be infinitely many blocks above $\rho^{\mathbf{M}}_i(h(\tau^i_{j(i)}))$. This implies that the number of slices $\tau^i_j$ with $j > j(i)$ goes to $\infty$ as $i \to \infty$. Since we assumed that the lengths of the main geodesics are bounded, this is possible only when there is a number $j_1$ independent of $i$ such that for any $n_0$, there is $i$ such that all the slices $\tau^i_{j(i)+j_1}, \ldots, \tau^i_{j(i)+j_1+n_0}$ share the same pair $(g_{h_i(B)}, w_i)$ in $R_i$. Let $V''_i$ be the Margulis tube whose core curve is vertically homotopic to a simple closed curve which is a component of $w_i$. Since $V''_i$ is reached within $j_1$ steps from $\tau^i_{j(i)}$ intersecting $b_i$, the tube is within bounded distance from $b_i$. Therefore, passing to a subsequence $\{V''_i\}$ converge to a cusp neighbourhood $V$ in $B$. This contradicts the fact shown above that the number of slices whose corresponding split level surfaces intersect $(\rho^{\mathbf{M}}_i)^{-1}(V)$ is bounded as $i \to \infty$. □

Now, we shall complete the proof of Proposition 4.8. By our definition of $h_i(B_i)$, we see that there is a geodesic $\gamma_i$ in $h_i$ with support $\Sigma$ which contains the main geodesic $g_{h_i(B_i)}$ as a subgeodesic. Therefore, we have already proved the conditions (1) and (2). It remains to show the conditions (3) and (4).

The main geodesic of $h_i(B_i)$ penetrates $B_i$ by Lemma 4.14. Since either $g_{h_i(B_i)}$ is a ray or the length of $g_{h_i(B_i)}$ goes to $\infty$, by Lemma 4.14 again, the first vertex of $g_{h_i(B_i)}$ coincides with that of $\gamma_i$ for large $i$. This means that we can identify the limit of $\gamma_i$ with that of $g_{h_i(B_i)}$.

As was remarked before, $(\rho^{\mathbf{M}}_i)^{-1} \circ g'$ is homotopic to $\Phi^{\mathbf{M}}_i$. For any $i$, there is a number $K(i)$ with $K(i) \to \infty$ such that the core curves of Margulis tubes in $B$ within the distance $K(i)$ from the basepoint $w_\infty$ are homotopic into $g'(S)$ within the range of the approximate isometry $\rho^{\mathbf{M}}_i$. It follows that there is a number $n(i)$ which goes to $\infty$ such that for each $v^i_j$ among the first $n(i)$-simplices $v^i_j$ of $\gamma_i$, a homotopy between $(\rho^{\mathbf{M}}_i)^{-1} \circ g'(v^i_j)$ and the corresponding core curves of Margulis tubes in $B_i$ can be pushed forward to a homotopy between $g'(v^i_j)$ and core curves of Margulis tubes in $B$. Note that any sequence of core curves of Margulis tubes going to the end of $B$ converges to the ending lamination of the simply degenerate end. This implies that $\gamma_i$ converges to a geodesic ray whose endpoint is the ending lamination uniformly on each one in an ascending exhausting sequence of compact sets in $[0,\infty)$, after passing to a subsequence, and we get the conditions (3) and (4) using Lemma 2.2.

4.4. **Non-algebraic simply degenerate ends.** In the proof of Proposition 4.8, we used the assumption that $B$ is algebraic only to show that the support of $\gamma_i$ is $\Sigma$. Even in the case when $B$ may not be algebraic, the argument above shows that we still have a geodesic as $\gamma_i$ in $h_i$, and its support is the preimage of $\Sigma$, which may depend on $i$. Thus we get the following corollary.

**Corollary 4.18.** *Let $B = \Sigma \times I$ be a simply degenerate brick of $\mathbf{M}$. Let $\mathcal{V}$ be the union of all boundary components of $\mathbf{M}$ touching the vertical boundary of $B$, and $\mathcal{V}_i$ the union of Margulis tubes corresponding to $(\rho_i^{\mathbf{M}})^{-1}(\mathcal{V} \cap B_{K_i r_i}(\mathbf{M}, x_\infty))$. Then, there is a geodesic $\gamma_i$ in $h_i$ satisfying the following conditions.*

1. *For sufficiently large $i$, the preimage $(\rho_i^{\mathbf{M}})^{-1}(B)$ is contained in a brick $B_i = \Sigma_i \times J_i$ in the standard brick decomposition of $\mathbf{M}_i \setminus \mathcal{V}_i$.*
2. *The geodesic $\gamma_i$ is supported on $\Sigma_i$. Passing to a subsequence, we can assume that all $\gamma_i$ either have finite lengths or are geodesic rays.*
3. *If the $\gamma_i$ have finite lengths, their lengths go to $\infty$ as $i \to \infty$.*
4. *Let $\partial_{\mathrm{real}} B$ be the real front of $B$. Let $k_i : \Sigma \to \Sigma_i$ be a homeomorphism induced from $(\rho_i^{\mathbf{M}})^{-1}|\partial_{\mathrm{real}} B$. If $\gamma_i$ has finite length, for the last vertex $v_i$ of $\gamma_i$, its image $k_i^{-1}(v_i)$ on $\Sigma$ converges to the ending lamination of the simply degenerate end of $B$. If $\gamma_i$ is a ray, then for the endpoint $e_i$ of $\gamma_i$ at infinity, $k_i^{-1}(e_i)$ converges to the ending lamination of the end in $B$ in $\mathcal{EL}(\Sigma)$.*

## 5. Limits of end invariants and ends of models

In this section, we consider the situation where $\{(G_i, \phi_i) = qf(m_i, n_i)\}$ converges to $(\Gamma, \psi)$ and $\{G_i\}$ converges geometrically to $G_\infty$. We assume that $\{m_i\}$ converges to $[\mu^-]$ and $\{n_i\}$ converges to $[\mu^+]$ in the Thurston compactification of the Teichmüller space. Let $\Sigma^-$ and $\Sigma^+$ be the boundary components of the convex core of $M_i = \mathbb{H}^3/G_i$ facing the upper and lower ends respectively. We shall first recall the following fact, which follows from the continuity of length function.

**Lemma 5.1.** *Let $\nu$ be a component of either $\mu^-$ or $\mu^+$. If $\nu$ is a weighted simple closed curve, then $\Psi(|\nu|)$ represents a parabolic class of $\Gamma$. Otherwise, its image $\Psi(\nu)$ represents the ending lamination for an end of $M_0'$.*

*Proof.* This is just a combination of Thurston's theorem and the continuity of the length function proved by Brock [7] in general form. We can assume that $\nu$ is a component of $\mu^-$ since the argument for the case of $\mu^+$ is exactly the same. Thurston's Theorem 2.2 in [39] (whose proof can be found in [18] and [40]) shows that there is a sequence of simple closed curve $r_i\gamma_i$ converging to $\mu^+$ such that $\mathrm{length}_{m_i}(r_i\gamma_i)$ goes to 0. By Bers' inequality [4], this implies that $\mathrm{length}_{\Sigma_i^-}(r_i\gamma_i)$ also goes to 0. By the continuity of the length function with respect to the algebraic topology (see Brock [7]), we have $\mathrm{length}_{M'}\Psi(\mu^-) = 0$, which means every component of $\Psi(\mu^-)$ represents either a parabolic element or an ending lamination. $\square$

We shall refine this lemma to show that the components of the limit of $\{m_i\}$ appear as lower parabolics or lower ending laminations whereas those of $\{n_i\}$ appear as upper parabolics or upper ending laminations.

**Theorem 5.2.** *Let $c_{m_i}$ and $c_{n_i}$ be shortest (hyperbolic) pants decompositions of $(S, m_i)$ and $(S, n_i)$ respectively. Let $\nu^-, \nu^+$ be the Hausdorff limits of $\{c_{m_i}\}$ and $\{c_{n_i}\}$ respectively. Then the minimal components of $\nu^+$ that are not simple closed curves coincide with the ending laminations of upper algebraic simply degenerate ends of $\mathbf{M}$. Moreover, no upper algebraic parabolic curve of $\mathbf{M}$, regarded as a curve on $S$, can intersect a minimal component of $\nu^+$ transversely. Similarly the minimal components of $\nu^-$ that are not simple closed curves coincide with the ending laminations of lower algebraic simply degenerate ends of $\mathbf{M}$, and no lower algebraic parabolic curve can intersect a minimal component of $\nu^-$ transversely.*

*Proof.* We shall only deal with $\nu^+$. The argument for $\nu^-$ is obtained only by turning $\mathbf{M}$ upside down. Let $h_i$ be a hierarchy corresponding to $qf(m_i, n_i)$, and consider the model manifold $\mathbf{M}_i$ such that $M_i[k]$ converges geometrically to $\mathbf{M}[k]$ as before. We regard $\mathbf{M}$ as being embedded in $S \times [0,1]$.

We shall first show that the ending lamination of any upper algebraic simply degenerate end $e$ of $\mathbf{M}$ is a minimal component of $\nu^+$. Let $B = \Sigma \times [s,t)$ be a simply degenerate brick of $\mathbf{M}$ containing the end $e$. By Proposition 4.8, the hierarchy $h_i$ contains a geodesic $\gamma_i$ supported on $\Sigma$ whose last vertex converges to the ending lamination $\lambda_e$ of $e$ in $\mathcal{UML}(S)$. Now, as was shown in §6 in Masur-Minsky [27] using Theorem 3.1 in the same paper, the distance between the last vertex of $\gamma_i$ and the projection of the terminal marking of $h_i$ to $\Sigma$ is uniformly bounded. In particular, for the shortest pants decomposition $c_{n_i}$, which consists of the base curves of $T(h_i)$, its projection to $\Sigma$ converges to $\lambda_e$ in $\mathcal{UML}(\Sigma)$. Since the Hausdorff limit of $c_{n_i}|\Sigma$ contains the limit of $c_{n_i}$ in $\mathcal{UML}(\Sigma)$, this shows that the ending lamination of any upper algebraic simply degenerate end is contained in $\nu^+$.

Secondly, we shall show that no upper algebraic parabolic curve can intersect a minimal component of $\nu^+$ transversely. Let $c$ be a upper parabolic locus. There are three cases which we have to consider. The first is the case (a) when $g'(c)$ is homotopic to a curve on a torus boundary of $\mathbf{M}$, the second is the case (b) when $g'(c)$ is homotopic to a core curve of an open annulus boundary component of $\mathbf{M}$ touching a geometrically finite end, and the third is the case (c) when $g'(c)$ is homotopic to a core curve of an open annulus boundary component meeting only simply degenerate ends.

(a) We first consider the case when the curve $c$ lies on a torus component $T$ of $\mathbf{M}$. Let $V_i$ be the Margulis tube bounded by $(\rho_i^{\mathbf{M}})^{-1}(T)$ in $\mathbf{M}_i$ for large $i$. Then $\Re\omega_{\mathbf{M}_i}(V_i)$ goes to $\infty$ whereas the imaginary part is bounded as $i \to \infty$. Let $c_i$ be a simple closed curve on $S$ whose image by $\Phi_i^{\mathbf{M}}$ is homotopic to the longitude of $V_i$. Since the longitude of $\partial V_i$ converges to that of $T$ which is homotopic to the image of a simple closed curve under $g'$, the homotopy class of $c_i$ is constant for large $i$. Therefore, by taking a subsequence, we can assume that $c_i$ is constantly $c$. Let $A$ be an annulus on $S$ which is a regular neighbourhood of $c$. Since $\Re\omega_{\mathbf{M}_i}(V_i) \to \infty$, there is a geodesic $\gamma_i$ in $h_i$ supported on $A$ whose length goes to $\infty$ as $i \to \infty$. Let $a(i)$ and $b(i)$ be the first and last vertices of $\gamma_i$, and let $n(a)_i$ and $n(b)_i$ be the (signed) number of times which $a(i)$ and $b(i)$ go around $c$ compared to the transversal of the marking determined by $\Phi_i^{\mathbf{M}}$. Then $|n(b)_i - n(a)_i|$ goes to $\infty$ as $i \to \infty$. We set $n(i)$ to be $n(b)_i - n(a)_i$.

By the definition of hierarchy, there is a vertex of a geodesic $g_i$ in $h_i$ with $\xi(D(g_i)) = 4$ which represents $c$ (and is denoted also by $c$), satisfying $\pi_A(\mathrm{prec}(c)) =$

$a(i), \pi_A(\mathrm{succ}(c)) = b(i)$. As was shown above, the distance between $\pi_A(\mathrm{prec}(c)))$ and $\pi_A(\mathrm{succ}(c))$ goes to $\infty$. Since these $\mathrm{prec}(c)$ and $\mathrm{succ}(c)$ may depend on $i$, we denote $\mathrm{prec}(c)$ in $g_i$ by $v_i$ and $\mathrm{succ}(c)$ in $g_i$ by $w_i$.

Since there is an elementary move changing $v_i$ to $c$, there is a block $b_i$ in $\mathbf{M}_i$ realising this elementary move by the definition of model manifolds by Minsky [29]. Let $V_i$ be the Margulis tube in $\mathbf{M}_i$ whose core curve represent $v_i$, and $u_i$ its horizontal longitude. Recall that the block decomposition of $\mathbf{M}_i$ converges geometrically to that of $\mathbf{M}$ as $i \to \infty$. Therefore, the block can be pushed forward to a block $b_\infty$ in $\mathbf{M}$ for large $i$, and hence there is either a Margulis tube or a torus boundary in $\mathbf{M}$ whose core curve or longitude, which we denote by $u_\infty$ is homotopic to $\rho_i^{\mathbf{M}}(u_i)$ for every large $i$. First suppose that $g'$ does not go around $T$. Then $g'$ can be homotoped so that it passes $b_\infty$ horizontally, and consequently, there is a simple closed curve $u$ on $S$ such that $g'(u)$ is homotopic to $u_\infty$. By pulling back this to $\mathbf{M}_i$, we see $\Phi_i^{\mathbf{M}}(u)$ is homotopic to $\mathrm{prec}(c)$ for large $i$. This means that $n(a)_i$ is bounded as $i \to \infty$, and hence $|n(b)_i|$ goes to $\infty$.

We next consider the case when $g'$ goes $k$-times around $T$ for $k \in \mathbb{Z} \setminus \{0\}$. Then, we can homotope $(\rho_i^{\mathbf{M}})^{-1} \circ g'(u)$ to $u_i$ after passing $k$-times through $V_i$. This means that the $kn_i$-time Dehn twist of $u_i$ around $c$ represents a constant homotopy class for large $i$. Therefore, $|n(a)_i|$ grows in the order of $|kn(i)|$ and $|n(b_i)|$ in the order of $|(k+1)n(i)|$ in this case.

