Stratifying derived module categories
Stratification de catégories dérivées de modules

Lidia Angeleri Hügel, Steffen Koenig, Qunhua Liu, Dong Yang

Abstract. We use the concept of a recollement to obtain a stratification of the derived module category of a ring which may be regarded as an analogue of a composition series for groups or modules. This analogy raises the problem whether a 'derived' Jordan Hölder theorem holds true; that is, are such stratifications unique up to ordering and equivalence? We explain that this is indeed the case for several classes of rings, including semi-simple rings, commutative noetherian rings, group algebras of finite groups, and finite dimensional algebras which are piecewise hereditary.

Résumé. Nous utilisons la notion de recollement pour obtenir celle d'une stratification de la catégorie dérivée de la catégorie des modules sur un anneau. Ces stratifications sont des analogues des suites de composition pour les groupes et les modules. Nous sommes ainsi amenés à chercher un analogue "dérivé" du théorème de Jordan Hölder: les stratifications sont-elles uniques à l'ordre des facteurs et aux équivalences près? Nous allons expliquer que c'est effectivement le cas pour plusieurs classes d'anneaux, y compris les anneaux semi-simples, les anneaux commutatifs noethériens, les algèbres de groupes de groupes finis et les algèbres de dimension finie qui sont héréditaires par morceaux.

Keywords: derived module category, recollement, piecewise hereditary algebra, group algebra, commutative noetherian ring. **Mot-clés:** catégorie dérivée de modules sur un anneau, recollement, algèbre héréditaire par morceaux, algèbre d'un groupe, anneau commutatif noethérien.

1. Introduction

Recollements of triangulated categories provide a tool for deconstructing the derived category of a ring A into smaller pieces. A recollement of a triangulated category $\mathcal{D}$ is given by six functors arranged in a diagram of the form

$$\mathcal{Y} \quad \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} \quad \mathcal{D} \quad \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} \quad \mathcal{X}$$

where $\mathcal{Y}$ and $\mathcal{X}$ are a triangulated subcategory and a triangulated quotient category of $\mathcal{D}$ with the property that $\mathcal{D}$ is obtained by glueing $\mathcal{Y}$ and $\mathcal{X}$; see below for a precise definition. We will regard recollements as analogues of short exact sequences.

In our case $\mathcal{D}$ will either be the unbounded derived category $D(\text{Mod}-A)$ of the category of all (right) A-modules for some ring A, or the bounded derived category $D^b(\text{mod}-A)$ of the category of finitely generated A-modules, and we will consider the following two kinds of recollements:

(a) recollements of $D(\text{Mod}-A)$ where the two outer terms are again unbounded derived categories of some rings B and C,
(b) recollements of $D^b(\text{mod}-A)$ where the two outer terms are again bounded derived categories of the category of finitely generated modules over some rings B and C.

Typically, the rings B and C in the two outer terms are less complicated than A. For example, in a recollement of type (b) where A, B, C are finite dimensional algebras, the algebras B and C have smaller Grothendieck rank. One can then study A by investigating the two outer rings. This reduction can be employed to discuss homological properties. Indeed, since recollements induce long exact sequences for various cohomology theories, A and the outer terms share several homological invariants. For instance, A has finite global or finitistic dimension if and only if B and C have so as well, see [17, 31, 22, 4]. A recollement of the derived category of A by derived categories of rings that are well understood can then be used to compute homological invariants inductively, see for example [20, 21, 23].

Recollements are analogues of short exact sequences, deconstructing $\mathcal{D}$ into $\mathcal{X}$ and $\mathcal{Y}$. Continuing this procedure by deconstructing $\mathcal{X}$ and $\mathcal{Y}$, and so on, can be seen as stratifying $\mathcal{D}$ by 'smaller' derived categories. If this process ends with derived categories that cannot be stratified further, these 'simple' strata can be seen as composition factors of the original derived category $\mathcal{D}$. At this point, there are basic questions to be asked, for any given class of rings or algebras:

(1) Is there a finite stratification? Are then all stratifications finite?
(2) Are the strata unique, up to ordering and derived equivalence?
(3) Which derived categories are simple?

