DEGENERATE FLAG VARIETIES OF TYPE A: FROBENIUS SPLITTING AND BWB THEOREM

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Abstract. Let $\mathcal{F}^a_\lambda$ be the PBW degeneration of the flag varieties of type A_{n-1}. These varieties are singular and are acted upon with the degenerate Lie group SL^a_n. We prove that $\mathcal{F}^a_\lambda$ have rational singularities, are normal and locally complete intersections, and construct a desingularization R_λ of $\mathcal{F}^a_\lambda$. The varieties R_λ can be viewed as towers of successive $\mathbb{P}^1$-fibrations, thus providing an analogue of the classical Bott-Samelson-Demazure-Hansen desingularization. We prove that the varieties R_λ are Frobenius split. This gives us Frobenius splitting for the degenerate flag varieties and allows to prove the Borel-Weyl-Bott type theorem for $\mathcal{F}^a_\lambda$. Using the Atiyah-Bott-Lefschetz formula for R_λ, we compute the q-characters of the highest weight $\mathfrak{sl}_n$-modules.

Introduction

Let $\mathfrak{g}$ be a simple Lie group and G be the Lie group of $\mathfrak{g}$. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$. Let $\mathfrak{g}^a$ and G^a be the degenerate Lie algebra and Lie group (see [Fe1], [Fe2]). Namely, $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is an abelian ideal isomorphic to $\mathfrak{n}^-$ as a vector space and $\mathfrak{b}$ acts on $(\mathfrak{n}^-)^a$ via the isomorphism $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$. The Lie group G^a is a semi-direct product of the Borel subgroup B and the normal abelian group $\mathbb{G}_a^{\dim \mathfrak{n}}$, where $\mathbb{G}_a = (\mathbb{C}, +)$ is the additive group of the field.

Consider the complete flag variety $\mathcal{F} = G/B$. This variety has a degenerate version $\mathcal{F}^a$ (see [Fe1], [Fe2]). In this paper we are concerned with the case $G = SL_n$. We denote the corresponding classical flag variety by $\mathcal{F}_n$ and the degenerate version by $\mathcal{F}^a_n$. For simplicity, we consider only the case of the complete flag varieties in the Introduction. However, in the main body of the paper we work out the case of general (parabolic) flag varieties as well. The varieties $\mathcal{F}^a_n$ are singular projective algebraic varieties, which can be explicitly described as follows. Fix a basis $w_1, \dots, w_n$ in an n-dimensional vector space W and define the projection operators $pr_d : W \to W$, $pr_d(\sum_{i=1}^n c_i w_i) = \sum_{i \neq d} c_i w_i$. Let us denote by $Gr(d, n)$ the Grassmannian of d-dimensional subspaces in W. Then $\mathcal{F}^a_n$ is the variety of collections $(V_1, \dots, V_{n-1})$ of subspaces, $V_d \in Gr(d, n)$ such that

$$pr_{d+1} V_d \subset V_{d+1},\ d = 1, \dots, n-2.$$

The group G^a acts on $\mathcal{F}^a_n$ with an open $\mathbb{G}_a^{\dim \mathfrak{n}}$-orbit. The varieties $\mathcal{F}^n_a$ are flat degenerations of the classical flags $\mathcal{F}_n$. Our first theorem is as follows:

Theorem 0.1. *The varieties $\mathcal{F}^a_n$ are normal locally complete intersections (in particular, Cohen-Macaulay and even Gorenstein).*

Recall (see [FFL1], [FFL2]) that for each dominant integral $\mathfrak{g}$-weight λ there exists a $\mathfrak{g}^a$-module V^a_λ,. which is the associated graded to V_λ with respect to the PBW filtration. Similar to the classical situation (see [K]), there exists a map $\imath_\lambda$ from $\mathcal{F}^a_n$ to the projectivization $\mathbb{P}(V^a_\lambda)$ (this map is an embedding if λ is regular). Therefore, one can pull back the line bundles $\mathcal{O}(1)$ from the projective space to $\mathcal{F}^a_n$. We prove the following theorem, which is the degenerate analogue of the Borel-Weyl-Bott theorem:

Theorem 0.2. *Let $\mathfrak{g} = \mathfrak{sl}_n$. For any dominant integral weight λ one has:*

$$H^0(\mathcal{F}^a_n, \imath^*_\lambda \mathcal{O}(1))^* \simeq V^a_\lambda, \ H^{>0}(\mathcal{F}^a_n, \imath^*_\lambda \mathcal{O}(1)) = 0.$$

We note that this theorem agrees with the fact that the varieties $\mathcal{F}^a_n$ are flat degenerations of the classical flags $\mathcal{F}_n$. Our main tool for the proof of Theorems 0.1 and 0.2 is an explicit construction for desingularization of $\mathcal{F}^a_n$. Namely, consider the variety R_n consisting of collections of subspaces $V_{i,j}$, $1 \le i \le j \le n-1$ such that $V_{i,j} \in Gr(i,n)$ and the following conditions hold:

- $V_{i,j} \subset \mathrm{span}(w_1, \dots, w_i, w_{j+1}, \dots, w_n)$,
- $V_{i,j} \subset V_{i+1,j}, \ V_{i,j} \subset V_{i,j+1} \oplus \mathbb{C}w_{j+1}$.

We show that R_n is a successive tower of $\mathbb{P}^1$ fibrations (and thus smooth) and the map $\pi_n : \ R_n \to \mathcal{F}^a_n$ sending $(V_{i,j})_{1\le i\le j<n}$ to $(V_{i,i})_{i=1}^{n-1}$ is a birational isomorphism. We would like to emphasize the following peculiarity of the resolution $\pi_n : \ R_n \to \mathcal{F}^a_n$ which is in a sharp contrast with the classical Bott-Samelson-Demazure-Hansen resolution. The computer verification shows that π_n is *small* for $n \le 7$, and *semismall* for $n \le 10$. We conjecture that π_n is always semismall. Now the degenerate Borel-Weyl-Bott theorem follows from the following result:

Theorem 0.3. *The varieties $\mathcal{F}^a_n$ and R_n over $\overline{\mathbb{F}}_p$ are Frobenius split. The varieties $\mathcal{F}^a_n$ have rational singularities.*

For the proof we use the Mehta-Ramanathan criterion [MR]. Using the Atiyah-Bott-Lefschetz formula ([AB], [T]) we deduce from Theorem 0.2 a q-character formula for the characters of V^a_λ. The formula is a sum of contributions of the $2^{\dim \mathfrak{n}}$ torus fixed points in R_n.

Our paper is organized as follows:
In Section 1 we recall main definitions and fix notations.
In Section 2 we construct the desingularizations R_λ for the degenerate flag varieties.
In Section 3 we prove that the varieties $\mathcal{F}^a_\lambda$ are normal locally complete intersections.
In Section 4 we prove that the varieties $\mathcal{F}^a_\lambda$ and their desingularizations are Frobenius split.
In Section 5 we prove that the varieties $\mathcal{F}^a_\lambda$ have rational singularities, and

use the results of the previous sections to deduce the analogue of the Borel-Weyl-Bott theorem and the q-character formula for V^a_λ.

1. Definitions and notations

Let $\mathfrak{g}$ be a simple Lie algebra. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Let R_+ be the set of positive roots for $\mathfrak{g}$ and α_d, ω_d, $d = 1, \dots, \mathrm{rk}(\mathfrak{g})$ be the simple roots and fundamental weights (see [FH]). For a positive root α we sometimes write $\alpha > 0$ instead of $\alpha \in R_+$. Let f_α, $\alpha > 0$ be an $\mathfrak{h}$-eigenbasis of $\mathfrak{n}^-$ and, similarly, e_α for $\mathfrak{n}$. We denote by G, B, N, T, N^- the Lie groups of $\mathfrak{g}, \mathfrak{b}, \mathfrak{n}, \mathfrak{h}, \mathfrak{n}^-$.

Let $(\mathfrak{n}^-)^a$ be an abelian Lie algebra with the underlying vector space $\mathfrak{n}^-$. The degenerate Lie algebra $\mathfrak{g}^a$ is isomorphic to $\mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where both $\mathfrak{b}$ and $(\mathfrak{n}^-)^a$ are subalgebras, $(\mathfrak{n}^-)^a$ is an abelian ideal and the structure of the $\mathfrak{b}$-module on $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$ is induced by the adjoint action (see [Fe1], [Fe2]). We denote the corresponding degenerate group by G^a. Thus, $G^a \simeq B \ltimes (N^-)^a$, where $(N^-)^a$ is an abelian Lie group with the Lie algebra $(\mathfrak{n}^-)^a$, $(N^-)^a \simeq \mathbb{G}_a^M$, where $\mathbb{G}_a = (\mathbb{C}, +)$ is the abelian group of the field and $M = \dim \mathfrak{n}$ is the number of positive roots.

Let λ be a dominant integral weight of $\mathfrak{g}$ and let V_λ be the corresponding irreducible $\mathfrak{g}$-module with a highest weight vector v_λ. We have $\mathfrak{n} v_\lambda = 0$, $h v_\lambda = \lambda(h) v_\lambda$ and $V_\lambda = U(\mathfrak{n}^-) v_\lambda$. We denote by $\mathcal{F}_\lambda$ the generalized flag variety:

$$\mathcal{F}_\lambda = G \cdot \mathbb{C} v_\lambda = \overline{N^- \cdot \mathbb{C} v_\lambda} \subset \mathbb{P}(V_\lambda).$$

For example, for $\mathfrak{g} = \mathfrak{sl}_n$ the varieties $\mathcal{F}_{\omega_d}$ are isomorphic to the Grassmannians $Gr(d, n)$ and for regular λ ($(\lambda, \omega_d) > 0$ for all d) the corresponding flag variety $\mathcal{F}_\lambda$ is isomorphic to the variety of complete flags in $\mathbb{C}^n$. Denote by $U(\mathfrak{n}^-)_k$ the PBW (standard) filtration of the universal enveloping algebra $U(\mathfrak{n}^-)$:

$$U(\mathfrak{n}^-)_k = \mathrm{span}(x_1 \dots x_l, x_i \in \mathfrak{n}^-, l \le k).$$

The PBW filtration $U(\mathfrak{n}^-)_k v_\lambda$ on V_λ is induced by the degree filtration. We denote by V^a_λ the associated graded module:

$$V^a_\lambda = \bigoplus_{k \ge 0} V^a_\lambda(k) = \bigoplus_{k \ge 0} U(\mathfrak{n}^-)_k v_\lambda / U(\mathfrak{n}^-)_{k-1} v_\lambda.$$

The q-character of V_λ (the character of V^a_λ) is defined by the formula

$$\mathrm{ch}_q V^a_\lambda = \bigoplus_{k \ge 0} q^k \mathrm{ch} V_\lambda(k).$$

It is easy to see that the structure of $\mathfrak{g}$-module on V_λ induces the structures of $\mathfrak{g}^a$- and G^a-module on V^a_λ. In particular, $V^a_\lambda = \mathbb{C}[f_\alpha]_{\alpha > 0} v_\lambda$. The corresponding degenerate flag variety $\mathcal{F}^a_\lambda \subset \mathbb{P}(V^a_\lambda)$ is defined as the closure of the orbit of the line containing v_λ:

$$\mathcal{F}^a_\lambda = \overline{G^a \cdot \mathbb{C} v_\lambda} = \overline{(N^-)^a \cdot \mathbb{C} v_\lambda}.$$

In particular, $\mathcal{F}^a_\lambda$ are the $\mathbb{G}_a^M$-varieties (see [A], [AS], [HT]).

It is convenient to consider an extension $\mathfrak{g}^a \oplus \mathbb{C}d$ of the algebra $\mathfrak{g}^a$, where d is the PBW grading operator, i.e. $[d, \mathfrak{b}] = 0$ and $[d, f_\alpha] = f_\alpha$ for any positive α. All the $\mathfrak{g}^a$-modules V_λ^a can be made into the $\mathfrak{g}^a \oplus \mathbb{C}d$-modules by setting $d = k$ on $V_\lambda^a(k)$. The corresponding extended group is $G^a \rtimes \mathbb{C}^*$. In particular, the torus acting on $\mathbb{P}(V_\lambda^a)$ is of dimension $rk(\mathfrak{g}) + 1$.

