KAZHDAN-LUSZTIG COMBINATORICS IN THE MOMENT GRAPH SETTING

MARTINA LANINI

Abstract. Motivated by the multiplicity conjecture on the stalks of the canonical sheaf on a Bruhat graph, we lift some equalities concerning (parabolic) Kazhdan-Lusztig polynomials to this moment graph setting. Our proofs hold also in positive characteristic, under some technical assumptions.

1. Introduction

In 1979 Kazhdan and Lusztig ([18]) associated to a given Coxeter group $\mathcal{W}$ a family of polynomials $\{P_{y,w}(q)\}$ indexed by pairs of elements in $\mathcal{W}$. In the case $\mathcal{W}$ was a Weyl group, then $P_{y,w}(q)$ was related to the intersection cohomology of the corresponding Schubert variety (cf. Appendix A of [18] and [19]). Some years later, Deodhar in [5] introduced the parabolic analogue of Kazhdan-Lusztig polynomials. Namely, if $(\mathcal{W},\mathcal{S})$ is a Coxeter system, $J\subseteq S$, and $\mathcal{W}^J$ is the set of minimal representatives of the equivalence classes of $\mathcal{W}/\langle J\rangle$, he defined two families of polynomials $\{P^{J,-1}_{y,w}(q)\}$ and $\{P^{J,q}_{y,w}(q)\}$, where $y,w\in\mathcal{W}^J$. The $\{P^{J,-1}_{y,w}(q)\}$ are just a generalization of the polynomials defined by Kazhdan and Lusztig and for $J=\emptyset$ they coincide. As in the regular case, if $\mathcal{W}$ is a Weyl group, then these polynomials have a geometrical meaning, and, in particular, they are related to the intersection cohomology of the corresponding partial Schubert variety.

Kazhdan and Lusztig ([18]), resp. Lusztig ([23]), conjectured that the Kazhdan-Lusztig polynomials played a very important role in the representation theory of complex Lie algebras, resp. of semisimple, simply connected, reductive algebraic groups over a field of positive characteristic. The characteristic zero setting is now well understood, while the positive characteristic analogue is not. Actually Lusztig's conjecture was *almost* proved in the Nineties via the joint work of Kazhdan-Lusztig ([20]), Kashiwara-Tanisaki ([21]) and Andersen-Jantzen-Soergel ([1]). Here *almost* means that it was possible to prove the conjecture only if the characteristic of the base field is big enough, since it was obtained as a limit of the characteristic zero case. A new approach to Lusztig's conjecture is due to Fiebig ([7],[12]) and it is based on the theory of sheaves on moment graphs.

Moment graphs were introduced by Goresky, Kottwitz and MacPherson ([15]), in order to study the equivariant cohomology of a complex algebraic variety equipped with a torus action and having some nice properties. In 2001 Braden and MacPherson in [2] were able to describe the equivariant intersection cohomology of such a variety via sheaves on a given moment graph. In particular, if $\mathcal{W}$ is a Weyl group with $\mathcal{S}$, set of simple reflections, and $J\subseteq\mathcal{S}$, Braden and MacPherson associated to $w\in\mathcal{W}^J$ a sheaf $\mathscr{B}^J_w$: the canonical sheaf. Such an object describes the intersection cohomology of the corresponding (partial) Schubert variety. Braden-MacPherson's construction was performed in characteristic zero, but it is possible to develop such a theory in any characteristic. Fiebig and Williamson proved in [14] that, with certain technical assumptions, $\mathscr{B}^J_w$ in positive characteristic computes the stalks of indecomposable parity sheaves (introduced in [17]). It is now natural to ask whether it is possible to connect the canonical sheaf to Kazhdan-Lusztig polynomials; actually there is a conjecture on the stalks of such a sheaf, stating that, under some assumptions on the characteristic of the base field,

Conjecture 4.1. *if $y, w \in \mathcal{W}^J$, then $\underline{rk}(\mathscr{B}_w^J)^y = P_{y,w}^{J,-1}$.*

Such a conjecture is proved in characteristic zero and in this case it is equivalent to Kazhdan-Lusztig's conjecture (cf.[7]). In characteristic p it is proved for p bigger than a huge (but explicit) lower bound and it implies Lusztig's conjecture (cf.[12],[10]). Anyway, this conjecture motivates our work, since it now makes sense to interpret some equalities concerning (parabolic) Kazhdan-Lusztig polynomials in terms of stalks of the canonical sheaves. In order to lift properties of KL-polynomials to the level of canonical sheaves, we will use two different techniques: pullback of canonical sheaves (see Section 5) and an action of the Weyl group on the set of global sections of the BMP-sheaf (see Section 6).

Let k be a local ring with $2 \in k^*$. We define the notion of k-homomorphism between two moment graphs and of pullback sheaves. These will provide a useful tool, namely, under some assumptions on k,

Lemma 5.1. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two moment graphs, both with a unique maximal vertex, w resp. w', and let f be a k-isomorphism between them. If $\mathscr{B}_w$ and $\mathscr{B}'_{w'}$ are the corresponding canonical sheaves, then $\mathscr{B}_w \cong f^*\mathscr{B}'_{w'}$ as k-sheaves on $\mathcal{G}$.*

Thanks to this result, in some good situations it will be enough to study the combinatorics of the underlying moment graphs that in our case are just labeled, oriented Bruhat graphs (see §2.2). Such is the case in the following theorem:

Theorem 5.1. *Let $y, w \in \mathcal{W}$ be such that $y \leq w$, then*

(i): $\mathscr{B}_w^y \cong \mathscr{B}_{w^{-1}}^{y^{-1}}$.

Let $s \in \mathcal{S}$ be such that $ws < w$, then

(ii): *if $y \not\leq ws$, $\mathscr{B}_w^y \cong \mathscr{B}_{ws}^{ys}$,*

where we write $\mathscr{B}_w$ instead of $\mathscr{B}_w^{\emptyset}$.

The last part of the paper is devoted to the study of an action of a certain subgroup of the Weyl group $\mathcal{W}$ on the module of global sections of the canonical sheaf and prove that the data we need to build the canonical sheaf are contained in the module of invariants with respect to this action. This result, together with some combinatorics of the corresponding Bruhat graph, gives us the categorical analogue of a result due to Kazhdan and Lusztig (cf. [19]):

Theorem 6.1. *Under some assumpions on k, if $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ are such that $y \leq w$ and $ws < w$, then $\mathscr{B}_w^y \cong \mathscr{B}_w^{ys}$.*

Inspired by a theorem of Deodhar ([5]) we prove a relation between the canonical sheaf on a regular Bruhat graph $\mathcal{G}$ and the ones on the corresponding parabolic Bruhat graphs $\mathcal{G}^J$, for J such that the subgroup $\langle J \rangle$ is finite. Let w_J be the longest element of $\langle J \rangle$ then, under some assumptions on k, we have

Theorem 6.2. *if $y, w \in \mathcal{W}^J$, then $(\mathscr{B}_w^J)^y \cong (\mathscr{B}_{ww_J}^{\emptyset})^{yw_J}$.*

In order to get this result we consider again the module of invariants with respect to the above action. The claim follows from the connection between this module and the parabolic canonical sheaf.

Structure of the paper. Section 2 and 3 are about moment graphs and sheaves on them. In Section 4 we introduce Braden-MacPherson sheaves and recall some of their properties. We develop and apply the technique of pullbacks in Section 5, while the one of invariants is used in the last section.

2. Moment graphs

In this section we recall the definition of moment graphs on a lattice and we define the notion of k-homomorphism between two moment graphs.

Let k be from now on a local ring such that 2 is an invertible element. Let $Y \cong \mathbb{Z}^r$ be a lattice of finite rank and denote by $Y_k := Y \otimes_{\mathbb{Z}} k$.

Definition 2.1. *A* moment graph *$\mathcal{G}$ on Y is given by $(\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, where:*
(i) *$(\mathcal{V}, \mathcal{E})$ is a directed graph without directed cycles nor multiple edges,*
(ii) *$\trianglelefteq$ is a partial order on $\mathcal{V}$ such that if $x, y \in \mathcal{V}$ and $E : x \to y \in \mathcal{E}$ then $x \trianglelefteq y$,*
(iii) *$l : \mathcal{E} \to Y \setminus \{0\}$ is a map called the* label function.