The first two questions are asking for a version of the Jordan Hölder theorem for derived categories.

The questions above have come up about twenty years ago in the context of viewing categories of algebraic Lie theory as highest weight categories and as module categories of quasi-hereditary algebras, see [11, 22, 31] for first results. Only recently, however, the technology available for triangulated and derived categories has advanced far enough to allow answers to such questions. These answers are being surveyed here. For general rings, questions (1) and (2) have negative answers; this rules out the possibility of obtaining results by purely formal arguments. Derived Jordan Hölder theorems have, however, been established for piecewise hereditary algebras, including quiver algebras and 'canonical' algebras belonging to weighted projective lines, as well as for group algebras of finite groups, in any characteristic. In the latter case the result extends Maschke's theorem by proving that in any characteristic, group algebras are 'derived semisimple'. The large class of 'derived simple' rings moreover includes all indecomposable commutative rings.

There is another emerging application of recollements, hidden behind some results to be discussed below: a close connection to tilting theory that can be employed to classify tilting objects, see [1, 5] for further information.

2. Recollements

Let $\mathcal{X}, \mathcal{Y}$ and $\mathcal{D}$ be triangulated categories. $\mathcal{D}$ is said to be a *recollement* of $\mathcal{X}$ and $\mathcal{Y}$ if there are six triangle functors as in the following diagram

$$\mathcal{Y} \quad \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{i_*=i_!} \\ \xleftarrow[\;i^!\;]{} \end{array} \quad \mathcal{D} \quad \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{j^!=j^*} \\ \xleftarrow[\;j_*\;]{} \end{array} \quad \mathcal{X}$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$, (j^*, j_*) are adjoint pairs;
(2) i_*, j_*, $j_!$ are full embeddings;
(3) $i^! \circ j_* = 0$ (and thus also $j^! \circ i_! = 0$ and $i^* \circ j_! = 0$);
(4) for each $C \in \mathcal{D}$ there are canonical triangles given by the adjunction morphisms

$$i_! i^!(C) \to C \to j_* j^*(C) \to$$

$$j_! j^!(C) \to C \to i_* i^*(C) \to$$

Recollements were introduced by Beilinson, Bernstein and Deligne [8] in 1982. In algebraic geometry, they form a natural habitat for Grothendieck's six functors relating sheaves on a topological space with sheaves on a closed subspace and its open complement. Recollements of derived module categories occurred first around 1990 in the work of Cline, Parshall and Scott on highest weight categories [11]: the main examples were derived module categories of quasi-hereditary algebras, which admit iterated recollements by derived categories of vector spaces.

Example 2.1. *The standard recollement.* Let A be a ring and $e \in A$ an idempotent. According to [12], the corresponding idempotent ideal $J = AeA$ is *stratifying*, that is, $Ae \otimes^{\mathbf{L}}_{eAe} eA \cong AeA$, if and only if the canonical ring epimorphism $A \to A/J$ induces an embedding of derived categories $D(\mathrm{Mod}{-}A/J) \hookrightarrow D(\mathrm{Mod}{-}A)$. In this case we obtain a recollement

$$D(\mathrm{Mod}{-}A/J) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} D(\mathrm{Mod}{-}A) \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} D(\mathrm{Mod}{-}eAe)$$

In particular, every heredity ideal is stratifying. Recall that a two-sided ideal J of a finite dimensional algebra A is a *heredity ideal* if $J = AeA$ for some idempotent e such that eAe is a semi-simple algebra and J is projective as an A-module. Further, A is *quasi-hereditary* if it admits a *heredity chain*, that is, a chain $0 = J_0 \subset J_1 \subset \ldots \subset J_s = A$ of two-sided ideals such that J_i/J_{i-1} is a heredity ideal of A/J_{i-1} for all $i \geq 1$. We then obtain a sequence of iterated recollements, that is, a stratification of $D(\mathrm{Mod}{-}A)$ with strata being derived categories of simple algebras.