From now on we fix $\mathfrak{g} = \mathfrak{sl}_n$, $G = SL_n$. Then all positive roots are of the form $\alpha_{i,j} = \alpha_i + \cdots + \alpha_j$, $1 \le i \le j < n$.

Example 1.1. Let $\lambda = \omega_d$. Then $V_{\omega_d}^a = \bigoplus_{k=0}^{\min(d,n-d)} V_{\omega_d}^a(k)$. The space $V_{\omega_d}^a(k)$ has a basis $w(S)$ labeled by collections $S = (l_1 < \cdots < l_d)$ such that $1 \le l_i \le n$ and $\#\{i: \ l_i > d\} = k$. We note that $w(S)$ are the images of the wedges $w_{l_1} \wedge \cdots \wedge w_{l_d}$. The operators $f_{i,j}$ act trivially on $V_{\omega_d}^a$ unless $i \le d \le j$. If this condition is satisfied, then $f_{i,j}$ acts via the usual formula for the action on a wedge power. Similarly, the operators $e_{i,j}$ act trivially unless $i > d$ or $j < d$. The non-trivial operators act by the usual formula.

In contrast with the classical situation, a representation $V_{\omega_d}^a$ is no longer isomorphic to $\bigwedge^d(V_{\omega_1}^a)$. However, $V_{\omega_d}^a$ can be constructed as a wedge power of another $\mathfrak{g}^a$-module. Namely, let $W^{(d)}$ be an n-dimensional vector space with a basis $w_1, \dots, w_n$. We define a structure of $\mathfrak{g}^a$-module on $W^{(d)}$ as follows: $f_{\alpha_{i,j}}$ acts trivially unless $i \le d \le j$ and $e_{\alpha_{i,j}}$ acts trivially unless $j < d$ or $i > d$. The non-trivial operators act by the usual formulas:

$$f_{\alpha_{i,j}} w_k = \delta_{i,k} w_{j+1},\ e_{\alpha_{i,j}} w_k = \delta_{j+1,k} w_i.$$

Then $V_{\omega_d}^a \simeq \bigwedge^d(W^{(d)})$. The following simple lemma will be important for us:

Lemma 1.2. *For all* $1 \le i \le j < n$ *the subspaces* $\mathrm{span}(w_{i+1}, \dots, w_j) \subset W^{(i)}$ *are* $\mathfrak{g}^a$*-invariant, making the quotients*

$$W_{i,j} = W^{(i)}/\mathrm{span}(w_{i+1}, \dots, w_j)$$

into $\mathfrak{g}^a$*- and* G^a*-modules.*

In what follows we denote the images in $W_{i,j}$ of the basis vectors w_k by the same symbols w_k. For instance, (the images of) $w_1, \dots, w_i, w_{j+1}, \dots, w_n$ form a basis of $W_{i,j}$.

Example 1.3. Let $\lambda = \omega_d$. Then $\mathcal{F}_{\omega_d}^a \simeq \mathcal{F}_{\omega_d} \simeq Gr(d, n)$ (since the radical in $\mathfrak{sl}_n$ corresponding to any fundamental weight is abelian, i.e. fundamental representations are cominuscule). The torus T acts on $Gr(d, n)$ with a finite number of fixed points, which are labeled by collections $S = (l_1, \dots, l_d)$ with $1 \le l_1 < \cdots < l_d \le n$. Let $p(S) \in Gr(d, n)$ be the corresponding point, i.e. $p(S) = \mathbb{C}w(S) \in \mathbb{P}(V_{\omega_d}^a)$. Then $Gr(d, n)$ is the disjoint union of affine cells $G^a \cdot p(S)$. We note however that these cells are different from the classical ones $B \cdot p(S)$. Namely, let k be a number such that $l_k \le d < l_{k+1}$ and let $T_d : W \to W$ be an isomorphism given by

$$T_d w_1 = w_{d+1}, \dots, T_d w_{n-d} = w_n, T_d w_{n-d+1} = w_1, \dots, T_d w_n = w_d.$$

Then

$$G^a \cdot p(S) = T_d \left(B \cdot p(l_{k+1} - d, \dots, l_d - d, l_1 - d + n, \dots, l_k - d + n) \right),$$

where B acts on $Gr(d,n)$ classically (i.e. as a subgroup of SL_n). We note that B considered as a subgroup of G^a acts on $Gr(d,n)$, but this action is different from the classical one (for instance, for $n = 2$ the subgroup $B \subset SL_2^a$ acts trivially on $\mathbb{P}^1$). We denote a cell $G^a \cdot p(S)$ by $C(S)$.

For general λ the varieties $\mathcal{F}_\lambda^a$ are not isomorphic to the classical flag varieties. These varieties enjoy an explicit description as subvarieties inside the product of Grassmannians. We first consider the case of the complete flag varieties, corresponding to the case of regular λ. These varieties do not depend on (regular) λ. We denote them by $\mathcal{F}_n^a$.

Let $w_1, \dots, w_n$ be the standard basis of the fundamental vector representation $W = V_{\omega_1}$. We denote by $pr_d : W \to W$ the projection operators defined by $pr_d(\sum_{i=1}^n c_i w_i) = \sum_{i \neq d} c_i w_i$. In what follows we will need the following properties of $\mathcal{F}_n^a$ (see [Fe1],[Fe2]).

Proposition 1.4. 1). *The degenerate complete flag varieties $\mathcal{F}_n^a$ are flat degenerations of the classical flag varieties $\mathcal{F}_n$.*
2). *The variety $\mathcal{F}_n^a$ can be realized inside the product of Grassmannians $\prod_{d=1}^{n-1} Gr(d,n)$ as a subvariety of collections $(V_d)_{d=1}^{n-1}$ satisfying:*

$$pr_d V_d \subset V_{d+1}, \ d = 1, \dots, n-2.$$

3). *The variety $\mathcal{F}_n^a$ has a cell decomposition*

$$\bigsqcup_{S_1, \dots, S_{n-1}} \left(\mathcal{F}_n^a \cap \prod_{i=1}^{n-1} C(S_i) \right),$$

where the disjoint union is taken over the collections $S_1, \dots, S_{n-1}$ of subsets $S_i \subset \{1, \dots, n\}$ such that $\# S_i = i$ and $S_i \subset S_{i+1} \cup \{i+1\}$.

There is an analogue of Proposition 1.4 for the degenerate partial flag varieties. First we note that $\mathcal{F}_\lambda^a \simeq \mathcal{F}_\mu^a$ if and only if $(\lambda, \omega_d) > 0$ is equivalent to $(\mu, \omega_d) > 0$ for any d. Therefore, it suffices to consider the weights $\lambda = \omega_{d_1} + \dots + \omega_{d_k}$ with $1 \le d_1 < \dots < d_k < n$ and the corresponding degenerate flag varieties $\mathcal{F}_\lambda^a$, which we denote by $\mathcal{F}_{(d_1, \dots, d_k)}^a$, or simply by $\mathcal{F}_{\mathbf{d}}^a$, where $\mathbf{d} = (d_1, \dots, d_k)$. We recall that the classical analogues $\mathcal{F}_{\mathbf{d}}$ are isomorphic to the partial flag varieties, i.e. to the varieties consisting of collections of subspaces $V_1, \dots, V_k$ such that $\dim V_i = d_i$ and $V_i \subset V_{i+1}$. For $1 \le p \le q < n$ we define the operators $pr_{p,q} : W \to W$ via the formula

$$pr_{p,q}(\sum_{j=1}^n c_j w_j) = \sum_{j<p} c_j w_j + \sum_{j \ge q} c_j w_j.$$

Then the following proposition holds:

Proposition 1.5. 1). *The degenerate partial flag varieties* $\mathcal{F}^a_{\mathbf{d}}$ *are flat degenerations of the classical partial flag varieties* $\mathcal{F}_{\mathbf{d}}$.
2). *The variety* $\mathcal{F}^a_{\mathbf{d}}$ *can be realized inside the product of Grassmannians* $\prod_{i=1}^k Gr(d_i,n)$ *as a subvariety of collections* $(V_{d_i})_{i=1}^k$ *satisfying:*

$$pr_{d_i+1,d_{i+1}} V_{d_i} \subset V_{d_{i+1}},\ i=1,\dots,k-1.$$

3). *The variety* $\mathcal{F}^a_{\mathbf{d}}$ *has a cell decomposition*

$$\bigsqcup_{S_1,\dots,S_k} \left(\mathcal{F}^a_{\mathbf{d}} \cap \prod_{i=1}^k C(S_i)\right),$$

where the disjoint union is taken over the collections $S_1,\dots,S_k$ *of subsets* $S_i\subset\{1,\dots,n\}$ *such that* $\#S_i=d_i$ *and* $S_i\subset S_{i+1}\cup\{d_i+1,\dots,d_{i+1}\}$.

Remark 1.6. The image of the embedding $\mathcal{F}^a_{\mathbf{d}}\subset\prod_{i=1}^k Gr(d_i,n)$ can be also described in terms of the degenerate Plücker relations ([Fe1]), similar to the classical ones ([Fu]).

2. Desingularization

2.1. **Definition.** We define a desingularization R_n of the complete degenerate flag varieties $\mathcal{F}^a_n$ as follows. Let $W_{i,j}\subset W$ be the linear span of the vectors $w_1,\dots,w_i,w_{j+1},\dots,w_n$.

Definition 2.1. The variety R_n consists of collections $\mathbf{V}$ of subspaces $V_{i,j}\subset W$, $1\le i\le j\le n-1$ satisfying the following properties:

(i) $\dim V_{i,j}=i$,

(ii) $V_{i,j}\subset W_{i,j}$,

(iii) $pr_{j+1}V_{i,j}\subset V_{i,j+1}\subset V_{i+1,j+1}$ for all $1\le i\le j\le n-2$.

Remark 2.2. Since the subspace $V_{i,j}$ in embedded into $W_{i,j}$, the condition $pr_{j+1}V_{i,j}\subset V_{i,j+1}$ is equivalent to the condition

$$V_{i,j}\subset V_{i,j+1}\oplus\mathbb{C}w_{j+1}.$$

Remark 2.3. In what follows we often identify pairs (i,j) with positive roots of $\mathfrak{sl}_n$, $(i,j)\to\alpha_{i,j}$. We also sometimes consider a space $V_{i,j}$ as being attached to the root $\alpha_{i,j}$ and we write $V_{\alpha_{i,j}}$ for $V_{i,j}$.

We note that R_n is naturally embedded into the product of Grassmannians

$$R_n\hookrightarrow\prod_{1\le i\le j\le n-1} Gr(i,W_{i,j}),$$

where $Gr(i,W_{i,j})$ is the Grassmannian of i-dimensional subspaces in $W_{i,j}$. Define the map $\pi_n: R_n\to\prod_{i=1}^{n-1}Gr(i,n)$ by the formula

$$\mathbf{V}=(V_{i,j})_{1\le i\le j\le n-1}\mapsto(V_{1,1},\dots,V_{n-1,n-1}). \tag{2.1}$$

Proposition 2.4. *The image of* π_n *is equal to* $\mathcal{F}_n^a$. *The variety* R_n *is smooth and the map* $\pi_n : R_n \to \mathcal{F}_n^a$ *is a birational isomorphism.*

Proof. We note that if $\mathbf{V} \in R_n$ then

$$pr_{i+1} V_{i,i} \subset V_{i,i+1} \subset V_{i+1,i+1}$$

and thus $\pi_n(R_n) \subset \mathcal{F}_n^a$. Now given an element $(V_1, \dots, V_{n-1}) \in \mathcal{F}_n^a$, we define a collection $\mathbf{V}$ via the following inductive procedure: $V_{i,i} = V_i$ and

$$V_{i,j+1} = \begin{cases} pr_{j+1} V_{i,j}, & \text{if } \dim pr_{j+1} V_{i,j} = i, \\ pr_{j+1} V_{i,j} \oplus \mathbb{C} w_m, & \text{if } \dim pr_{j+1} V_{i,j} = i-1, \end{cases}$$

where $m \in \{1, \dots, i\}$ is the minimal number such that $w_m \notin pr_{j+1} V_{i,j}$. Then it is easy to see that $\mathbf{V} = (V_{i,j})$ belongs to R_n. Hence π_n surjects R_n to $\mathcal{F}_n^a$.

