Universal length bounds for non-simple closed geodesics on hyperbolic surfaces

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Abstract

We investigate the relationship, in various contexts, between a closed geodesic with self-intersection number k (for brevity called a k-geodesic) and its length. We show that for a fixed compact hyperbolic surface, the short k-geodesics have length comparable to the square root of k. On the other hand, if the fixed hyperbolic surface has a cusp and is not the punctured disc, then the short k-geodesics have length comparable to $\log k$.

The length of a k-geodesic on any hyperbolic surface is known to be bounded from below by a constant that goes to infinity with k. In this paper, we show that the optimal constants $\{M_k\}$ are comparable to $\log k$ leading to a generalization of the well-known fact that length less than $4\log(1+\sqrt{2})$ implies simple. Finally, we show that for each natural number k, there exists a hyperbolic surface where the constant M_k is realized as the length of a k-geodesic. This was previously known for $k = 1$, where M_1 is the length of the figure eight on the thrice punctured sphere.

1 Introduction and results.

In this paper, we investigate the relationship between the length of a closed geodesic and its self-intersection number on a hyperbolic surface.

The length of a closed geodesic ω is denoted $\ell(\omega)$ and its self-intersection number (counting multiplicity) is denoted $|\omega \cap \omega|$. See section (2) for precise definitions. The *moduli space* of the hyperbolic surface S, denoted $\mathcal{M}_S$, is the space of hyperbolic surfaces that are quasiconformally equivalent to S. This space inherits a topology from its branched universal cover the Teichmüller space.

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Theorem 1.1. *Let S be a compact hyperbolic surface with (possibly empty) geodesic boundary. There exists a constant c_S depending on the hyperbolic structure of S so that if ω is a non-simple closed geodesic on S then*

$$\ell(\omega) \geq c_S \sqrt{|\omega \cap \omega|}. \tag{1}$$

c_S is a continuous function of S. If S is closed then c_S goes to zero as S goes to the boundary of $\mathcal{M}_S$.

In order to state the next two theorems, we first define some auxiliary functions that arise in a geometric context. For $x > 0$, define the collar function $c(x) = \log \coth \frac{x}{4}$, $A(x) = c(2x) = \sinh^{-1}\left[\frac{1}{\sinh \frac{x}{2}}\right]$, and $h(x) = \frac{x}{A(x)}$. Note that h is a smooth increasing function. Let $p(x) = \frac{1}{2}(x^2 - x - 3)$ and set $L = p \circ h$. See lemma (2.1) for the properties of these functions.

Theorem 1.2. *If ω is a non-simple closed geodesic on a hyperbolic surface then $|\omega \cap \omega| \leq L(\ell(\omega))$. In particular,*

$$\ell(\omega) \geq \frac{1}{4} \log 2|\omega \cap \omega|. \tag{2}$$

Denote the set of closed geodesics on a hyperbolic surface S with crossing number k by $\mathcal{C}_k(S)$. For simplicity we call these geodesics *k-geodesics.* We are interested in the growth rate of short k-geodesics as a function of k. This problem can be considered at three levels. The first is to look for short k-geodesics on a fixed hyperbolic surface S. The next is to look for such geodesics in the quasiconformal equivalence class of S (that is, the Moduli space of S) and the last is to look for universal growth rate of short k-geodesics without any restriction on the hyperbolic surface used. With this in mind we consider three quantities. For $k \in \mathbb{N}$ and S a hyperbolic surface with non-abelian fundamental group define,

- $s_k = s_k(S) := \inf\{\ell(\omega) : \omega \in \mathcal{C}_k(S)\}$,
- $m_k = m_k(\mathcal{M}_S) := \inf\{\ell(\omega) : \omega \in \mathcal{C}_k(Y), Y \in \mathcal{M}_S\}$, and
- $M_k := \inf\{\ell(\omega) : \omega \in \mathcal{C}_k(Y)$ for some hyperbolic surface Y$\}$.

Clearly, $M_k \leq m_k \leq s_k$.

Corollary 1.3. *Let S be a hyperbolic surface.*

1. *If S is compact with (possibly empty) geodesic boundary, then there exist constants c_S and L_S so that,*

$$c_S\sqrt{k} \leq s_k(S) \leq L_S\sqrt{k} + L_S. \tag{3}$$

where c_S is the constant in inequality (1) and L_S is the length of the shortest figure eight on S.

2. *If S has at least one cusp and is not the punctured disc, then for $k = 2, 3, \ldots,$*

$$\frac{1}{4}\log 2k \leq s_k(S) \leq 2\sinh^{-1} k + d_S + 1, \tag{4}$$

where d_S is the shortest orthogonal distance from the length one boundary of a cusp in S to itself.

If S is closed then using the Bers constant one can bound L_S and d_S from above by a constant that only depends on the genus of S.

It was shown in [1] that the $\{M_k\}$ go to infinity with k. As a corollary of our work we achieve bounds on their growth rate.