Schur algebras of classical groups are known to be quasi-hereditary, as well as blocks of the BGG-category $\mathcal{O}$ of a semisimple complex Lie algebra. In the latter case, the derived category is equivalent to a derived category of sheaves on a flag variety G/B, which itself by the Bruhat decomposition is geometrically stratified into Schubert cells BwB/B. This has motivated the definition of recollements of triangulated categories and of perverse sheaves in [8].

Example 2.2. *Recollements induced by large tilting modules.* Let T_C be a tilting module of projective dimension one over a ring C and let $A = \mathrm{End}(T_C)$. When T_C is finitely presented, the functors $j^* = - \otimes^{\mathbf{L}}_A T$ and $j_* = \mathbf{R}\,\mathrm{Hom}_C(T, -)$ define an equivalence between $D(\mathrm{Mod}{-}A)$ and $D(\mathrm{Mod}{-}C)$ by a well known result from [15]. In general, however, $D(\mathrm{Mod}{-}C)$ is only equivalent to the Verdier quotient of $D(\mathrm{Mod}{-}A)$ with respect to the kernel of the functor j^* (provided T_C is a 'good' tilting module, see [6]). Recently, Chen and Xi [9] have shown that j^* and j_* actually belong to a recollement

$$D(\mathrm{Mod}{-}B) \quad \begin{matrix}\xleftarrow{\;i^*\;}\\ \xrightarrow{i_*=i_!}\\ \xleftarrow[\;i^!\;]{}\end{matrix} \quad D(\mathrm{Mod}{-}A) \quad \begin{matrix}\xleftarrow{\;j_!\;}\\ \xrightarrow{j^!=j^*}\\ \xleftarrow[\;j_*\;]{}\end{matrix} \quad D(\mathrm{Mod}{-}C)$$

where the ring B on the left hand side can be computed as universal localisation (see [30]) of A at a certain map between finitely generated projective A-modules.

Examples 2.1 and 2.2 are special cases of the following construction.

Example 2.3. *Recollements induced by homological ring epimorphisms.* Let $\lambda\colon A \to B$ be a ring epimorphism, that is, an epimorphism in the category of rings. Following [14], we say that λ is a *homological epimorphism* if $\mathrm{Tor}^A_i(B, B) = 0$ for all $i > 0$, or equivalently, if λ induces a fully faithful functor $i_* = \lambda_* : D(\mathrm{Mod}{-}B) \to D(\mathrm{Mod}{-}A)$. The ring epimorphism $A \to A/J$ given by a stratifying ideal in 2.1 is thus an example of a homological epimorphism. Also universal localisation is often a homological epimorphism.
It is shown in [26, 27] that recollements are closely related to the more general notion of differential graded homological epimorphism. In particular, every homological ring epimorphism $\lambda : A \to B$ gives rise to a recollement

$$D(\mathrm{Mod}{-}B) \quad \begin{matrix}\xleftarrow{\;i^*\;}\\ \xrightarrow{i_*=i_!}\\ \xleftarrow[\;i^!\;]{}\end{matrix} \quad D(\mathrm{Mod}{-}A) \quad \begin{matrix}\xleftarrow{\;j_!\;}\\ \xrightarrow{j^!=j^*}\\ \xleftarrow[\;j_*\;]{}\end{matrix} \quad \mathcal{X}$$

where $i^* = - \otimes^{\mathbf{L}}_A B$, $i^! = \mathbf{R}\,\mathrm{Hom}_A(B, -)$, and $j^* = - \otimes^{\mathbf{L}}_A X$ with X given by the triangle $X \to A \xrightarrow{\lambda} B \to$. The triangulated category $\mathcal{X}$ on the right hand side, however, need not be a derived category of a ring.