In either case we see that $|n(b)_i|$ goes to $\infty$. Therefore, by §6 of [27] again, the projection of the shortest pants decomposition $c_{n_i}$ to $\mathcal{CC}(A)$ also goes around $n'(i)$ times around $c$ with $n'(i) \to \infty$. This shows that the Hausdorff limit $\nu^+$of $c_{n_i}$ contains $c$ as a minimal component.

(b) Next we consider the second case when $g'(c)$ is homotopic to a core curve of an open annulus boundary component $T$ which touches a geometrically finite end $e$ of $\mathbf{M}$. The end $e$ corresponds to $\Sigma \times \{1\}$ for some essential subsurface $\Sigma$ of $S$ and has some conformal structure $n_\Sigma$. By the definition of the conformal structures on the geometrically finite bricks of $\mathbf{M}$, there is a subsurface $\Sigma_i$ in $S$ such that $n_i|\Sigma_i$ converges to $n_\Sigma$. Since $T$ is assumed to be algebraic, a frontier component $c$ of $\Sigma$ is pulled back to a frontier component $c_i$ of $\Sigma_i$ such that $\Phi_i^{\mathbf{M}}(c)$ is homotopic to $c_i$. This implies that the length of $c$ with respect to $n_i$ goes to 0, and the pants decomposition $c_{n_i}$ must contain $c$. Therefore $c$ is contained in the Hausdorff limit of $c_{n_i}$ also in this case.

(c) Now, we consider the third case when $g'(c)$ is homotopic to a core curve of an open annulus boundary component $T$ which meets only simply degenerate ends. There are either one or two such ends. Since the argument is quite similar in both cases, we assume that there are two simply degenerate ends $e_1$ and $e_2$ which $T$ touches. Let $B_1 \cong \Sigma^1 \times J_1$ and $B_2 \cong \Sigma^2 \times J_2$ be simply degenerate bricks of $\mathbf{M}$ containing $e_1$ and $e_2$ respectively. By Corollary 4.17, there are bricks $B_i^1 \cong \Sigma_i^1 \times J_i^1$ and $B_i^2 \cong \Sigma_i^1 \times J_i^2$ containing $(\rho_i^{\mathbf{M}})^{-1}(B_1)$ and $(\rho_i^{\mathbf{M}})^{-1}(B_2)$ respectively, and geodesics $\gamma_i^1$ and $\gamma_i^2$ supported on $\Sigma_i^1$ and $\Sigma_i^2$ whose lengths go to $\infty$ as $i \to \infty$. The approximate isometry induces homeomorphisms $f_i^1 : \Sigma^1 \to \Sigma_i^1$ and $f_i^2 : \Sigma^2 \to \Sigma_i^2$. Also, we know that for the last vertices $t_i^1, t_i^2$ of $\gamma_i^1, \gamma_i^2$, their images $(f_i^1)^{-1}(t_i^1), (f_i^2)^{-1}(t_i^2)$ converge to the ending laminations of $e_1$ and $e_2$, which we denote by $\lambda_1$ and $\lambda_2$ respectively. By the same argument as before using §6 of [27], we see that $(f_i^1)^{-1}\pi_{\Sigma_i^1}(c_{m_i})$ and $(f_i^2)^{-1}\pi_{\Sigma_i^2}(c_{m_i})$ converge to $\lambda_1$ and $\lambda_2$ respectively.

By renaming $e_1$ and $e_2$ if necessary, we can assume that $\sup J_1 \leq \sup J_2$. By the structure of brick manifolds, we have $\inf J_1 = \inf A = \inf J_2$. For $t$ between $\inf J_1$ and $\sup J_1$, we denote by $A_t$ the horizontal annulus bounded by the two components of $T \cap S \times \{t\}$ on $S \times \{t\}$. Since $c$ lies on the frontiers of $\Sigma^1$ and $\Sigma^2$ and $T$ is algebraic, there is an essential subsurface $F$ of $S$ containing $c$ as a non-peripheral curve such that for $t \in (\inf J_1, \sup J_1)$, the surface $F \times \{t\} \setminus A_t$ is homotopic into $g'(F)$. This implies that $f_i^1$ and $f_i^2$ must be homotopic to the identity in $\Sigma^1 \cap F$ and $\Sigma^2 \cap F$ for large $i$.

Suppose, seeking a contradiction, that there is a minimal component $d$ of $\nu^+$ intersecting $c$ transversely. Note that $(f_i^1)^{-1}\pi_{\Sigma^1}(\nu^+)$ converges to $\lambda_1$ in the Hausdorff topology since $(f_i^1)^{-1}\pi_{\Sigma^1}(c_{m_i})$ converges to $\lambda_1$. Therefore, $(f_i^1)^{-1}(\pi_{\Sigma_i^1}(d))$ must also converge to $\lambda_1$. On the other hand, since $d$ is a minimal component, starting from a point $c \cap d$ and going into $\Sigma^1$, it either returns to another point of $c \cap d$ or goes out from $\Sigma^1$. Let $a$ be such an arc in $d \cap \overline{\Sigma^1}$. Then by the definition of the map $\pi_{\Sigma^1}$, there are only finitely many possibilities for $\pi_{\Sigma^1}(a)$. Since $f_i^1$ is the identity on $F \cap \Sigma^1$, we see that both the number of the leaves of $(f_i^1)^{-1}\pi_{\Sigma^1}(d)|F$ and their homotopy classes are bounded. Therefore its Hausdorff limit has only isolated leaves in $F$. This contradicts the fact that $\lambda_1$ is arational. Thus we have completed the proof of the fact that no algebraic parabolic curve can intersect a minimal component of $\nu^+$ transversely.

To complete the proof, it remains to show that $\nu^+$ has no minimal component that is not a compact leaf and is disjoint from the minimal supporting surfaces of the ending laminations of upper algebraic simply degenerate ends. We also recall that no minimal component of $\nu^+$ intersects an upper parabolic locus regarded as lying on $S$, as has been shown above. Let $\Sigma_1, \ldots, \Sigma_{j_0}$ be the minimal supporting surfaces of the ending laminations of upper algebraic simply degenerate ends. (We take these so that their boundaries are totally geodesic.) Let $\Sigma'$ be a component of the complement of the union of $\cup_{j=1}^{j_0}\Sigma_j$ and all upper parabolic loci, which we take to be geodesics. What we have to show is that every minimal component of $\nu^+$ contained in $\Sigma'$ is a simple closed curve.

Before dealing with the general situation, we begin with considering special cases when $g'(\Sigma')$ is homotopic into an end lying above $g'(S)$. Then there is domain $S'$ containing $\Sigma'$ such that $g'(\Sigma')$ is homotopic into an end $S' \times \{s\}$, where $s = 1$ if the end is geometrically finite. We first consider the case when $s = 1$ and $S' \times \{1\}$ is geometrically finite. By our definition of geometrically finite ends of $\mathbf{M}$, the surface $S' \times \{1\}$ has a hyperbolic metric $n_\infty$ which is a geometric limit of $(S, n_i)$ with base point lying in the thick part of $(S'_i, n_i|S'_i)$ for some subsurface $S'_i$ homeomorphic to $S'$. Since $\Sigma' \times \{1\}$ in $\mathbf{M}$ is homotopic to $g'(\Sigma')$, we see that $S'_i$ contains $\Sigma'$ for large $i$ and that $n_i|\Sigma'$ converges to $n_\infty|\Sigma'$ preserving the markings. Note that $\nu_\infty|\Sigma'$ is a Hausdorff limit of $c_{n_i}|\Sigma'$. Since $n_i|\Sigma'$ converges to $n_\infty$, we see that a minimal component of the Hausdorff limit of $c_{n_i}$ contained in $\Sigma'$ must be a compact leaf.

Next suppose that there is a simply degenerate end $e$ of $\mathbf{M}$ of the form $S' \times \{s\}$ lying above $g'(S)$ into which $g'(\Sigma')$ is homotopic. Then this end cannot be algebraic since $\Sigma'$ lies in the complement of the minimal supporting surfaces of ending laminations of upper algebraic simply degenerate ends. Let $B$ be a simply degenerate brick containing $S' \times \{s\}$, and $\lambda_e$ the ending lamination of $e$. Since $S' \times \{s\}$ is not algebraic, $\Sigma'$ is a proper subsurface of $S'$. Now, by Corollary 4.17, there is a brick of the form $S'_i \times [s_i, t_i]$ in $\hat{\mathbf{M}}_i$ containing $(\rho_i^{\mathbf{M}})^{-1}(B \cap B_{K_i r_i}(\mathbf{M}, x_\infty))$ in which a tube union corresponding to a geodesic $\gamma_i$ is put. Also, for a homeomorphism $f_i : S' \to S'_i$ induced from $(\rho_i^{\mathbf{M}})^{-1}|\partial_{\mathrm{real}}B$, the image of the last vertex or the lamination corresponding to the endpoint at infinity of $\gamma_i$ under $f_i^{-1}$ converges to the ending lamination $\lambda_e$. By the same argument as before using §6 of [27], this implies that $f_i^{-1}(\pi_{S'_i}(c_{n_i}))$ converges to $\lambda_e$. Now, since $g'(\Sigma')$ is homotopic into $S' \times \{s\}$ and $\lambda_e$ is arational, this shows that the Hausdorff limit of $c_{n_i}|\Sigma'$ consists only of arcs. Therefore, $\Sigma'$ cannot contain a minimal component of $\nu^+$ in this case.

In general, the surface $\Sigma'$ is decomposed into subsurfaces $S''_1, \ldots, S''_p$ intersecting each other only at their boundaries such that for each $S''_k$, the surface $g'(S''_k)$ is homotopic into an end lying above $g'(S)$ and $S''_k$ is maximal up to isotopies among surfaces having this property. As was shown above if $g'(S''_k)$ is homotopic into simply degenerate end, then the Hausdorff limit of the restriction of $v_\infty(i)$ to $S''_k$ does not have a minimal component in $\mathrm{Int}S''_i$. On the other hand, if $g'(S''_k)$ is homotopic into a geometrically finite end, then our argument for the case when $\Sigma'$ is homotopic into $S \times \{1\}$ within $\mathbf{M} \cup S \times \{1\}$ shows that every minimal component of the Hausdorff limit of $v_\infty(i)|S''_k$ is a compact leaf. Thus we have shown that every minimal component of the Hausdorff limit of $v_\infty(i)$ contained in $\Sigma'$ is a compact leaf. This completes the proof of Theorem 5.2. □

Theorem 2 in §3 is obtained as a corollary of Theorem 5.2, as follows.

*Proof of Theorem 2.* Each simply degenerate end of $\mathbf{M}$ is mapped to that of $(M_\infty)_0$. Let $p : M' \to M_\infty$ be a covering associated to the inclusion of the algebraic limit $\Gamma$ into the geometric limit $G_\infty$. By the covering theorem of Thurston [38] and Canary [14], each simply degenerate end of $(M')_0$ has a neighbourhood which is mapped homeomorphically to a neighbourhood of a simply degenerate end of $(M_\infty)_0$. Furthermore the ending lamination of an end of $(M')_0$ is identified with that of the corresponding end of $(M_\infty)_0$ by $p$, which follows immediately by the definition of ending laminations. Therefore the algebraic simply degenerate ends of $\mathbf{M}$ correspond to simply degenerate ends of $(M')_0$ one-to-one preserving the ending laminations. It is also obvious that upper (resp. lower) ends of $\mathbf{M}$ correspond to upper (resp. lower) ends of $(M')_0$. Similarly the algebraic parabolic curves correspond to the core curves of the parabolic loci of $M'$. These show that Theorem 2 follows from Theorem 5.2. □

**Lemma 5.3.** *Let $\{g_i\}$ be a sequence in the Teichmüller space $\mathcal{T}(S)$ which converges to a projective lamination $[\mu]$ in the Thurston compactification. Let $c_i$ be a pants decomposition on $(S, g_i)$ whose total length is uniformly bounded. Then the Hausdorff limit of any subsequence of $\{c_i\}$ contains all the components of $|\mu|$ as minimal components.*

*Proof.* Let $\lambda$ be the Hausdorff limit of a convergent subsequence of $\{c_i\}$. Since $c_i$ is pants decomposition of $S$, there is no measured lamination on $S$ which is disjoint from $c_i$, other than the components of $c_i$. Therefore there is no measured lamination on $S$ which is disjoint from $\lambda$, other than minimal components of $\lambda$. Let $\mu_0$ be a component of $\mu$, and suppose that its support is not a minimal component of $\lambda$. Then $\mu_0$ must intersect $\lambda$ essentially.

We shall first consider the case when $\mu_0$ is not a simple closed curve. Let $\Sigma$ be the minimal supporting surface of $\mu_0$. We consider a sequence of essential arcs and simple closed curves $c_i \cap \Sigma$ on $\Sigma$. Note that $c_i \cap \Sigma$ converges to $\lambda \cap \Sigma$ with respect to the Hausdorff topology, which is non-empty. If $\lambda \cap \Sigma$ contains a minimal component contained in Int$\Sigma$, then there is a sequence of positive numbers $r_i$ going to 0 such that $r_i c_i \cap \Sigma$ converges to a measured lamination $\gamma$ in $\Sigma$. Otherwise we can find a bounded sequence of positive numbers $r_i$ such that $\{r_i c_i\}$ converges to a union $\gamma$ of essential arcs. (The convergence is taken in the space of weighted essential curves in $\Sigma$ with the weak topology.) In either case, let $R$ be $\max_i r_i$. Now, $\text{length}_{g_i}(c_i) \geq r_i \text{length}_{g_i}(c_i)/R$, where the right hand goes to $\infty$ since $i(\mu_0, \gamma) > 0$ and an arc with non-zero intersection with the Thurston limit has length going to $\infty$ if we consider hyperbolic structures on $\Sigma$ with geodesic boundaries. This implies that $\text{length}_{g_i}(c_i)$ must also go to $\infty$, which is a contradiction.

Next suppose that $\mu_0$ is a simple closed curve. If the length of $\mu_0$ with respect to $n_i$ goes to 0, then we can take an annular neighbourhood $A_i(\mu_0)$ of $\mu_0$ whose width (with respect to $n_i$) goes to $\infty$ as $i \to \infty$. Since $\lambda$ intersects $\mu_0$ essentially, $c_i$ passes through $A_i(\mu_0)$ for large $i$. This implies the length of $c_i$ in $(S, n_i)$ goes to $\infty$, which is a contradiction. Next suppose that the length of $\mu_0$ is bounded from both above and below by positive constants. Then we can take an annular neighbourhood $A_i(\mu_0)$ whose width is bounded away from 0. Consider the shortest essential arc $\alpha_i$ in $A_i(\mu_0)$. Since $\mu_0$ is contained in the limit lamination $[\mu]$ of $\{g_i\}$, the shortest arc $\alpha_i$ must spirals around $\mu_0$ more and more as $i \to \infty$. Since $\lambda$ does not contain $\mu_0$, the number of spiralling of $c_i$ around $\mu_0$ is bounded. This means twisting number between $\alpha_i$ and $c_i|A_i(\mu_0)$ goes to $\infty$. Therefore, the length of $c_i|A_i(\mu_0)$ goes to $\infty$ as $i \to \infty$. In the case when the length of $\mu_0$ goes to $\infty$, we take $A_i(\mu_0)$ whose width goes to 0 as $i \to \infty$. Also in this case, the shortest essential arc $\alpha_i$ spirals around $\mu_0$ more and more as $i \to \infty$. Since the twisting number between $\alpha_i$ and $c_i|A_i(\mu_0)$ goes to $\infty$ also in this case, we see that the length of $c_i|A_i(\mu_0)$ goes to $\infty$. This is a contradiction. □

Combining this lemma with Theorem 5.2, we get the following corollary.