Now we show that R_n can be viewed as a tower of $\mathbb{P}^1$-fibrations. Let us order all positive roots of $\mathfrak{sl}_n$ as follows:

$$\beta_1 = \alpha_{1,n-1}, \beta_2 = \alpha_{1,n-2}, \beta_3 = \alpha_{2,n-1}, \beta_4 = \alpha_{1,n-3}, \beta_5 = \alpha_{2,n-2}, \dots.$$

Let $R_n(k)$, $k = 1, \dots, n(n-1)/2$ be the variety of collections $(V_{\beta_l})_{l=1,\dots,k}$, satisfying properties (i), (ii), (iii) from Definition 2.1 (conditions (i), (ii) and (iii) are applied only to those $V_{i,j}$ which show up in $R_n(k)$, i.e. for $\beta_l = \alpha_{i,j}$ one has $l \le k$). Then $R_n(n(n-1)/2) = R_n$ and there exist obvious projections $R_n(k) \to R_n(k-1)$. We prove that for all $k \ge 1$ the projections $R_n(k) \to R_n(k-1)$ are fibrations with fibers $\mathbb{P}^1$ (we set $R_n(0) = pt$).

For $k = 1$ we have $\beta_1 = \alpha_{1,n-1}$ and V_{β_1} is a one-dimensional space embedded into two-dimensional space $\mathrm{span}(w_1, w_{n-1})$. Therefore $R_n(1) \simeq \mathbb{P}^1$. Now fix some k. Let $\beta_k = \alpha_{i,j}$. First, let $i \ne 1$, $j \ne n-1$. Then the i-dimensional subspace $V_{i,j}$ has to satisfy the conditions

$$V_{i-1,j} \subset V_{i,j} \subset V_{i,j+1} \oplus \mathbb{C} w_{j+1} \tag{2.2}$$

(see Remark 2.2). Suppose we have fixed all subspaces V_{β_l} with $l < k$ (i.e. a point of $R_n(k-1)$). Since $V_{i-1,j}$ and $V_{i,j+1}$ are already fixed, conditions (2.2) say that the possible choices of $V_{i,j}$ are labeled by points of

$$\mathbb{P}^1 \simeq \mathbb{P}\left(\frac{V_{i,j+1} \oplus \mathbb{C} w_{j+1}}{V_{i-1,j}}\right).$$

Second, let $i = 1$. Then we have to fix a one-dimensional subspace $V_{1,j}$ living in a two-dimensional space $V_{1,j+1} \oplus \mathbb{C} w_{j+1}$. This gives us again a $\mathbb{P}^1$-fibration. Finally, let $j = n-1$. Then we need to fix an i-dimensional subspace $V_{i,n-1}$ subject to the conditions

$$V_{i-1,n-1} \subset V_{i,n-1} \subset \mathrm{span}(w_1, \dots, w_i, w_n),$$

which again produces $\mathbb{P}^1$.

It remains to prove that the map $\pi_n : R_n \to \mathcal{F}_n^a$ is a birational isomorphism. Consider the subvariety $U \subset \mathcal{F}_n^a$ consisting of all collections of subspaces $(V_i)_{i=1}^{n-1}$ such that $\dim V_i = i$ and

$$\dim pr_{i+1} \dots pr_{n-1} V_i = i$$

(i.e. the composition of the projections as above has no kernel on V_i). First note that these conditions cut out an open subvariety in $\mathcal{F}^a_n$ (in fact it is easy to see that U is an affine cell). In addition, the preimage $\pi_n^{-1}(V_i)_{i=1}^{n-1}$ consists of a single point, since

$$V_{i,j} \subset pr_j \dots pr_{i+1} V_{i,i}$$

and both spaces are i-dimensional. □

Remark 2.5. As we have seen in the proof of Proposition 2.4, the variety R_n can be constructed as a tower of successive $\mathbb{P}^1$-fibrations $\rho_k : R_n(k) \to R_n(k-1)$. We can make this statement a bit stronger. Let us write $\rho_k = \rho_{i,j}$ if $\beta_k = \alpha_{i,j}$. Then it is easy to see that the maps $\rho_{i,j}$ with fixed $j-i$ "commute", i.e. for each $m = n-1, \dots, 1$ there exist maps

$$\bar\rho_m : R_n(m(m+1)/2) \to R_n((m-1)m/2),$$

which are the $(\mathbb{P}^1)^m$ fibrations, and $\bar\rho_m = \prod_{i=1}^m \rho_{i,i+n-m-1}$.

Denote by $\xi_{i,j} : R_n \to Gr(i, W_{i,j})$ the projection given by $\mathbf{V} \mapsto V_{i,j}$.

Lemma 2.6. *For any $k = 0, \dots, n-2$ the image of the map $\prod_{j-i=k} \xi_{i,j}$ is isomorphic to $\mathcal{F}^a_{n-k}$.*

Proof. Recall that $V_{i,j} \subset W_{i,j}$. Consider an isomorphism

$$A_{i,j} : W_{i,j} \to \mathrm{span}(w_1, \dots, w_{n-j+i})$$

defined by

$$A_{i,j}(c_1w_1 + \dots + c_iw_i + c_{j+1}w_{j+1} + \dots + c_nw_n) = \\ c_1w_1 + \dots + c_iw_i + c_{j+1}w_{i+1} + \dots + c_nw_{n-j+i}.$$

Then it is easy to see that this map induces the isomorphism stated in our lemma. □

Corollary 2.7. *We have an embedding $R_n \hookrightarrow \prod_{k=1}^{n-1} \mathcal{F}^a_k$.*

Corollary 2.8. *The varieties $R_n(k(k+1)/2)$ and R_k are isomorphic.*

Proof. We note that the spaces involved in the construction of $R_n(k(k+1)/2)$ are exactly $V_{i,j}$ with $j - i \ge n-k-1$. Now the maps $A_{i,j}$ as above induce the desired isomorphism. □

We now consider the case of partial flag varieties $\mathcal{F}^a_{\mathbf{d}}$ (recall the notation $\mathbf{d} = (d_1, \dots, d_k)$). Let $P_{(d_1,\dots,d_k)} = P_{\mathbf{d}}$ be the subset of the set of positive roots of $\mathfrak{sl}_n$ corresponding to the radical of the parabolic subalgebra defined by the simple roots $\alpha_{d_1}, \dots, \alpha_{d_k}$, i.e.

$$P_{\mathbf{d}} = \{\alpha_{i,j} : \ \exists\, l \text{ such that } (\alpha_{i,j}, \omega_{d_l}) > 0\}.$$

We sometimes consider $P_{\mathbf{d}}$ as a subset of $\mathbb{N}^2$ identifying $\alpha_{i,j}$ with the pair (i,j). Let $R_{\mathbf{d}}$ be the image of the map

$$\prod_{(i,j)\in P_{\mathbf{d}}} \xi_{i,j} : R_n \to \prod_{(i,j)\in P_{\mathbf{d}}} Gr(i, W_{i,j}).$$

More concretely, $R_{\mathbf{d}}$ is the variety of collections $V_{i,j} \subset W$ with $(i,j) \in P_{\mathbf{d}}$ satisfying conditions

(i) $\dim V_{i,j} = i$,

(ii) $V_{i,j} \subset W_{i,j}$,

(iii) $pr_{j+1} V_{i,j} \subset V_{i,j+1}$ if $(i,j), (i,j+1) \in P_{\mathbf{d}}$,

(iv) $V_{i,j+1} \subset V_{i+1,j+1}$ if $(i,j+1), (i+1,j+1) \in P_{\mathbf{d}}$

from Definition 2.1. Obviously, R_n surjects to $R_{\mathbf{d}}$ by forgetting all components $V_{i,j}$ but those with $(i,j) \in P_{\mathbf{d}}$.

Proposition 2.9. *Foe any* $\mathbf{d}$ *the variety* $R_{\mathbf{d}}$ *is smooth and a natural map* $R_{\mathbf{d}} \to \mathcal{F}^a_{\mathbf{d}}$ *defined by forgetting the off-diagonal* $(i \neq j)$ *subspaces* $V_{i,j}$ *is a desingularization (a birational isomorphism).*

Proof. The proof is very similar to the proof for the complete flag varieties. □

Remark 2.10. In the Introduction the varieties $R_{\mathbf{d}}$ are denoted by R_λ with $\lambda = \omega_{d_1} + \ldots + \omega_{d_k}$.

2.2. **Cell decomposition for R_n.** In this section we construct a cell decomposition for R_n which is compatible with the cell decomposition for $\mathcal{F}^a_n$ (i.e. the map π_n is cellular).

Lemma 2.11. *The group* G^a *acts naturally on each* $Gr(i, W_{i,j})$. *The number of* G^a*-orbits is finite and the orbits are labeled by torus fixed points. Each orbit is an affine cell.*

Proof. Fix a pair $1 \le i \le j \le n-1$. Recall that $V_{i,j} \subset W_{i,j}$. Therefore, $V_{i,j}$ can be considered as a point in $\mathbb{P}(\bigwedge^i (W_{i,j}))$. The spaces $\bigwedge^i (W_{i,j})$ carry a natural structure of $\mathfrak{g}^a$- and G^a-modules (see Lemma 1.2). This produces a G^a-action on $\mathbb{P}(\bigwedge^i (W_{i,j}))$ and thus on the variety $Gr(i, W_{i,j})$ of i-dimensional subspaces of $W_{i,j}$.

Let us consider the smaller group SL^a_{n-j+i}. Using the maps $A_{i,j}$ from Lemma 2.6 we endow $W_{i,j}$ and all its wedge powers with the standard structure of SL^a_{n-j+i}-modules (we identify $\bigwedge^k (W_{i,j})$ with the SL^a_{n-j+i}-modules $V^a_{\omega_k}$). Let $\phi : SL^a_{n-j+i} \to GL(\bigwedge^k (W_{i,j}))$ be the representation map and also let $\psi : SL^a_n \to GL(\bigwedge^k (W_{i,j}))$ be the map defining the G^a action on $\bigwedge^k (W_{i,j})$. It is easy to see that the images of ϕ and ψ coincide. Therefore, Example 1.3 (applied to the group SL^a_{n-j+i}) implies the statement of our Lemma. □

We note that the torus fixed points in the Grassmannian of i-dimensional subspaces in $W_{i,j}$ are labeled by the sequences

$$S = (l_1 < \cdots < l_i) \subset \{1, \ldots, i, j+1, \ldots, n\}.$$

In what follows we denote the corresponding point by $p(S)$. We also denote the corresponding orbit $G^a \cdot p(S) \subset Gr(i, W_{i,j})$ by $C(S)$.

Recall that R_n sits inside $\prod_{1\le i\le j\le n-1} Gr(i, W_{i,j})$. The group G^a acts on this product via the action on each factor.

Lemma 2.12. *The variety R_n is invariant with respect to this action and $\pi_n : R_n \to \mathcal{F}_n^a$ is G^a-equivariant.*

Proof. First, take $b \in B \subset G^a$ and fix a point $\mathbf{V} = (V_{i,j}) \in R_n$. We need to show that for any $1 \le i \le j \le n-1$

$$bV_{i-1,j} \subset bV_{i,j} \subset bV_{i,j+1} \oplus \mathbb{C}w_{j+1}. \tag{2.3}$$

We note that $W_{i-1,j} \subset W_{i,j}$ and the B-action on $W_{i-1,j}$ is a restriction of the action on $W_{i,j}$. Therefore, the first embedding in (2.3) follows. To prove the second embedding we note that $\mathbb{C}w_{j+1}$ is a $\mathfrak{b}$-submodule in $W_{i,j}$ and the quotient module is isomorphic to $W_{i,j+1}$.