For example, consider the Kronecker algebra A over an algebraically closed field k. The derived category $D(\text{Mod}-A)$ has a geometric interpretation as category of quasi-coherent sheaves on a projective line [7], and it has the standard recollement from 2.1

$$D(\text{Mod}-k) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-A) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-k)$$

On the other hand, there is a homological ring epimorphism $A \to B$ where B is a simple artinian ring not Morita equivalent to k, which is obtained as universal localisation of A at the union $\mathbf{t}$ of all tubes in the Auslander Reiten quiver, cf. [13]. This homological epimorphism gives rise to a recollement

$$D(\text{Mod}-B) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-A) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} \mathcal{X}$$

where the right hand side $\mathcal{X}$ is the smallest localising subcategory of $D(\text{Mod}-A)$ containing $\mathbf{t}$. Notice that $\mathcal{X}$ is not equivalent to the derived category of a ring. Moreover, since there are no maps nor extensions between different tubes, $\mathcal{X}$ can be decomposed further, producing an infinite stratification of $D(\text{Mod}-A)$ by triangulated categories.

The last example shows that if we aim at a uniqueness result for stratifications of derived categories, we have to put some restriction on the triangulated categories allowed to occur in the recollements. Henceforth, we will focus on recollements of type (a) or (b) as in the introduction.

3. Derived simple rings

We now investigate the simple objects in a stratification. We say that a ring A is

- *derived simple* if it does not admit a non-trivial recollement of type (a),
- D^b(mod)-*derived simple* if it does not admit a non-trivial recollement of type (b).

To discuss derived simpleness, we need the following result that characterizes the existence of a recollement of type (a) in terms of a suitable pair of exceptional objects. Recall that $X \in D(\text{Mod}-A)$ is *exceptional* if $\text{Hom}(X, X[n]) = 0$ for all non-zero integers n. Further, X is *compact* if the functor $\text{Hom}(X, -)$ preserves small coproducts, or equivalently, X is quasi-isomorphic to a bounded complex consisting of finitely generated projective modules. Finally, X is *self-compact* if $\text{Hom}(X, -)$ preserves small coproducts inside the localising subcategory of $D(\text{Mod}-A)$ generated by X.

Theorem 3.1. *(*[22, 26, 27]*) There are rings A, B, C with a recollement of the form*

$$D(\text{Mod}-B) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-A) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-C)$$

if and only if there are exceptional objects $X, Y \in D(\text{Mod}-A)$ such that

(i) *X is compact, Y is self-compact, and* $\text{Hom}(X[n], Y) = 0$ *for all* $n \in \mathbb{Z}$,
(ii) *an object $D \in D(\text{Mod}-A)$ is zero whenever* $\text{Hom}(X \oplus Y, D[n]) = 0$ *for every integer n.*

In particular, $X = j_!(C)$, $Y = i_*(B)$, and $C \cong \text{End}(X)$, $B \cong \text{End}(Y)$.

To see an example, suppose now that A is a local ring and $X \in D(\text{Mod}-A)$ is a compact object. Let P be a representative of X in the homotopy category $K^b(\text{proj}-A)$ of bounded complexes of finitely generated projective A-modules. We can choose P such that it has no direct summands of the form $X \xrightarrow{\text{Id}} X$ or its shifts for some finitely generated projective A-module X. Moreover, we can assume that P is of the form $\dots 0 \to P^{-n} \to \dots \to P^0 \to 0 \dots$ where P^0 has degree 0. Since P^{-n} and P^0 are actually free modules, there is a non-zero map $P^{-n} \to P^0$, which, written in matrix form, has zero entries except in one position, where the entry is an isomorphism. It gives rise to a chain map $X \to X[n]$ which is not homotopic to zero. This shows that all compact exceptional objects in $D(\text{Mod}-A)$ are projective modules up to shift. From Theorem 3.1 we deduce that A is derived simple, cf. [2, 4.10].