**Corollary 5.4.** *Let* $[\mu^+]$ *and* $[\mu^-]$ *be projective laminations to which* $\{m_i\}$ *and* $\{n_i\}$ *converge in the Thurston compactification of* $\mathcal{T}(S)$ *after taking subsequences. Then each minimal component of* $|\mu^+|$ *is either of the ending lamination of an upper algebraic simply degenerate end or an upper algebraic parabolic curve of* $\mathbf{M}$. *Similarly, each component of* $\mu^-$ *is either the support of the ending lamination of a lower algebraic simply degenerate end or a lower algebraic parabolic curve of* $\mathbf{M}$.

*Proof.* Each component of $|\mu^+|$ that is not a simple closed curve is the ending lamination of an upper simply degenerate end by Theorem 5.2 and Lemma 5.3. Let $c$ be a component of $|\mu^+|$ which is a simple closed curve. Then by Lemma 5.1, $\psi(c)$ is parabolic. Therefore $c$ is an algebraic parabolic locus in $\mathbf{M}$. It remains to show that $c$ is upper.

Suppose that $c$ is not upper, seeking a contradiction. This assumption implies, in particular, that if $c$ is isolated and is homotopic to a curve lying on a torus boundary $T$ of $\mathbf{M}$, then the standard algebraic immersion $g'$ does not go around $T$. There are only three possibilities for the curve $g'(c)$: (1) the first is when $c$ lies in a domain $F$ of $S$ as a non-peripheral curve and $g'(F)$ is homotopic into some simply degenerate end above $g'(S)$, (2) there exists $F$ containing $c$ as above such that $g'(F)$ is homotopic into geometrically finite end, lying on $S \times \{1\}$, and (3) the third is when there is an upper algebraic parabolic curve intersecting $c$ essentially on $S$.

(1) Suppose that $g'(F)$ is homotopic into a simply degenerate end $e$. Then its ending lamination $\lambda_e$ intersects $c$ essentially. We shall argue as in the proof of Theorem 5.2, where we dealt with the case when $g'(\Sigma')$ is homotopic into a simply degenerate end. Let $B = S' \times [s,t)$ be a simply degenerate brick of $\mathbf{M}$ containing $e$. By Corollary 4.17, there are a brick $B_i = S'_i \times [s_i, t_i]$ in $\hat{\mathbf{M}}_i$ containing $(\rho_i^{\mathbf{M}})^{-1}(B \cap B_{K_i r_i}(\mathbf{M}, x_\infty))$ and a geodesic $\gamma_i$ in $h_i$ which has a tube union in $B_i$. As before there is a homeomorphism $f_i : S' \to S'_i$ induced from $(\rho_i^{\mathbf{M}})^{-1}|\partial_{\mathrm{real}} B$, such that for the last vertex $v_i$ of $\gamma_i$ its image $f_i^{-1}(v_i)$ converges to $\lambda_e$.

Let $A(c)$ be an annulus with core curve $c$. Since $F$ is homotopic into $e$, we see that $f_i|F$ is the identity. In particular, we see that $f_i|A(c)$ is the identity. Therefore $v_i|A(c) = f_i^{-1}(v_i)|A(c)$ converges to some vertex in $\mathcal{CC}(A(c))$, corresponding to $\pi_{A(c)}(\lambda_e)$. This implies, again by the argument of §6 of Masur-Minsky [27], that $c_{n_i}|A(c)$ converges to some vertex in $\mathcal{CC}(A(c))$ after passing to a subsequence. On the other hand, by Lemma 5.3, $c_{n_i}$ must converge to a lamination containing $c$ in the Hausdorff topology. This is a contradiction.

(2) In the second case, $g'(F)$ is homotopic into an upper geometrically finite end of $\mathbf{M}$. Take a simple closed curve $\delta$ on $F$ intersecting $c$ essentially. Since $g'(\delta)$ is homotopic to a curve in an upper geometrically finite end, which is a geometric limit of $(S, m_i)$ with base point in $F$, we see that $\mathrm{length}_{m_i}(\delta)$ is bounded as $i \to \infty$. This contradicts the assumption that $c$ is contained in $\mu^+$.

(3) In the third case, let $d$ be an upper algebraic parabolic curve with $i(c,d) > 0$. Recall that $c$ is a minimal component of the Hausdorff limit $\nu^+$ of $c_{m_i}$ by Lemma 5.3. Therefore, $d$ intersects a minimal component of $\nu^+$ transversely in this situation, contradicting Theorem 5.2. Thus we have completed the proof. □

## 6. Proofs of Theorem 1, Theorem 3 and Theorem 4

We can now prove Theorem 1, Theorem 3 and Theorem 4 making use of our results in the previous section.

### 6.1. **Proof of Theorem 1.**

We consider the geometric limit $M_\infty$ of $M_i$ and the model $\mathbf{M}$ of its non-cuspidal part as before. By the definition of algebraic simply degenerate ends of $\mathbf{M}$, they are mapped to simply degenerate ends of $(M_\infty)_0$ which lift to those of the algebraic limit $(M')_0$. Upper ends are mapped to those lifted to upper ends of $(M')_0$, and lower ones to those lifted to lower ends of $(M')_0$. Now, by Corollary 5.4, every component of $|\mu^+|$ that is not a simple closed curve is an ending lamination of an upper algebraic simply degenerate end. Therefore, it is an ending lamination of a upper simply degenerate end of $(M')_0$. The same argument works for $|\mu^-|$.

The second paragraph of the statement also follows immediately from Corollary 5.4.

### 6.2. **Proof of Theorem 3.**

Suppose, seeking a contradiction, that $\{qf(m_i, n_i)\}$ as in the statement of Theorem 3 converges after taking a subsequence. Then by Theorem 1, $\mu_0^+$ is an ending lamination of an upper end of $(M')_0$ and $\mu_0^-$ is that of a lower end of $(M')_0$. Let $\Sigma^+\Sigma^-$ be the minimal supporting surfaces of $\mu_0^+$ and $\mu_0^-$ respectively, which were assumed to share at least one boundary component $c$. Since $c$ lies on the boundary of both $\Sigma^-$ and $\Sigma^+$, it represents a $\mathbb{Z}$-cusp both above and below $\Psi(S)$. This is impossible since no two distinct cusps have homotopic core curves.

6.3. **Proof of Theorem 4.** As in the statement, let $\mu^-$ and $\mu^+$ be two measured laminations on $S$ whose supports share only simple closed curves $c_1, \ldots, c_r$. By our assumption, there is a component $\mu_0$ of either $\mu^+$ or $\mu^-$ whose minimal supporting surface $\Sigma(\mu_0)$ has some $c_j$ among $c_1, \ldots, c_r$ on its frontier. We can assume that $\mu_0$ is a component of $\mu^+$ since the other case can be deal with in the same way just turning everything upside down.

By Corollary 5.4 a curve $c_j$ shared by $\mu^-$ and $\mu^+$ is both upper and lower algebraic parabolic curve. On the other hand, by Corollary 5.4, there is an upper algebraic simply degenerate end of $\mathbf{M}$ in the form $\Sigma(\mu_0) \times \{t\}$ which has $|\mu_0|$ as the ending lamination. This implies that there is a boundary component of $\mathbf{M}$ which is an open annulus with core curve homotopic to $c_j$. By Lemma 4.4, this shows that $c_j$ cannot be a lower algebraic parabolic curve. This is a contradiction, and we have completed proof of Theorem 4.

## 7. Proof of Theorem 5

7.1. **Necessity.** We shall first show that the condition (1) is necessary. Suppose, on the contrary, that there is $c_j$ among $c_1, \ldots, c_r$ such that $\text{length}_{n_i}(c_j)$ goes to 0 whereas $\{(G_i, \phi_i) = qf(m_i, n_i)\}$ converges. (The argument for the case when $\text{length}_{m_i}(c_j)$ goes to 0 is quite the same.) Let $G_\infty$ be the geometric limit of a subsequence of $\{G_i\}$ as before, and set $M_i = \mathbb{H}^3/G_i$ and $M_\infty = \mathbb{H}^3/G_\infty$. Consider the model manifold $\mathbf{M}$ of $(M_\infty)_0$. By Corollary 5.4, $c_j$ is an upper algebraic parabolic curve. Let $g : S \to \mathbf{M}$ be a standard algebraic immersion. Since $\text{length}_{n_i}(c_j) \to 0$, the boundary blocks of $\mathbf{M}_i$ corresponding to the upper boundary are pinched along an annulus with core curve $c_j$ and are split in the geometric limit. Therefore, there is an open annulus component of $\partial\mathbf{M}$ whose core curve is homotopic to $g'(c_j)$. By Lemma 4.4, this shows $g'(S)$ cannot go around a torus boundary component whose longitude corresponds to $c_j$. In particular, $c_j$ cannot be a lower algebraic parabolic curve. This contradicts, by way of Corollary 5.4, the fact that $|\mu_-|$ also contains $c_j$. This completes the proof of the necessity of the condition (1).

Next we turn to show the necessity of the condition (2). By Corollary 5.4 again, we see that if $\{(G_i, \phi_i)\}$ converges algebraically, each of $c_1, \ldots, c_r$ must be both upper and lower algebraic parabolic curve. By Lemma 4.4, this is possible only when $g'(S)$ goes around a torus boundary component $T_j$ of $\mathbf{M}$ whose longitude is homotopic to $g'(c_j)$ for each $j = 1, \ldots, r$. Suppose that $g'(S)$ goes $a_j$ times around $T_j$ for $a_j \in \mathbb{Z}\setminus\{0\}$, where we define the counter-clockwise rotation to be the positive direction in Figure 1. As a convention we define the condition $a_j = 0$ means that $g'(S)$ passes below $T_j$. If $g'(S)$ passes above $T_j$ not going around it, we define $a_j$ to be $-1$.

Let $\mathbf{M}_i$ be a model manifold of $(M_i)_0$. Since $\mathbf{M}_i[0]$ converges to $\mathbf{M}[0]$ geometrically, there is a torus boundary $T_j(i)$ of $\mathbf{M}_i[0]$ which is mapped to $T_j$ by the approximate isometry $\rho_i^{\mathbf{M}}$. The torus $T_j(i)$ consists of two horizontal annuli and two vertical annuli. We choose an oriented meridian-longitude system of $T_j$ in such a way that the longitude $l_j$ lies on a horizontal annulus and the meridian $m_j$ is shortest among all the simple closed curves on $T_j$ intersecting the longitude at one point. We define the orientation of $l_j$ so that the positive direction of $l_j$ is induced by the right-hand twist around $c_j$. By pulling back this system using the approximate isometry $\rho_i^{\mathbf{M}}$ between $\mathbf{M}_i[0]$ and $\mathbf{M}[0]$, we get a longitude $l_j(i)$ and a meridian

$m_j(i)$ on $T_j(i)$. There is a Margulis tube $V_j(i)$ attached to $T_j(i)$ in $\mathbf{M}_i$. The compressing curve of $V_i(j)$ intersects the longitude $l_j(i)$ only at one point. Therefore we can express the homology class of the compressing curve as $k_i[l_j(i)] + [m_j(i)]$. Since $T_j(i)$ converges geometrically to a torus boundary component of $\mathbf{M}$, we have $|k_i| \to \infty$.

Fix some $j$ and consider $c_j$. In $\mathbf{M}$, there is a block $B^+$ intersecting $T_j$ by an annulus $A^+$ containing the upper horizontal annulus of $T_j$ in the middle. Similarly, there is a block $B^-$ intersecting $T_j$ by an annulus $A^-$ containing the lower horizontal annulus of $T_j$. One or both of these may be boundary blocks. We shall only consider the case when $a_j > 0$, i.e. the case when $g'(S)$ goes around $T_j$ counter-clockwise in Figure 1. Since $c_j$ is algebraic parabolic curve, the standard immersion passes through both $B^-$ and $B^+$.

Now, consider a simple closed curve $\gamma^-$ in $B^-$ which is homotopic to a core curve the lower horizontal annulus if $B^-$ is an internal block. When $B^-$ is a boundary block, we consider horizontal upper boundary components adjacent to $A^-$. If $c_j$ is separating, there are two such surfaces, whose union we denote by $\Delta^-$, and if $c_j$ is non-separating there is only one such surface, which we denote by $\Delta^-$. We take a simple closed curve $\gamma^-$ which lies in $A^- \cup \Delta^-$ and intersect the core curve of $A^-$ at two points when $c_j$ is separating and at one point when $c_j$ is non-separating. We define $\gamma^+$ in the same way. By pulling back $\gamma^+$ and $\gamma^-$ by $(\rho_i^{\mathbf{M}})^{-1}$, we get simple closed curves $\gamma^+(i)$ and $\gamma^-(i)$. Using the vertical projection to $S$ in $\mathbf{M}_i$, we regard $\gamma^+(i)$ and $\gamma^-(i)$ also as curves on $S$.

Let $g_i' : S \to \mathbf{M}_i$ be a pull-back of the standard immersion $g'$ obtained by composing $(\rho_i^{\mathbf{M}})^{-1}$. We consider to homotope $g_i'$ to unwrap it around $T_j(i)$ and make the surface pass under $T_j(i)$, by making it pass $a_j$ times through $V_j(i)$. Let $g_i''$ be a surface obtained by modifying the part of $g_i'$ going around $T_j(i)$ to a horizontal surface and giving a natural marking coming from the structure of $S \times I$, which is equal to a marking determined by a pull-back of a horizontal surface in $\mathbf{M}$ obtained by removing the parts of $g'$ going around torus boundaries. Note that $g_i''$ and $g_i'$ are not homotopic as maps because of the difference of markings. (This means that $g_i''$ is not homotopic to $\Phi_i^{\mathbf{M}}$.) Each time $g_i'(S)$ passes through $V_j(i)$, a curve $\delta$ on $S$ intersecting $c_j$ is twisted $k_i$-times around $c_j$. (It goes $|k_i|$-times around $c_i$ in the same direction as $l_i$ if $k_i > 0$ and in the opposite direction if $k_i < 0$.) Since $\gamma^-(i)$ is homotopic to $g_i''(\gamma^-)$, we see that $g_i'(\tau_{c_j}^{k_i a_j}(\gamma^-))$ is homotopic to $\gamma^-(i)$. Similarly, since $\gamma^+(i)$ is homotopic to $g_i''(\tau^{k_i}(\gamma^+))$, we see that $g_i'(\tau_{c_j}^{k_i(a_j+1)}(\gamma^+))$ is homotopic to $\gamma^+(i)$.