Corollary 1.4.

$$\frac{1}{4}\log 2k \leq M_k \leq m_k(\mathcal{M}_S) \leq 2\cosh^{-1}(1+2k). \tag{5}$$

In particular, $m_k \asymp \log k$ and $M_k \asymp \log k$.

The notation $f(k) \asymp g(k)$ means f/g is bounded from above and below by positive constants.

Although we do not know whether the $\{M_k\}$ form an increasing sequence, since the lower bound in (5) is increasing, we immediately have a generalization of the well-known fact that $\ell(\omega) < 4\log(1+\sqrt{2})$ implies ω is simple.

Corollary 1.5. *If ω is a closed geodesic of length ℓ satisfying $\ell < \frac{1}{4}\log 2k$, then $|\omega \cap \omega| < k$.*

Finally we ask if the infima in the definitions of s_k, m_k, and M_k are realized. It was shown independently by Hempel, Nakanishi, and Yamada ([14], [19], [23], [24]) that the length of the shortest non-simple closed geodesic is the figure eight on the thrice punctured sphere. Thus $M_1 = 4\log(1+\sqrt{2})$.

Theorem 1.6 (Realization)**.** *For each $k \in \mathbb{N}$, M_k is realized as the length of a k-geodesic on some hyperbolic surface.*

Remark 1.7. *Since the length spectrum goes to infinity on a topologically finite hyperbolic surface S, the $\{s_k\}$ are trivially realized as lengths of k-geodesics on S. On the other hand, in section (5) we show that $m_k(\mathcal{M}_S)$ is not realized.*

Define the collar function $c(\ell) = \log\coth\frac{\ell}{4}$ and $A(\ell) = \sinh^{-1}\left[\frac{1}{\sinh\frac{\ell}{2}}\right] = c(2\ell)$. A crucial tool in our proof of theorem (1.2) is the density lemma.

Lemma 1.8 (Density Lemma)**.** *Let ω be a non-simple closed geodesic of length ℓ and let J be a geodesic arc where $\ell(J) \leq A(\ell)$. Then*

$$|J \cap \omega| \leq \begin{cases} \frac{\ell+\ell(J)}{A(\ell)} - 3, & \text{if } J \subset \omega. \\ \frac{\ell}{A(\ell)}, & \text{if } J \not\subset \omega. \end{cases} \tag{6}$$

The methods in this paper are strictly hyperbolic geometric. For a combinatorial group theory approach relating word length in the fundamental group with intersection number, see the papers [8], [9], [10], and [16]. In particular, in [8] and [9], Chas and Phillips make a detailed, complete study of the relationship between word length and its intersection number for the fundamental group of the torus with a hole and a pair of pants. In the paper [17] it is shown that there is a relationship between the depth of an element in the lower central series of a surface group and its intersection number. Other papers on lengths of self-intersecting closed geodesics include, [2], [5], [6], [11], [12], [13], [15], [20], [21], and [22]. For the basics on hyperbolic geometry, we refer the reader to Beardon [3], Buser [7], and Maskit [18].

In section (2) we prove the density lemma, and in section (3) we prove theorems (1.1) and (1.2). In section (4), we produce upper bounds for short k-geodesics and finally in section (5), we take up the realization question. As a matter of convention, we denote a subsequence with the same notation as the original sequence.

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2 Preliminaries and the proof of the density lemma.

Throughout this paper by abuse of language we use curve to mean both a parametrized curve and the set of points that make up the image of a parametrized curve. All our closed geodesics are primitive.

Let X be a smooth surface and let $\omega : \mathbb{S}^1 \rightarrow X$ be a smooth (not necessarily closed) curve. Orient ω and assume that all intersections are transverse. At an intersection point p, we say that two unit tangent vectors to ω form a *transverse pair* if they span the tangent plane of X at p. If n is the number of unit tangent vectors at p, then the number of transverse pairs at p is $\binom{n}{2}$. The total number of transverse pairs, denoted $|\omega \cap \omega|$, is called the *self-intersection (crossing) number* of ω. When ω is a closed geodesic in a hyperbolic surface then $|\omega \cap \omega|$ realizes the minimal crossing number among all curves freely homotopic to ω. In this sense, for a geodesic in the free homotopy class of a closed curve its crossing number is constant as a function on moduli space. A curve is in *general position* if its intersection points are double points. The crossing number of a closed geodesic ω is the same as the total number of double points of a curve in general position which is freely homotopic to ω and realizes the minimum number of intersections.