A similar argument works for simple artinian rings. Moreover, using that over a commutative ring vanishing of $\mathrm{Hom}(X, Y[n])$ is determined locally, it is not hard to see that every indecomposable commutative ring is derived simple [4]. Notice, however, that the derived category $D(\mathrm{Mod}-A)$ may still admit recollements of triangulated categories: indeed, for commutative noetherian rings such recollements are parametrized by the subsets of $\mathrm{Spec}A$ that are closed under specialization [25].

There are also examples of indecomposable finite dimensional algebras with two or more non-isomorphic simple modules that are derived simple [31, 16, 4]. The easiest example of this kind is given by the quiver $1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$ with relations $\alpha\beta = \beta\alpha = 0$.

A further class of examples of derived simple rings is provided by symmetric algebras. It is shown in [24] that a finite dimensional indecomposable symmetric algebra over a field k is derived simple whenever it satisfies the following condition:

($\sharp$) for any finitely generated non-projective A-module M there are infinitely many integers n with $\mathrm{Ext}^n_A(M, M) \neq 0$.

This applies for instance to group algebras of finite groups or to symmetric algebras of finite representation type. Here the main ingredient of the proof is a Calabi-Yau property of the homotopy category $K^b(\mathrm{proj}-A)$. More precisely, given $P \in K^b(\mathrm{proj}-A)$ and $M \in D(\mathrm{Mod}-A)$, there is a bifunctorial isomorphism $D\,\mathrm{Hom}_A(P, M) \cong \mathrm{Hom}_A(M, P)$ where $D = \mathrm{Hom}_k(-, k)$, cf. [28]. Actually, one can even show that every indecomposable triangulated category which is d-Calabi-Yau for some integer d is *simple*, in the sense that it does not admit non-trivial recollements of triangulated categories.

We summarize the results mentioned above.

Theorem 3.2. *The following rings are derived simple:*

- *local rings,*
- *simple artinian rings,*
- *indecomposable commutative rings,*
- *blocks of group algebras of finite groups.*

Let us now turn to $D^b(\mathrm{mod}\ )$-derived simpleness. In [3, 4] we carry out a detailed analysis of lifting and restriction of recollements from $D^b(\mathrm{mod}\ )$ to $D(\mathrm{Mod}\ )$ and vice versa when A is a finite dimensional algebra. In particular, we show that any recollement of type (b) can be lifted to a recollement of type (a). The reason is that the objects $i_*(B)$ and $j_!(C)$ in a recollement of type (b) are always compact and thus yield a pair of exceptional objects as required by Theorem 3.1. On the other hand, restriction of recollements is not always possible. However, if A has finite global dimension, then any recollement of type (a) can be restricted to a recollement of type (b). As a consequence, we obtain

Proposition 3.3. *A finite dimensional algebra over a field is D^b(mod)-derived simple if it is derived simple. The converse holds true provided A has finite global dimension.*

Example 3.4. The radical square zero algebra given by the quiver

$$\gamma \circlearrowright 1 \xleftarrow{\ \alpha\ } 2 \circlearrowleft \beta$$

with relations $\gamma^2 = 0$, $\beta^2 = 0$ and $\alpha \circ \beta = \gamma \circ \alpha = 0$ is an example of a $D^b(\mathrm{mod}\ )$-derived simple algebra which is not derived simple. Here the two indecomposable projective modules P_1 and P_2 and $\mathrm{Cone}(P_2 \overset{\alpha}{\to} P_1)$ are the only indecomposable exceptional compact objects, up to shift and up to isomorphism. They give rise to non-trivial recollements of type (a) that do not restrict to $D^b(\mathrm{mod}\ )$-level. For details, we refer to [4].