Let $h_i$ be a hierarchy of tight geodesics for $\mathbf{M}_i$ as before. Since $V_j(i)$ appears as a Margulis tube in $\mathbf{M}_i$, we see that $c_j$ supports a geodesic $\gamma_i(c_j)$ in $h_i$. By the definition of gluing blocks in Minsky [29], we see that $\pi_{c_j}(\gamma^-(i))$ is the initial marking and $\pi_{c_j}(\gamma^+(i))$ is the terminal marking of $\gamma_i(c_j)$ if both $B^-$ and $B^+$ are interior blocks. By §6 of Masur-Minsky [27], this implies that $\pi_{c_j}(I(h_i))$ is in a uniformly bounded distance from $\pi_{c_j}(\gamma^-(i))$ and $\pi_{c_j}(T(h_i))$ is in a uniformly bounded distance from $\pi_{c_j}(\gamma^+(i))$. Even when $B^-$ or $B^+$ is a boundary block, we have the same property: for, since the length of $\gamma^-(i)$ or $\gamma^+(i)$ is bounded, its projection to $A(c_j)$ is within bounded distance from those of $I(h_i)$ or $T(h_i)$. Since $g_i'$ is homotopic to $\Phi_i$ for large $i$, we see that $\pi_{c_j}(I(h_i))$ is within a uniformly bounded distance from $\pi_{c_j}(\tau_{c_j}^{k_i a_j}(\gamma^-))$, whereas $\pi_{c_j}(T(h_i))$ is within a bounded distance from

$\pi_{c_j}(\tau_{c_j}^{k_i(a_j+1)}(\gamma^+))$. This shows that if we consider the pushed-forward metrics $(\tau_{c_j}^{-k_i a_j})_* m_i$ and $(\tau_{c_j}^{-k_i(a_j+1)})_* n_i$ instead of $m_i$ and $n_i$, then $\pi_{c_j}(\gamma^+)$ is within a bounded distance from the projection of the terminal marking and $\pi_{c_j}(\gamma^-)$ is within a bounded distance from that of the initial marking. By Lemma 5.3, this implies that the limits of $(\tau_{c_j}^{-k_i a_j})_* m_i$ and $(\tau_{c_j}^{-k_i(a_j+1)})_* n_i$ in the Thurston compactification of $\mathcal{T}(S)$ do not contain $c_j$ as a leaf. We repeat the same argument for every $c_j$, and let $p_i^j$ and $q_i^j$ be $-k_i a_j$ and $-k_i(a_j+1)$ respectively. This completes the proof of the necessity.

It is clear that if $a_j \leq 0$, then $c_j$ is an upper parabolic curve since $g'$ goes around the torus containing $c_j$ in the counter-clockwise and $f(c_j)$ is lifted to a curve lying above the core surface obtained the lift of $f \circ g'$ in $M'$. This shows that $c_j$ is a core curve of an upper parabolic locus if $a_j \geq 0$. We can argue in the same way also when $a_j < 0$.

7.2. **Existence.** We shall next show that the existence of limits of quasi-Fuchsian groups satisfying the conditions (1) and (2). Our construction just follows the argument of Anderson-Canary [2].

We first construct a geometrically finite Kleinian group $\Gamma_0$ such that $N = \mathbb{H}^3/\Gamma_0$ is homeomorphic to the complement of $c_j \times \{1/2\}$ $(j = 1, \ldots, r)$ in $S \times [0,1]$, and the conformal structures corresponding to the ends $S \times \{0\}$ and $S \times \{1\}$ are the same point $m_0 \in \mathcal{T}(S)$. (Here we identify $S$ with $S \times \{0\}$ and $S \times \{1\}$ by the natural inclusions.) We consider an immersion $g_0 : S \to N_0$ which is in the standard form in the sense of Lemma 4.4, and wraps $a_j$ times around each $c_j \times \{1/2\}$ counted counter-clockwise as in Figure 1.

Next, we consider a quasi-conformal deformation of $\Gamma_0$. Let $\mu_1^-, \ldots, \mu_s^+$ and $\mu_1^+, \ldots \mu_t^+$ be the components of $\mu^-$ and $\mu^+$ that are not the shared simple closed curves. We define a Kleinian group $\Gamma_k$ to be one obtained by deforming the conformal structures $m_0$ on $S \times \{0\}$ by the earthquake with respect to $k(\cup_{j=1}^s \mu_j^-)$ and $m_0$ on $S \times \{1\}$ by the earthquake with respect to $k(\cup_{j=1}^t \mu_j^+)$. Let $N_k$ be $\mathbb{H}^3/\Gamma_k$ and $h_k : N \to N_k$ a natural homeomorphism derived from the quasi-conformal deformation. We regard $N_k$ also as embedded in $S \times [0,1]$ in such a way that the image of drilled out curves lie on $S \times \{1/2\}$. Then we get an immersion $g_k : S \to N_k$ which is defined to be the composition $h_k \circ g_0$.

In the manifold $N$, every essential annulus either intersects a torus cusp or $\mu^+ \cup \mu^-$, or has boundary contained in a component of $S \times \{0\} \setminus \mu^-, S \times \{1\} \setminus \mu^+$, where the conformal structure is not deformed, by the conditions (1*), (2*), and (3*). Therefore, by the main theorem of [31], we see that $N_k$ with marking determined by $h_k$ converges algebraically to a hyperbolic 3-manifold $N_\infty = \mathbb{H}^3/\Gamma_\infty$ with a homeomorphism $h_\infty : N \to N_\infty$. The laminations $\mu_1^-, \ldots, \mu_s^-; \mu_1^+, \ldots, \mu_t^+$ represent ending laminations of simply degenerate ends. The convergence is strong, by the covering theorem of Thurston and Canary ([38], [14]) combined with Abikoff's Lemma 3 in [1]. Let $g_\infty : S \to N_\infty$ be an immersion which is defined to be $h_\infty \circ g_0$.

Let $l_j$ and $m_j$ be a longitude a tubular neighbourhood of $c_j \times \{1/2\}$ lying on a level surface along $c_j \times \{1/2\}$ and any meridian intersecting $l_j$ at one point. Let $m_j(k)$ and $l_j(k)$ be a meridian and longitude in $N_k$ obtained by pulling back $m_j$ and $l_j$ using approximate isometries. We orient them so that $(l_j, m_j)$ determines an orientation of a torus around $c_j \times \{1/2\}$ whose normal vectors point to the inside of $N_\infty$. Now, we consider a hyperbolic Dehn filling of $N_k$ such that the compressing disc is attached along a curve represented by $-k[l_j(k)]+[m_j(k)]$. Since $N_k$ converges geometrically to $N_\infty$, we see, by passing to a subsequence, that the filling corresponding to $-k[l_j(k)] + [m_j(k)]$ is hyperbolic. We define $M_k$ to be thus obtained geometrically finite hyperbolic 3-manifold, which is homeomorphic to $S \times (0,1)$. We let $G_k$ be the corresponding quasi-Fuchsian group and $\phi_k : \pi_1(S) \to \pi_1(M_k)$ an isomorphism derived from the pull-back of $g_k$ by an approximate isometry between $N_k$ and $M_k$. By the same argument as in [2], we see that the conformal structure at infinity of $M_k$ on the end corresponding to $S \times \{0\}$, denoted by $m_k$, is obtained by performing the $ka_j$-time (right-hand) Dehn twist around the $c_j$ and the earthquake along $k(\mu_1^- \cup \cdots \cup \mu_s^-)$ from $m_0$ and that on the end corresponding to $S \times \{1\}$, denoted by $n_k$, by performing the $k(a_j + 1)$-Dehn twist around the $c_j$ and the earthquake along $k(\mu_1^+ \cup \cdots \cup \mu_t^+)$. This shows that the limits in the Thurston compactification of the conformal structures $m_k$ and $n_k$ are $[\mu^-]$ and $[\mu^+]$ respectively and that those of $(\tau_{c_1}^{-ka_1} \circ \cdots \circ \tau_{c_r}^{-ka_r})_*(m_k)$ and $(\tau_{c_1}^{-k(a_1+1)} \circ \cdots \circ \tau_{c_r}^{-k(a_r+1)})_*(n_k)$ are $[\mu_1^- \cup \cdots \cup \mu_s^-]$ and $[\mu_1^+ \cup \cdots \cup \mu_t^+]$ respectively. In the case when one of these latter two projective laminations is empty, the corresponding conformal structure stays in a compact set of the Teichmüller space.

By the diagonal argument, we see that $(G_k, \phi_k)$ converges algebraically to a subgroup of $\Gamma_\infty$ corresponding to the covering of $N_\infty$ associated to $(g_\infty)_*\pi_1(S)$. Thus we have obtained a sequence of quasi-Fuchsian groups $(G_k, \phi_k)$ as we wanted.

Here we have constructed an example such that if either $\mu^-$ or $\mu^+$ consists only of $c_1, \ldots, c_r$, then $(\tau_{c_1}^{-ka_1} \circ \cdots \circ \tau_{c_r}^{-ka_r})_*(m_k)$ or $(\tau_{c_1}^{-k(a_1+1)} \circ \cdots \circ \tau_{c_r}^{-k(a_r+1)})_*(n_k)$ stays in a compact set of the Teichmüller space. We can make it converge to a projective lamination $\nu^-$ or $\nu^+$ not containing $c_1, \ldots, c_r$ as leaves, by composing the earthquake along $\sqrt{k}\nu^-$ or $\sqrt{k}\nu^+$.

## 8. Non-existence of exotic convergence

We shall prove Theorem 6.

Let $\Gamma$ be a b-group as in the statement and $\psi : \pi_1(S) \to \Gamma$ an isomorphism giving the marking. Let $\{(G_i, \phi_i)\}$ be a sequence of quasi-Fuchsian groups converging to $(\Gamma, \psi)$. What we need to show is that the conformal structures at infinity of the bottom ideal boundaries of the $M_i = \mathbb{H}^3/G_i$ are bounded in the Teichmüller space then.

Let $M_\infty$ be a geometric limit (a subsequence) of $M_i$ with basepoint coming from a fixed basepoint in $\mathbb{H}^3$ as usual. Let $\mathbf{M}$ be a model manifold of $M_\infty$ with a model map $f_\infty : \mathbf{M} \to (M_\infty)_0$. Let $g' : S \to \mathbf{M}_\infty$ be a standard algebraic immersion. By our assumption, there is no isolated algebraic parabolic loci in $\mathbf{M}$. This implies by Lemma **??** that there is no torus boundary around which $g'$ can go. If there is an algebraic simply degenerate end below $g'(S)$, it is mapped to a simply degenerate end of $(M_\infty)_0$ which is lifted to a lower simply degenerate end of $(\mathbb{H}^3/\Gamma)_0$. This is a contradiction since $\Gamma$ is a b-group. Similarly, there is no lower algebraic parabolic locus. Since $g'$ does not go around a torus boundary component, an end closest to $g'(S)$ among those below $g'(S)$ must be algebraic. These imply that the only possible end below $g'(S)$ is a geometrically finite end corresponding to the entire $S \times \{0\}$.

The diameter of the manifold cobounded by $f_\infty \circ g'(S)$ and the lower boundary of the convex core in $(M_\infty)_0$ is finite since the manifold cobounded by $g'(S)$ and $S \times \{0\}$ in $\mathbf{M}$ has finite diameter. This cobordism in $(M_\infty)_0$ can be pulled back to

$(M_i)_0$ for large $i$. It follows that the lower convex core of $M_i$ converges geometrically to that of $M_\infty$, and hence the marked hyperbolic structure on the lower boundary component of the convex core of $M_i$ converges to that of $M_\infty$. This shows that the lower conformal structure at infinity of $M_i$ is bounded in the Teichmüller space as $i \to \infty$ by Sullivan's theorem (see Epstein-Marden [17]).

## 9. Self-bumping

In this section, we shall prove Theorem 7 and Corollary 8. For that, we shall show that for $\{qf(m_i, n_i)\}$ as in the statement of Theorem 7, there is a continuous deformation to a strong convergent sequence whose algebraic limit is a quasi-conformal deformation of $\Gamma$. This will be done by using a model manifold of $qf(m_i, n_i)$ with a special property, which we shall construct in Lemma 9.2. Let us state the existence of a deformation as a proposition.

**Proposition 9.1.** *We consider quasi-Fuchsian groups $qf(m_i, n_i)$ and their algebraic limit $(\Gamma, \psi)$ as in Theorem 7. Let $c_1, \dots, c_s$ be core curves of the upper parabolic loci and $c'_1, \dots, c'_t$ be those of the lower parabolic loci on $S$ of $(\Gamma, \psi)$. Then, passing to a subsequence of $\{(m_i, n_i)\}$, there is an arc $\alpha_i : [0,1] \to QF(S)$ with the following properties. Let $(\bar{m}_i, \bar{n}_i)$ denote a point in $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ such that $qf(\bar{m}_i, \bar{n}_i) = \alpha_i(1)$.*

*(1) $\alpha_i(0) = qf(m_i, n_i)$.*
*(2) $\{\alpha_i(1) = qf(\bar{m}_i, \bar{n}_i)\}$ converges strongly to a quasi-conformal deformation $(\Gamma', \psi')$ of $(\Gamma, \psi)$ as $i \to \infty$.*
*(3) The length of each of $c_1, \dots, c_s$ with respect to $\bar{n}_i$ goes to $0$ and the length of each of $c'_1, \dots, c'_t$ with respect to $\bar{m}_i$ also goes to $0$ as $i \to \infty$.*
*(4) In the case when $\Gamma$ is a b-group, the lower conformal structure at infinity of $\alpha_i(t)$ is constant with respect to $t$, for every $i$.*
*(5) For any neighbourhood $U$ in $AH(S)$ of the quasi-conformal deformation space $QH(\Gamma, \psi)$, there exists $i_0$ such that for $i > i_0$, the arc $\alpha_i$ is contained in $U$.*

9.1. **Proof of Proposition 9.1.** Set $(G_i, \phi_i) = qf(m_i, n_i)$ and $M_i = \mathbb{H}^3/G_i$. We consider the geometric limit $M_\infty$ of $M_i$ with basepoints coming from some fixed basepoint in $\mathbb{H}^3$. Let $\mathbf{M}_i$ be a bi-Lipschitz model of $(M_i)_0$ and $\mathbf{M}$ that of $(M_\infty)_0$ as before. Let $g' : S \to \mathbf{M}$ be a standard algebraic immersion as in Lemma 4.4. Let $E^1, \dots, E^p$ be the algebraic simply degenerate ends of $\mathbf{M}$. We shall consider to deform $\{(G_i, \phi_i)\}$ to make all of these ends lie on $S \times \{0\} \cup S \times \{1\}$ in the new geometric limit.