Definition 2.2. *Let $\mathcal{G}$ be a moment graph on the lattice Y, then*

- *$\mathcal{G}$ is a k-*moment graph *on Y if all labels are non-zero in Y_k*
- *$(\mathcal{G}, k)$ is a GKM-*pair *if all pairs of distinct edges containing a common vertex have labels linearly independent in Y_k.*

Observe that if $(\mathcal{G}, k)$ is a GKM-pair, then $\mathcal{G}$ is a k-moment graph. These properties are very important and in the sequel they will give a restriction on the ring k.

2.1. k-homomorphisms of moment graphs.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$ and $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l')$ be two moment graphs on Y. Since a moment graph is given by an oriented and ordered graph plus some other data coming from Y, we define a k-homomorphism as a homomorphism of graphs plus a collection of endomorphisms of the k-module Y_k satisfying certain requirements. More precisely,

Definition 2.3. $f = (f_{\mathcal{V}}, f_{\mathcal{E}}, \{f_{l,x}\}_{x \in \mathcal{V}}) : \mathcal{G} \to \mathcal{G}'$ *is a* k-homomorphism of moment graphs *if*
(i) *$f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$ is such that whenever u —— $v \in \mathcal{E}$ then $f_{\mathcal{V}}(u)$ —— $f_{\mathcal{V}}(v) \in \mathcal{E}'$,*
(ii) *for $E : u$ —— $v \in \mathcal{E}$, $f_{\mathcal{E}}(E) := f_{\mathcal{V}}(u)$ —— $f_{\mathcal{V}}(v)$,*
(iii) *for any $x \in \mathcal{V}$, $f_{l,x} : Y_k \to Y_k$ is a homomorphism of k-modules such that if there exists an edge $E : x$ —— $y \in \mathcal{E}$, then*
(a) *$f_{l,x}(l(E)) = h \cdot l'(f_{\mathcal{E}}(E))$ for some invertible element $h \in k^*$*
(b) *$\pi \circ f_{l,x} = \pi \circ f_{l,y}$, where π is the canonical quotient map $\pi : Y_k \to Y_k / l'(f_{\mathcal{E}}(E)) Y_k$.*

Definition 2.4. $f = (f_{\mathcal{V}}, f_{\mathcal{E}}, \{f_{l,x}\}_{x \in \mathcal{V}}) : \mathcal{G} \to \mathcal{G}'$ *is a* k-isomorphism of moment graphs *if $f : \mathcal{G} \to \mathcal{G}'$ is a k-homomorphism and*
(i) *$f_{\mathcal{V}}$ is a isomorphism of posets between $\mathcal{V}$ and $\mathcal{V}'$,*
(ii) *$f_{\mathcal{E}}$ is a bijection between $\mathcal{E}$ and $\mathcal{E}'$,*
(iii) *for any $x \in \mathcal{V}$ $f_{l,x} : Y_k \longrightarrow Y_k$ is an automorphism of the k-module Y_k.*

If f is a k-isomorphism from the moment graph $\mathcal{G}$ to itself, we say that it is a *k-automorphism* of $\mathcal{G}$.

2.2. **Bruhat graphs.** Here we describe a class of moment graphs, that is for our purposes the most important.

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{t}$ be a symmetrizable Kac-Moody algebra, a standard Borel subalgebra and a Cartan subalgebra. These data uniquely determine a root system R, its set of positive roots and of simple roots. Let $\mathcal{W}$ be the associated Weyl group and $\mathcal{S}$ be its set of simple reflections. For any $J \subseteq \mathcal{S}$, denote by $\mathcal{W}_J := \langle J \rangle$ and by $\mathcal{W}^J$ the set of minimal representatives of the equivalence classes of $\mathcal{W}/\mathcal{W}_J$. Thus the *Bruhat graph* $\mathcal{G}^J$ is the moment graph (on the weight lattice X if $\mathcal{W}$ is finite or on the affine weight lattice $\widehat{X} := X \oplus \mathbb{Z}\delta$ if $\mathcal{W}$ is affine) having set of vertices the elements of $\mathcal{W}^J$, equipped with the (induced) Bruhat order. The set of edges is

$$\mathcal{E} := \{x \rightarrow y |\, \exists\, \alpha \in R^+,\, w \in \mathcal{W}_J \text{ s.t. } y = s_\alpha x w \text{ and } x < y\},$$

where R^+ denotes the set of positive (real) roots and s_α is the reflection corresponding to the root $\alpha \in R^+$. Recall that the set of all reflections $\mathcal{T}$ of $\mathcal{W}$ is indexed by R^+, so

$$\mathcal{E} = \{x \rightarrow y |\, \exists\, t \in \mathcal{T},\, w \in \mathcal{W}_J \text{ s.t. } y = txw \text{ and } x < y\}.$$

Finally, the label function is $l(x \rightarrow s_\alpha x w) := \alpha$.

Such a moment graph has an important geometric meaning. Indeed, there is a partial flag variety Y corresponding to $\mathcal{W}$ and J as above (see [22]) and it carries an action of a torus T and a (T-invariant) stratification with certain good properties (see [2]). The Bruhat graph encodes the action of this torus, in particular, the vertices are the 0-dimensional orbits, while the edges represent the 1-dimensional orbits (cf.§2.1 of [14]). The partial order on the set of vertices is induced by the stratification coming from the decomposition $Y = \bigsqcup_{w \in \mathcal{W}^J} Y_w$, where, indeed, $\overline{Y_w} = \bigsqcup_{\substack{y \leq w \\ y \in \mathcal{W}^J}} Y_y$.

3. Category of k-sheaves on a moment graph

Consider a moment graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$ on a lattice Y. Recall that for any local ring k with $2 \in k^*$ we denoted by $Y_k := Y \otimes_{\mathbb{Z}} k$.

Let $S_k := Sym(Y_k)$ be the symmetric algebra of Y_k. We provide S_k with a $\mathbb{Z}$-grading such that $(S_k)_{\{2\}} = Y_k$. From now on every S_k-module will be finitely generated and $\mathbb{Z}$-graded and every morphism between S_k-modules will be of degree zero.

Definition 3.1. *A k-sheaf $\mathcal{F}$ on $\mathcal{G}$ is given by $(\{\mathcal{F}^x\}, \{\mathcal{F}^E\}, \{\rho_{x,E}\})$, where:*
(i) *for all $x \in \mathcal{V}$, $\mathcal{F}^x$ is an S_k-module;*
(ii) *for all $E \in \mathcal{E}$, $\mathcal{F}^E$ is an S_k-module such that $l(E) \cdot \mathcal{F}^E = \{0\}$;*
(iii) *for $x \in \mathcal{V}$, $E \in \mathcal{E}$, $\rho_{x,E} : \mathcal{F}^x \rightarrow \mathcal{F}^E$ is a homomorphism of S_k-modules defined if x is in the border of the edge E.*

Definition 3.2. *A* homomorphism $\varphi : \mathcal{F} \longrightarrow \mathcal{F}'$ *between the k-sheaves on the moment graph $\mathcal{G}$ is given by the following data*
(i) *for all $x \in \mathcal{V}$, $\varphi^x : \mathcal{F}^x \rightarrow \mathcal{F}'^x$ is a homomorphism of S_k-modules*

(ii) *for all $E \in \mathcal{E}$, $\varphi^E : \mathcal{F}^E \to \mathcal{F}'^E$ is a homomorphism of S_k-modules such that for any $x \in \mathcal{V}$ on the border of $E \in \mathcal{E}$ the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F}^x & \xrightarrow{\rho_{x,E}} & \mathcal{F}^E \\ \downarrow{\scriptstyle \varphi^x} & & \downarrow{\scriptstyle \varphi^E} \\ \mathcal{F}'^x & \xrightarrow{\rho'_{x,E}} & \mathcal{F}'^E \end{array}$$

We denote by $\mathfrak{Sh}^k_{\mathcal{G}}$ the category of *k-sheaves* on $\mathcal{G}$ having as objects the k-sheaves on $\mathcal{G}$ and as morphisms the homomorphisms between them.

3.1. **Pullback sheaves.** Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, resp. $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l')$, be a moment graph on Y and fix $f : \mathcal{G} \longrightarrow \mathcal{G}'$ a k-homomorphism of moment graphs (cf.§2.1) and $\mathcal{F} \in Ob(\mathfrak{Sh}^k_{\mathcal{G}'})$.