Let $J \subset \omega$ and $p \in J \cap \omega$ and again assume that ω (and hence J) is oriented. A *transverse pair* (between J and ω) at p is a pair of unit tangent vectors based at p, one tangent to J and the other tangent to ω. The total number of such transverse pairs for all points in $J \cap \omega$ is denoted $|J \cap \omega|$. This is of course the same as the total number of unit tangent vectors to ω based in J but not tangent to J. If the closed geodesic $\omega : [a, b] \to S$ is the union of m geodesic segments, $J_i = [\omega(t_{i-1}), \omega(t_i)]$, whose interiors are embedded arcs, then

$$2|\omega \cap \omega| = \sum_{i=1}^{m} |J_i \cap \omega|.$$

Recall that $c(x) = \log \coth \frac{x}{4}$, $A(x) = c(2x), h(x) = \frac{x}{A(x)}$, $p(x) = \frac{1}{2}(x^2 - x - 3)$, and $L = p \circ h$. Since p is an increasing function for $x \geq 1$, it is easy to see that $p \circ h$ is a smooth increasing function for $\{x > 0 : h(x) \geq 1\}$.

We start with some elementary facts about these functions.

Lemma 2.1 (Geometric functions)**.** *Regarding the functions just defined,*

1. $(A \circ A)(x) = x$ *for all* $x > 0$.
2. $h(x) \leq e^{2x}$ *for* $x > 0$.
3. $A(x) \sim \exp(-x)$, *as* $x \to \infty$.
4. $h(x) \sim x \exp(x)$, *as* $x \to \infty$.
5. $L(x) \sim \frac{x^2 e^{2x}}{2}$, *as* $x \to \infty$.

Proof. The proofs of items (1) and (3)-(5) are routine. For the proof of item (2),

$$A(x) = \log \frac{1 + e^{-x}}{1 - e^{-x}} \geq \log(1 + e^{-x}) \geq e^{-2x}(e^x - \frac{1}{2}) \geq e^{-2x}(e^x - 1) \qquad (7)$$

Now divide the above inequality by x and use the fact that $\frac{e^x - 1}{x} \geq 1$ to get the desired result.

□

Lemma 2.2 (Geodesic segments)**.** *Suppose I and J are distinct geodesic segments with common endpoints z and w. Furthermore, assume I and J intersect transversely at w and that I is contained in a non-simple closed geodesic of length ℓ. Then*

$$\ell(I) + \ell(J) \geq 2A(\ell) \qquad (8)$$

Remark 2.3. *As a consequence of the above lemma, a segment of length less than* $2A(\ell)$ *in a closed geodesic must be embedded.*

Proof of 2.2. The hyperbolic surface $S = \mathbb{H}^2/G$, where G is a torsion-free Fuchsian group. Orient I from z to w and J from w to z. Choose a lift to $\mathbb{H}^2$ of the oriented loop $I \cup J$ based at z. This lift is a piecewise geodesic arc $\tilde{I} \cup \tilde{J}$ where the intersection point $\tilde{I} \cap \tilde{J}$ is a lift of w and the endpoints are lifts of z. Hence there exists $f \in G$ which identifies these lifts. Moreover, $\tilde{I}$ is part of the axis of a hyperbolic element say $g \in G$ which projects to the closed geodesic in S. Denote the lift of z which is the endpoint of $\tilde{I}$ by $\tilde{z}$. Consider the subgroup $< f, g > \leq G$. This is a torsion-free discrete subgroup of isometries of the hyperbolic plane. Since the intersection point $\tilde{I} \cap \tilde{J}$ is not smooth, we may conclude that the group $< f, g >$ is non-elementary. One of the quantative versions of the Margulis lemma (see Beardon [3]) for torsion-free Fuchsian group tells us,

$$\sinh\left\{\frac{\rho(\tilde{z}, f(\tilde{z}))}{2}\right\}\sinh\left\{\frac{\rho(\tilde{z}, g(\tilde{z}))}{2}\right\} \geq 1. \tag{9}$$

Now, since $\tilde{z}$ is on the axis of g, $\rho(\tilde{z}, g(\tilde{z})) = \ell$ and $\ell(I)+\ell(J) \geq \rho(\tilde{z}, f(\tilde{z}))$ by the triangle inequality. Using these two relations and plugging into (9) and rearranging yields the desired result. □

Remark 2.4. *Lemma 2.2 implies that a geodesic loop in a closed geodesic of length* ℓ *has length at least* $2A(\ell)$*. In particular, a non-simple closed geodesic of length* ℓ *satisfies,* $\ell \geq 4A(\ell)$*. Hence, equation (6) for* $J \not\subset \omega$ *trivially holds for* $|J \cap \omega| \leq 2$*. Similarly since*

$$\frac{\ell + \ell(J)}{A(\ell)} - 3 \geq 1 \tag{10}$$

for $J \subset \omega$*, we may conclude that equation (6) holds when* $|J \cap \omega| \leq 1$*.*

We next prove the density lemma.

Proof of the density lemma. Let $A = A(\ell)$, where $\ell = \ell(\omega)$. Put an orientation on $\omega : [0, 1] \to S$. Given the comments in remark 2.4, we may assume that $|J \cap \omega| \geq 2$ if $J \subset \omega$, and $|J \cap \omega| \geq 3$ if $J \not\subset \omega$.