We renumber $E^1, \dots, E^p$ so that $E^1, \dots, E^q$ are upper ends whereas $E^{q+1}, \dots, E^p$ are lower. (It may be possible that $p = 0$ or $q = p$.) We let $\Sigma^j$ be a subsurface of $S$ such that $E^j$ is contained in a brick of the form $\Sigma^j \times J^j$. We shall show that we can modify model manifolds $\mathbf{M}_i$ and $\mathbf{M}$ so that there are two horizontal surfaces in a new model manifold, one containing all the upper ends $E^1, \dots, E^q$ and the other containing all the lower ends $E^{q+1}, \dots, E^p$.

**Lemma 9.2.** *There are uniform bi-Lipschitz model manifolds $\mathbf{M}'_i$ for $(M_i)_0$ and $\mathbf{M}'$ for $(M_\infty)_0$, both of which are embedded in $S \times [0,1]$ preserving horizontal and vertical foliations, and have the following properties.*

*(1) $\mathbf{M}'_i[k]$ converges geometrically to $\mathbf{M}'[k]$ for every $k \in \{0, 1, \dots, \}$.*

(2) *There is a homeomorphism from* $\mathbf{M}$ *to* $\mathbf{M}'$ *taking algebraic loci of* $\mathbf{M}$ *to those of* $\mathbf{M}'$.
(3) *We use the same symbol* $E^j$ $(j = 1, \ldots, p)$ *to denote the end of* $\mathbf{M}'$ *corresponding to* $E^j$ *of* $\mathbf{M}$. *Then,* $E^j$ *is contained in a brick* $B^j = \Sigma^j \times J^j$ *of* $\mathbf{M}'$ *such that* $\inf B^1 = \cdots = \inf B^q, \sup B^1 = \cdots = \sup B^q, \sup B^{q+1} = \cdots = \sup B^p$, *and* $\inf B^{q+1} = \cdots = \inf B^p$, *with* $\inf B^1 > \sup B^p$ *unless* $q = 0$ *or* $p = q$.
(4) *An algebraic locus in* $\mathbf{M}'$ *can be homotoped to a horizontal surface lying between* $\sup B^p$ *and* $\inf B^1$.

*Proof.* For each component $c$ of $\mathrm{Fr}\Sigma^j$ for $\Sigma_j$ among $\Sigma^1, \ldots, \Sigma^q$, we consider a solid torus $V(c)$ in $S \times (0,1)$ which has the form of $A(c) \times J_c$, where $A(c)$ is an annulus with core curve $c$ and $J_c$ is a closed interval in $[0,1]$ such that $\inf J_c = \inf B^j$ and $\sup J_c < \sup B^j$. Let $\bar{\mathcal{T}}^+$ be the union of all $V(c)$ for all frontier components of $\Sigma^1, \ldots, \Sigma^q$. Even if a curve $c$ is homotopic to frontier components of two distinct $\Sigma^j$ and $\Sigma^{j'}$, we take only one solid torus. In such a case, we take $J_c$ so that $\sup J_c < \min\{\sup B^j, \sup B^{j'}\}$. Similarly, we take $\bar{\mathcal{T}}^-$ for $\Sigma^{q+1}, \ldots, \Sigma^p$.

Now for each $V(c) = A(c) \times J_c$ in $\bar{\mathcal{T}}^+$, we let $V'(c)$ be $A(c) \times [5/8, 3/4]$ in $S \times [0,1]$, and denote the union of such solid tori by $\mathcal{T}^+$. Similarly, for each $V(c)$ in $\bar{\mathcal{T}}^-$, we let $V'(c)$ be $A(c) \times [1/4, 3/8]$, and denote the union of such solid tori by $\mathcal{T}^-$. Then, we see that there is a homeomorphism from $S \times (0,1) \setminus (\bar{\mathcal{T}}^- \cup \bar{\mathcal{T}}^+)$, in which $\mathbf{M}$ is embedded, to $S \times (0,1) \setminus (\mathcal{T}^- \cup \mathcal{T}^+)$ taking a standard algebraic immersion $g'$ to $S \times \{1/2\}$ since $\mathbf{M}$ has no torus boundary around which $g'$ can go (Lemma **??**) and $\bar{\mathcal{T}}^-$ lies in the lower component of $S \times [0,1] \setminus g'(S)$ whereas $\bar{\mathcal{T}}^+$ lies in the upper component.

We let a new brick manifold $\bar{\mathbf{M}}$ be the one obtained by the standard brick decomposition of $S \times [1/8, 7/8] \setminus (\mathcal{T}^- \cup \mathcal{T}^+)$. Then $\bar{\mathbf{M}}$ consists of bricks lying on five levels: the top one $S \times [3/4, 7/8]$, those touching $\bar{\mathcal{T}}^+$ along vertical boundaries, the middle one $S \times [3/8, 5/8]$, those touching $\bar{\mathcal{T}}^-$ along vertical boundaries, and the bottom one $S \times [1/8, 1/4]$. For each $i$, we define a labelled brick manifold $\check{\mathbf{M}}_i'$ to be the one obtained by attaching two geometrically finite bricks (i.e. boundary blocks) $S \times (0, 1/8]$ and $S \times [7/8, 1)$, one on the top and the other on the bottom to $\bar{\mathbf{M}}$. We give conformal structures $m_i$ on the bottom and $n_i$ to the top.

Corresponding to $\check{\mathbf{M}}_i'$, we shall construct a geometrically finite hyperbolic manifolds from $M_i$ using the drilling theorem of Bromberg [11] and Brock-Bromberg [8]. Recall that by $f_i \circ (\rho_i^{\mathbf{M}})^{-1}$, each boundary component of $\mathcal{T}^+$ and $\mathcal{T}^-$ is mapped to the boundary of a Margulis tube in $M_i$. We denote the Margulis tubes in $M_i$ corresponding to $\mathcal{T}^+$ by $V_1^i, \ldots, V_s^i$ and those corresponding to $\mathcal{T}^-$ by ${V_1^i}', \ldots, {V_t^i}'$.whose core geodesics $c_1^i, \ldots, c_s^i; {c_1^i}', \ldots, {c_t^i}'$ correspond to $\Phi_i(c_1), \ldots, \Phi_i(c_s); \Phi_i(c_1') \ldots, \Phi_i(c_t')$. Note that the length of each of $c_1^i, \ldots, c_s^i; {c_1^i}', \ldots, {c_t^i}'$ goes to 0 as $i \to \infty$. Moreover, for sufficiently large $i$, the map $f_i \circ (\rho_i^{\mathbf{M}})^{-1}$ induces a homeomorphism from $S \times (0,1) \setminus (\bar{\mathcal{T}}^+ \cup \bar{\mathcal{T}}^-)$ to $M_i$. The complement of these curves $\check{M}_i = M_i \setminus (\cup_{j=1}^s c_j^i \cup \cup_{j=1}^t {c_j^i}')$ admits a geometrically finite hyperbolic structure with conformal structures $m_i$ at the bottom and $n_i$ at the top, by Thurston's uniformisation theorem. Therefore, for sufficiently large $i$, we can apply the drilling theorem to see that for some $K$ independent of $i$, there is a $K$-bi-Lipschitz homeomorphism $\check{f}_i$ between $M_i \setminus (\cup_{j=1}^s V_j^i \cup \cup_{j=1}^t {V_j^i}')$ and $(\check{M}_i)_0$.

By the general theory which we developed in §3.4 of [35], we can decompose $\check{\mathbf{M}}'_i$ into blocks, which define a metric on $\check{\mathbf{M}}'_i$. Theorem 4.1 in the same paper shows that we have a $L$-bi-Lipshitz model map $f'_i : \check{\mathbf{M}}'_i \to (\check{M}_i)_0$, with $L$ independent of $i$. Then, by composing $(\check{f}_i)^{-1}$ with $f'_i$, we get a $K'$-bi-Lipschitz embedding of $\check{\mathbf{M}}'_i$ into $M_i$, with $K'$ depending only on $K$ and $L$. By filling appropriate Margulis tubes into $\check{\mathbf{M}}'_i$, we get a $K''$-bi-Lipschitz model manifold $\mathbf{M}'_i$ for $(M_i)_0$, with $K''$ independent of $i$, for sufficiently large $i$.

It remains to verify that this $\mathbf{M}'_i$ has the desired properties. Since $\mathbf{M}'_i$ has decomposition into blocks, and is a uniform bi-Lipschitz model for $(M_i)_0$, by the argument of §5 of [35], there is a labelled brick manifold $\mathbf{M}'$ which is a bi-Lipschitz model manifold of $(M_\infty)_0$ such that $\mathbf{M}'_i[k]$ converges to $\mathbf{M}'[k]$ geometrically. This shows the conditions (1) and (2). Since the geometric convergence of $\mathbf{M}'_i[k]$ to $\mathbf{M}'[k]$ preserves horizontal foliations, the bottom annuli of all the boundary components of $\mathbf{M}'$ corresponding to $\bar{\mathcal{T}}^+$ lie on the same horizontal surface $S \times \{t_1\}$, the top annuli of all the components corresponding to $\bar{\mathcal{T}}^-$ lie on the same horizontal surface $S\times\{t_2\}$, and $t_1 > t_2$, with respect to the embedding of $\mathbf{M}'$ preserving the horizontal and vertical foliations. Therefore, we have $\inf B^1 = \cdots = \inf B^q, \sup B^{q+1} = \cdots = \sup B^p$, and we can modify the embedding by making the height of these bricks sufficiently small to make the condition (3) $\sup B^1 = \cdots = \sup B^q, \inf B^{q+1} = \cdots = \inf B^p$ hold. Finally, we verify the condition (4). Since $E^1, \ldots, E^q$ are upper ends and $E^{q+1}, \ldots, E^p$ are lower ends, an algebraic locus must pass under $B^1, \ldots, B^q$ and above $B^{q+1}, \ldots, B^q$. By our assumption, there are no other algebraic ends, neither torus boundary components containing algebraic parabolic curves. Therefore, there is no obstruction to homotope an algebraic locus to a horizontal surface above $B^{q+1}, \ldots, B^p$ and below $B^1, \ldots, B^q$. □

We shall use the symbol $g'$ to denote the horizontal algebraic locus in $\mathbf{M}'$ as in (4) of the above lemma. (This is the same symbol as the standard immersion in $\mathbf{M}$, but there is no fear of confusion since we can distinguish them by model manifolds in which they are lying.)

Take $s, t, s'$ and $t'$ such that $B^j = \Sigma^j \times [s,t)$ for $j = 1, \ldots, q$ and $B^j = \Sigma^j \times (s', t']$ for $j = q+1, \ldots, p$. Let $\rho_i^{\mathbf{M}'}$ denote an approximate isometry between $\mathbf{M}'_i$ and $\mathbf{M}'$ which is associated to the geometric convergence of $\mathbf{M}'_i[k]$ to $\mathbf{M}'[k]$. We denote by $x'_i$ and $x'_\infty$ basepoints in the thick parts of $\mathbf{M}'_i$ and $\mathbf{M}'$, which we used for the convergence. By our construction of $\mathbf{M}'_i$, there are bricks $B_i^j \cong \Sigma^j \times [s, u]$ of $\check{\mathbf{M}}_i$ which contains $(\rho_i^{\mathbf{M}'})^{-1}(B^j \cap B_{K_i r_i}(\mathbf{M}', x_\infty))$.

Now, we shall construct two sequences of markings on $S$ starting from $c_{m_i}$ and $c_{n_i}$ respectively, and a sequence of markings on $\Sigma^j$ for $j = 1, \ldots, p$, in both of which markings advance by elementary moves. Recall that $\check{\mathbf{M}}'_i$ consists of interior bricks lying on five levels and two geometrically finite bricks. Since $\check{\mathbf{M}}'_i$ is a labelled brick manifold, there are tube unions in $\check{\mathbf{M}}'_i$ which decompose it into blocks. As was shown in §3.4 in [35], a front of every brick intersects tubes so that its complement is a disjoint union of thrice-punctured spheres.

We first consider the bottom interior brick corresponding to $S \times [1/8, 1/4]$. We denote this brick by $\bar{B}_i$. Lemma 4.4 in [35] shows that we have a 4-complete hierarchy $\check{h}(\bar{B}_i)$ on $S$ corresponding to tube unions in $\bar{B}_i$, whose initial marking is $c_{m_i}$. In each annular component domain in $\check{h}(\bar{B}_i)$ except for those corresponding to tubes intersecting the upper front of $\bar{B}_i$, we can put a tight geodesic, which is uniquely determined. Such a geodesic has length which determines the $\omega_{\mathbf{M}_i}$ of the Margulis tube which was filled in the corresponding torus boundary of $\check{\mathbf{M}}'_i[0]$. For each annular component domain corresponding to a tube intersecting the upper front, we put a geodesic of length 0 at this stage, for the terminal markings for such geodesics are not determined if we look only at $\bar{B}_i$. We denote by $h(\bar{B}_i)$ the hierarchy obtained by adding such annular geodesics to $\check{h}(\bar{B}_i)$. A resolution $\tau(\bar{B}_i) = \{\tau(\bar{B}_i)_k\}$ of $h(\bar{B}_i)$ gives rise to a sequence of split level surfaces starting from one lying on the lower front and ending at one lying on the upper front. On the other hand, forward steps in the resolution $\tau(\bar{B}_i)$ corresponds to elementary moves of markings which can be assumed to be clean. (See Minsky [29].) Next we consider the top interior brick, which corresponds to $S \times [3/4, 7/8]$, and we denote by $\hat{B}_i$. Similarly to the case of $\bar{B}_i$, we have a hierarchy $h(\hat{B}_i)$. Reversing the order of slices of this hierarchy, we get a resolution, which we denote by $\tau(\hat{B}_i) = \{\tau(\hat{B}_i)_k\}$, giving rise to split level surfaces starting from one on the upper front and ending at one on the lower front, and a sequence of clean markings on $S$ advancing by elementary moves. Since $\mathbf{M}'_i$ is obtained from $\check{\mathbf{M}}'_i$ by filling Margulis tubes, the split level surfaces as above can be also regarded as lying in $\mathbf{M}'_i$.