Define $f^*\mathcal{F} = (\{(f^*\mathcal{F})^x\}, \{(f^*\mathcal{F})^E\}, \{(f^*\rho)_{x,E}\})$ in the following way:

(i): for all $x \in \mathcal{V}$ $(f^*\mathcal{F})^x := \mathcal{F}^{f_{\mathcal{V}}(x)}$;
(ii): for all $E \in \mathcal{E}$ $(f^*\mathcal{F})^E := \mathcal{F}^{f_{\mathcal{E}}(E)}$;
(iii): for all $x \in \mathcal{V}$, $E \in \mathcal{E}$ such that x is on the border of E, $(f^*\rho)_{x,E} := \rho_{f_{\mathcal{V}}(x), f_{\mathcal{E}}(E)}$.

Lemma 3.1. *$f^*\mathcal{F} \in Ob(\mathfrak{Sh}^k_{\mathcal{G}})$.*

Proof. The structure of S_k-module on $(f^*\mathcal{F})^x$ is given by the map $f_{l,x} : Y_k \to Y_k$. Indeed, $f_{l,x}$ induces a map that we denote with the same symbol $f_{l,x} : S_k \to S_k$ and so $g \in S_k$ acts on $(f^*\mathcal{F})^x = \mathcal{F}^{f_{\mathcal{V}}(x)}$ via $f_{l,x}(g)$.

Suppose $E : x \text{ --- } y$, then $g \in S_k$ acts on on $(f^*\mathcal{F})^{E:x\text{---}y}$ via $f_{l,x}(g)$, and, in particular, $f_{l,x}(l(E)) = h \cdot l'(f_{\mathcal{E}}(E))$ for $h \in k^*$ and so $f_{l,x}(l(E))(f^*\mathcal{F})^E = h \cdot l'(f_{\mathcal{E}}(E))\mathcal{F}^{f_{\mathcal{E}}(E)} = 0$. This structure of S_k-module is well defined since $\pi \circ f_{l,x} = \pi \circ f_{l,y}$, where $\pi : Y_k \to Y_k / l'(f_{\mathcal{E}}(E))Y_k$.

□

We say that $f^*\mathcal{F}$ is the *pullback* of $\mathcal{F}$. In the sequel the notion of pullback sheaf will be very important in order to compare k-sheaves on different moment graphs.

3.2. **Sections of sheaves.** Even if the k-sheaves on a moment graph are not sheaves, since we did not mention topology, we still have a notion of sections (cf.[2]). For each $\mathcal{I} \subset \mathcal{V}$ we can consider the *set of local sections* of a k-sheaf $\mathcal{F} \in Ob(\mathfrak{Sh}^k_{\mathcal{G}})$ over $\mathcal{I}$:

$$\Gamma_k(\mathcal{I}, \mathcal{F}) := \Big\{ (m_x) \in \prod_{x \in \mathcal{I}} \mathcal{F}^x \mid \rho_{x,E}(m_x) = \rho_{y,E}(m_y) \, \forall E : x \text{ --- } y \in \mathcal{E} \Big\}.$$

We denote by $\Gamma_k(\mathcal{F}) = \Gamma_k(\mathcal{V}, \mathcal{F})$ the set of global sections of the k-sheaf $\mathcal{F}$.

We call *k-structure algebra* of the moment graph $\mathcal{G}$ the set

$$\mathcal{Z}_k = \mathcal{Z}_k(\mathcal{G}) := \Big\{ (z_x) \in \prod_{x \in \mathcal{V}} S_k \mid z_x - z_y \in l(E) S_k \; \forall E : x \text{ --- } y \in \mathcal{E} \Big\}.$$

It is easy to check that for any $\mathcal{F} \in Ob(\mathfrak{Sh}^k_{\mathcal{G}})$ the k-structure algebra $\mathcal{Z}_k$ acts on $\Gamma_k(\mathcal{F})$ via componentwise multiplication.

3.3. **Flabby sheaves.** We use the order on the set of vertices of a moment graph $\mathcal{G}$ to define a topology on it: the *Alexandrov topology*. We say that $\mathcal{I}$ is open if for any $x \in \mathcal{I}$ and any $y \in \mathcal{V}$ such that $x \trianglelefteq y$ then $y \in \mathcal{I}$ as well.

A classical question in sheaf theory is to ask whether a sheaf is flabby or not, that is whether or not any local section over an open set extends to a global one or not.

Let $\mathcal{F} \in Ob(\mathfrak{Sh}^k_{\mathcal{G}})$. We fix a vertex $x \in \mathcal{V}$ and we denote

$$\mathcal{E}_{\delta x} := \big\{ E \in \mathcal{E} \mid E : x \to y \big\}$$

$$\mathcal{V}_{\delta x} := \big\{ y \in \mathcal{V} \mid \exists E \in \mathcal{E}_{\delta x} \text{ such that } E : x \to y \big\}.$$

Now we define $\mathcal{F}^{\delta x}$ as the image of $\Gamma_k(\{\triangleright x, \mathcal{F}\}) := \Gamma_k(\{y \in \mathcal{V} \mid y \triangleright x\}, \mathcal{F})$ under the composition of the following functions:

$$u_x : \Gamma_k(\{\triangleright x\}, \mathcal{F}) \longrightarrow \bigoplus\nolimits_{y \triangleleft x} \mathcal{F}^y \longrightarrow \bigoplus\nolimits_{y \in \mathcal{V}_{\delta x}} \mathcal{F}^y \xrightarrow{\oplus \rho_{y,E}} \bigoplus\nolimits_{E \in \mathcal{E}_{\delta x}} \mathcal{F}^E$$

Fiebig shows in [7] the following,

Proposition 3.1. *Let $\mathcal{F} \in Ob(\mathfrak{Sh}^k_{\mathcal{G}})$. Then the following are equivalent:*
(i) *$\mathcal{F}$ is flabby with respect to the Alexandrov topology, i.e. for any open $\mathcal{I} \subseteq \mathcal{V}$ the restriction map $\Gamma_k(\mathcal{F}) \to \Gamma_k(\mathcal{I}, \mathcal{F})$ is surjective.*
(ii) *For any vertex $x \in \mathcal{V}$ the restriction map $\Gamma_k(\{\trianglerighteq x\}, \mathcal{F}) \to \Gamma_k(\{\triangleright x\}, \mathcal{F})$ is surjective.*
(iii) *For any vertex $x \in \mathcal{V}$ the map $\oplus_{E \in \mathcal{E}_{\delta x}} \rho_{x,E} : \mathcal{F}^x \to \bigoplus_{E \in \mathcal{E}_{\delta x}} \mathcal{F}^E$ contains $\mathcal{F}^{\delta x}$ in its image.*

In the proof of such a proposition the following fact, that will be useful later, plays a role:

Lemma 3.2. *(cf.[7]) Let $\mathcal{F} \in Ob(\mathfrak{Sh}^k_{\mathcal{G}})$, $\mathcal{I} \subseteq \mathcal{V}$ be an open set and $x \in \mathcal{I}$ a minimal element. Then $m = (m_z) \in \Gamma_k(\mathcal{I} \setminus \{x\}, \mathcal{F})$ if and only if $((m_z), m_x) \in \Gamma_k(\mathcal{I}, \mathcal{F})$, for any m_x such that $\oplus_{E \in \mathcal{E}_{\delta x}} \rho_{x,E}(m_x) = u_x((m_z)_{z > x})$.*

4. Braden-MacPherson sheaves.

In this section we introduce the most important object of our paper, namely the canonical sheaf. It was at first defined by Braden and MacPherson only in characteristic zero in order to compute certain intersection cohomology complexes. Despite this, their algorithm makes sense in any characteristic and Fiebig and Williamson proved in [14] that it computes multiplicities of parity sheaves (see [17]) in positive characteristic if $(\mathcal{G}, k)$ is a *GKM*-pair. The following theorem allows us to consider this important sheaf

Theorem 4.1. *([2], char k=0; [7]) Let $\mathcal{G}$ be a finite k-moment graph over Y with highest vertex w. There exists exactly one (up to isomorphism) indecomposable k-sheaf $\mathscr{B}_w$ on $\mathcal{G}$ with the following properties:*
(i) *$\mathscr{B}^w_w \cong S_k$;*
(ii) *If $x, y \in \mathcal{V}$, $E : x \to y \in \mathcal{E}$, then the map $\rho_{y,E} : \mathscr{B}^y_w \to \mathscr{B}^E_w$ is surjective with kernel $l(E)\mathscr{B}^y_w$;*
(iii) *If $x, y \in \mathcal{V}$, $E : x \to y \in E$, then $\rho_{\delta x} := (\bigoplus_{E \in \mathcal{E}_{\delta x}} \rho_{x,E})^T : \mathscr{B}^x_w \to \mathscr{B}^{\delta x}_w$ is a projective cover in the category of graded S_k-modules.*

We call $\mathscr{B}_w$ the *Braden-MacPherson sheaf* or the *canonical sheaf*. We will also refer to it as the *BMP-sheaf*.