Now, if $J \subset \omega$, then J inherits an orientation from ω. Let $\omega(t_0)$ be the endpoint of J after ω has traversed J. Let t_1 be the first time ω transversely returns to J, t_2 the second, and so on to t_m, where $m = |J \cap \omega|$. Letting $\omega(t_{m+1})$ be the beginning endpoint of J we have,

$$\omega = J \cup [\omega(t_0), \omega(t_1)] \cup ... \cup [\omega(t_{m-1}), \omega(t_m)] \cup [\omega(t_m), \omega(t_{m+1})] \tag{11}$$

In the rest of the proof we make repeated use of Lemma 2.2.

Note that the interval $[\omega(t_0), \omega(t_1)]$ is contained in a geodesic loop based at $\omega(t_0)$. Since this loop has length at least $2A$, we must have that $[\omega(t_0), \omega(t_1)]$ has length at least $2A - \ell(J)$. Similarly, the interval $[\omega(t_m), \omega(t_{m+1})]$ has length at least $2A - \ell(J)$.

The fact that $\ell(J) \leq A$ implies (by lemma 2.2) that the segment $[\omega(t_{k-1}), \omega(t_k)]$ for $k = 2, ..., m$ has length at least A. Using these lower bounds and equation (11), we have

$$\ell(\omega) \geq \ell(J) + 4A - 2\ell(J) + (m-1)A. \quad (12)$$

Solving for m and noting that $m = |J \cap \omega|$, inequality (6) now follows for $J \subset \omega$.

If $J \not\subset \omega$, let $\omega(t_1)$ be a point on J. Let t_2 be the second time ω returns to J, and so on until t_m, where $m = |J \cap \omega|$. We have

$$\omega = [\omega(t_1), \omega(t_2)] \cup ... \cup [\omega(t_{m-1}), \omega(t_m)] \cup [\omega(t_m), \omega(t_1)]. \quad (13)$$

Therefore, by lemma 2.2, $\ell(\omega) \geq mA$ and inequality (6) now follows for $J \not\subset \omega$. This finishes the proof of the density lemma. □

3 The proofs of theorems 1.1 and 1.2

Proof of theorem 1.1. Consider ω a closed geodesic on S with an orientation. We assume that $|\omega \cap \omega| \geq 1$.

Step (1). [Hexagon decomposition] Choose a pants decomposition of S and then further decompose each pair of pants into a pair of isometric right-angled hexagons glued along three common sides called *seams*. The other sides of the hexagons are called *half-holes*. Hence the sides of each such hexagon alternate between being half-holes and seams. Of course the half-holes join up to be the pants curves of the pants decomposition of the surface. There are -2χ hexagons, where χ is the Euler characteristic of S. Denote the hexagons in this decomposition by $\{H_i\}$.

Two Hexagons in this decomposition are either disjoint, isometric and glued along their seams to form a pair of pants, or meet skewed along half-holes. Furthermore, a hexagon may have two half-hole sides that get glued to each other as happens in a handle. Note that a hexagon can not have more than two of its sides identified.

Step (2). Define a *thread* (of ω) to be a smooth connected component of $\omega \cap H_i$, and assume that there are n threads. The orientation on ω as well as a choice of base point $p \in \omega \cap H_1$ induces an ordering on the threads. Denote this ordered set by $\{t_1, t_2, ..., t_n\}$. Since the gluing of two hexagons along a half-hole can be skewed, it is possible for two successive threads to be short. On the other hand,

Claim: For any three consecutive threads $\{t_m, t_{m+1}, t_{m+2}\}$,

$$\ell(t_m) + \ell(t_{m+1}) + \ell(t_{m+2}) \geq a,$$

where a is the shortest distance between any two non-adjacent sides of the hexagons. That is, if a_i is the shortest distance between non-adjacent sides of H_i, then $a = \min\{a_i : i = 1, ..., -2\chi\}$.

proof of claim. Any thread that goes between non-adjacent sides of a hexagon has length bigger than a. Hence we need only consider the case where $\{t_m, t_{m+1}, t_{m+2}\}$ each go between adjacent sides. The triple of threads then prescribes a length four alternating sequence of half-holes and seams. Note that two consecutive threads that meet in a seam have to be as long as the seam and thus bigger than a. Since this must occur for three consecutive threads we are done. □

Using the above claim we have,

$$\ell(\omega) = \sum_{i=1}^{-2\chi} \ell(\omega \cap H_i) = \sum_{i=1}^{-2\chi} \ell(t_i) \geq \frac{n}{3}a. \tag{14}$$

Step (3). Now, let n_i be the number of threads in H_i, and let k_i be the intersection number of ω in H_i. For counting purposes, If a thread corresponds to a seam (and hence is in two hexagons) by convention we assume it is only in the hexagon to its left (recall that ω and therefore the threads are oriented). Similarly, If a self-intersection point occurs on the boundary of more than one hexagon make an (admittedly) arbitrary choice and put the point in a particular hexagon. With this proviso, $\sum k_i = |\omega \cap \omega|$.