Related results can also be found in [2, Section 4]. Finally, let us remark that condition ($\sharp$) is not needed at $D^b(\text{mod })$-level: every finite dimensional indecomposable symmetric algebra over a field k is $D^b(\text{mod })$-derived simple [24].

4. Stratifications

By a *stratification* of $D(\text{Mod}-A)$ we mean a sequence of iterated recollements

$$D(\text{Mod}-B) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-A) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-C)$$

$$D(\text{Mod}-B_1) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-B) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-B_2)$$

$$D(\text{Mod}-C_1) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-C) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\text{Mod}-C_2)$$

$$\vdots$$

that is either infinite or ends when we reach derived simple rings at all positions. That is, a stratification is given by a binary tree with derived simple rings at the leaves.
$D^b(\text{mod })$-*stratifications* are defined correspondingly by using recollements of type (b).

Question 4.1. *Jordan Hölder Theorem for derived categories:* Given a ring A, does $D(\text{Mod}-A)$ (or $D^b(\text{mod}-A)$) have a finite ($D^b(\text{mod })$-)stratification which is unique up to ordering and derived equivalence of the strata?

In general, the answer will be negative. Indeed, a counterexample for the existence of finite stratifications is provided by the countable product $A = k^{\mathbb{N}}$ of a field k, see [2, 5.2]. The question of uniqueness is much more subtle. Chen and Xi exhibit in [9, 10] rather sophisticated examples of hereditary non-artinan rings where uniqueness fails by using the constructions described in 2.1 and 2.2. We will see below, however, that the Jordan Hölder Theorem holds true for derived categories of finite dimensional hereditary algebras.

We start with the following observation. If a ring A has a block decomposition $A = A_1 \oplus \ldots \oplus A_s$ with derived simple blocks, then the Jordan Hölder theorem holds true. Indeed, the simple factors of any stratification are exactly the derived categories of $A_1, \ldots, A_s$. For details, we refer to [4]. Combining this with Theorem 3.2, we obtain:

Corollary 4.2. *Let A be a semisimple ring, or a commutative noetherian ring, or the group algebra of a finite group. Then $D(\text{Mod}-A)$ has a finite stratification whose factors are the derived categories of the blocks of A. Any stratification of $D(\text{Mod}-A)$ has precisely these factors, up to ordering and equivalence. The corresponding result holds true for $D^b(\text{mod })$-stratifications.*

The stratifications occuring here are just direct sum decompositions. Thus, the corollary can be restated as saying that group algebras of finite groups are derived semisimple. This can be seen as a categorical analogue of Maschke's theorem, holding also in the modular case where the characteristic of the ground field divides the group order.

Group algebras, when not being semisimple, have plenty of cohomology, and we have seen that the proof makes strong use of that. A very different class of algebras are path algebras of quivers; these are hereditary and there is no cohomology in degrees bigger than one. For hereditary algebras, or more generally for piecewise hereditary algebras, there are many non-trivial recollements, but the answer to Question 4.1 is still positive.

Recall that a finite dimensional algebra A over a field k is called *piecewise hereditary* if there exists a hereditary and abelian category $\mathcal{H}$ such that the bounded derived categories $D^b(\text{mod}-A)$ and $D^b(\mathcal{H})$ are equivalent as triangulated categories. In other words, there exists a tilting complex T in $D^b(\mathcal{H})$ with endomorphism ring being A. By a result of Happel [18], extended by Happel and Reiten [19], $\mathcal{H}$ is, up to derived equivalence, either the category mod-H for some finite dimensional hereditary k-algebra H, or the category $\mathrm{coh}(X)$ of coherent sheaves on an exceptional curve X (which is a weighted projective line in the sense of [14] when k is algebraically closed). In the second case, there is a 'standard' tilting object T in $\mathcal{H} = \mathrm{coh}(X)$ with endomorphism ring being a canonical algebra in the sense of [29]. So, the algebra A is derived equivalent to a hereditary algebra or to a canonical algebra.