Now take $g \in (N^-)^a$. We need to prove that for any $1 \le i \le j \le n-1$

$$gV_{i-1,j} \subset gV_{i,j} \subset gV_{i,j+1} \oplus \mathbb{C}w_{j+1}.$$

The proof is very similar to the proof of (2.3) and we omit it. □

Let $\mathbf{S} = (S_{i,j})_{1\le i\le j\le n-1}$ be a collection of sets such that $\#S_{i,j} = i$ and $S_{i,j} \subset \{1,\dots,i,j+1,\dots,n\}$. We call such a collection admissible if

$$S_{i-1,j} \subset S_{i,j} \subset S_{i,j+1} \cup \{j+1\}. \tag{2.4}$$

The following lemma is simple, but important for us.

Lemma 2.13. *A point $p(\mathbf{S}) = \prod_{1\le i\le j\le n-1} p(S_{i,j})$ belongs to R_n if and only if $\mathbf{S}$ is admissible. If a point $p = \prod_{1\le i\le j<n} p_{i,j}$, $p_{i,j} \in C(S_{i,j})$ belongs to R_n, then the collection $\mathbf{S} = (S_{i,j})$ is admissible.* □

For an admissible collection $\mathbf{S}$ we introduce the notation

$$C(\mathbf{S}) = R_n \cap \prod_{1\le i\le j\le n-1} C(S_{i,j}).$$

We have the decomposition

$$R_n = \bigcup_{\text{admissible } \mathbf{S}} C(\mathbf{S}).$$

Our next goal is to show that $C(\mathbf{S})$ is an affine cell and to compute it dimension.

For a number l, $-n < l \le n$ we set $\bar{l} = l$ if $l > 0$ and $\bar{l} = l + n$ otherwise. So $1 \le \bar{l} \le n$.

Theorem 2.14. *$C(\mathbf{S})$ is an affine cell for any admissible $\mathbf{S}$. The map π_n is cellular, mapping $C(\mathbf{S})$ to $C(S_{1,1},\dots,S_{n-1,n-1})$.*

Proof. We need to do two things: first, to construct coordinates on a cell $C(\mathbf{S})$ and, second, to construct coordinates on each fiber of the map

$$C(\mathbf{S}) \to C(S_{1,1}, \dots, S_{n-1,n-1})$$

such that this map becomes a trivial fibration with an affine fiber. We start with the first part.

We want to construct coordinates on $C(\mathbf{S})$. Namely, we need to attach coordinates to collections of subspaces $(V_{i,j})_{1\le i\le j<n} \in R_n$. We do it by decreasing induction on $j-i$. We start with $j-i=n-2$, i.e. $i=1$, $j=n-1$. Then either $S_{1,n}=(n)$ or $S_{1,n}=(1)$. In the first case the cell $C((n))$ is a point and in the second case $V_{1,n-1}$ is spanned by a single vector v_1+av_n and a is our first coordinate. Assume that we have attached coordinates to all subspaces $V_{i,j}$ with $j-i>k$ and we proceed with $j-i=k$. We consider three cases.

Let $i=1$. Then the only condition we have is $V_{1,j} \subset V_{1,j+1} \oplus \mathbb{C}v_{j+1}$. Let $S_{1,j}=(l)$. There are two cases: $l=j+1$ and $l\neq j+1$. In the first case we do not have to add any coordinates, since $C((j+1)) \subset Gr(1, W_{1,j})$ is a point. Let $l \neq j+1$ and let $v \in V_{1,j+1}$ be a basis vector. Then a basis vector for $V_{1,j}$ is of the form $v+aw_{j+1}$ and therefore we have added one more coordinate.

Let $j=n-1$. Then we have the condition $V_{i-1,n-1} \subset V_{i,n-1}$. We know that $S_{i,n-1}=S_{i-1,n-1}\cup\{l\}$. There are two cases: $l=i$ and $l\neq i$. First, let $l=i$. Let $m=\{1,\dots,i-1,n\}\setminus S_{i-1,n}$. Since $V_{i-1,n}$ is $(i-1)$-dimensional, we need to specify one more basis vector in $V_{i,n-1}$ in order to fix it. This basis vector has to be of the form

$$w_i + c_{i-1}w_{i-1} + \dots + c_1w_1 + c_nw_n,\ c_k \in \mathbb{C}$$

We note that by adding an appropriate vector from $V_{i-1,n-1}$, any vector of the form as above can be reduced to $w_i + aw_m$. This gives one additional coordinate. Second, let $l \neq i$. Then $l=\{1,\dots,i-1,n\}\setminus S_{i-1,n}$. A basis vector we have to add to $V_{i-1,n-1}$ in order to fix $V_{i,n-1}$ is of the form

$$w_l + c_{l-1}w_{l-1} + \dots + c_1w_1 + c_nw_n,\ c_k \in \mathbb{C}.$$

Since w_i never appears in the decomposition as above, such a vector (modulo $V_{i-1,n-1}$) is equal to w_l and we do not have to add a coordinate.

Let $i>1$, $j<n-1$. Then we have

$$S_{i-1,j} \subset S_{i,j} \subset S_{i,j+1} \cup \{j+1\},\ V_{i-1,j} \subset V_{i,j} \subset V_{i,j+1} \oplus \mathbb{C}w_{j+1}.$$

First, let $S_{i,j}=S_{i,j+1}$, i.e. $j+1 \notin S_{i,j}$. Let $l=S_{i,j}\setminus S_{i-1,j}$. Then a basis vector we have to add to $V_{i-1,j}$ in order to fix $V_{i,j}$ is of the form

$$w_l + c_{l-1}w_{l-1} + \dots + c_1w_1 + c_nw_n + \dots + c_{j+1}w_{j+1}.$$

Since this vector has to belong to $V_{i,j+1}$, the only freedom we have is a coefficient c_{j+1} (note that $l \neq j+1$). Therefore, we have to add one additional coordinate in this case. Second, let $S_{i,j} \neq S_{i,j+1}$, i.e. $j+1 \in S_{i,j}$. Then

$S_{i,j} = S_{i,j+1} \setminus \{m\} \cup \{j+1\}$. Recall $l = S_{i,j} \setminus S_{i-1,j}$. A basis vector we have to add to $V_{i-1,j}$ in order to fix $V_{i,j}$ is of the form

$$w_l + c_{l-1}w_{l-1} + \dots + c_1 w_1 + c_n w_n + \dots + c_{j+1}w_{j+1}. \tag{2.5}$$

Recall that for a number l, $-n < k \le n$ we set $\overline{l} = l$ if $l > 0$ and $\overline{l} = l + n$ otherwise. There are two cases now: $\overline{l-j} < \overline{m-j}$ and $\overline{l-j} > \overline{m-j}$. Let $\overline{l-j} < \overline{m-j}$. Then the vector w_m never appears in the decomposition (2.5) and therefore there exists a single vector in $V_{i,j+1}$ of the form (2.5). Thus no new coordinates have to be added. Finally, let $\overline{l-j} > \overline{m-j}$. Then a vector w_m is present in (2.5). Therefore, there exists exactly one-parameter family of vectors in $V_{i,j+1}$ of the form (2.5). Thus one additional coordinate has to be added.

To complete the proof of the theorem we need to construct coordinates on the fibers of the map $C(\mathbf{S}) \to C(S_{1,1}, \dots, S_{n-1,n-1})$. To do this, one need to fix a collection of subspaces $V_{i,i} \in C(S_{i,i})$ such that $(V_{i,i})_{i=1}^{n-1} \in \mathcal{F}_n^a$ and then start looking at all possible values of other $V_{i,j} \in C(S_{i,j})$ moving from lower values of $j - i$ to higher ones. The procedure is very similar to the one worked out above, so we omit the details. □

Corollary 2.15. *For an admissible* $\mathbf{S}$ *the dimension of the cell* $C(\mathbf{S})$ *is equal to the sum of* $n(n-1)/2$ *terms* $g_{i,j}$ *labeled by pairs* $1 \le i \le j \le n-1$*. Each summand is either* 0 *or* 1 *and is given by the following rule:*

- *Let* $i = 1$, $j = n-1$. *If* $S_{1,n} = (1)$, *then* $g_{1,n-1} = 1$. *Otherwise* $g_{1,n-1} = 0$.
- *Let* $i = 1$ *and* $S_{1,j} = (l)$. *If* $l \ne j+1$, *then* $g_{i,j} = 1$. *Otherwise* $g_{i,j} = 0$.
- *Let* $j = n-1$. *Let* $\{l\} = S_{i,n-1} \setminus S_{i-1,n-1}$. *If* $l = i$, *then* $g_{i,j} = 1$. *Otherwise* $g_{i,j} = 0$.
- *Let* $i > 1$ *and* $j < n-1$.
  If $j+1 \notin S_{i,j}$, *then* $g_{i,j} = 1$.
  If $j+1 \in S_{i,j}$, *set* $l = S_{i,j} \setminus S_{i-1,j}$, $m = S_{i,j+1} \setminus S_{i,j}$. *If* $\overline{l-j} > \overline{m-j}$, *then* $g_{i,j} = 1$. *Otherwise* $g_{i,j} = 0$.

Proof. Follows from the explicit construction of the coordinates on $C(\mathbf{S})$. □

Corollary 2.16. *The relative dimension* $\dim C(\mathbf{S}) - \dim C(S_{i,i})_{i=1}^{n-1}$ *is equal to the sum of* $(n-1)(n-2)/2$ *terms* $h_{i,j}$ *labeled by pairs* $1 \le i < j \le n-1$*. Each summand is either* 0 *or* 1 *and is given by the following rule. Let* $l = S_{i,j} \setminus S_{i,j-1}$, $m = S_{i+1,j} \setminus S_{i,j}$. *Then* $h_{i,j} = 0$ *if and only if* $\overline{m-j} < \overline{l-j}$ *and* $j \in S_{i,j-1}$.

Proof. Follows from the explicit construction of the coordinates on $C(\mathbf{S})$. □

We propose the following conjecture (supported by the computer experiments):

Conjecture 2.17. The desingularization π_n is semismall, i.e. for any cell $C_1 \subset R_n$ mapped to a cell $C_2 \subset \mathcal{F}_n^a$ one has

$$\dim C_1 + (\dim C_1 - \dim C_2) \le \dim R_n = n(n-1)/2. \tag{2.6}$$

Computer experiments show that for $n \le 7$ the resolution $R_n \to \mathcal{F}_n^a$ is *small*. However, for $\mathfrak{sl}_8$ there exist cells C_1 and C_2 such that (2.6) is an equality.

Example 2.18. As we have explained above, the cells are labeled by the collections $S_{i,j}$ satisfying (in particular) $S_{i,j} \subset S_{i+1,j}$. Therefore, in order to specify an admissible collection $\mathbf{S} = (S_{i,j})$, it suffices to fix the numbers $x_{i,j} = S_{i,j} \setminus S_{i-1,j}$. Let $n = 8$ and let us consider a cell $\mathbf{S}$, where the numbers $x_{i,j}$ are as follows:

$$\begin{array}{ccccccc}
5 & 5 & 5 & 5 & 1 & 1 & 1 \\
 & 1 & 1 & 1 & 7 & 7 & 2 \\
 & & 7 & 7 & 6 & 3 & 3 \\
 & & & 3 & 3 & 2 & 8 \\
 & & & & 5 & 5 & 5 \\
 & & & & & 6 & 6 \\
 & & & & & & 7
\end{array}$$

Then using Lemmas 2.15 and 2.16 one can check that $\dim C(\mathbf{S}) = 24$ and the relative dimension $\dim C(\mathbf{S}) - \dim \pi_8(C(\mathbf{S})) = 4$. Therefore the sum is equal to 28, which is exactly the dimension of the flag variety for $\mathfrak{sl}_8$.