We next relate the number of threads with the number of intersections in a hexagon. Now given n_i threads the most number of intersections one can have on a simply connected convex domain is when each pair of lines intersect, that is $k_i \leq \binom{n_i}{2}$. Solving this for n_i, we get

$$n_i \geq \sqrt{2k_i + \frac{1}{4}} + \frac{1}{2} \geq \sqrt{2}\sqrt{k_i}. \tag{15}$$

Therefore,

$$n = \sum n_i \geq \sqrt{2} \sum \sqrt{k_i} \geq \sqrt{2}\sqrt{\sum k_i} = \sqrt{2}\sqrt{|\omega \cap \omega|} \tag{16}$$

where the sums range from $i = 1, ..., -2\chi$.

Step (4). Putting together inequalities (14) and (16) and setting $c_S = \frac{a\sqrt{2}}{3}$ we have

$$\ell(\omega) \geq c_S\sqrt{|\omega \cap \omega|}. \tag{17}$$

□

We next prove theorem 1.2.

Proof of theorem 1.2. Let $\omega : [0, \ell] \to S$ be a closed geodesic where ℓ is the length of ω. Cut ω into segments of equal length $A = A(\ell)$ with the possible exception of the last segment which has length less than $A(\ell)$. Without loss of generality, we can assume that none of the cuts are at intersection points. Thus we have cut ω into $\left[\frac{\ell}{A}\right]$ segments of length A and possibly one segment of length less than A. If n is the number of segments then $n \leq \left[\frac{\ell}{A}\right] + 1$. Let $J_1, ..., J_n$ be these segments and note that each segment is embedded and has length A except J_n. Using the fact that $2|\omega \cap \omega| = \sum_{i=1}^{n} |J_i \cap \omega|$, applying the density lemma to each interval, and adding we have,

$$2|\omega \cap \omega| < \sum_{i=1}^{n} (\frac{\ell + \ell(J_i)}{A} - 3) = \frac{1}{A} \sum_{i=1}^{n} (\ell + \ell(J_i)) - 3n = \tag{18}$$

$$= \frac{1}{A}(n\ell + \ell) - 3n = \frac{\ell}{A}(n+1) - 3n = n(\frac{\ell}{A} - 3) + \frac{\ell}{A} \tag{19}$$

$$\leq (\frac{\ell}{A} + 1)(\frac{\ell}{A} - 3) + \frac{\ell}{A}, \tag{20}$$

where the last inequality follows from $n \leq \frac{\ell}{A} + 1$.

Simplifying and rearranging we have,

$$|\omega \cap \omega| \leq \frac{1}{2}(\frac{\ell^2}{A^2} - \frac{\ell}{A} - 3). \tag{21}$$

The right-hand side of (21) is $L(\ell)$, proving the first part of theorem 1.2.

Finally, to prove inequality (2) it is easy to see from inequality (21) that, $2|\omega \cap \omega| < 2|\omega \cap \omega| + 1 < \frac{\ell^2}{A^2} < e^{4\ell}$, where we have used lemma 2.1 for the last inequality. Taking logarithms we obtain,

$$\frac{1}{4} \log |\omega \cap \omega| + \frac{\log 2}{4} \leq \ell \tag{22}$$

□

4 Examples: the proofs of corollaries 1.3 and 1.4

Theorems 1.1 and 1.2 supply the lower bounds for corollaries 1.3 and 1.4. In this section we describe the examples that lead to the upper bounds.

Throughout this section X denotes the complex plane minus the points 0 and 1. Let a and b be generators for the fundamental group of X which wind counterclockwise around 0 and 1, resp., so that the free homotopy classes of a, b and ba (read from right to left) represent the boundary components of X. We are interested in the lengths of the geodesics in the homotopy classes of the curves $b^k a^{-1}$ for various hyperbolic structures on the pair of pants. Note that $|b^k a^{-1} \cap b^k a^{-1}| = k$.

Let S be a compact hyperbolic surface. To get the upper bound for s_k in inequality (3), let η be the shortest figure eight on S. Then η is the figure eight for an embedded pair of pants in S ([7]). Using the notation above, we set $\eta = b^{-1}a$. For $n \in \mathbb{N}$, consider the curve $\omega = b^{-1}\eta^n$. If L_S is the length of the figure eight η then the length of this geodesic is at most $nL_S + L_S$. Moreover, $|\omega \cap \omega| = n(n+1)$. Hence $\ell(\omega) \leq L_S\sqrt{|\omega \cap \omega|} + \ell(\beta) \leq L_S\sqrt{|\omega \cap \omega|} + L_S$, and we have the desired upper bound for s_k in inequality (3).