Remark 4.1. *By Theorem 4.1 and Proposition 3.1, the canonical sheaf is flabby for the Alexandrov topology. Such a property of the BMP-sheaf is fundamental and we will use it in the sequel.*

4.1. **The multiplicity conjecture.** For $j \in \mathbb{Z}$ and M a graded S-module we denote by $M\{j\}$ the graded S-module obtained from M by shifting the grading by j, i.e. $M\{j\}_{\{i\}} = M_{\{j+i\}}$. If $M = \bigoplus_{i=1}^{n} S_k\{j_i\}$, then its graded rank is $\underline{\mathrm{rk}}\, M = \sum_{i=1}^{n} q^{-\frac{j_i}{2}} \in \mathbb{Z}_{\geq 0}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$.

Let $\mathcal{G}^J$ be the Bruhat graph we defined in §2.2. Thus for any $w \in \mathcal{W}^J$ we can consider the subgraph $\mathcal{G}^J_w := \mathcal{G}^J_{|\{\leq w\}}$. It is a finite k-moment graph (for any k) with highest vertex w, hence we may build the corresponding Braden-MacPherson sheaf $\mathscr{B}^J_w \in Ob(\mathfrak{Sh}^k_{\mathcal{G}^J_w})$ and we have:

Conjecture 4.1. *If* $(\mathcal{G}^J_w, k)$ *is a* GKM*-pair and* $y \leq w$ *then* $\underline{rk}\,(\mathscr{B}^J_w)^y = P^{J,-1}_{y,w}(q)$,

where $P^{J,-1}_{y,w}$ is the parabolic Kazhdan-Lusztig polynomial (corresponding to the parameter $u = -1$) introduced by Deodhar in [5]. If k is $\mathbb{Q}$, the conjecture follows from [18], [19] and [2]. Moreover, Fiebig proved that this is equivalent to a conjecture due to Kazhdan and Lusztig (see [7]). If k is a field of characteristic $p > 2$ Conjecture 4.1 implies a conjecture by Lusztig (see [12],[10]). In this case the conjecture is known to be true for p bigger than an explicit bound depending on $\mathcal{W}$ (see [13]).

Conjecture 4.1 gives us an explicit formula connecting canonical sheaves and parabolic Kazhdan-Lusztig polynomials and motivates our work. We will indeed translate in terms of stalks of BMP-sheaves some well-known identities concerning these polynomials.

5. Pullback of canonical sheaves.

The following lemma tells us that the pullback functor f^* preserves canonical sheaves if f is a k-isomorphism.

Lemma 5.1. *Let* $\mathcal{G}$ *and* $\mathcal{G}'$ *be two* k*-moment graphs on* Y*, both with a unique maximal vertex,* w *resp.* w'*, and let* $f : \mathcal{G} \longrightarrow \mathcal{G}'$ *be a* k*-isomorphism. If* $\mathscr{B}_w$ *and* $\mathscr{B}'_{w'}$ *are the corresponding canonical sheaves, then* $\mathscr{B}_w \cong f^*\mathscr{B}'_{w'}$ *as* k*-sheaves on* $\mathcal{G}$.

Proof. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \trianglelefteq, l)$, $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \trianglelefteq', l')$ and $f = (f_{\mathcal{V}}, f_{\mathcal{E}}, \{f_{l,x}\})$.

Notice that $I \subseteq \mathcal{V}$ is an open subset if and only if $I' := f_{\mathcal{V}}(I) \subseteq \mathcal{V}'$ is an open subset. We prove our Lemma by induction on $|I| = |I'|$, for I open.

If $|I| = |I'| = 1$, we have $I = \{w\}$ and $I' = \{w'\}$. In this case $\mathscr{B}^w_w = S_k$, $\mathscr{B}'^{w'}_{w'} = S_k$ and the isomorphism $\varphi^w : \mathscr{B}^w_w \to \mathscr{B}'^{w'}_{w'}$ is just the automorphism of S_k induced by the automorphism $f_{l,w}$ of Y_k.

Now let $|I| = |I'| = n > 1$ and $y \in I$ be a minimal element. Obviously, $y' := f_{\mathcal{V}}(y)$ is also a minimal element for I'. Moreover, for any $E \in \mathcal{E}$ we set $E' := f_{\mathcal{E}}(E)$.

First of all, observe that $z \in \mathcal{V}_{\delta y}$ if and only if $z' := f_{\mathcal{V}}(z) \in \mathcal{V}'_{\delta y'}$. By the inductive hypothesis, for all $x \triangleright y$ there exists an isomorphism $\varphi^x : \mathscr{B}^x_w \xrightarrow{\sim} \mathscr{B}'^{x'}_{w'}$ such that $\varphi^x(s\cdot m) = f_{l,x}(s)\cdot\varphi^x(m)$, for $s\in S_k$ and $m\in\mathscr{B}^x_w$. Moreover, if $E\notin\mathcal{E}_{\delta y}$ and x is on the border of E with $x\rhd y$, by the inductive hypothesis we have an isomorphism $\varphi^E:\mathscr{B}^E_w\xrightarrow{\sim}\mathscr{B}'^{E'}_{w'}$ such that $\varphi^E(s\cdot n)=f_{l,x}(s)\cdot\varphi^E(n)$, for $s\in S_k$ and $n\in\mathscr{B}^E_w$ and such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{B}^x_w & \xrightarrow{\rho_{x,E}} & \mathscr{B}^E_w \\ \downarrow{\scriptstyle\varphi^x} & & \downarrow{\scriptstyle\varphi^E} \\ \mathscr{B}'^{x'}_{w'} & \xrightarrow{\rho'_{x',E'}} & \mathscr{B}'^{E'}_{w'} \end{array}$$

Now, if $E: y\longrightarrow x$ and $E': y'\longrightarrow x'$, then

$$\mathscr{B}^E_w\cong\mathscr{B}^x_w/l(E)\mathscr{B}^x_w \quad\text{and}\quad \mathscr{B}'^{E'}_{w'}\cong\mathscr{B}'^{x'}_{w'}/l'(E')\mathscr{B}'^{x'}_{w'}.$$

By assumption, $f_{l,x}(l(E)) = h\cdot l'(E')$ for some invertible element $h\in k^*$ and $\varphi^x(l(E)\mathscr{B}^x_w)=f_{l,x}(l(E))\mathscr{B}^{x'}_{w'}=l'(E')\mathscr{B}^{x'}_{w'}$. Thus the quotients are also isomorphic and so there exists $\varphi^E:\mathscr{B}^E_w\xrightarrow{\sim}\mathscr{B}'^{E'}_{w'}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{B}^x_w & \xrightarrow{\rho_{y,E}} & \mathscr{B}^E_w \\ \downarrow{\scriptstyle\varphi^x} & & \downarrow{\scriptstyle\varphi^E} \\ \mathscr{B}'^{x'}_{w'} & \xrightarrow{\rho'_{y',E'}} & \mathscr{B}'^{E'}_{w'} \end{array}$$