Next we consider a brick $B_i^j$ for $j = q+1, \ldots, p$. Again, we have a complete hierarchy $h(B_i^j)$ supported on $\Sigma^j$ corresponding to the decomposition of $B_i^j$ into blocks and filled-in Margulis tubes. We can take a resolution $\tau(B_i^j) = \{\tau(B_i^j)_k\}$ starting from the restriction of the last slice of $\tau(\bar{B}_i)$ to $\Sigma^j$. Similarly, for each brick $B_i^j$ for $j = 1, \ldots, p$, we have a complete hierarchy $h(B_i^j)$. We reverse the order of slices in this case, and consider a resolution $\tau(B_i^j) = \{\tau(B_i^j)_k\}$ starting from the restriction of the last slice of $\tau(\hat{B}_i)$. The slices $\tau(\bar{B}_i)$ and $\tau(\hat{B}_i)$ have pairs $(g, v)$ with annular geodesics $g$. We can ignore such pairs since they do not affect our construction below.

For each $n \in \mathbb{N}$, we consider the $n$-th slice $\tau(B_i^j)_{-n}$ counting from the last one of each $\tau(B_i^j)$ for $j = q+1, \ldots, p$. Corresponding to the slice $\tau(B_i^j)_{-n}$, we have a split level surface $S(B_i^j)(n)$ embedded in $B_i^j$. Taking the union of all $S(B_i^j)(n)$ and the thrice-punctured spheres lying on $\partial_+ \bar{B}_i \setminus (\cup_{j=p+1}^q B_i^j)$, we get a split level surface $S_i^-(n)$. We construct an *extended level surface* $\hat{S}_i^-(n)$ as follows. Let $\mathbf{T}_-$ union of Margulis tubes intersecting $S_i^-(n)$. Each torus $T$ in $\partial \mathbf{T}_-$ is split into two annuli by cutting along $T \cap S_i^-(n)$, which we call upper annulus and lower annulus depending on their locations. We paste the upper one to $S_i^-(n)$ for each $T$ and get a surface homeomorphic to $S$, which we define to be $\hat{S}_i^-(n)$. We also regard a union of slices, taken one from each $\tau(B_i^j)$ $(j = q+1, \ldots, p)$, as a marking on $S$ by defining its restriction to $S \setminus \cup_{j=q+1}^p \Sigma^j$ to be the marking defined by the last slice of $\tau(\bar{B}_i)$, and setting a transversal of a component $c$ of $\mathrm{Fr}\Sigma^j$ to be the one determined by the first vertex of a geodesic supported on $c$ which is uniquely determined by the Margulis tube of $\mathbf{M}'_i$ corresponding to $c$. We can consider the same construction for $B_i^j$ with $j = 1, \ldots, q$, and get a split level surface $S_i^+(n)$ and an extended split level surface $\hat{S}_i^+(n)$, this time using lower annuli. Also, we can regard a union of slices taken one from each $\tau(B_i^j)$ $(j = 1, \ldots, q)$ as a marking on $S$ in the same way, by setting a transversal of each component of $\mathrm{Fr}\Sigma_j$ to be the last vertex this time.

Recall that in the construction of model manifolds, Minsky defined the initial and terminal markings as the shortest markings for the conformal structures at infinity. We consider a correspondence in the opposite direction. We can choose some positive constant $\delta_0$ such that for any clean marking, there is a marked conformal structure on $S$ in which all the curves (both base curves and transversals) of the marking have hyperbolic lengths greater than $\delta_0$. (The existence of such constant is easy to see since there are only finitely many configurations of clean markings up to homeomorphisms of $S$.) For each marking $\mu$, we define the marked conformal structure $m(\mu)$ to be one for which a clean marking $\mu'$ compatible with $\mu$ is a shortest marking and such that every curve of $\mu'$ has hyperbolic length greater than $\delta_0$. Although there are more than one such structures, we just choose any one. Then there is a universal constant $K$ depending only on $\delta_0$ (and $S$) such that if $\mu'$ is obtained from $\mu$ by one step in the resolutions $\tau(\bar{B}_i)$ or $\tau(\hat{B}_i)$ or $\tau(B_i^j)$, which corresponds to an elementary move of markings on $S$ or $\Sigma_j$, then the Teichmüller distance between $m(\mu)$ and $m(\mu')$ is bounded by $K$, whatever choices of $m(\mu)$ and $m(\mu')$ we make.

Now, we shall consider two sequences of markings. The first one is obtained by combining a sequence of markings corresponding to $\tau(\bar{B}_i)$ with those corresponding to the $\tau(B_i^j)\,(j = q+1,\ldots,p)$. By our definition of markings corresponding to slices in $\tau(B_i^j)$, if we choose first slice from every $\tau(B_i^j)$, the corresponding marking coincides with the one corresponding to the last slice of $\tau(\bar{B}_i)$, where vertices on annular geodesics are regarded as the first ones. Therefore after the sequence of markings for $\tau(\bar{B}_i)$, we can append the one obtained by advancing slices in the $\tau(B_i^j)\,(j = q+1,\ldots,p)$, so that we proceed at each step by advancing a slice in $\tau(B_i^j)$ which is farthest from the goal, i.e. the $n$-th slice counted from the last one, up to the point where all the slices are the $n$-th counting from the last one. We denote thus obtained sequence of markings by $\mu^-(i,n) = \{\mu_k^-(i,n)\}$. In the same way, we define a sequence of markings obtained by combining one corresponding to $\tau(\hat{B}_i)$ with those corresponding to $\tau(B_i^j)\,(j = 1,\ldots,q)$. We denote this sequence by $\mu^+(i,n) = \{\mu_l^+(i,n)\}$. To simplify the notation, we denote the last markings in $\mu^-(i,n)$ and $\mu^+(i,n)$ by $\mu_\infty^-(i,n)$ and $\mu_\infty^+(i,n)$ respectively.

For these sequences, we can construct arcs in the Teichmüller space as follows. Starting from $\mu_1^-(i,n)$, which is a shortest marking for $m_i$, we consider for each step $\mu_k^-(i,n) \to \mu_{k+1}^-(i,n)$ in $\mu^-(i,n)$, a Teichmüller geodesic arc connecting $m(\mu_k^-(i,n))$ with $m(\mu_{k+1}^-(i,n))$, then construct a broken geodesic arc $\alpha^-(i,n)$ connecting $m(\mu_1^-(i,n))$ to $m(\mu_\infty^-(i,n))$ by joining them. Both $m(\mu_1^-(i,n))$ and $m_i$ have $\mu_1^-(i,n)$ as a shortest marking, but the hyperbolic lengths of base curves of $\mu_1^-(i,n)$ may be different in $m_i$ and in $m(\mu_1^-(i,n))$. We can connect $m_i$ with $m(\mu_1^-(i,n))$ by a Teichmüller quasi-geodesic so that base$(\mu_1^-(i,n))$ remains a shortest pants decomposition throughout the points on the geodesic. (The quasi-geodesic constant depends only on $S$.) We define $\bar{\alpha}^-(i,n)$ to be a broken quasi-geodesic arc obtained by joining this quasi-geodesic with $\alpha^-(i,n)$. In the same way, we construct a broken quasi-geodesic arc $\bar{\alpha}^+(i,n)$ connecting $n_i$ with $m(\mu_\infty^+(i,n))$. In the case when either lower or upper algebraic simply degenerate ends do not exist, we define the corresponding $\bar{\alpha}^-(i,n)$ or $\bar{\alpha}^+(i,n)$ to be a constant map.

For any $n$, if we take a sufficiently large $i$, the component of $(\rho_i^{\mathbf{M}'})^{-1}(\mathbf{M}' \cap B_{K_i r_i}(\mathbf{M}', x_\infty))$ containing $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ contains all $\hat{S}_-(k)$ and $\hat{S}_+(k)$ with $k \leq$

$n$, since the distance between $B_i^j$ and $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ is uniformly bounded, and so are the diameters of extended level surfaces. Now, for each $i$, let $\mathbf{n}_i$ be the largest number such that both $S_-(\mathbf{k})$ and $S_+(\mathbf{k})$ with all $k \leq \mathbf{n}_i$ are contained in the component of $(\rho_i^{\mathbf{M}'})^{-1}(\mathbf{M}' \cap B_{K_i r_i}(\mathbf{M}', x_\infty))$ containing $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$ and are homotopic to $(\rho_i^{\mathbf{M}'})^{-1}(g'(S))$. By the above observation, we have $\mathbf{n}_i \to \infty$ as $i \to \infty$. Let $\alpha_i^\pm : [0, k_i^\pm] \to \mathcal{T}(S)$ be broken quasi-geodesic arcs $\bar{\alpha}^\pm(i, \mathbf{n}_i)$ connecting $m(\mu_\infty^\pm(i, \mathbf{n}_i))$ with $m_i, n_i$ which are constructed as above by joining Teichmüller geodesics and one quasi-geodesic, setting $n$ above to be $\mathbf{n}_i$. We parametrise $\alpha_i^\pm$ so that the integral points in $[0, k_i^\pm]$ correspond to the endpoints of the Teichmüller geodesics. We define $\beta_i : [0, \bar{k}_i] \to QF(S)$ by setting $\beta_i(t)$ to be $qf(\alpha_i^-(t), \alpha_i^+(t))$, where $\bar{k}_i = \max\{k_i^+, k_i^-\}$ and we assume the arcs $\alpha_i^\pm$ stay at the endpoint after $t$ gets out of the domain.

**Claim 9.3.** A sequence of quasi-Fuchsian group $\{qf(m(\mu_\infty^-(i, \mathbf{n}_i)), m(\mu_\infty^+(i, \mathbf{n}_i))\}$ converges strongly to a quasi-conformal deformation of $(\Gamma, \psi)$ after passing to a subsequence.

*Proof.* By our definition of the function $m$ and the argument in the proof of Lemma 9.2, a uniformly bi-Lipschitz model manifold for the quasi-Fuchsian group $qf(m(\mu_\infty^-(i, \mathbf{n}_i)), m(\mu_\infty^+(i, \mathbf{n}_i)))$, which we denote by $\mathbf{M}'(\mathbf{n}_i)$, can be obtained from the submanifold of $\mathbf{M}_i'[0]$ cobounded by $S_-(\mathbf{n}_i)$ and $S_+(\mathbf{n}_i)$ by pasting boundary blocks corresponding to $m(\mu_\infty^-(i, \mathbf{n}_i))$ and $m(\mu_\infty^+(i, \mathbf{n}_i))$ respectively and filling Margulis tubes. This model manifold contains $(\rho_i^{\mathbf{M}'})^{-1} \circ g'(S)$. This implies that for a fixed generator system of $\pi_1(S)$ with base point on $(\rho_i^{\mathbf{M}'})^{-1} \circ g'(S)$, the length of the shortest curve in $\mathbf{M}'(\mathbf{n}_i)$ representing each generator is bounded as $i \to \infty$, and hence that $qf(m(\mu_\infty^-(i, \mathbf{n}_i)), m(\mu_\infty^+(i, \mathbf{n}_i)))$ converges algebraically after passing to a subsequence. Let $(\Gamma', \psi')$ be the algebraic limit, and denote $\mathbb{H}^3/\Gamma'$ by $M_{\Gamma'}$.

The geometric limit $\mathbf{M}'(\mathbf{n}_\infty)$ of the complement of boundary blocks in $\mathbf{M}'(\mathbf{n}_i)$ is embedded in $\mathbf{M}'$ as a submanifold, by our definition of $\mathbf{M}'(\mathbf{n}_i)$. For each $j = 1, \ldots, q$, the intersection of $\rho_i^{\mathbf{M}'}(S_+(\mathbf{n}_i))$ with $B^j$ goes deeper and deeper into $B^j$ to the direction of the end $E_j$ as $i \to \infty$ since $\mathbf{n}_i \to \infty$. The same holds for $\rho_i^{\mathbf{M}'}(S_-(\mathbf{n}_i)) \cap B^j$ for $j = q+1, \ldots p$. Therefore the entire $B^j$ is contained in $\mathbf{M}'(\mathbf{n}_\infty)$ for every $j = 1, \ldots, p$.

Let $\mathbf{V}_i^k$ $(k = 1, \ldots, s)$ be the Margulis tube in $\mathbf{M}_i'$ corresponding to $V_i^k$ in $M_i$, and let $F_i$ be a component of $\hat{S}_+(\mathbf{n}_i) \setminus (\cup_{j=1}^q B_i^j \cup \cup_{k=1}^s \mathbf{V}_i^k)$. The boundary of $F_i$ lies on a component of $\cup_{k=1}^s \partial \mathbf{V}_i^k$, which converges geometrically to an annular boundary component of $\mathbf{M}'$. Since $F_i \cap \cup_{k=1}^s \partial \mathbf{V}_i^k$ does not go into $B_i^j$, its image under $\rho_i^{\mathbf{M}'}$ stays in a compact subset of the annular boundary component. Furthermore, there is a positive lower bound for the injectivity radii of the points on $F_i$ since its geometric limit cannot intersect a parabolic locus of $\mathbf{M}'$ other than those in $\mathcal{T}$ by the assumption that there is no isolated algebraic parabolic locus. These imply that $F_i$ converge geometrically to a surface $F_\infty$ which is homeomorphic to $F_i$ for large $i$. The same holds for each component of $\hat{S}_-(\mathbf{n}_i) \setminus (\cup_{j=q+1}^p B_I^j \cup \cup_{k=1}^t \mathbf{V}_i^k)$, where $\mathbf{V}_i^k$ is the Margulis tube in $\mathbf{M}_i$ corresponding to $V_i^{k'}$. Let $F_\infty^1, \ldots, F_\infty^r$ be such geometric limits for the components of $\hat{S}_+(\mathbf{n}_i) \setminus (\cup_{j=1}^q B_i^j \cup \cup_{k=1}^s \mathbf{V}_i^k)$ and $\hat{S}_-(\mathbf{n}_i) \setminus (\cup_{j=q+1}^p B_I^j \cup \cup_{k=1}^t \mathbf{V}_i^k)$. Then, $\mathbf{M}'(\mathbf{n}_\infty)$ is obtained by cutting $\mathbf{M}'$ along $\cup_{j=1}^r F_\infty^j$, has simply degenerate ends $E_1, \ldots, E_p$, and is homotopy equivalent to $S$ in particular. A model manifold of the geometric limit of $m(\mu_\infty^-(i, \mathbf{n}_i)), m(\mu_\infty^+(i, \mathbf{n}_i)))$

is obtained by pasting boundary blocks to $\mathbf{M}'(\mathbf{n}_\infty)$ along $F^1_\infty, \ldots, F^r_\infty$, and hence is homeomorphic to $S \times (0,1)$. This shows that the geometric limit has fundamental group isomorphic to $\pi_1(S)$, and hence that the convergence is strong. The model manifold shows that the ends of $M_{\Gamma'}$ consists of simply degenerate ends corresponding to $E_1, \ldots, E_p$ and geometrically finite ends and that every parabolic locus touches one of $E_1, \ldots, E_p$. Since the ending laminations of $E_1, \ldots, E_p$ are the same as those of the corresponding ends of $M' = \mathbb{H}^3/\Gamma$, by the ending lamination theorem due to Brock-Canary-Minsky [10], we see that $(\Gamma', \psi')$ is a quasi-conformal deformation of $(\Gamma, \psi)$. □

**Claim 9.4.** For any sequence $\{t_i \in [0, \bar{k}_i]\}$, the sequence $\{\beta_i(t_i) \in QF(S)\}$ converges algebraically to a quasi-conformal deformation of $(\Gamma, \psi)$.