Next, suppose S is a hyperbolic surface, not the punctured disc and containing at least one cusp. To get the upper bound for s_k in inequality (4), let $\mathcal{B} \subset S$ be the punctured region bounded by the horocycle of length 1. It is well-known that a cusp always contains such a region.

Lemma 4.1. *If $\beta \subset \mathcal{B}$ is a geodesic segment that starts on $\partial\mathcal{B}$, winds around the cusp m-times, and returns to $\partial\mathcal{B}$ then*

$$m = |\beta \cap \beta| + 1 = \left[2\sinh\frac{\ell(\beta)}{2}\right] \tag{23}$$

The symbol $[\cdot]$ is the greatest integer function.

Proof. In the upper half-plane model for the hyperbolic plane, we may normalize so that the region $\mathcal{B}$ has lift the region above the horocycle of height one and the parabolic element (identified with the element of the fundamental group that winds once around $\mathcal{B}$) is $f : z \mapsto z + 1$. The lift of the geodesic segment is a segment $\tilde{\beta}$ in the upper half-plane whose endpoints are in the height one horocycle. $\tilde{\beta}$ extends to a semicircle which we can further normalize so that the origin is its center. Denoting the endpoints of this semicircle by $-r$ and r, a computation yields,

$$\ell(\beta) = 2\cosh^{-1} r \text{ or equivalently } r = \cosh\left(\ell(\beta)/2\right). \tag{24}$$

Now if m is the number of times β winds around the cusp then it must be that the f-translates of $\tilde{\beta}$ intersect $\tilde{\beta}$ exactly $(m-1)$-times in $\mathcal{B}$ and hence $m - 1 = |\beta \cap \beta|$. This is the same as saying,

$$m = \max\left\{n \in \mathbb{N} : -r + n \leq -r + 2\sqrt{r^2 - 1}\right\} = \left[2\sqrt{r^2 - 1}\right] \tag{25}$$

Plugging in equation (24) and simplifying finishes the proof. □

Proposition 4.2. *Assume S is a hyperbolic surface which is not the punctured disc and has at least one cusp. For each $k = 2, 3, \ldots$, there exists a k-geodesic ω on S for which*

$$\ell(\omega) \leq 2\sinh^{-1} k + (d_S + 1). \tag{26}$$

where d_S is the length of the shortest orthogonal from the boundary of a cusp to itself.

Proof. We construct the geodesic ω. Let $\mathcal{B}$ be as in the proof of lemma 4.1. Denote by β a geodesic curve that starts on the boundary $\partial\mathcal{B}$, winds around the cusp $k+1$-times, and returns to $\partial\mathcal{B}$; let $p, q \in \partial\mathcal{B}$ be the initial and terminal points of β, resp. Next let δ_1 be the shortest orthogonal from $\partial\mathcal{B}$ to itself. After putting an orientation on δ_1, denote its initial and terminal points by x and y, respectively. Finally, call the shortest segment along $\partial\mathcal{B}$ from q to x, δ_2, and similarly the shortest segment from y to p, δ_3. Each has length at most $1/2$. Then define ω to be the closed geodesic in the free homotopy class of the piecewise geodesic closed curve, $\delta_3 * \delta_1 * \delta_2 * \beta$ (read from right to left). First observe that $|\omega \cap \omega| = |\beta \cap \beta| = k$ by lemma 4.1. Now $\ell(\omega) \leq \ell(\beta) + d_S + 1$ and using Lemma 4.1, we can bound $\ell(\beta)$ by $2\sinh^{-1}(\frac{k+2}{2})$. Hence,

$$\ell(\omega) \leq \ell(\beta)+d_S+1 \leq 2\sinh^{-1}\left(\frac{k+2}{2}\right)+d_S+1 \leq 2\sinh^{-1}(k)+d_S+1, \quad (27)$$

where the last inequality is true for $k \geq 2$. □

Our next set of examples verify the upper bound in inequality (5) of corollary 1.4. Though strictly speaking it is enough to supply upper bounds for $m_k(\mathcal{M}_S)$ and thus attain upper bounds for M_k, we choose to first give upper bounds for M_k to emphasize the fact that they all come from geodesics on the thrice punctured sphere. We then use the fact that arbitrarily close (in the geometric topology) to a closed geodesic in the thrice punctured sphere is a geodesic in a pair of pants with short geodesic boundary. Since every hyperbolic surface with non-abelian fundamental group contains a pair of pants the conclusion will follow.