Finally, we note that Theorem 2.14 as well as Corollaries 2.15 and 2.16 have their obvious parabolic analogues. Namely, let us call a collection $\mathbf{S} = (S_{i,j})_{(i,j)\in P_{\mathbf{d}}}$ $\mathbf{d}$-admissible, if condition (2.4) holds provided the corresponding pairs of indices are in $P_{\mathbf{d}}$. Then the following theorem holds:

Proposition 2.19. 1). *$R_{\mathbf{d}}$ is a disjoint union of the cells*

$$\bigsqcup_{\mathbf{d}-admissible\ \mathbf{S}} \left(R_{\mathbf{d}} \cap \prod_{(i,j)\in P_{\mathbf{d}}} C(S_{i,j}) \right).$$

2). *The map $R_{\mathbf{d}} \to \mathcal{F}^a_{\mathbf{d}}$ is cellular.*
3). *The dimensions and relative dimensions are equal to the sum of terms $g_{i,j}$ and $h_{i,j}$ from Corollaries 2.15 and 2.16 with $(i,j) \in P_{\mathbf{d}}$.*

3. Normality

3.1. **Complete flag varieties.** We first construct a quiver realization of the complete degenerate flag varieties. Let $W_1, \dots, W_{n-1}, W_n$ be a collection of fixed spaces with $\dim W_i = i$. Additionally, we fix a basis $e_1, \dots, e_n$ in W_n and the projections pr_k along e_k. We now construct a scheme Q_n as follows. A point of Q_n is a collection of linear embeddings

$$A_i : W_i \to W_n,\ i = 1, \dots, n-1, \qquad B_j : W_j \to W_{j+1},\ j = 1, \dots, n-2$$

subject to the relations

$$A_{i+1}B_i = pr_{i+1}A_i,\ i = 1,\dots,n-2. \tag{3.1}$$

The following picture illustrates the construction:

$$\begin{array}{ccccccc}
W_n & \xrightarrow{pr_2} & W_n & \dots\dots\dots & W_n & \xrightarrow{pr_{n-1}} & W_n \\
\uparrow{\scriptstyle A_1} & & \uparrow{\scriptstyle A_2} & & \uparrow{\scriptstyle A_{n-2}} & & \uparrow{\scriptstyle A_{n-1}} \\
W_1 & \xrightarrow{B_1} & W_2 & \dots\dots\dots & W_{n-2} & \xrightarrow{B_{n-2}} & W_{n-1}
\end{array}$$

We note that the group $\Gamma = \prod_{i=1}^{n-1} GL(W_i)$ acts freely on Q_n via the change of bases.

Lemma 3.1. *Γ acts freely on Q_n and $Q_n/\Gamma \simeq \mathcal{F}_n^a$, i.e. Q_n is a Γ-torsor over $\mathcal{F}_n^a$. The dimension of Q_n is equal to $n(n-1)/2+1^2+2^2+\dots+(n-1)^2$.*

We note that Q_n is a subscheme in the affine space

$$\prod_{i=1}^{n-1} Hom(W_i, W_n) \times \prod_{i=1}^{n-2} Hom(W_i, W_{i+1}). \tag{3.2}$$

Lemma 3.2. *Q_n is a complete intersection.*

Proof. We note that the condition $A_{i+1}B_i = pr_{i+1}A_i$ produces $n \times i$ equations (the number of equations is equal to $\dim Hom(W_i, W_n)$). Now our lemma follows from the equality

$$\dim Q_n = \sum_{i=1}^{n-1} ni + \sum_{i=1}^{n-2} i(i+1) - \sum_{i=1}^{n-2} ni.$$

□

Theorem 3.3. *The degenerate flag varieties $\mathcal{F}_n^a$ are normal locally complete intersections (in particular, Cohen-Macaulay and even Gorenstein).*

Proof. Since $Q_n \to \mathcal{F}_n^a$ is a torsor, it suffices to prove that Q_n is a normal reduced scheme (i.e. a variety). Since Q_n is a complete intersection, the property of being reduced (resp. normality) of Q_n follows from the fact that the singularities of Q_n are contained in the subvariety of codimension at least two by the virtue of Proposition 5.8.5 (resp. Theorem 5.8.6) of [EGA]. Using again that $Q_n \to \mathcal{F}_n^a$ is a torsor, it suffices to prove that the codimension of the variety of singular points of $\mathcal{F}_n^a$ is at least two. We prove this statement in a separate lemma. □

Lemma 3.4. *$\mathcal{F}_n^a$ is smooth off codimension two.*

Proof. We use the desingularization $\pi_n : R_n \to \mathcal{F}_n^a$. Since R_n is smooth, it suffices to show that that the map π_n is one-to-one on all cells of (complex) codimension one. Dimension counting from Corollary 2.15 implies that the codimension one cells are labeled by pairs $1 \le a \le b \le n-1$ and the collection $\mathbf{S} = (S_{i,j})$ corresponding to a pair (a,b) is as follows:

$$S_{i,j} = \begin{cases} \{1,2,\dots,i\} \text{ if } (i < a \text{ or } j > b), \\ \{1,2,\dots,i\} \setminus \{a\} \cup \{b+1\}, \text{otherwise}. \end{cases} \tag{3.3}$$

It is easy to see from Corollary 2.16 that the resolution map π_n is one-to-one on such cells (i.e. the relative dimension is zero). □

3.2. **Parabolic flag varieties.** Our goal is to generalize the results from the previous subsection to the case of the general parabolic degenerate flag varieties. So let $\mathbf{d} = (d_1,\dots,d_k)$ be a collection with $1 \le d_1 < \dots < d_k \le n$. We define a scheme $Q_\mathbf{d}$ as follows. As above, we fix the spaces W_{d_i}, $i = 1,\dots,k$ with $\dim W_{d_i} = d_i$. A point of $Q_\mathbf{d}$ is a collection of linear embeddings

$$A_i : W_{d_i} \to W_n,\ i = 1,\dots,k, \qquad B_j : W_{d_j} \to W_{d_{j+1}},\ j = 1,\dots,k-1$$

subject to the relations

$$A_{i+1}B_i = pr_{d_i+1}\dots pr_{d_{i+1}}A_i,\ i = 1,\dots,n-2. \tag{3.4}$$

The group $\Gamma_\mathbf{d} = \prod_{i=1}^k GL(W_{d_i})$ acts freely on $Q_\mathbf{d}$ via the change of bases and $Q_\mathbf{d}/\Gamma_\mathbf{d} \simeq \mathcal{F}^a_\mathbf{d}$, i.e. $Q_\mathbf{d}$ is a $\Gamma_\mathbf{d}$-torsor over $\mathcal{F}^a_\mathbf{d}$. Moreover, explicit computation as above shows that $Q_\mathbf{d}$ is a complete intersection. Now the following theorem holds:

Theorem 3.5. *The degenerate flag varieties $\mathcal{F}^a_\mathbf{d}$ are normal locally complete intersections (in particular, Cohen-Macaulay and even Gorenstein).*

Again, as in the complete case, we only need to prove that each variety $\mathcal{F}^a_\mathbf{d}$ is smooth outside of the codimension two subvariety.

Proposition 3.6. *$\mathcal{F}^a_\mathbf{d}$ is smooth off codimension two.*

Proof. As in Lemma 3.4 it suffices to construct a desingularization $Y_\mathbf{d}$ of $\mathcal{F}^a_\mathbf{d}$ such that the map $\tau_\mathbf{d} : Y_\mathbf{d} \to \mathcal{F}^a_\mathbf{d}$ is one-to-one off codimension two. Unfortunately, $R_\mathbf{d}$ does not do the job (it is too big). We refine it in the following way. Let $Y_\mathbf{d}$ be the variety of subspaces V_{d_i,d_j}, $1 \le i \le j \le k$ satisfying the following properties:

$$\dim V_{d_i,d_j} = d_i, \quad V_{d_i,d_j} \subset W_{d_i,d_j},$$
$$V_{d_i,d_j} \subset V_{d_{i+1},d_j}, \quad pr_{d_j+1}\dots pr_{d_{j+1}} V_{d_i,d_j} \subset V_{d_i,d_{j+1}}.$$

The projection map $\tau_\mathbf{d}$ is defined by $(V_{d_i,d_j})_{i,j=1}^k \mapsto (V_{d_i,d_i})_{i=1}^k$ (i.e. simply forgetting the off-diagonal entries).

The varieties $Y_\mathbf{d}$ are smooth and can be viewed as towers of fibrations with fibers isomorphic to the Grassmann varieties. More precisely, these towers are constricted as follows. First, the subspace V_{d_1,d_k} varies in $Gr(d_1, W_{d_1,d_k})$. Second, we consider $V_{d_1,d_{k-1}}$ and V_{d_2,d_k}. For the former, the only condition is

$$V_{d_1,d_{k-1}} \subset V_{d_1,d_k} \oplus \operatorname{span}(w_{d_{k-1}+1},\dots,w_{d_k}),$$

which produces the fibration over $Gr(d_1, W_{d_1,d_k})$ with a fiber $Gr(d_1, d_1+d_k-d_{k-1})$. Now the conditions for V_{d_2,d_k} are $V_{d_1,d_k} \subset V_{d_2,d_k} \subset W_{d_2,d_k}$, producing a fibration over $Gr(d_1, W_{d_1,d_k})$ with a fiber $Gr(d_2-d_1, n-d_k+d_2-d_1)$. Proceeding further, we see that $Y_{\mathbf{d}}$ is a tower of fibrations with fibers being Grassmannians.

As in the case of complete flag varieties, the varieties $Y_{\mathbf{d}}$ possess a cellular decomposition. Namely, the cells are labeled by collections $\mathbf{S} = (S_{d_i,d_j})$, $1 \le i \le j \le k$ satisfying the usual properties

$$\#S_{d_i,d_j} = d_i, \quad S_{d_i,d_j} \subset \{1,\dots,d_i, d_j+1, n\},$$
$$S_{d_i,d_j} \subset S_{d_{i+1},d_j} \subset S_{d_{i+1},d_{j+1}} \cup \{d_j+1,\dots,d_{j+1}\}.$$

A cell $C(\mathbf{S})$ is defined as the intersection $Y_{\mathbf{d}} \cap \prod_{i,j} C(S_{d_i,d_j})$. For example, the big cell in $Y_{\mathbf{d}}$ is given by $S_{d_i,d_j} = \{1,\dots,d_i\}$. It is easy to see that $\tau_{\mathbf{d}}$ is one-to-one on this cell. In order to prove the proposition it suffices to show that $\tau_{\mathbf{d}}$ is still bijective on all cells of codimension one. Let us describe these cells.