*Proof.* Recall that $\beta_i(t_i) = qf(\alpha_i^-(t_i), \alpha_i^+(t_i))$, and that $\alpha^-$ and $\alpha^+$ are broken quasi-geodesic arcs consisting of Teichmüller geodesics with bounded lengths except for the first quasi-geodesics which may be long. We first assume that neither $\alpha_i^-(t_i)$ nor $\alpha^+(t_i)$ lies on the first quasi-geodesics. In this case, since $\alpha_i^-(t_i)$ and $\alpha^+(t_i)$ lie on Teichmüller geodesics with bounded length, we have only to consider the case when both $\alpha^+(t_i)$ and $\alpha_i^-(t_i)$ are endpoints of Teichmüller geodesic arcs constituting $\alpha^-$ and $\alpha^+$, i.e. the case when $t_i$ is an integer. The marking $\alpha^-(t_i)$ corresponds to either a slice in $\tau(\bar{B}_i)$ or a union of slices, taken one from each of $\tau(B_i^j)\,(j = q+1, \ldots, p)$. This corresponds in turn to an extended level surface $\hat{S}_-(t_i)$. Similarly, we have an extended level surface $\hat{S}_+(t_i)$ from $\alpha^+(t_i)$. Then, we can see that a uniform bi-Lipschitz model manifold $\mathbf{M}'(\beta_i(t_i))$ for $\beta_i(t_i)$ is obtained from the submanifold of $\mathbf{M}'_i$ cobounded by $S_-(l_i^-)$ and $S_+(l_i^+)$ by pasting boundary blocks and filling Margulis tubes by the argument of Lemma 9.2. Then by the same argument as in the previous claim, we see that $\{\beta_i(t_i)\}$ converges algebraically after passing to a subsequence, and the geometric limit $\mathbf{M}'_\infty(\beta)$ of $\mathbf{M}'(\beta_i(t_i))$ contains $B^1, \ldots, B^p$. Since the interior blocks of $\mathbf{M}'_\infty(\beta)$ lies in $\mathbf{M}'$, the ends of the algebraic limit other than those corresponding to $E_1, \ldots, E_p$ are geometrically finite and there are no extra parabolic loci. This implies that the algebraic limit is a quasi-conformal deformation of $(\Gamma, \psi)$ as in the previous claim.

Next suppose that $\alpha^+(t_i)$ lies in the first quasi-geodesic. Then all the interior blocks of $\mathbf{M}'_i$ above $(\rho_i^{\mathbf{M}'})^{-1} \circ g'(S)$ lie also in a model manifold $\mathbf{M}'(\beta(t_i))$ for $\beta(t_i)$ defined in the same way as above. Therefore, the algebraic ends of $\mathbf{M}'_i$ lie also in the model manifold $\mathbf{M}'(\beta(t_i))$. The argument for showing that there is no extra parabolic loci works also in this case. We can argue in the same way also in the case when $\alpha^-(t_i)$ lies (or both $\alpha^-(t_i)$ and $\alpha^+(t_i)$ lie) in the first geodesic. □

Now, we shall complete the proof of Proposition 9.1 by setting $\alpha_i(t) = \beta_i(\bar{k}_i t)$ for $\beta_i$ defined above and showing thus constructed arc $\alpha_i$ satisfies the conditions in the statement. The conditions (1) and (2) have already been shown. Let us show (3). By our construction, the model manifold $\mathbf{M}'_i(\beta(1))$ has Margulis tubes $\mathbf{V}^i_1, \ldots, \mathbf{V}^i_s$ and ${\mathbf{V}^i_1}', \ldots, {\mathbf{V}^i_t}'$ with core curves corresponding to $c^i_1, \ldots, c^i_s$ and ${c^i_1}', \ldots, {c^i_t}'$. Because of the existence of the bricks $B_i^j$ which supports $\gamma_i^j$ with length going to $\infty$, for each tube $V$ among $\mathbf{V}^i_1, \ldots, \mathbf{V}^i_s; {\mathbf{V}^i_1}', \ldots, {\mathbf{V}^i_t}'$, we have $\Im\omega_{\mathbf{M}_i(\beta_i(0))}(V)$ goes to $\infty$. Since $\mathbf{V}^i_1, \ldots, \mathbf{V}^i_s$ are also attached to the upper boundary block and ${\mathbf{V}^i_1}', \ldots, {\mathbf{V}^i_t}'$ to the lower boundary block and the geometric limit coincides the algebraic limit, this is possible only when the upper boundary block split along the curves corresponding to $c^i_1, \ldots, c^i_n$ and the lower one along curves corresponding to ${c^i_1}', \ldots {c^i_t}'$ in the geometric limit. This means that the length of each of $c^i_1, \ldots, c^i_s$ with respect to $n'_i$ and that of each of ${c^i_1}', \ldots, {c^i_t}'$ with respect to $m'_i$ goes to 0.

Next we shall show the condition (4). In the case when $\Gamma$ is a b-group, there is no lower algebraic simply degenerate ends for $\mathbf{M}'$. By our definition of $\beta_i$, in this case the lower conformal structure $\alpha^-_i(t)$ is the same as $m_0$ for every $t$.

Finally, we shall verify the condition (5). Suppose, seeking a contradiction, that the condition (5) does not hold. Then, there exist a neighbourhood $U$ of $QH(\Gamma, \psi)$ and $t_i \in [0, k_i]$ such that $\beta_i(t_i)$ stays outside $U$ for every $i$ after passing to a subsequence. Now, we apply Claim 9.4 to see that $\beta(t_i)$ converges to a point in $QH(\Gamma, \psi)$ after passing to a subsequence. This is a contradiction. Thus we have completed the proof of Proposition 9.1.

### 9.2. **Proofs of Theorem 7 and Corollaries 8, ??.**

*Proof of Theorem 7.* Having proved Proposition 9.1, to prove Theorem 7, it remains to show that two sequences strongly converging to groups in the quasi-conformal deformation space of $(\Gamma, \psi)$ as constructed in the proof of Proposition 9.1 can be joined by arcs in small neighbourhoods of the deformation space, fixing the conformal structure at bottom when $\Gamma$ is a b-group.

Let $\{(G_i, \phi_i)\}$ and $\{(H_i, \varphi_i)\}$ be two sequences of quasi-Fuchsian groups both of which converge algebraically to quasi-conformal deformations of $(\Gamma, \psi)$. Let $\mathbf{M}'_i$ and $\mathbf{N}'_i$ be model manifolds for $\mathbb{H}^3/G_i$ and $\mathbb{H}^3/H_i$ constructed as in Lemma 9.2, converging geometrically to model manifolds $\mathbf{M}'$ and $\mathbf{N}'$ for the geometric limits of $\{G_i\}$ and $\{H_i\}$ respectively. Since the algebraic limits of $\{(G_i, \phi_i)\}$ and $\{(H_i, \varphi_i)\}$ are quasi-conformally equivalent, $\mathbf{M}'$ and $\mathbf{N}'$ have the same number of algebraic simply degenerate ends with the same ending laminations $\lambda_1, \ldots, \lambda_p$. As in Proposition 9.1, we renumber them so that those having $\lambda_1, \ldots, \lambda_q$ as ending laminations are upper whereas the rest are lower. In particular, we can assume that the union of the boundary components $\mathcal{T}$ touching these ends, which was defined in the proof of Proposition 9.1, is the same for $\mathbf{M}'$ and $\mathbf{N}'$ as open annuli in $S \times [0, 1]$.

Let $\{(G'_i, \phi'_i) = qf(m'_i, n'_i)\}$ and $\{(H'_i, \varphi'_i) = qf(\mu'_i, \nu'_i)\}$ be two strongly convergent sequences as constructed in Proposition 9.1 for two sequences $\{(G_i, \phi_i)\}$ and $\{(H_i, \varphi_i)\}$. Recall that in the construction of such a sequence in Proposition 9.1, we took a number $\mathbf{n}_i$. Note that the construction works if we take a number smaller than $\mathbf{n}_i$ for each $i$ provided that the number goes to $\infty$. Therefore, we can let the number $\mathbf{n}_i$ be common to both $G'_i$ and $H'_i$. Let $\mathbf{M}'(\mathbf{n}_i)$ and $\mathbf{N}'(\mathbf{n}_i)$ be model manifolds for them as in the proof of the proposition. Then both $\mathbf{M}'(\mathbf{n}_i)$ and $\mathbf{N}'(\mathbf{n}_i)$, regarded as embedded in $S \times [0, 1]$, have the following properties. There are unions of Margulis tubes $\mathbf{V}_i$ and $\mathbf{V}'_i$ coming from $\mathcal{T}$ in $\mathbf{M}'(\mathbf{n}_i)$ and $\mathbf{N}'(\mathbf{n}_i)$ respectively, which can be assumed to correspond to the same union of solid tori in $S \times [0, 1]$. As in the proof of Proposition 9.1, the complements $\mathbf{M}'(\mathbf{n}_i) \setminus \mathbf{V}_i$ and $\mathbf{N}'(\mathbf{n}_i) \setminus \mathbf{V}'_i$ have decompositions into bricks among which there are ${B^j_i}' = \Sigma^j \times {J^j_i}'$ in $\mathbf{M}'(\mathbf{n}_i)$ and ${B^j_i}'' = \Sigma^j \times {J^j_i}''$ in $\mathbf{N}'(\mathbf{n}_i)$ on the same side of the preimages of the standard algebraic immersions. Recall that they were obtained from $\check{\mathbf{M}}'$ and $\check{\mathbf{N}}'_i$ by throwing away top and bottom interior bricks, cutting bricks vertically touching $\mathbf{V}_i$ and $\mathbf{V}'_i$, and pasting boundary blocks. The bricks ${B^j_i}'$ and ${B^j_i}''$, obtained by this procedure, contain tube unions corresponding to geodesics ${\gamma_i^j}'$ and ${\gamma_i^j}''$ supported on $\Sigma^j$ whose lengths go to $\infty$. Furthermore, for each of ${\gamma_i^j}'$ and ${\gamma_i^j}''$, one of the endpoints stays in a compact set and the other endpoint (which we denote by ${b_i^j}'$ for ${\gamma_i^j}'$ and ${b_i^j}''$ for ${\gamma_i^j}''$) goes to the ending lamination $\lambda_j$ as $i \to \infty$. Also, the tube unions put in ${B_i^j}'$ and ${B_i^j}''$ induce hierarchies ${h_i^j}'$ and ${h_i^j}''$ on $\Sigma^j$ with main geodesics ${\gamma_i^j}'$ and ${\gamma_i^j}''$ respectively.

In the following, we only consider the case when the end of $\mathbf{M}'$ (hence also that of $\mathbf{N}'$) having $\lambda_j$ as the ending lamination is an upper end, i.e. $j = 1, \dots q$. The case when it is an lower end can be dealt with just by turning everything upside down as usual. Let ${\mu_\infty^+}'(i, \mathbf{n}_i)$ be a sequence of markings which we constructed as in the proof of Proposition 9.1 for $\mathbf{N}'$, corresponding to $\mu^+(i, \mathbf{n}_i)$ there for $\mathbf{M}'$. In the construction of $\mathbf{M}'(\mathbf{n}_i)$ and $\mathbf{N}'(\mathbf{n}_i)$ in the proof of Proposition 9.1, the last slices of ${h_i^j}'$ and ${h_i^j}''$ correspond to the restrictions of the last terms $\mu_\infty^+(i, \mathbf{n}_i)$ and ${\mu_\infty^+}'(i, \mathbf{n}_i)$ of the sequences of markings $\mu^+(i, \mathbf{n}_i)$ and ${\mu^+}'(i, \mathbf{n}_i)$.

Since the length of the frontier of $\Sigma^j$ goes to 0 with respect to both $n_i'$ and $\nu_i'$, we can assume that the length of each component of $\Sigma^j$ with respect to $n_i'$ is equal to that with respect to $\nu_i'$ without changing the algebraic limit and the structure of the model manifolds except for the boundary blocks.

Now, we connect the endpoints ${b_i^j}'$ and ${b_i^j}''$ by a tight sequence $\gamma_i = \{c_1^i, \dots, c_{u_i}^i\}$ in $\mathcal{CC}(\Sigma_i^j)$. By the hyperbolicity of $\mathcal{CC}(\Sigma_i^j)$, for every integer $s_i$ between 1 and $u_i$, the simplex $c_{s_i}^i$ also converges to the lamination $\lambda_j$ as $i \to \infty$. By letting $\mu_\infty^+(i, \mathbf{n}_i)$ and ${\mu_\infty^+}'(i, \mathbf{n}_i)$ be the initial and terminal markings respectively, we can regard $\gamma_i$ as a tight geodesic, and there is a hierarchy $h(\gamma_i)$ on $\Sigma_j$ which has $\gamma_i$ as its main geodesic. Considering a resolution of this hierarchy $h(\gamma_i)$, we can connect $\mu_\infty^+(i, \mathbf{n}_i)$ with ${\mu_\infty^+}'(i, \mathbf{n}_i)$ by elementary moves of markings, and correspondingly $n_i'$ with $n_i'|(S \setminus \Sigma^j) \cup \nu_i'|\Sigma^j$ by a piecewise Teichmüller geodesic arc $\delta_i : [0, w_i] \to \mathcal{T}(S)$. Since every $c_{s_i}^i$ converges to $\lambda_j$ and the structure of $\mathbf{M}_i'(\mathbf{n}_i)$ outside $B^j$ is unchanged, we see that $qf(m_i', \delta_i(t_i))$ converges to $(\Gamma, \psi)$ strongly. By the same argument as in the proof of Proposition 9.1, we see that for any neighbourhood $U$ of $QH(\Gamma, \psi)$, the arc $qf(m_i', \delta_i[0, w_i])$ is contained in $U$ for large $i$.