Theorem 4.3. [2, 3] *Let A be a piecewise hereditary finite dimensional algebra over a field k, and let $S_1, \dots, S_n$ be representatives of the isomorphism classes of simple A-modules. Then $D(\mathrm{Mod}{-}A)$ has a stratification whose factors are the derived categories of $\mathrm{End}_A(S_1), \dots, \mathrm{End}_A(S_n)$. Any stratification of $D(\mathrm{Mod}{-}A)$ has precisely these factors, up to ordering and equivalence. The corresponding result holds true for D^b(mod)-stratifications.*

The proof of uniqueness relies on the connection between recollements and exceptional objects. Recall from Theorem 3.1 that every recollement of type (a) provides us with a compact exceptional object $X = j_!(C) \in D(\mathrm{Mod}{-}A)$. When A is piecewise hereditary, we also have a converse.

Proposition 4.4. *Let A be a piecewise hereditary finite dimensional algebra over a field k with n simple modules, and let $X \in D(\mathrm{Mod}{-}A)$ be a compact exceptional object. Then there is a recollement*

$$D(\mathrm{Mod}{-}B) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\mathrm{Mod}{-}A) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\mathrm{Mod}{-}C)$$

where B and C are piecewise hereditary algebras with at most $n-1$ simple objects. More precisely, $C \cong \mathrm{End}(X)$, and there is a homological ring epimorphism $A \to B$.

As a consequence of Proposition 4.4, we obtain the following stronger version of Theorem 4.3 establishing a 'normal form' for stratifications of derived categories of piecewise hereditary algebras.

Proposition 4.5. *Let A be a piecewise hereditary finite dimensional algebra over a field k with n simple modules. Any stratification of $D(\mathrm{Mod}{-}A)$ can be rearranged into a chain of increasing derived module categories*

$$D(\mathrm{Mod}{-}A_n) \hookrightarrow \dots \hookrightarrow D(\mathrm{Mod}{-}A_2) \hookrightarrow D(\mathrm{Mod}{-}A)$$

corresponding to a chain of homological epimorphisms

$$A = A_1 \to A_2 \to \dots \to A_n,$$

where for each $1 \leq i < n$ there is a recollement

$$D(\mathrm{Mod}{-}A_{i+1}) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\mathrm{Mod}{-}A_i) \overset{\longleftarrow}{\underset{\longleftarrow}{\longrightarrow}} D(\mathrm{Mod}{-}C_i)$$

with $C_1, \dots, C_{n-1}$ and A_n being derived simple.

References

[1] L. Angeleri Hügel, S. Koenig, Q. Liu, Recollements and tilting objects, J. Pure Appl. Algebra 215 (2011), 420–438. Also arXiv:0908.1988.

[2] L. Angeleri Hügel, S. Koenig, Q. Liu, On the uniqueness of stratifications of derived module categories, preprint (2009), arXiv:1006.5301.

[3] L. Angeleri Hügel, S. Koenig, Q. Liu, Jordan Hölder theorems for derived module categories of piecewise hereditary algebras, preprint (2011), arXiv:1104.3418.

[4] L. Angeleri Hügel, S. Koenig, Q. Liu, D. Yang, On derived simple algebras, in preparation.

[5] L. Angeleri Hügel, J. Sánchez, Tilting modules over tame hereditary algebras, preprint (2010).

[6] S. Bazzoni, Equivalences induced by infinitely generated tilting modules, Proc. Amer. Math. Soc. 138 (2010), 533–544.

[7] A. A. Beilinson, Coherent sheaves on P^n and problems in linear algebra. (Russian) Funktsional. Anal. i Prilozhen. 12 (1978), no. 3, 68–69.

[8] A. A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque 100 (1982).