First consider a single Grassmannian $Gr(d,n)$. The unique codimension one cell is $C(S)$ with $S = \{2,\dots,d,n\}$. Using this observation and the construction of $Y_{\mathbf{d}}$ as a tower of successive fibrations with fibers being Grassmanians, we obtain the following description of codimension one cells in $Y_{\mathbf{d}}$. These cells are labeled by pairs $1 \le a \le b \le k$ and a collection $\mathbf{S}$ corresponding to such a pair is given by

$$S_{d_i,d_j} = \begin{cases} \{1,2,\dots,d_i\}, \text{ if } (i<a \text{ or } j>b), \\ \{1,2,\dots,d_i\} \setminus \{d_{a-1}+1\} \cup \{d_{b+1}\}, \text{ otherwise} \end{cases}$$

(compare with (3.3)). It is easy to check that the map $\tau_{\mathbf{d}}$ is one-to-one on such cells. $\square$

4. Frobenius splitting

The goal of this section is to show that the varieties $\mathcal{F}_n^a$ are Frobenius split. The general references are [MR], [BK]. We first recall the definition. Let X be an algebraic variety over an algebraically closed field of characteristics $p > 0$. Let $F : X \to X$ be the Frobenius morphism, i.e. an identity map on the underlying space X and the p-th power map on the space of functions. Then X is called Frobenius split if there exists a projection $F_*\mathcal{O}_X \to \mathcal{O}_X$ such that the composition $\mathcal{O}_X \to F_*\mathcal{O}_X \to \mathcal{O}_X$ is the identity map. The Frobenius split varieties enjoy the following important property (see. e.g. Proposition 1 of [MR]):

Proposition 4.1. *Let X be a Frobenius split projective variety with a line bundle $\mathcal{L}$ such that for some i and all large enough m $H^i(X, \mathcal{L}^m) = 0$. Then $H^i(X, \mathcal{L}) = 0$.*

In order to prove Frobenius splitting of $\mathcal{F}_n^a$, we use two statements from [MR], which we recall now. The first one is Proposition 4 of [MR]:

Proposition 4.2. *Let $f : Z \to X$ be a proper morphism of algebraic varieties such that $f_*\mathcal{O}_Z = \mathcal{O}_X$. Then if Z is Frobenius split, then X is also Frobenius split.*

Corollary 4.3. *If R_n is Frobenius split, then $\mathcal{F}_n^a$ is Frobenius split as well.*

Proof. The normality of $\mathcal{F}_n^a$ implies $\pi_n^*\mathcal{O}_{R_n} = \mathcal{O}_{\mathcal{F}_n^a}$. □

In order to prove that R_n is Frobenius split we use the Mehta-Ramanathan theorem (Proposition 8 of [MR]) which we recall now:

Theorem 4.4. *Let Z be a smooth projective variety of dimension M and let $Z_1, \dots, Z_M$ be codimension one subvarieties satisfying the following conditions:*

- (i) *For any $I \subset \{1, \dots, M\}$ the intersection $\cap_{i \in I} Z_i$ is smooth of codimension $\#I$.*
- (ii) *There exists a global section s of the anti-canonical bundle K^{-1} on Z such that the zero divisor of s equals $Z_1 + \dots + Z_M + D$ for some effective divisor D with $\cap_{i=1}^M Z_i \notin suppD$.*

Then Z is Frobenius split and for any subset $I \subset \{1, \dots, M\}$ the intersection $Z_I = \cap_{i \in I} Z_i$ is Frobenius split as well.

In our situation $Z = R_n$ over a field $\mathsf{k} = \overline{\mathbb{F}}_p$ and $M = n(n-1)/2$ is the number of positive roots. Let us construct the divisors $Z_1, \dots, Z_M$. For convenience, we denote them by $Z_{i,j}$, $1 \le i \le j \le n-1$. Recall that we have a tower of successive $\mathbb{P}^1$-fibrations $\rho_l : R_n(l) \to R_n(l-1)$ such that $R_n(M) = R_n$. For each l we construct a section s_l of ρ_l as follows. We note that in order to specify an element in the fiber $\rho_l^{-1}\mathbf{V}$ for some $\mathbf{V} \in R_n(l-1)$ it suffices to determine the space $(\rho_l(\mathbf{V}))_{i,j}$, where $\beta_l = \alpha_{i,j}$. We consider three cases. First, let $i = 1$. Then we put

$$(s_l(\mathbf{V}))_{1,j} = \mathsf{k}w_{j+1}.$$

Second, let $j = n-1$. Then

$$(s_l(\mathbf{V}))_{i,n-1} = V_{i-1,n-1} \oplus \mathsf{k}w_i.$$

Finally, let $i \neq 1$ and $j \neq n-1$. Then we set

$$(s_l(\mathbf{V}))_{i,j} = V_{i-1,j+1} \oplus \mathsf{k}w_{j+1}.$$

It is easy to check that with such a definition the resulting element belongs to $R_n(l)$. In what follows we denote the image $s_l(R_n(l-1))$ by s_l or by $s_{i,j}$ (recall $\beta_l = \alpha_{i,j}$).

Let $f_l = \rho_{l+1} \dots \rho_M : R_n \to R_n(l)$. Define

$$Z_l = Z_{i,j} = \{\mathbf{V} \in R_n : \ f_l\mathbf{V} \subset s_l\}.$$

In other words, the divisor Z_l can be constructed step by step compatibly with the fibrations $\rho_\bullet$ in such a way that at the l-th step one takes not the whole preimage, but the section s_l only.

Let $\mathcal{L}_{i,j}$, $1 \le i \le j \le n-1$ be the i-dimensional bundle on R_n with fiber $V_{i,j}$ at a point $\mathbf{V}$. We set $\omega_{i,j} = \det^{-1}\mathcal{L}_{i,j}$.

Theorem 4.5. *We have*

$$K_{R_n}^{-1} = \mathcal{O}\left(\sum_{l=1}^{M} Z_l\right) \otimes \bigotimes_{i=1}^{n-1} \omega_{i,i} \otimes \bigotimes_{i=1}^{n-2} \omega_{i,i+1}.$$

We first prove a lemma. Let B be a smooth projective variety and let $\mathcal{L}_2$ be a two-dimensional bundle on B with a line subbundle $\mathcal{L}_1$. Let $\rho : E \to B$ be a $\mathbb{P}^1$-fibration with $E = \mathbb{P}(\mathcal{L}_2)$. Let $s : B \to E$ be a section of ρ defined by $\mathcal{L}_1$. In what follows we denote the section $s(B) \subset E$ simply by s.

Lemma 4.6. *For a line bundle $\mathcal{F}$ on E such that the restriction of $\mathcal{F}$ to a fiber of ρ is equal to $\mathcal{O}(k)$ one has*

$$\mathcal{F} = \mathcal{O}(ks) \otimes \rho^*(\mathcal{F}|_s \otimes \mathcal{O}(-ks)|_s).$$

Proof. We note that $\mathcal{F} \otimes \mathcal{O}(-ks)$ restricts trivially to a fiber of ρ and therefore can be pulled back from some line bundle on the section. □

We apply this lemma to the case $B = R_n(l-1)$, $E = R_n(l)$, $\rho = \rho_l$, s being a section constructed above. The bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ in our situation are described as follows: the fiber of $\mathcal{L}_1$ at a point $\mathbf{V} \in R_n(l-1)$ is equal to

$$\frac{V_{i-1,j+1} \oplus \mathsf{k} w_{j+1}}{V_{i-1,j}} \tag{4.1}$$

and the fiber of $\mathcal{L}_2$ at a point $\mathbf{V}$ is equal to

$$\frac{V_{i,j+1} \oplus \mathsf{k} w_{j+1}}{V_{i-1,j}}. \tag{4.2}$$

For $\mathcal{F}$ we first take $K_{R_n(l)}^{-1}$ and then $\omega_{i,j}$, where $\beta_l = \alpha_{i,j}$.

Lemma 4.7. *Let $i \neq 1$ and $j \neq n-1$. Then*

$$K_E^{-1} = \mathcal{O}(2s_{i,j}) \otimes \rho^*(K_B^{-1}) \otimes \rho^*(\omega_{i,j+1} \otimes (\omega_{i-1,j+1}^*)^{\otimes 2} \otimes \omega_{i-1,j}).$$

Let $i = 1$. Then

$$K_E^{-1} = \mathcal{O}(2s_{1,j}) \otimes \rho^*(K_B^{-1}) \otimes \rho^*(\omega_{1,j+1}).$$

Let $j = n-1$. Then

$$K_E^{-1} = \mathcal{O}(2s_{i,n-1}) \otimes \rho^*(K_B^{-1}) \otimes \rho^*(\omega_{i-1,n-1}).$$

Proof. We prove the first formula (the rest of the proof is very similar). Our main tool is Lemma 4.6. Let $s = s_{i,j}$. We note that the restriction of K_E to the fibers of the map ρ equals $\mathcal{O}(-2)$. Also

$$\mathcal{O}(s)|_s \simeq Hom(\mathcal{L}_1, \mathcal{L}_2/\mathcal{L}_1) \simeq T_{E/B},$$

where $T_{E/B}$ is the normal line bundle to $s \simeq B$. Consider the exact sequence

$$0 \to T_{E/B} \to T_E \to \rho^* T_B \to 0.$$

Since $K_E = \det T_E^*$, we obtain $K_E = \det T_{E/B}^* \otimes \det \rho^* T_B^*$. Therefore, Lemma 4.6 gives

$$K_E^{-1} = \mathcal{O}(2s) \otimes \rho^*(K_B^{-1}) \otimes \rho^*(T_{E/B}^*).$$

Now explicit computation of $T_{E/B} = \mathcal{L}_1^* \otimes (\mathcal{L}_2/\mathcal{L}_1)$ (using (4.1) and (4.2)) gives the desired formula. □

We now take $\mathcal{F} = \omega_{i,j}$ (recall $\beta_l = \alpha_{i,j}$ and $\rho = \rho_l : R_n(l) \to R_n(l-1)$).

Lemma 4.8. *Let $i \neq 1$ and $j \neq n-1$. Then*

$$\omega_{i,j} = \mathcal{O}(s_{i,j}) \otimes \rho^*(\omega_{i,j+1} \otimes \omega_{i-1,j+1}^* \otimes \omega_{i-1,j}).$$

Let $i = 1$. Then

$$\omega_{1,j} = \mathcal{O}(s_{1,j}) \otimes \rho^*(\omega_{1,j+1}).$$

Let $j = n-1$. Then

$$\omega_{i,n-1} = \mathcal{O}(s_{i,n-1}) \otimes \rho^*(\omega_{i-1,n-1}).$$

Proof. We note that the restriction of $\omega_{i,j}$ to the fibers of ρ equals to $\mathcal{O}(1)$. Now the formula can be proved by an explicit computation using Lemma 4.6. □

Corollary 4.9. $K_{R_n}^{-1} = \bigotimes_{i=1}^{n-1} \omega_{i,i}^{\otimes 2}$.

Proof. We substitute the expression for $\mathcal{O}(s_{i,j})$ in terms of $\omega_{i,j}$ from Lemma 4.8 into the formulas from Lemma 4.7. □

Corollary 4.10. *Theorem 4.5 holds.*

Proof. Recall the $(\mathbb{P}^1)^m$-fibrations $\bar{\rho}_m : R_n(m(m+1)/2) \to R_n(m(m-1)/2)$, where $n-1 \geq m \geq 1$. Using Lemmas 4.7 and 4.8 we obtain

$$K_{R_n}^{-1} = \bigotimes_{i=1}^{n-1} \mathcal{O}(Z_{i,i}) \otimes \bigotimes_{i=1}^{n-1} \omega_{i,i} \otimes \bigotimes_{i=1}^{n-3} \omega_{i,i+2}^* \otimes \bar{\rho}_{n-1}^* K_{R_n((n-1)(n-2)/2)}^{-1}.$$

Using Corollary 2.8 and Lemmas 4.7 and 4.8 again, we rewrite further

$$K_{R_n}^{-1} = \bigotimes_{i=1}^{n-1} \mathcal{O}(Z_{i,i}) \otimes \bigotimes_{i=1}^{n-2} \mathcal{O}(Z_{i,i+1}) \otimes$$
$$\bigotimes_{i=1}^{n-1} \omega_{i,i} \otimes \bigotimes_{i=1}^{n-2} \omega_{i,i+1} \otimes \bigotimes_{i=1}^{n-4} \omega_{i,i+3}^* \otimes \bar{\rho}_{n-2}^* \bar{\rho}_{n-1}^* K_{R_n((n-2)(n-3)/2)}^{-1}.$$

Continuing further, we arrive at the desired formula. □

Corollary 4.11. *The varieties R_n and $\mathcal{F}_n^a$ are Frobenius split.*

Proof. According to Theorem 4.4 it suffices to find a section of the line bundle

$$\bigotimes_{i=1}^{n-1} \omega_{i,i} \otimes \bigotimes_{i=1}^{n-2} \omega_{i,i+1}.$$

which does not vanish at the point $\cap_{l=1}^M Z_l$. But it is easy to see that this line bundle does not have any base points at all. □

Theorem 4.12. *All the degenerate partial flag varieties $\mathcal{F}_{\mathbf{d}}^a$ are Frobenius split.*

Proof. We note the the resolution $R_{\mathbf{d}}$ can be realized inside R_n as an intersection $\bigcap_{(i,j)\notin P_{\mathbf{d}}} Z_{i,j}$. Therefore, Theorem 4.4 guaranties the Frobenius splitting for $R_{\mathbf{d}}$. Now the normality of $\mathcal{F}^a_{\mathbf{d}}$ implies the desired Frobenius splitting for $\mathcal{F}^a_{\mathbf{d}}$. □

We close this section with the following remark. The divisors $Z_{i,j}$ produce a cell decomposition for R_n. Namely, introduce the following notations: $Z_{i,j} = Z_\beta$ if $\beta = \alpha_{i,j}$; and for a subset $I \subset R_+$: $Z_I = \bigcap_{\beta\in I} Z_\beta$. We set $\overset{\circ}{Z}_I := Z_I \setminus \left(\bigcup_{J\supsetneq I} Z_J\right)$. Then we have a cell decomposition

$$R_n = \bigsqcup_{I\subset R^+} \overset{\circ}{Z}_I.$$

We note however that the map $R_n \to \mathcal{F}^a_n$ is not cellular with respect to this decomposition and the cell decomposition of $\mathcal{F}^a_n$ defined in Section 2. For example, let $n = 3$. Then the projection $R_n \to Gr(2, W_{2,2})$ induces a cell decomposition of $Gr(2, 3)$. The zero-dimensional cell is given by $\mathrm{span}(w_2, w_3)$. However, the zero-dimensional cell in $Gr(2, W_{2,2})$ coming from the G^a-action as in Section 2 is equal to $\mathrm{span}(w_1, w_3)$.