Now we have to construct $\mathscr{B}^{\delta y}_w$ and $\mathscr{B}'^{\delta y'}_{w'}$. Observe that $(\varphi^x)_{x\rhd y}$ induces an isomorphism of S_k-modules between the sets of sections $\Gamma(\{\rhd y\},\mathscr{B}_w)\cong\Gamma(\{\rhd' y'\},\mathscr{B}'_{w'})$ and, from what we have observed above, the following diagram commutes:

$$\begin{array}{ccccccc} \Gamma(\{\rhd y\},\mathscr{B}_w) & \longrightarrow & \bigoplus_{x\rhd y}\mathscr{B}^x & \longrightarrow & \bigoplus_{x\in\mathcal{V}_{\delta y}}\mathscr{B}^x_w & \xrightarrow{\oplus\rho_{x,E}} & \bigoplus_{E\in\mathcal{E}_{\delta y}}\mathscr{B}^E_w \\ \downarrow{\scriptstyle\oplus_{x\rhd y}\varphi^x} & & \downarrow{\scriptstyle\oplus_{x\rhd y}\varphi^x} & & \downarrow{\scriptstyle\oplus_{x\in\mathcal{V}_{\delta y}}\varphi_x} & & \downarrow{\scriptstyle\oplus_{E\in\mathcal{E}_{\delta y}}\varphi^E} \\ \Gamma(\{\rhd' y'\},\mathscr{B}'_{w'}) & \longrightarrow & \bigoplus_{x'\rhd' y'}\mathscr{B}^{x'}_{w'} & \longrightarrow & \bigoplus_{x'\in\mathcal{V}_{\delta y'}}\mathscr{B}'^{x'}_{w'} & \xrightarrow[\oplus\rho'_{x',E'}]{} & \bigoplus_{E\in\mathcal{E}_{\delta y'}}\mathscr{B}'^{E'}_{w'} \end{array}$$

It follows that there exists an isomorphism of S_k-modules $\mathscr{B}^{\delta y}_w\cong\mathscr{B}'^{\delta y'}_{w'}$ and by the unicity of the projective cover we obtain $\mathscr{B}^y_w\cong\mathscr{B}'^{y'}_{w'}$. This proves the Lemma. □

Remark 5.1. *Let $y,x,z,w\in\mathcal{W}$ be such that $y\le w$ and $x\le z$. If one could show that any isomorphism of posets $[y,w]\cong[x,z]$ induces a k-isomorphism of moment graphs $f:\mathcal{G}_{|[y,w]}\to\mathcal{G}_{|[x,z]}$ (at least for $k=\mathbb{Q}$), then, by Lemma 5.1, the Lusztig-Dyer* Combinatorial Invariance Conjecture *(stated in* [6]*) would follow. See* [4] *for partial results on this conjecture.*

5.1. **Two KL-properties of the canonical sheaf.** Here we apply Lemma 5.1 in order to lift some equalities concerning KL-polynomials to the moment graph setting.

From now on we denote by $\mathcal{G}=(\mathcal{V},\mathcal{E},l,\le)$ the Bruhat graph corresponding to a Weyl group $\mathcal{W}$ and $J=\emptyset$. As in §2.2 we denote by $\mathcal{S}$ and $\mathcal{T}$ the set of simple reflections and of reflections, respectively, of $\mathcal{W}$. Recall that $\mathcal{G}$ is a moment graph on the lattice $Y = X$ (the weight lattice), resp. $\widehat{X}$ (the affine weight lattice) if $\mathcal{W}$ is finite, resp. affine. Recall, moreover, that there is a linear $\mathcal{W}$-action on Y in both cases.

5.1.1. *Inverses.* Kazhdan and Lusztig gave an inductive formula to calculate the KL-polynomials ((2.2.c) of [18]). From such a formula it follows easily by induction (cf. Ex.12, Chap.5 of [3]) that for any pair $y, w \in \mathcal{W}$ such that $y \leq w$ one has

$$P_{y,w} = P_{y^{-1},w^{-1}}. \tag{1}$$

We translate such an equality in a k-isomorphism of stalks of canonical sheaves.

Lemma 5.2. *Let $\mathcal{W}$ be a Weyl group. The anti-involution on $\mathcal{W}$ defined by the mapping $x \mapsto x^{-1}$ induces a k-automorphism of the Bruhat graph $\mathcal{G}$ for any k.*

Proof. $f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ defined by the mapping $x \mapsto x^{-1}$ is obviously a bijection. Moreover, for each $x, y \in \mathcal{W}$, $x \leq y$ if and only if $x^{-1} \leq y^{-1}$. So $f_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ is a bijection of posets.

Observe that there exists a reflection $t \in \mathcal{T}$ such that $y = tx$ if and only if $y^{-1} = rx^{-1}$, where $r = x^{-1}tx \in \mathcal{T}$. So $x \text{---} y \in \mathcal{E}$ if and only if $x^{-1} \text{---} y^{-1} \in \mathcal{E}$.

Thus, for every $x \in \mathcal{W}$ and any $v \in Y_k$, we set $f_{l,x}(v) := x^{-1}(v)$ and observe that if $E : x \text{---} y = tx$, we have

(a) $f_{l,x}(l(x \text{---} tx)) = x^{-1}(\alpha_t) = \pm l(x^{-1} \text{---} y^{-1})$, where $\pm x^{-1}(\alpha_t) \in R^+$

(b)

$$\begin{aligned} f_{l,y}(v) = y^{-1}(v) = x^{-1}(tv) = x^{-1}(v) - \langle x^{-1}(v), \alpha_t{}^\vee \rangle x^{-1}(\alpha_t) \equiv \\ \equiv x^{-1}(v) = f_{l,x}(v) \pmod{x^{-1}(\alpha_t)} \end{aligned}$$

This proves that we have a k-automorphism of the moment graph $\mathcal{G}$ for any k. ☐

Thus it follows

Corollary 5.1. *Let $y, w \in \mathcal{W}$ be such that $y \leq w$. Denote by $\mathcal{G}$ the corresponding Bruhat graph and let f be as in Lemma 5.2. Then $\mathscr{B}_w \cong f^*\mathscr{B}_{w^{-1}}$ as k-sheaves on $\mathcal{G}$ for any k.*

Proof. By Lemma 5.2, $f_{\mathcal{V}} : x \mapsto x^{-1}$ induces a k-isomorphism between the two complete subgraphs $\mathcal{G}_w$ and $\mathcal{G}_{w^{-1}}$, so we may apply Lemma 5.1 and obtain the statement. ☐

5.1.2. *Multiplying by a simple reflection. Part I.* Let $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ such that $y \leq w$, $ws < w$ and $y \not\leq ws$. In these hypothesis Kazhdan and Lusztig observed (proof of Theor. 4.2 of [18]) that

$$P_{y,w} = P_{ys,ws}. \tag{2}$$

In order to interpret (2) in our moment graph setting we need a standard combinatorial result (that actually holds for any Coxeter group):

Lemma 5.3 ([16], Lemma 7.4)**.** *Let $s \in \mathcal{S}$ and $v, u \in \mathcal{W}$ be such that $vs < v$ and $u < v$.*

(i) *If $us < u$, then $us < vs$.*

(ii) *If $us > u$, then $us \leq v$ and $u \leq vs$.*

Thus, in both cases, $us \leq v$.

We are now able to define for any k a k-isomorphism of Bruhat (sub)graphs:

Lemma 5.4. *Let $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ such that $y \leq w$, $ws < w$ and $y \not\leq ws$, then for any k there is a k-isomorphism of moment graphs $\mathcal{G}_{|_{[y,w]}} \xrightarrow{\sim} \mathcal{G}_{|_{[ys,ws]}}$.*

Proof. We show that $f_{\mathcal{V}} : [y, w] \to [ys, ws]$, $x \mapsto xs$ is a bijection of posets inducing the identity map on the labels.

We verify that if $x \in [y, w]$ then $xs \in [ys, ws]$. We see that $xs < x$; indeed, if it were not the case, by Lemma 5.3 (ii) it should be $x \leq ws$ but this implies that $y \leq ws$. In particular this holds for y, i.e. $ys < y$. Now by Lemma 5.3 (i):

$$xs < x \ ,\ ws < w \ \Rightarrow\ xs \leq ws$$

$$ys < y \ ,\ xs < x \ \Rightarrow\ ys \leq xs.$$

We now show that if $z \in [ys, ws]$ then $zs \in [y, w]$. Observe that $zs > z$; indeed, $ys < z$, $y = (ys)s > ys$ and if $zs < z$, then by Lemma 5.3 (ii), with $u = ys$ and $v = z$, we would get $y = (ys)s \leq z \leq ws$.

Moreover, $z \leq ws < w$ and, by Lemma 5.3 (ii),

$$zs > z \ ,\ ws < w \ \Rightarrow\ zs \leq w.$$

$$y = (ys)s > ys \ ,\ z = (zs)s < zs \ \Rightarrow\ y \leq zs.$$

This completes the proof that $f_{\mathcal{V}}$ maps $[y, w]$ to $[ys, ws]$.