We repeat the same procedure for each $j = 1, \dots, p$. Then, we get an arc $\alpha_i'$ connecting $qf(m_i', n_i')$ and $qf(m_i'|(S \setminus \cup_{j=q+1}^p \Sigma^j) \cup \mu_i'| \cup_{j=q+1}^p \Sigma_i^j, n_i'|(S \setminus \cup_{j=1}^q \Sigma^j) \cup \nu_i'| \cup_{j=1}^p \Sigma^j)$ such that for any neighbourhood $U$ of $QH(\Gamma, \psi)$, the arc $\alpha_i'$ is contained in $U$ for sufficiently large $i$. Since all of the $m_i'|(S \setminus \cup_{j=q+1}^p \Sigma^j)$, the $\mu_i'|(S \setminus \cup_{j=q+1}^p \Sigma^j)$, $n_i'|(S \setminus \cup_{j=1}^p \Sigma_i^j)$, and $\nu_i'|(S \setminus \cup_{j=1}^p \Sigma^j)$ are bounded in the Teichmüller spaces, we can deform $qf(m_i'|(S \setminus \cup_{j=q+1}^p \Sigma^j) \cup \mu_i'| \cup_{j=q+1}^p \Sigma^j, n_i'|(S \setminus \cup_{j=1}^q \Sigma^j) \cup \nu_i'| \cup_{j=1}^p \Sigma_i^j)$ to $qf(\mu_i', \nu_i')$ by a uniformly bounded quasi-conformal deformation. Thus, we have shown that we can connect $(G_i', \phi_i') = qf(m_i', n_i')$ with $(H_i', \psi_i') = qf(\mu_i', \nu_i')$ with an arc $\alpha_i$ in $QF(S)$ such that for any neighbourhood $U$ of $QH(\Gamma, \psi)$, the arc $\alpha_i$ is contained in $U$ large $i$. On the other hand, by our definition of $(m_i', n_i')$ and $(\mu_i', \nu_i')$, there are arcs with such a property connecting $(G_i, \phi_i)$ with $(G_i', \phi_i')$ and $(H_i, \psi_i)$ with $(H_i', \psi_i')$. Therefore, connecting these three arcs, we get an arc as we wanted.

In the case when $\Gamma$ is a b-group, by Theorem 6, the lower conformal structures at infinity of both $(G_i, \phi_i)$ and $(H_i, \psi_i)$ converge to $m_0$. Therefore, we can construct an arc as above keeping the lower conformal structures in an arbitrarily small neighbourhood of $m_0$ for large $i$. This shows the second paragraph of our theorem. □

Corollary 8 follows easily from this as follows.

*Proof of Corollary 8.* Suppose that there is any small neighbourhood of $(\Gamma, \psi)$ intersects more than one component of $QF(S)$. Then there are sequences $\{qf(m_i, n_i)\}$ and $\{qf(\hat{m}_i, \hat{n}_i)\}$ both converging to $(\Gamma, \psi)$ such that $qf(m_i, n_i)$ and $qf(\hat{m}_i, \hat{n}_i)$ belong to different component of $U \cap QF(S)$ for every small neighbourhood $U$ of $(\Gamma, \psi)$ in $QF(S)$. Applying Theorem 7, we see that $qf(m_i, n_i)$ and $qf(\hat{m}_i, \hat{n}_i)$ must be connected in a small neighbourhood of $QH(\Gamma, \psi)$. In the case when every component of $\Omega_\Gamma/\Gamma$ is a thrice-punctured sphere, this is a contradiction since $QH(\Gamma, \psi)$ consists of only $(\Gamma, \psi)$ then.

In the case when there is a component of $\Omega_\Gamma/\Gamma$ which is homeomorphic to $S$, the Kleinian group $\Gamma$ is a b-group. Then the second paragraph of Theorem 7 shows that we can connect $qf(m_i, n_i)$ to $qf(\hat{m}_i, \hat{n}_i)$ keeping the lower conformal structure within a small neighbourhood. This means that they can be connected in a small neighbourhood of $(\Gamma, \psi)$ in $QF(S)$. This again is a contradiction. □

To get Corollary **??**, we need to review the argument of the proof of Theorem 7.

*Proof of Corollary* **??**. In the proof of Theorem 7, we used the assumption that $\Gamma$ has no isolated parabolic loci in two places, first to show the standard immersion $g'$ can be isotoped to a horizontal embedding, and secondly in the proofs of Claims 9.3 and 9.4 to show that the limit is a quasi-conformal deformation of $(\Gamma, \psi)$. We shall show how to modify the argument in the latter part first.

In the proofs of the claims, we used the fact that all the parabolic loci of the algebraic limit $\Gamma'$ of the new sequence touch simply degenerate ends, hence the parabolic loci of $\Gamma'$ and $\Gamma$ are the same. In our setting now, if we do the same construction, this is not the case any more. To preserve the parabolic loci throughout the modification of quasi-Fuchsian groups, we consider to modify the model manifolds $\mathbf{M}'$ and $\mathbf{M}'_i$ constructed in Lemma 9.2 as follows.

Let $c_1, \ldots, c_u$ be core curves of the isolated parabolic loci of $\Gamma$. We renumber them so that $c_1, \ldots, c_v$ are upper and $c_{v+1}, \ldots, v_u$ are lower. These curves correspond to algebraic parabolic curves in the original model manifold $\mathbf{M}$ lying on the boundary components. For each of them we set $V(c_j) = A(c_j) \times [3/4, 7/8]$ for $j = 1, \ldots, v$ and $V(c_j) = A(c_j) \times [1/8, 1/4]$ for $j = v+1, \ldots, u$. We define $\mathcal{U}^+$ to be $\cup_{j=1}^v V(c_j)$ and $\mathcal{U}^-$ to be $\cup_{j=v+1}^u V(c_j)$. We then define $\bar{M}$ to be $S \times [1/8, 7/8] \setminus (\mathcal{T}^- \cup \mathcal{T}^+ \cup \mathcal{U}^- \cup \mathcal{U}^+)$. This new $\bar{M}$ also have standard brick decomposition having five levels. Still, the bottom and top levels are changed: the bottom level consists of disjoint union of the $\Sigma \times [1/8, 1/4]$ for the components $\Sigma$ of $S \setminus \cup_{j=v+1}^u A(c_j)$, and the top level consists of the $\Sigma \times [3/4, 7/8]$ for the components $\Sigma$ of $S \setminus \cup_{j=1}^v A(c_j)$. By the same construction as in the proof of Lemma 9.2, we can construct model manifolds $\mathbf{M}'_i$.

This modification affects the construction of resolutions $\tau(\bar{B}_i)$ and $\tau(\hat{B}_i)$. In the present situation, $\bar{B}_i$ is a disjoint union of bricks of the form $B_1^- = \Sigma_1^- \times [1/8, 1/4], \ldots, B_w^- = \Sigma_w^- \times [1/8, 1/4]$, where $\Sigma_1^-, \ldots, \Sigma_w^-$ are the components of $S \setminus \cup_{j=v+1}^u A(c_j)$. Similarly, $\hat{B}_i$ is a disjoint union of bricks of the form $B_1^- = \Sigma_1^+ \times [3/4, 7/8], \ldots, B_y^- = \Sigma_y^+ \times [3/4, 7/8]$, where $\Sigma_1^+, \ldots, \Sigma_y^+$ are the components of $S \setminus \cup_{j=1}^{v} A(c_j)$. The tube unions put into the $B_j^{\pm}$ in $\mathbf{M}'_i$ induce hierarchies on the $B_j^{\pm}$. Then we get a resolution $\tau_i(B_j^{\pm})$ supported on $\Sigma_j^{\pm}$.

For these resolutions, we can define as follows the first parts of sequences of markings $\mu^-(i,n)$ and $\mu^+(i,n)$ replacing sequences determined by $\tau(\bar{B}_i)$ and $\tau(\hat{B}_i)$. Each slice in $\tau_i(B_j^-)$ determines a marking on $\Sigma_j^-$. We extend markings determined by slices in $\tau_i(B_1^-), \ldots, \tau_i(B_w^-)$ by letting $c_{v+1}, \ldots, c_u$ also be base curves and setting their transversals to be the first vertex on the geodesics supported on $A(c_1), \ldots, A(c_v)$. We move markings by one step by advancing one of the slices in the resolutions of $\tau_i(B_1^-), \ldots, \tau_i(B_w^-)$, not touching $c_{v+1}, \ldots, c_u$. We append a sequence of markings determined by $\tau(B_i^{q+1}), \ldots \tau(B_i^p)$ as in the proof of Proposition 9.1. Similarly, we define $\mu^+(i,n)$ by defining its first part using $\tau_i(B_1^+), \ldots, \tau_i(B_y^+)$, and last vertices on geodesics supported on $A(c_1), \ldots, A(c_v)$.

We also need to modify the definition of the function $m$. For each $\mu$ in $\mu^-(i,n)$, we define $m(\mu)$ to be a point in $\mathcal{T}(S)$ such that a clean marking compatible with $\mu$ is a shortest marking in $(S, m(\mu))$, and *the lengths of $c_{v+1}, \ldots, c_u$ are equal to those with respect to $m_i$*. We define $m$ for $\mu \in \mu^+(i,n)$ similarly. Then the rest of the construction in the proof of Proposition 9.1 works without any change. The algebraic limits appearing Claims 9.3 and 9.4 has $c_1, \ldots, c_u$ as parabolic elements since their lengths go to 0 as $i \to \infty$ whatever point you choose on $\alpha^-$ or $\alpha^+$ by our definition of $m$ as above.

It remains to show that the standard immersion $g'$ does not go around a torus boundary component in $\mathbf{M}'$ in our setting. We consider first the case when $\Gamma$ is a b-group. Let $\{(G_i, \phi_i) = qf(m_0, n_i)\}$ and $\{(\bar{G}_i, \bar{\phi}_i) = qf(m_0, \bar{n}_i)\}$ be sequences in the Bers slice converging to $(\Gamma, \psi)$. Let $\mathbf{M}'$ and $\bar{\mathbf{M}}'$ be model manifolds for geometric limits $G_\infty$ and $G'_\infty$ of $\{G_i\}$ and $\{G'_i\}$, constructed as above respectively. Let $g' : S \to \mathbf{M}'$ and $\bar{g}' : S \to \bar{\mathbf{M}}'$ be standard algebraic immersions. Since the lower geometrically finite ends must be lifted to the algebraic limits, neither $g'$ nor $\bar{g}'$ can go around torus boundary components, hence homotopic to horizontal surfaces in $\mathbf{M}'$ and $\bar{\mathbf{M}}'$ respectively. Therefore the argument of the proof of Theorem 7 works, and we see that for any neighbourhood $U$ of $(\Gamma, \psi)$, there is an arc $\alpha_i$ connecting $(G_i, \phi_i)$ and $(\bar{G}_i, \bar{\phi}_i)$ in the Bers slice $U \cap B_{m_0}$.

Next suppose that $\Gamma$ is not a b-group. Let $\{(G_i, \phi_i) \in QF(S)\}$ be a sequence converging to $(\Gamma, \psi)$. Again, we have only to show that a model manifold $\mathbf{M}$ of the geometric limit of $G_i$, a standard algebraic immersion does not go around a torus boundary component. Suppose that $g'$ goes around a torus component $T$ of $\mathbf{M}$ counter-clockwise. Let $c$ be a longitude of $T$. Since $g'$ goes around $T$ counter-clockwise, $c$ represents on $S$ an upper parabolic locus of $M_\Gamma = \mathbb{H}^3/\Gamma$.

Since we assumed that $\Omega_\Gamma/\Gamma$ consists of thrice-punctured spheres, there is either a lower parabolic locus $d$ or an ending lamination $\lambda$ of a lower end, intersecting $c$ essentially. If there is a lower parabolic locus $d$, then there is a boundary component $T'$ of $\mathbf{M}$ whose core curve or a longitude is homotopic to $g'(d)$ and which is situated below $g'(S)$. This is impossible since $g'$ goes around $T$ whose longitude intersects $d$ essentially on $S$. If there is an ending lamination $\lambda$, then there is a algebraic simply degenerate end $\Sigma \times (s,t]$ having $\lambda$ as the ending lamination. Then $\Sigma \times \{s + \epsilon\}$ must be homotopic to $g'(\Sigma)$. Again this is impossible since $g'$ goes around $T$ whose longitude intersects $\lambda$ essentially on $S$. Thus, we are lead to a contradiction in either case, and see that $g'$ cannot go around a torus boundary component counter-clockwise. The case when $g'$ goes around a torus boundary component clockwise can be deal with in the same way just by turning everything upside down. □

## 10. Proof of Theorems 9 and 10

10.1. **Proof of Theorem 9.** After passing to a subsequence, $\{G_i\}$ can be assumed to converge geometrically to a Kleinian group $G_\infty$ containing $\Gamma$ as before. This induces a pointed Gromov convergence of $(M_i, y_i)$ to $(M_\infty, y_\infty)$ with $M_\infty = \mathbb{H}^3/G_\infty$. Let $\mathbf{M}$ be a model manifold of $(\mathbb{H}^3/G_\infty)_0$, and $\mathbf{M}_i$ that of $(M_i)_0 = (\mathbb{H}^3/G_i)_0$ as before. As in Lemma 4.4, we take a standard algebraic immersion $g' : S \to \mathbf{M}$. Then $\mathbf{M}_i[0]$ converges geometrically to $\mathbf{M}[0]$ by taking a basepoint $x_i$ in $\mathbf{M}_i$ which is mapped to a point within uniformly bounded distance from the basepoint $y_i$ of $M_i$.

Let $E$ be an algebraic simply degenerate end of $\mathbf{M}$ with ending lamination $\lambda_E$, and $B = \Sigma \times J$ a brick of $\mathbf{M}$ containing $E$. We assume that $E$ is an upper end. The case when $E$ is a lower end can be argued in the same way by just turning everything upside down as usual. Then by Proposition 4.8, there is a geodesic $\gamma_i$ supported on $\Sigma$ one of whose endpoints converges to $\lambda_E$ as $i \to \infty$. In the case when $\gamma_i$ is a geodesic ray, its endpoint at infinity is an ending lamination of an upper end of $\mathbf{M}_i$, which is contained in $e_+(i)$. Therefore, we see that $\lambda_E$ is contained in the Hausdorff limit of $e_+(i)$.

Next suppose that $\gamma_i$ is a finite geodesic. Then, by Theorem 3.1 and §6 of Masur-Minsky [27], the last vertex and the $\pi_\Sigma(e_+(i))$ are within uniformly bounded distance, and hence the endpoint of $\gamma_i$ and $\pi_\Sigma(e_+(i))$ converge to the same lamination $\lambda_E$ with respect to the topology of $\mathcal{UML}(\Sigma)$. Since $\lambda_E$ is arational, we see that the Hausdorff limit of $e_+(i)|\Sigma$ contains $\lambda_E$.

The converse can be shown by the same argument as the proof of Theorem 5.2. Also, we can show that every upper isolated parabolic locus cannot intersect a minimal component of $e_+(\infty)$ in the same way as in the proof of Theorem 5.2.

10.2. **Proof of Theorem 10.** Suppose, seeking a contradiction, that a sequence $\{(G_i, \phi_i)\}$ as in the statement converges. Let $\mathbf{M}$ be a model of the geometric limit $(\mathbb{H}^3/G_\infty)_0$ as usual. Then by Theorem 9, $\lambda$ is an ending lamination of an upper end and $\mu$ is an ending lamination of a lower end. This implies the shared boundary component of the minimal supporting surface of $\mu$ is a parabolic locus which is both upper and lower. This is impossible, and we have completed the proof of Theorem 10.

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