We first construct the thrice punctured sphere group. Consider the group generated by $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & -1 \\ 4 & -3 \end{pmatrix}$, this group acts on the upper half-plane with quotient the thrice punctured sphere. To see this, note that the ideal polygon with vertices at $\{0, \frac{1}{2}, 1, \text{and} \infty\}$ is a fundamental polygon for $< f, g >$. Moreover, using f and g to glue the sides of this polygon and using the natural isomorphism from the fundamental group of the quotient to $< f, g >$, it is not difficult to see that g represents the curve a and f represents the curve b from our model sphere with three holes, X. Then for each $k \in \mathbb{N}$, the element $f^k g^{-1}$ represents the free homotopy class of a closed geodesic with crossing number k. The element of the fundamental group is thus $b^k a^{-1}$ and we denote the length of the closed geodesic in its free homotopy class $\ell(b^k a^{-1})$. To compute this length, first note that the trace, $tr(f^k g^{-1}) = -2(1+2k)$. Using the fact that the length of the corresponding closed geodesic is equal to $2\cosh^{-1}\frac{|tr(f^k g^{-1})|}{2}$, we have

$$\ell(b^k a^{-1}) = 2\cosh^{-1}(1+2k) \quad (28)$$

Hence, we have realized the upper bound for the $\{M_k\}$ in (5) of corollary 1.4.

We next jazz up the above example to find bounds on the $\{m_k(S)\}$. Let S be a hyperbolic surface with non-abelian fundamental group. Such a surface contains at least one embedded pair of pants. With this in mind, for each $n \in \mathbb{N}$ construct a pair of pants P_n having the same analytic type as this embedded pants, all of whose boundary geodesics have length $\frac{1}{n}$. The $\{P_n\}$ geometrically converge to the thrice punctured sphere, P_∞. Thus for $k \in \mathbb{N}$ the length of the geodesic freely homotopic to $b^k a^{-1}$ in P_n, converges as $n \to \infty$ to the length of the corresponding geodesic (given by (28)) in P_∞. Next, we realize P_n as an embedded pair of pants on a hyperbolic surface S_n, where S_n is quasiconformally equivalent to S. Finally, there is a geodesic with k self-intersections whose length is arbitrarily close (for n large) to the length given by (28). This verifies the upper bound for m_k in (5).

Remark 4.3. *Note that in the above examples, if S is not a thrice punctured sphere then the upper bounds for $\{m_k(\mathcal{M}_S)\}$ are realized on different hyperbolic structures in $\mathcal{M}_S$.*

5 Filling geodesics and realizing the $M_k's$

The fact that the length spectrum goes to infinity for a hyperbolic surface of finite topological type implies that $s_k(S)$ is attained for each $k \in \mathbb{N}$. On the other hand, the quantity $m_k(\mathcal{M}_S)$ is not realizable. To see this, let S be a pair of pants with geodesic boundary and ω a figure eight on S. As the lengths of the boundary geodesics go to zero, the length of the figure eight converges to the length of the figure eight on the thrice punctured sphere which is $m_1(\mathcal{M}_S)$. Since the thrice punctured sphere is not in $\mathcal{M}_S$, and since any compact surface with geodesic boundary contains a pair of pants with geodesic boundary, this argument may be used to show that $m_1(\mathcal{M}_S)$ is not realized for any compact hyperbolic surface S with (possibly empty) geodesic boundary.

We say that the closed curve ω on the topological surface X is *filling* (or fills X), if $\omega \cap \eta \neq \emptyset$ for all non-peripheral simple closed curves η on X. Thus a filling curve in a hyperbolic surface S has complementary components in S that are either discs, punctured discs, or annuli.

Lemma 5.1. *Let ω be a filling geodesic on the hyperbolic surface S. Then*

1. $$Area(S) \leq 2\pi|\omega \cap \omega|. \quad (29)$$

2. $\frac{\ell(\gamma)}{\ell(\omega)} \leq \frac{1}{2}|\gamma \cap \omega|$, *for any non-peripheral simple closed geodesic γ.*

Proof. To prove item (1), First put ω in general position (that is, with all simple intersections); The number of intersections is then $|\omega \cap \omega|$. Next plug-up all boundary components and punctures of S by attaching discs to each boundary and points to each punctured disc to get the closed surface S'. Hence $\chi(S) = \chi(S') - n$, where n is the number of components of S. Put an orientation on ω and note that ω induces a cell decomposition of S'. The number of vertices of this decomposition is $|\omega \cap \omega|$ and since each vertex has exactly two outgoing edges, the number of edges is $2|\omega \cap \omega|$. Finally the fact that the number of faces is at least n gives us,

$$\chi(S) = \chi(S') - n \geq -|\omega \cap \omega|. \tag{30}$$

For the proof of item (2), consider the simply connected components $\{P_i\}_{i=1}^m$ of $S - \omega$. Since γ is simple and non-peripheral, these are the only complementary regions of ω that γ may enter. Put an orientation on γ and consider the threads of γ; that is, the smooth connected components of $\gamma \cap P_i$. Corresponding to each intersection of γ with ω is a thread. Let $\{t_i\}$ be the threads. Since each of the P_i are simply connected, $\ell(t_i) \leq \frac{1}{2}\ell(\partial P_i)$ and hence summing over the threads,

$$\ell(\gamma) = \sum_{i=1}^{|\gamma \cap \omega|} \ell(t_i) \leq \frac{1}{2} \sum_{i=1}^{|\gamma \cap \omega|} \ell(\partial P_i) \leq \frac{1}{2}|\gamma \cap \omega| \ell(\omega), \tag{31}$$

where the last inequality uses $\ell(\partial P_i) \leq \ell(\omega)$ for each i. □

For any natural number n the generalized figure eight on a sphere with $(n+2)$ punctures (and hence area $2\pi n$) has intersection number n, and thus inequality (29) is optimal.