Let $x, z \in [y, w]$, then $x \leq z$ if and only if $xs \leq zs$. Indeed, we have already proved that $xs < x$ and $zs < z$ so, by Lemma 5.3 (i), with $u = x$ and $v = z$, we have $xs \leq zs$. On the other hand, $x = (xs)s > xs$ and it follows from Lemma 5.3 (ii) with $u = xs$ and $v = z$ that $x = (xs)s \leq z$.

Finally from what we proved above, for each $t \in \mathcal{T}$ we have that $x, tx \in [y, w]$ if and only if $xs, txs \in [ys, ws]$.This means that we have a bijection between sets of edges such that $f_{\mathcal{E}}(x \xrightarrow{\alpha} tx) = xs \xrightarrow{\alpha} txs$.

Therefore $f = (f_{\mathcal{V}}, f_{\mathcal{E}}, \{Id_V\}_{x \in \mathcal{V}})$ is a k-isomorphism of moment graphs for any k.

□

So we have:

Corollary 5.2. *Consider $y, w \in \mathcal{W}$ such that $ws < w$, $y \not\leq ws$ for some $s \in \mathcal{S}$. Let f be as in Lemma 5.4, then $\mathscr{B}_w \cong f^*\mathscr{B}_{ws}$ as k-sheaves on $\mathcal{G}_{|_{[y,w]}}$ for any k.*

Proof. The statement follows by combining Lemma 5.4 and Lemma 5.1 . □

We recollect the results of this section:

Theorem 5.1. *Let $y, w \in \mathcal{W}$, then*

(i) $\mathscr{B}^y_w \cong \mathscr{B}^{y^{-1}}_{w^{-1}}$.

Let $s \in \mathcal{S}$ be such that $ws < w$, then

(ii) *if $y \not\leq ws$, $\mathscr{B}^y_w \cong \mathscr{B}^{ys}_{ws}$*

All isomorphisms are isomorphisms of S_k-modules, for any k.

Proof.

(i) This follows from Corollary 5.1, since two k-sheaves are isomorphic only if their stalks are pairwise isomorphic.

(ii) As before, the isomorphism descends from the k-isomorphism of k-sheaves we obtained in Corollary 5.2.

□

6. Invariants

Clearly not all equalities concerning Kazhdan-Lusztig polynomials come from k-isomorphisms of the underlying Bruhat graphs. In this section we develop another technique and, as in the previous section, we apply it in order to categorify two well-known properties of these polynomials.

6.1. **Multiplying by a simple reflection. Part II.** Another property that Kazhdan and Lusztig in [18] (2.3.g) proved is that if $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ are such that $y \leq w$ and $ws < w$, then

$$P_{y,w} = P_{ys,w} \tag{3}$$

It is clear that in this case there is no hope of finding any k-isomorphism of moment graphs, since the two Bruhat intervals $[y, w]$ and $[ys, w]$ obviously have different cardinality.

Recall that $\mathcal{T} = \{s_\alpha \,|\, \alpha \in R^+\} = \{wsw^{-1} \,|\, w \in \mathcal{W},\, s \in \mathcal{S}\}$ and denote

$$G_L(x, y) := \big\{t \in \mathcal{T} \,|\, tx \in (x, y]\big\}.$$

Lemma 6.1. *Let $w, y \in \mathcal{W}$ and $s \in \mathcal{S}$ be such that $y \leq w$, $ws < w$ and $ys < y$, then*

$$G_L(ys, w) = G_L(y, w) \cup \big\{ysy^{-1}\big\}.$$

Proof. We show that for all $t \in G_L(y, w)$ we have $ys < tys \leq w$ as well, i.e. $t \in G_L(ys, w)$. Indeed, if $tys > ty$, then $ys < y < ty < tys$ and, by Lemma 5.3 (ii), $tys \leq w$. Otherwise, $tys < ty \leq w$, $y < ty$, $ys < y$ and, by Lemma 5.3 (i), we obtain $ys < tys \leq w$.

Clearly, $ysy^{-1} \in G_L(ys, w)$.

Now we verify that if $t \in \mathcal{T}$, $tys \in [ys, w]$ and $ty \notin [y, w]$, so $t = ysy^{-1}$. Indeed, if $ty \notin [y, w]$, then $ty < y$. Moreover, $ys < y$ and so, by Lemma 5.3 (ii), $tys \leq y$. So $ys < tys \leq y$ and we know that $[ys, y] = \{ys, y\}$. Thus $tys = y$, that is, $t = ysy^{-1}$. □

Lemma 6.2. *Let $w, y \in \mathcal{W}$ and $s \in \mathcal{S}$ be such that $y \leq w$, $ys < y$ and $ws < w$, then the set $(ys, w] \setminus \{y\}$ is stabilized by the mapping $x \mapsto xs$.*

Proof. Notice that $ys < y \leq w$, so it makes sense to write $(ys, w]$. Let $\mathcal{I} := (ys, w] \setminus \{y\}$ and let $x \in \mathcal{I}$. If $xs > x$, then obviously $ys < xs$ and moreover, by Lemma 5.3 (ii) with $u = x$ and $v = w$, $xs \leq w$. If $xs < x$, then $xs < w$ and by applying Lemma 5.3 (i) with $u = y$ and $v = x$, $ys < xs$. Then, in both cases $xs \in (ys, w]$

Finally if $x \in \mathcal{I}$, then $xs \neq y$. Indeed $xs = y$ if and only if $x = ys \notin \mathcal{I}$. □

We want to prove that if $ws < w$, $y \leq w$, $E : y —— ty$ and $E' : ys —— tys$ then there are isomorphisms of S_k-modules $\varphi^y : \mathscr{B}^y_w \to \mathscr{B}^{ys}_w$ and $\varphi^E : \mathscr{B}^E_w \to \mathscr{B}^{E'}_w$ and they are such that the diagram:

$$\begin{array}{ccc} \mathscr{B}^y_w & \xrightarrow{\varphi^y} & \mathscr{B}^{ys}_w \\ \downarrow{\scriptstyle \rho_{y,E}} & & \downarrow{\scriptstyle \rho_{ys,E'}} \\ \mathscr{B}^E_w & \xrightarrow{\varphi^E} & \mathscr{B}^{E'}_w \end{array} \tag{4}$$

commutes.

Moreover we want that all these isomorphisms φ^y, φ^E are such that $\varphi^{ys} = (\varphi^y)^{-1}$ and $\varphi_{E'} = \varphi_E^{-1}$.

We will build these isomorphisms by induction on $n = l(w) - l(y)$.

If $n = 0$, then $y = w$ and $\mathscr{B}_w^w \cong S_k$. Moreover, if $\gamma = \alpha_{wsw^{-1}}$, we have that $S_k \to S_k/\gamma S_k$ is a projective cover and so $\mathscr{B}_w^{ws} \cong S_k$ as well.

Suppose by induction that we proved our claim for any $x \in \mathcal{W}$ with $l(w) - l(x) < n$. If $ys > y$ then $l(w) - l(ys) = n - 1$ and we may apply the inductive hypothesis: $\mathscr{B}_w^{ys} \cong \mathscr{B}_w^{(ys)s} = \mathscr{B}_w^y$.

Let $ys < y$, then from Lemma 6.2 it follows that the set $\mathcal{I} = (ys, w] \setminus \{y\}$ is stabilized by the mapping $x \mapsto xs$.

6.2. **Invariants.** In [12] Fiebig defined an involutive automorphism of the structure algebra of a Bruhat graph. Here in the same way we define an involution σ_s for a fixed simple reflection $s \in \mathcal{S}$ on the set of local sections of the canonical sheaf associated over $\mathcal{I} = (ys, w] \setminus \{y\} \subseteq \mathcal{W}$, where $y \leq w$ and $s \in \mathcal{S}$ is such that $ys < y$ and $ws < w$.

Let us now consider $(m_x) \in \Gamma_k(\mathcal{I}, \mathscr{B}_w)$. We define $\sigma_s((m_x)) := (m'_x)$ where $m'_x := \varphi^{xs}(m_{xs}) \in \mathscr{B}_w^x$ for any $x \in \mathcal{I}$. As the diagram (4) commutes, (m'_x) is still a a section. So σ_s preserves $\Gamma_k(\mathcal{I}, \mathscr{B}_w)$ and, moreover, thanks to the choice of φ^x we made, it is also an involution on $\Gamma_k(\mathcal{I}, \mathscr{B}_w)$. Denote by $\Gamma_k^s \subseteq \Gamma_k(\mathcal{I}, \mathscr{B}_w)$ the submodule of σ_s-invariant elements. Denote by Γ_k^{-s} the elements in $m \in \Gamma_k(\mathcal{I}, \mathscr{B}_w)$ such that $\sigma_s(m) = -m$.