5. The BWB-type theorem and graded character formula

5.1. **Rational singularities.** We prove that $\mathcal{F}^a_{\mathbf{d}}$ has rational singularities. Recall the desingularization $Y_{\mathbf{d}}$ introduced in the proof of Proposition 3.6.

Lemma 5.1. *The variety $\mathcal{F}^a_{\mathbf{d}}$ is Gorenstein, i.e. the dualizing complex $K_{\mathcal{F}^a_{\mathbf{d}}}$ is a line bundle. The resolution $\tau_{\mathbf{d}} : Y_{\mathbf{d}} \to \mathcal{F}^a_{\mathbf{d}}$ is crepant, i.e. $K_{Y_{\mathbf{d}}} = \tau^*_{\mathbf{d}} K_{\mathcal{F}^a_{\mathbf{d}}}$.*

Proof. We know that $\mathcal{F}^a_{\mathbf{d}}$ is a locally complete intersection. By the adjunction formula, it follows that $\mathcal{F}^a_{\mathbf{d}}$ is Gorenstein. According to the proof of Proposition 3.6, the map $\tau_{\mathbf{d}} : Y_{\mathbf{d}} \to \mathcal{F}^a_{\mathbf{d}}$ is one-to-one off codimension two in $Y_{\mathbf{d}}$. Hence, the canonical line bundle $K_{Y_{\mathbf{d}}}$ coincides with $\tau^*_{\mathbf{d}} K_{\mathcal{F}^a_{\mathbf{d}}}$ off codimension two. Hence the desired equality $K_{Y_{\mathbf{d}}} = \tau^*_{\mathbf{d}} K_{\mathcal{F}^a_{\mathbf{d}}}$. □

Remark 5.2. Corollary 4.9 says that the canonical line bundle of the complete degenerate flag variety is given by $K_{\mathcal{F}^a_{\mathbf{d}}} = \prod_{i=1}^{n-1} (\omega^*_{i,i})^{\otimes 2}$.

Theorem 5.3. *For the projection $\tau_{\mathbf{d}} : Y_{\mathbf{d}} \to \mathcal{F}^a_{\mathbf{d}}$ we have a canonical isomorphism $R(\tau_{\mathbf{d}})_*\mathcal{O} = \mathcal{O}$, i.e. $(\tau_{\mathbf{d}})_*\mathcal{O} = \mathcal{O}$ and $R^i(\tau_{\mathbf{d}})_*\mathcal{O} = 0$ for all $i > 0$.*

Proof. We know that $\mathcal{F}^a_{\mathbf{d}}$ is normal, so that $(\tau_{\mathbf{d}})_*\mathcal{O} = \mathcal{O}$. Recall the Grauert–Riemenschneider vanishing theorem ([GR]):

$$R^i(\tau_{\mathbf{d}})_* K_{Y_{\mathbf{d}}} = 0 \text{ for all } i > 0.$$

Since $K_{Y_{\mathbf{d}}} = \tau^*_{\mathbf{d}} K_{\mathcal{F}^a_{\mathbf{d}}}$ is the pull-back of a line bundle, the projection formula says

$$R^i(\tau_{\mathbf{d}})_*(\tau^*_{\mathbf{d}}\mathcal{L}) = (\mathcal{L} \otimes K^{-1}_{\mathcal{F}^a_{\mathbf{d}}}) \otimes R^i(\tau_{\mathbf{d}})_* K_{Y_{\mathbf{d}}} = 0$$

for any line bundle $\mathcal{L}$ on $\mathcal{F}^a_{\mathbf{d}}$. Using the projection formula again, we arrive at the desired vanishing of higher direct images of $\mathcal{O}$. □

5.2. **The BWB-type theorem.** We prove the following theorem, which is an analogue of the Borel-Weyl-Bott theorem. Let λ be a dominant integral weight. Consider the map $\imath_\lambda : \mathcal{F}^a_n \to \mathbb{P}(V^a_\lambda)$. Define a line bundle $\mathcal{L}_\lambda = \imath^*_\lambda \mathcal{O}(1)$ on $\mathcal{F}^a_n$.

Theorem 5.4. *We have*

$$H^0(\mathcal{F}^a_n, \mathcal{L}_\lambda)^* \simeq H^0(R_n, \pi^*_n \mathcal{L}_\lambda)^* \simeq V^a_\lambda,$$
$$H^{>0}(\mathcal{F}^a_n, \mathcal{L}_\lambda) = H^{>0}(R_n, \pi^*_n \mathcal{L}_\lambda) = 0.$$

Proof. First, we note that since $\mathcal{F}^a_n$ has rational singularities, we have the equalities

$$H^k(\mathcal{F}^a_n, \mathcal{L}_\lambda) \simeq H^k(R_n, \pi^*_n \mathcal{L}_\lambda)$$

for all $k \geq 0$.

Second, we prove that all non-zero cohomology $H^k(\mathcal{F}^a_n, \mathcal{L}_\lambda)$ vanish. In fact, first assume λ is regular. Then since the map $\mathcal{F}^a_n \to \mathbb{P}(V^a_\lambda)$ is an embedding, the line bundle $\mathcal{L}_\lambda$ is very ample. Therefore, for any k and big enough N one has $H^k(\mathcal{F}^a_n, \mathcal{L}^{\otimes N}_\lambda) = 0$. This implies $H^k(\mathcal{F}^a_n, \mathcal{L}_\lambda) = 0$, because $\mathcal{F}^a_n$ is Frobenius split over $\overline{\mathbb{F}}_p$ for any p.

Now consider a non regular λ. Let $\mathcal{F}^a_{\mathbf{d}}$ be the corresponding degenerate parabolic flag variety, which is embedded into $\mathbb{P}(V^a_\lambda)$. Then we have the following commutative diagram of projections:

$$\begin{array}{ccc} R_n & \xrightarrow{\ \eta\ } & R_{\mathbf{d}} \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{\mathbf{d}}} \\ \mathcal{F}^a_n & \xrightarrow{\ \phi\ } & \mathcal{F}^a_{\mathbf{d}} \end{array}$$

Let $\mathcal{L}'_\lambda$ be a line bundle on $\mathcal{F}^a_{\mathbf{d}}$ which is the pull back of the bundle $\mathcal{O}(1)$ on $\mathbb{P}(V^a_\lambda)$. Then $\mathcal{L}_\lambda = \phi^*\mathcal{L}'_\lambda$. Since $\mathcal{L}'_\lambda$ is very ample, and $\mathcal{F}^a_{\mathbf{d}}$ is Frobenius split over $\overline{\mathbb{F}}_p$ for any p, $H^k(\mathcal{F}^a_{\mathbf{d}}, \mathcal{L}'_\lambda) = 0$ (for positive k). Since $\mathcal{F}^a_{\mathbf{d}}$ has rational singularities, $H^k(R_{\mathbf{d}}, \pi^*_{\mathbf{d}}\mathcal{L}'_\lambda) = H^k(\mathcal{F}^a_{\mathbf{d}}, \mathcal{L}'_\lambda)$(= 0 for positive k). Now since η is a fibration with the fibers being towers of successive $\mathbb{P}^1$-fibrations, we obtain $H^k(R_n, \eta^*\pi^*_{\mathbf{d}}\mathcal{L}'_\lambda) = H^k(R_{\mathbf{d}}, \pi^*_{\mathbf{d}}\mathcal{L}'_\lambda)$(= 0 for positive k). Finally, since $\mathcal{F}^a_n$ has rational singularities, and $\eta^*\pi^*_{\mathbf{d}}\mathcal{L}'_\lambda = \pi^*_n\mathcal{L}_\lambda$, we arrive at $H^k(\mathcal{F}^a_n, \mathcal{L}_\lambda) = H^k(R_n, \pi^*_n\mathcal{L}_\lambda) = H^k(R_n, \eta^*\pi^*_{\mathbf{d}}\mathcal{L}'\lambda)$(= 0 for $k > 0$).

Third, we note that there exists an embedding $(V^a_\lambda)^* \hookrightarrow H^0(\mathcal{F}^a_n, \mathcal{L}_\lambda)$. In fact take an element $v \in (V^a_\lambda)^* \simeq H^0(\mathbb{P}(V^a_\lambda), \mathcal{O}(1))$. Then restricting to the embedded variety $\mathcal{F}^a_n$ we obtain a section of $\mathcal{L}_\lambda$. Assume that it is zero.

Then v vanishes on the open cell $(N^-)^a \cdot \mathbb{C}v_\lambda$. But the linear span of the elements of this cell coincides with the whole representation V_λ^a. Therefore, the restriction map $(V_\lambda^a)^* \to H^0(\mathcal{F}_n^a, \mathcal{L}_\lambda)$ is an embedding.

Finally, we recall that the varieties $\mathcal{F}_n^a$ are flat degenerations of the classical flag varieties. Since the higher cohomology of $\mathcal{L}_\lambda$ vanish, we arrive at the equality of the dimensions of $H^0(\mathcal{F}_n^a, \mathcal{L}_\lambda)$ and of V_λ. Therefore, the embedding $(V_\lambda^a)^* \to H^0(\mathcal{F}_n^a, \mathcal{L}_\lambda)$ is an isomorphism. □

Similarly one proves a parabolic version of the BWB-type theorem:

Theorem 5.5. *Let λ be a $\mathbf{d}$-dominant weight, i.e. $(\lambda, \omega_d) > 0$ implies $d \in \mathbf{d}$. Then there exists a map $\imath_\lambda : \mathcal{F}_\mathbf{d}^a \to \mathbb{P}(V_\lambda^a)$. We have*

$$H^0(\mathcal{F}_\mathbf{d}^a, \imath_\lambda^* \mathcal{O}(1))^* \simeq V_\lambda^a, \ \ H^{>0}((\mathcal{F}_\mathbf{d}^a, \imath_\lambda^* \mathcal{O}(1))) = 0.$$

5.3. **The q-character formula.** We now compute the q-character (PBW-graded character) of the modules V_λ^a (for combinatorial formula see [FFL1]). For this we use the Atiyah-Bott-Lefschetz fixed points formula applied to the variety R_n. Recall that the T-fixed points on R_n are labeled by the admissible collections $\mathbf{S} = (S_{i,j})$, i.e. those satisfying $S_{i,j} \subset \{1, \dots, i, j+1, \dots, n\}$, $\#S_{i,j} = i$ and

$$S_{i,j} \subset S_{i+1,j} \subset S_{i+1,j+1} \cup \{j+1\}. \tag{5.1}$$

In order to state the theorem we prepare some notations. Assume that we have fixed the sets $S_{i-1,j}$ and $S_{i,j+1}$. Then condition (5.1) says that there exist exactly two variants for $S_{i,j}$, namely

$$S_{i,j} = S_{i-1,j} \cup \{a\} \text{ or } S_{i,j} = S_{i-1,j} \cup \{b\},$$

where $\{a, b\} = S_{i,j+1} \cup \{j+1\} \setminus S_{i-1,j}$. Given a collection $\mathbf{S}$ we denote the numbers a, b as above by $a_{i,j}^\mathbf{s}$ and $b_{i,j}^\mathbf{s}$. We have:

$$S_{i,j} = S_{i-1,j} \cup \{a_{i,j}^\mathbf{s}\}, \quad S_{i,j+1} \setminus S_{i-1,j} = \{a_{i,j}^\mathbf{s}, b_{i,j}^\mathbf{s}\}.$$

We denote by $S'_{i,j}$ the set $S_{i,j} \setminus \{a_{i,j}^\mathbf{s}\} \cup \{b_{i,j}^\mathbf{s}\}$.