We next turn to the realization question for the M_k's. First note that $M_0 = 0$ and is thus not realizable. Next, $M_1 = 4\log(1+\sqrt{2})$ and is known to be the figure eight on the thrice punctured sphere.

Theorem 1.5 (realization). *For each $k \in \mathbb{N}$, M_k is realized as the length of a k-geodesic on some hyperbolic surface.*

We need a simple generalization of the Mumford compactness theorem.

Lemma 5.2. *Given a sequence of hyperbolic surfaces in the moduli space of a finite analytic type hyperbolic surface which go to the boundary of the moduli space, there exists a subsequence $\{S_i\}$ and closed geodesics $\{\gamma_i\}$, $\gamma_i \subset S_i$, so that either*

1. *$\ell(\gamma_i) \to 0$, or*

2. *$\ell(\gamma_i) \to \infty$, and γ_i is a boundary geodesic of S_i.*

Proof. Since S_i has finite topological type, it's double S_i^d (along its boundary geodesics) has finite area. Using the Mumford compactness theorem (see

[4]) on the sequence of doubles $\{S_i^d\}$, we know there exists a sequence of geodesics $\{\gamma_i\}$ on $\{S_i^d\}$ whose lengths go to zero. If infinitely many of these geodesics cross $\partial S_i \subset S_i^d$, then there exists β a component of the ∂S_i, so that $\ell(\beta) \to \infty$. Otherwise, infinitely many of the geodesics $\{\gamma_i\}$ are in $\{S_i\}$ and therefore $\ell(\gamma_i) \to 0$. $\square$

We use the convention that whenever we pass to a subsequence we continue to call the sequence by the same name.

Proof of Theorem 1.5. Fix $k \in \mathbb{N}$ throughout the proof. Realizing M_k will be done through a series of reductions, at each step passing to a subsequence. Let $\{\omega_i\}$ for $i = 1, 2, 3, ...$ be closed geodesics on the hyperbolic surfaces $\{S_i\}$ resp., with $|\omega_i \cap \omega_i| = k$ and $\ell(\omega_i)$ strictly decreasing to M_k. Since a closed geodesic is compact and hence is in a finite part of the surface, we can assume that the surfaces $\{S_i\}$ above have finite topological type.

Step (1). We can assume that the $\{\omega_i\}$ are filling k-geodesics on $\{S_i\}$, respectively. If ω_i is not filling then cut open S_i along a closed geodesic that does not intersect ω_i. Continue until there are not anymore closed geodesics to cut along. This process must stop since at each cutting stage we are never increasing the area of the surface and there are only a finite number of surfaces of area less than a fixed number.

Step (2). Using Lemma 5.1, and the fact that there are only finitely many signatures of surfaces having area bounded by a fixed number, we can pass to a subsequence so that the surfaces $\{S_i\}$ all have the same signature, say (g, n, m). That is, the $\{S_i\}$ are all in the same moduli space.

Step (3). A topologically finite surface has, up to homeomorphism, a finite number of homotopy classes of k-geodesics. Since the $\{\omega_i\}$ are all on the same underlying topological surface, we can assume that the $\{\omega_i\}$ are all in the same free homotopy class.

Step (4). If the metrics $\{S_i\}$ stay in a compact portion of the moduli space then, since the geodesic length function of a free homotopy class is continuous, we are done by passing to a convergent subsequence. Hence we can assume that the metrics $\{S_i\}$ go to the boundary of moduli space.

Step (5). If case (2) in lemma 5.2 occurs, since the $\{\omega_i\}$ are filling it would mean that their lengths go to infinity, a contradiction. If case (1) occurs, and the sequence that goes to 0 is not eventually a sequence of boundary geodesics then again $\ell(\omega_i) \to \infty$. Hence it must be that the boundary geodesics of the $\{S_i\}$ go to zero in length. Furthermore, no homotopy class of simple closed curve has geodesic length in S_i that goes to infinity by lemma 5.1. Thus $S_i \to S_0$, where S_0 is a surface with cusp on the boundary of the moduli space. Since the $\{\omega_i\}$ are in the same free homotopy class and since the geodesic length function on a free homotopy class is continuous on the closure of moduli space, it must be that $\ell(\omega_i) \to \ell(\omega)$, where ω is the k-geodesic on S_0 in the same free homotopy class as the $\{\omega_i\}$. On the other hand, by assumption $\ell(\omega_i) \to M_k$, and thus $\ell(\omega) = M_k$ for ω a k-geodesic on S_0. □

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