We define $c_s = (c_{s,x}) \in \bigoplus_{x \in \mathcal{W}} S_k$ by $c_{s,x} := x(\alpha_s)$; then $c_s \in \mathcal{Z}_k$ and so it acts on $\Gamma_k(\mathcal{I}, \mathscr{B}_w)$ by componentwise multiplication via $(c_{s,x})_{x \in \mathcal{I}}$.

Lemma 6.3. *Let $(\mathcal{G}_{|\mathcal{I}}, k)$ be a GKM-pair, then we have $\Gamma_k(\mathcal{I}, \mathscr{B}_w) = \Gamma_k^s \oplus c_s \cdot \Gamma_k^s$.*

Proof. (We follow [12], Lemma 2.4).

We have $\Gamma_k(\mathcal{I}, \mathscr{B}_w) = \Gamma_k^s \oplus \Gamma_k^{-s}$; indeed, by definition, σ_s is an involution and 2 is an invertible element in k.

Let $m \in \Gamma_k^s$, then $\sigma_s(c_s \cdot m) = -(c_s \cdot m)$, i.e. $c_s \cdot \Gamma_k^s \subseteq \Gamma_k^{-s}$. Indeed, $s(\alpha_s) = -\alpha_s$ and so for any $x \in I$ we have

$$(c_{s,x} \cdot m_x)' = xs(\alpha_s) \cdot m_x = x(-\alpha_s) \cdot m_x = -c_{s,x} \cdot m_x.$$

We have to prove the other inclusion, that is, every element $m \in \Gamma_k^{-s}$ can be divided by $(x(\alpha_s))_{x \in I}$ in $\Gamma_k(\mathcal{I}, \mathscr{B}_w)$.

If $m = (m_x) \in \Gamma_k^{-s}$ then $m_x = -\varphi^{xs}(m_{xs})$ and so $\rho_{xs,xs\to x}(m_{xs}) = -\rho_{x,xs\to x}(m_x)$, since the following diagram commutes:

(5)
$$\begin{array}{ccc} \mathscr{B}_w^{xs} & \xrightarrow{\varphi^{xs}} & \mathscr{B}_w^{x} \\ \downarrow{\scriptstyle \rho_{xs,xs\to x}} & & \downarrow{\scriptstyle \rho_{x,xs\to x}} \\ \mathscr{B}_w^{xs\to x} & \xrightarrow{\varphi^{xs\to x}} & \mathscr{B}_w^{xs\to x} \end{array}$$

But m is a section so $\rho_{xs,xs\to x}(m_{xs}) = \rho_{x,xs\to x}(m_x)$. It follows that $2\rho_{x,xs\to x}(m_x) = 0$, but, by definition of the canonical sheaf, $\ker \rho_{x,xs\to x} = \alpha_{xsx^{-1}} \mathscr{B}_w^x$, that is, $\alpha_{xsx^{-1}}$ divides m_x in $\mathscr{B}_w^x$.

Notice that $\alpha_{xsx^{-1}} = \pm x(\alpha_s) = \pm c_{s,x}$, i.e. $c_s^{-1} \cdot m \in \bigoplus_{x \in I} \mathscr{B}_w^x$. We have to verify that $\rho_{x,x—tx}(c_{s,x}^{-1} m_x) = \rho_{tx,x—tx}(c_{s,tx}^{-1} m_{tx})$ for all $t \in \mathcal{T}$:

$$(c_{s,tx}c_{s,x})(\rho_{tx,x—tx}(c_{s,tx}^{-1}m_{tx}) - \rho_{x,x—tx}(c_{s,x}^{-1}m_x)) = \tag{6}$$

$$= c_{s,x}(\rho_{tx,x—tx}(m_{tx})) - c_{s,tx}(\rho_{x,x—tx}(m_x)) = \tag{7}$$

$$= (c_{s,x} - c_{s,tx})\rho_{tx,x—tx}(m_{tx}) + c_{s,tx}(\rho_{tx,x—tx}(m_{tx}) - \rho_{x,x—tx}(m_x)). \tag{8}$$

The term on line (8) is divisible by α_t; indeed, $c_{s,x} - c_{s,tx} = x(\alpha_s) - x(\alpha_s) + \langle x(\alpha_s), \alpha_t{}^\vee \rangle \alpha_t \equiv 0 \pmod{\alpha_t}$ and $\rho_{tx,x—tx}(m_{tx}) - \rho_{x,x—tx}(m_x) = 0$.

Now for the GKM-property $c_{s,tx}c_{s,x} = tx(\alpha_s) \cdot x(\alpha_s)$ is a multiple of α_t if and only if $xsx^{-1} = t$, i.e. $xs = tx$. So, $m_x = -\varphi^{xs}(m_{tx})$, $c_{s,tx} = -c_{s,x}$ and, considering that diagram (5) commutes, we obtain

$$\begin{aligned}\rho_{x,x—tx}(c_{s,t}^{-1} m_x) = -c_{s,tx}^{-1}\, \rho_{x,x—tx}(m_x) = -c_{s,tx}^{-1}\,(-\rho_{tx,x—tx}(m_{tx})) = \\ = \rho_{tx,x—tx}(c_{s,tx}^{-1} m_{tx})\end{aligned}$$

Otherwise, $xsx^{-1} \neq t$ and α_t divides $\rho_{tx,x—tx}(c_{s,tx}^{-1}m_{tx}) - \rho_{x,x—tx}(c_{s,x}^{-1}m_x)$, i.e. $\rho_{x,x—tx}(c_{s,x}^{-1}m_x) = \rho_{tx,x—tx}(c_{s,tx}^{-1}m_{tx})$.

□

We will show that in order to build $\mathscr{B}_w^y$ we just need the invariant sections.

Let $x, w \in \mathcal{W}$, $x \leq w$, recall that (cf. §3.2)

$$u_x : \Gamma_k(\{> x\}, \mathscr{B}_w) \longrightarrow \bigoplus_{z > x} \mathscr{B}_w^z \longrightarrow \bigoplus_{z \in \mathcal{V}_{\delta x}} \mathscr{B}_w^x \xrightarrow{\oplus \rho_{z,E}} \bigoplus_{E \in \mathcal{E}_{\delta x}} \mathscr{B}_w^E$$

and if $y \leq w$ and $s \in \mathcal{S}$ is such that $ys < y$ and $ws < w$ denote by τ_{ys} the composition

$$\tau_{ys} : \Gamma_k(\{> ys\}, \mathscr{B}_w) \xrightarrow{u_{ys}} \bigoplus_{E \in \mathcal{E}_{\delta ys}} \mathscr{B}_w^E \longrightarrow \bigoplus_{E \in \mathcal{E}_{\delta ys} \setminus \{ys \to y\}} \mathscr{B}_w^E\ .$$

Using this notation we have:

Lemma 6.4. *Let $y, w \in \mathcal{W}$, $s \in \mathcal{S}$ be such that $y \leq w$, $ys < y$ and $ws < w$. If $m = (m_x) \in \Gamma_k(\{> ys\}, \mathscr{B}_w)$ is such that $(m_x)_{x \in I} \in \Gamma_k^s$, then*

$$\tau_{ys}(c_s \cdot (m_x)_{x \in I}, y(\alpha_s) \cdot m_y) = (ys(\alpha_s))_{E \in \mathcal{E}_{\delta ys} \setminus \{ys \to y\}} \cdot \tau_{ys}(m).$$

Proof. Let $x \in \mathcal{V}_{\delta ys} \setminus \{y\}$. Then there exists a reflection $t \in \mathcal{T}$ such that $x = tys$ and

$$x(\alpha_s) = tys(\alpha_s) = s_{\alpha_t}(ys(\alpha_s)) = ys(\alpha_s) - \langle ys(\alpha_s), \alpha_t{}^\vee \rangle \alpha_t \equiv ys(\alpha_s) \pmod{\alpha_t}.$$

□

Let $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ be such that $y \leq w$, $ys < y$ and $ws < w$. Denote by $(\mathscr{B}_w^{\delta y})^s$ the elements in $\mathscr{B}_w^{\delta y}$ coming from local sections on $\{> y\}$ that can be extended to elements in Γ_k^s. Observe that, thanks to Lemma 6.4, if $m \in \Gamma_k^s$ then $\tau_y(c_s \cdot m) \in (\mathscr{B}_w^{\delta y})^s$; indeed, it can be extended to the section $(y(\alpha_s)m_x)_{x \in \mathcal{I}} \in \Gamma_k^s$.