[9] H. X. Chen and C. C. Xi, Good tilting modules and recollements of derived modules categories, preprint (2010), arXiv:1012.2176
[10] H. X. Chen, C. C. Xi, Stratifications of derived categories from tilting modules over tame hereditary algebras, preprint (2011).
[11] E. Cline, B. Parshall and L. Scott, Finite-dimensional algebras and highest weight categories. J. Reine Angew. Math. 391 (1988), 85–99.
[12] E. Cline, B. Parshall, L. Scott, Stratifying endomorphism algebras. Mem. Amer. Math. Soc. 124, (1996), no. 591.
[13] W. Crawley-Boevey, Regular modules for tame hereditary algebras, Proc. London Math. Soc. 62 (1991) 490–508.
[14] W. Geigle, H. Lenzing, Perpendicular categories with applications to representations and sheaves, J. Algebra 144 (1991), 273–343.
[15] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Mathematical Society Lecture Note Series, 119. Cambridge University Press, 1988.
[16] D. Happel, A family of algebras with two simple modules and Fibonacci numbers. Arch. Math. (Basel) 57 (1991), no. 2, 133–139.
[17] D. Happel, Reduction techniques for homological conjectures, Tsukuba J. Math. 17 (1993), no.1, 115–130.
[18] D. Happel, A characterization of hereditary categories with tilting object. Invent. Math. 144 (2001), no. 2, 381–398.
[19] D. Happel, I. Reiten, Hereditary abelian categories with tilting object over arbitrary base fields, J. Algebra 256 (2002), no. 2, 414–432.
[20] B. Keller, Invariance and localization for cyclic homology of DG algebras, J. Pure Appl. Algebra 123 (1998), no. 1-3, 223–273.
[21] B. Keller, On the cyclic homology of exact categories, J. Pure Appl. Algebra 136 (1999), no. 1, 1–56.
[22] S. Koenig, Tilting complexes, perpendicular categories and recollements of derived module categories of rings, J. Pure Appl. Algebra 73 (1991), 211–232.
[23] S. Koenig, H. Nagase, Hochschild cohomology and stratifying ideals. J. Pure Appl. Algebra 213 (2009), no. 5, 886–891.
[24] Q. Liu, D. Yang, Blocks of group algebras are derived simple, preprint (2011), arXiv:1104.0500.
[25] A. Neeman, The chromatic tower for D(R). With an appendix by Marcel Bökstedt. Topology 31 (1992), no. 3, 519–532.
[26] P. Nicolás, On torsion torsionfree triples, PhD thesis (Murcia 2007), arXiv:0801.0507.
[27] P. Nicolás, M. Saorín, Parametrizing recollement data for triangulated categories. J. Algebra. 322 (2009), 1220–1250.
[28] J. Rickard, Equivalences of derived categories for symmetric algebras, J. Algebra 257 (2002), 460–481.
[29] C. M. Ringel, The canonical algebras, with an Appendix by W. Crawley-Boevey, Topics in Algebra, Banach center publications, vol. 26, part 1, PWN-Polish science publishers, Warsaw 1990.
[30] A. Schofield, Representations of rings over skew fields, London Mathematical Society Lecture Note Series 2, Cambridge University Press, Cambridge, 1985.
[31] A. Wiedemann, On stratifications of derived module categories, Canad. Math. Bull. 34 (1991), no. 2, 275–280.

Lidia Angeleri Hügel, Dipartimento di Informatica - Settore Matematica, Università degli Studi di Verona, Strada Le Grazie 15 - Ca' Vignal 2, I - 37134 Verona, Italy
E-mail address: lidia.angeleri@univr.it

Steffen Koenig, Qunhua Liu, Institute for Algebra and Number Theory, University of Stuttgart, Pfaffenwaldring 57, 70569 Stuttgart, Germany
E-mail address: skoenig@mathematik.uni-stuttgart.de, qliu@mathematik.uni-stuttgart.de

Dong Yang, Hausdorff Research Institute for Mathematics, Poppelsdorfer Allee 82, D-53115 Bonn, Germany
E-mail address: dongyang2002@gmail.com