Example 5.6. Let $n = 3$, $S_{1,1} = (2)$, $S_{1,2} = (1)$ and $S_{2,2} = (1, 3)$. Then

$$a_{1,1}^\mathbf{S} = 2, a_{1,2}^\mathbf{S} = 1, a_{2,2}^\mathbf{S} = 3 \text{ and } b_{1,1}^\mathbf{S} = 1, b_{1,2}^\mathbf{S} = 3, b_{2,2}^\mathbf{S} = 2.$$

Recall that the variety $\mathcal{F}_n^a$ sits inside the product of Grassmann varieties $\prod_{1 \le i \le j < n} Gr(i, W_{i,j})$. Each $\bigwedge^i (W_{i,j})$ is acted upon by $\mathfrak{g}^a \oplus \mathbb{C}d$ and therefore each Grassmannian carries a natural action of the group $G^a \rtimes \mathbb{C}^*$, where the additional $\mathbb{C}^*$ part corresponds to the PBW-grading operator. So we have an n-dimensional torus $T \rtimes \mathbb{C}^*$ acting on $Gr(i, W_{i,j})$. A T-fixed point $p(\mathbf{S}) \in R_n$ is a product of the fixed points $p(S_{i,j}) \in Gr(i, W_{i,j})$. We denote by $\gamma(S_{i,j}) \in \mathfrak{h}^* \oplus \mathbb{C}d$ the (extended) weight of the vector $p(S_{i,j}) \in W_{i,j}$. Explicitly, let $S_{i,j} = (l_1, \dots, l_i)$. Then

$$\gamma(S_{i,j}) = (\omega_{l_1} - \omega_{l_1 - 1}) + \dots + (\omega_{l_i} - \omega_{l_i - 1}) + \#\{r : \ l_r > i\} d.$$

(here $\omega_0 = \omega_n = 0$). For an element

$$\gamma = m_1\omega_1 + \dots + m_{n-1}\omega_{n-1} + md \in \mathfrak{h}^* \oplus \mathbb{C}d$$

we denote by e^γ the element $(e^{\alpha_1})^{m_1} \dots (e^{\alpha_{n-1}})^{m_{n-1}} q^m$ in the group algebra (so, $q = e^d$).

Example 5.7. Let $z_1 = e^{\alpha_1}, \dots, z_{n-1} = e^{\alpha_{n-1}}$. Then for $S_{i,j} = (l_1, \dots, l_i)$

$$e^{\gamma(S_{i,j})} = z_{l_1} z_{l_1-1}^{-1} \dots z_{l_i} z_{l_i-1}^{-1} q^{\#\{r:\ l_r > i\}}.$$

In particular, for $n = 3$, $i = j = 1$ we have

$$S_{1,1} = (1): \quad \gamma(1,1) = \omega_1,\ e^{\gamma(1,1)} = z_1;$$
$$S_{1,1} = (2): \quad \gamma(1,1) = \omega_2 - \omega_1 + d,\ e^{\gamma(1,1)} = z_1^{-1} z_2 q;$$
$$S_{1,1} = (3): \quad \gamma(1,1) = -\omega_2 + d,\ e^{\gamma(1,1)} = z_2^{-1} q.$$

Theorem 5.8. *The q-character of the representation V_λ^a is given by the sum over all admissible collections* $\mathbf{S}$ *of the summands*

$$\frac{e^{\gamma(S_{1,1}) + \dots + \gamma(S_{n-1,n-1})}}{\prod_{1 \le i \le j < n} \left(1 - e^{\gamma(S'_{i,j})} e^{-\gamma(S_{i,j})}\right)}. \tag{5.2}$$

Proof. Recall the Atiyah-Bott-Lefschetz formula (see [AB], [T]): let X be a smooth projective algebraic M-dimensional variety and let $\mathcal{L}$ be a line bundle on X. Let T be an algebraic torus acting on X with a finite set F of fixed points. Assume further that $\mathcal{L}$ is T-equivariant. Then for each $p \in F$ the fiber $\mathcal{L}_p$ is T-stable. We note also that since $p \in F$, the tangent space T_pX carries a natural T-action. Let $\gamma_1^p, \dots, \gamma_M^p$ be the weights of the eigenvectors of T-action on T_pX. Then the Atiyah-Bott-Lefschetz formula gives the following expression for the character of the Euler characteristics:

$$\sum_{k \ge 0} (-1)^k \mathrm{ch} H^k(X, \mathcal{L}) = \sum_{p \in F} \frac{\mathrm{ch}\mathcal{L}_p}{\prod_{l=1}^M (1 - e^{\gamma_l^p})}.$$

We apply this formula in our situation. Since $H^{>0}(R_n, \pi_n^* \mathcal{L}_\lambda) = 0$, the Euler characteristics coincides with the character of the zeroth cohomology. Further, the sum in (5.2) runs over the set of T-fixed points in R_n and for each summand the numerator in the $\mathbf{S}$-th term is exactly the character of the line $\pi_n p(\mathbf{S})$, or, equivalently, the character of the fiber $(\pi_n^* \mathcal{L}_\lambda)_{p(\mathbf{S})}$. We are only left to compute the torus action in the tangent space $T_{p(\mathbf{S})} R_n$.

Recall that R_n is a tower of successive $\mathbb{P}^1$-fibrations $R_n(l) \to R_n(l-1)$. Fix an admissible $\mathbf{S}$. Then the surjections $R_n \to R_n(l)$ define the T-fixed points $p(\mathbf{S}(l))$ in each $R_n(l)$ (note that $\mathbf{S}(l)$ consists of $S_{i,j}$ such that for $\beta_k = \alpha_{i,j}$ one has $k \le l$). For each $l = 1, \dots, M$ we denote by $v_l \in T_{p(\mathbf{S}(l))} R_n(l)$ a tangent vector to the fiber of the map $R_n(l) \to R_n(l-1)$ at the point $p(\mathbf{S}(l-1))$. Then it is easy to see that the weights of the eigenvectors of the T action in $T_{p(\mathbf{s})} R_n$ are exactly the weights of the vectors v_l, $l = 1, \dots, M$.

So let us fix an l, $1 \le l \le M$ and i, j with $\alpha_{i,j} = \beta_l$. Let us denote by Y_l the set of all pairs (t, u) such that for the root $\alpha_{t,u} = \beta_r$ one has $r \le l$. Then the fiber $\mathbb{P}^1$ of the map $R_n(l) \to R_n(l-1)$ at the point $p(\mathbf{S}(l-1))$ consists of all collections $(V_{t,u})$ with $(t, u) \in Y_l$ subject to the following conditions:

- $V_{t,u} = p(S_{t,u})$ if $\alpha_{t,u} \ne \beta_l$,
- $V_{i,j} \supset p(S_{i-1,j})$,
- $V_{i,j} \subset p(S_{i-1,j}) \oplus \mathbb{C}w_{a^{\mathbf{S}}_{i,j}} \oplus \mathbb{C}w_{b^{\mathbf{S}}_{i,j}}$.

Now it is easy to see that the character of the tangent vector to this fiber at the point $p(\mathbf{S}(l-1))$ is equal to $e^{\gamma(S'_{i,j})} e^{\gamma(S_{i,j})^{-1}}$ (recall $a^{\mathbf{S}}_{i,j} \in S_{i,j}$ and $S'_{i,j} = S_{i,j} \setminus \{a^{\mathbf{S}}_{i,j}\} \cup \{b^{\mathbf{S}}_{i,j}\}$). □

Example 5.9. Let $n = 2$. Then the formula above says

$$\mathrm{ch}_q V_{m\omega} = \frac{z^m}{1 - qz^{-2}} + \frac{z^{-m}}{1 - q^{-1}z^2} = z^m + qz^{m-2} + \cdots + q^m z^{-m}.$$

Example 5.10. Let $n = 3$. Then the contribution of a fixed point with $S_{1,1} = (2)$, $S_{1,2} = (1)$, $S_{2,2} = (1, 3)$ is given by

$$\frac{q^2}{(1 - z_1^{-1} z_2^{-1} q)(1 - z_1^2 z_2^{-1} q)(1 - z_1^{-1} z_2^2 q)}$$

and the contribution of a fixed point with $S_{1,1} = (2)$, $S_{1,2} = (3)$, $S_{2,2} = (1, 3)$ is given by

$$\frac{q^2}{(1 - z_1 z_2 q^{-1})(1 - z_1 z_2^{-2})(1 - z_1^{-2} z_2)}.$$

We note that these are exactly the points which are mapped by π_3 to the only singular point of $\mathcal{F}^a_3$, which is torus fixed and labeled by $S_{1,1} = (2)$, $S_{2,2} = (1, 3)$.

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References

[A] I. Arzhantsev, *Flag varieties as equivariant compactifications of* $\mathbb{G}_a^n$, arXiv:1003.2358.

[AS] I. Arzhantsev, E. Sharoiko, *Hassett-Tschinkel correspondence: modality and projective hypersurfaces*, arXiv:0912.1474.

[AB] M.F. Atiyah, R. Bott, *A Lefschetz fixed point formula for elliptic differential operators*, Bull. Amer. Math. Soc. **72** (1966), 245–250.

[BK] M. Brion, S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics **231**, Birkhäuser, Boston (2005), x+250pp.

[Fe1] E. Feigin, $\mathbb{G}_a^M$ *degeneration of flag varieties*, arXiv:1007.0646.

[Fe2] E. Feigin, *Degenerate flag varieties and the median Genocchi numbers*, arXiv:1101.1898.

[Fu] W. Fulton, *Young tableaux, with applications to representation theory and geometry.* Cambridge University Press, 1997.

[FH] W. Fulton, J. Harris, (1991), *Representation theory. A first course*, Graduate Texts in Mathematics, Readings in Mathematics **129**, New York.

[FFL1] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for irreducible modules in type* A_n, arXiv:1002.0674. To appear in Transformation Groups.

[FFL2] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for symplectic Lie algebras*, International Mathematics Research Notices 2011; doi: 10.1093/imrn/rnr014.

[EGA] A. Grothendieck, J. Dieudonné, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).

[GR] H. Grauert und O. Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Inv. Math. **11** (1970), 263–292.

[HT] B. Hassett, Yu. Tschinkel, *Geometry of equivariant compactifications of* $\mathbb{G}_a^n$, Int. Math. Res. Notices **20** (1999), 1211–1230.

[K] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory,* Progress in Mathematics **204**, Birkhauser, Boston (2002).

[MR] V. B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. of Math. (2) **122** (1985), no. 1, 27–40.

[T] R.W. Thomason, *Une formule de Lefschetz en K-théorie équivariante algébrique*, Duke Math. J. **68** (1992), 447–462.

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