Lemma 6.5. *Let $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ be such that $y \leq w$, $ys < y$ and $ws < w$. Then $\mathscr{B}_w^{\delta y} = (\mathscr{B}_w^{\delta y})^s$.*

Proof. Let $(\overline{m_x}) \in \mathscr{B}_w^{\delta y}$ and let $(m_x) \in \Gamma_k(\{> y\}, \mathscr{B}_w)$ be such that $\rho_{\delta_y}((m_x)_{x>y}) = (\overline{m_x})_{x\in\mathcal{V}_{\delta y}}$. By the flabbiness of $\mathscr{B}_w$, there exists a section over $\mathcal{I}$ $(m_x)_{x\in\mathcal{I}}$ which extends the previous one. Now, by Lemma 6.3, $(m_x)_{x\in\mathcal{I}} = (m'_x) + c_s \cdot (m''_x)$, where $(m'_x), (m''_x) \in \Gamma_k^s$. Let $(n_x) := (m'_x) + (y(\alpha_s) \cdot m''_x)$, then $(n_x) \in \Gamma_k^s$ and, by Lemma 6.4, $\rho_{\delta y}((n_x)_{x>y}) = \rho_{\delta_y}((m_x)_{x>y}) = (\overline{m_x}) \in (\mathscr{B}_w^{\delta y})^s$. □

We are finally able to categorify equality (3):

Theorem 6.1. *Let $(\mathcal{G}_{|_{[ys,w]}}, k)$ be a GKM-pair. Let $y, w \in \mathcal{W}$ and $s \in \mathcal{S}$ be such that $y \leq w$ and $ws < w$, then there is an isomorphism of S_k-modules $\mathscr{B}_w^y \cong \mathscr{B}_w^{ys}$.*

Proof. We prove the statement by induction on $n = l(w) - l(y)$. We have already considered the cases $n = 0$ and $ys > y$.

Now consider the case $ys < y$. Since $\mathscr{B}_w^{ys}$ is the projective cover of the S_k-module $\mathscr{B}^{\delta ys}$, and, since $\mathrm{rk}_{S_k}\mathscr{B}_w^y \leq \mathrm{rk}_{S_k}\mathscr{B}_w^{ys}$ (cf. Lemma 3.12. of [9]), the statement will follow from the unicity of the projective cover once we have defined a surjective map $\widetilde{\rho} : \mathscr{B}^y \to \mathscr{B}^{\delta ys}$.

Consider an element $(\overline{m_x}) \in \mathscr{B}_w^{\delta ys} = u_{ys}(\Gamma_k(\{> ys\}, \mathscr{B}_w)) = u_{ys}(\Gamma_k(\mathcal{I} \cup \{y\}, \mathscr{B}_w))$. We now know by Lemma 3.2 that it comes from a section $m = ((m_x)_{x\in\mathcal{I}}, m_y)$, where $(m_x)_{x\in\mathcal{I}} \in \Gamma_k(\mathcal{I}, \mathscr{B}_w)$ and m_y is such that $(\oplus_{E\in\mathcal{E}_{\delta y}}\rho_{y,E})^T(m_y) = (\rho_{x,E}(m_x))_{x\in\mathcal{V}_{\delta y}}$.

Moreover, thanks to Lemma 6.5, we may suppose $(m_x)_{x\in\mathcal{I}} \in \Gamma_k^s$. This means that $(\oplus_{E\in\mathcal{E}_{\delta y}}\rho_{y,E})^T(m_y) = (\rho_{x,E}(\varphi^{xs}(m_{xs})))$.

But diagram (5) had to commute, so

$$(\oplus_{E\in\mathcal{E}_{\delta y}}\rho_{y,E})^T(m_y) = (\varphi^{ys\to xs}(\rho_{xs,ys\to xs}(m_{xs})) = (\varphi^{ys\to xs}(\overline{m_{xs}})).$$

Since $\varphi^{y\to x} \circ \varphi^{ys\to xs} = Id$ by definition, we may finally set $\widetilde{\rho}(m_y) := (\varphi^E \circ \rho_{y,E}(m_y), \rho_{y,ys\to y}(m_y))$.

This gives us the desired surjection and hence the claim.

□

6.3. **Rational smoothness and p-smoothness of the flag variety.** We have an easy corollary of Theorem 6.1:

Corollary 6.1. *Let $\mathcal{W}$ be a finite Weyl group and w_0 its longest element. Then $\mathscr{B}_{w_0}^y \cong S_k$ for any $y \in \mathcal{W}$ and any k.*

Proof. By induction on $n = l(w_0) - l(y)$. If $n = 0$, by definition, $\mathscr{B}_{w_0}^{w_0} \cong S_k$. If $n \geq 1$ then there exists a simple reflection $s \in \mathcal{S}$ such that $ys > y$ (so, $l(w_0) - l(ys) = n - 1$). Actually, $w_0 s < w_0$ for any $s \in \mathcal{S}$ and, by Theorem 6.1 and inductive hypothesis, we have $\mathscr{B}_{w_0}^y \cong \mathscr{B}_{w_0}^{ys} \cong S_k$. □

Remark 6.1. *If $k = \mathbb{Q}$ the result above corresponds to the (rational) smoothness of flag varieties, while if k is a field of characteristic p it gives their p-smoothness (cf.* [14]*). Our proof is based only on the definition of canonical sheaf; we do not use Fiebig's multiplicity one results (see* [11]*), nor the geometry of the corresponding flag varieties.*

Remark 6.2. *Let k be a local principal domain. Fiebig and Williamson proved in* [14] *that if $(\mathcal{G}, k)$ is a GKM-pair, then the Braden-MacPherson algorithm computes the stalks of indecomposable parity sheaves (cf.* [17]*). Actually all results of this section hold for any local ring k with 2 an invertible element.*

6.4. **Parabolic setting.** Let $J \subseteq \mathcal{S}$ be such that $\mathcal{W}_J = \langle J \rangle$ is finite with longest element w_J. Let $\mathcal{W}^J$ be the set of minimal representatives of the equivalence classes $\mathcal{W}/\mathcal{W}_J$. For $w \in \mathcal{W}^J$, denote by $\mathscr{B}_{ww_J}$, resp.$\mathscr{B}^J_w$, the canonical sheaf on $\mathcal{G}_{ww_J}$, resp. on $\mathcal{G}^J_w$. It is now easy to see that:

Lemma 6.6. *Let W_J and w_J be as above and consider $x, w \in \mathcal{W}^J$ such that $y \leq w$, then $\mathscr{B}^x_{ww_J} \cong \mathscr{B}^{xu}_{ww_J}$ for any $u \in W_J$.*

Proof. By induction of $l(u)$. Clearly there is nothing to prove if $l(u) = 0$. If $l(x) > 0$ then there exists an $s \in \mathcal{S}$ such that $us < u$ and so by the inductive hypothesis, we get $\mathscr{B}^x_{ww_J} = \mathscr{B}^{xus}_{ww_J}$. Now for any $s \in J$, $ww_Js < ww_J$ and by Theorem 6.1 we obtain the claim. □

Theorem 6.2. *Let $(\mathcal{G}_{ww_J}, k)$ be a GKM-pair and let $\mathcal{W}^J$ and w_J be as above. If $y, w \in \mathcal{W}^J$ and $y \leq w$, then there is an isomorphism of S_k-modules*

$$(\mathscr{B}_{ww_J})^{yw_J} \cong (\mathscr{B}^J_w)^y.$$

Proof. We proceed by induction on $n = l(w) - l(y)$. If $n = 0$ the statement is trivial. Suppose we have a collection of isomorphisms of S_k-modules $\eta_x : (\mathscr{B}^J_w)^x \to (\mathscr{B}_{ww_J})^{xw_J}$ for any x such that $l(w) - l(x) < n$.

There is a natural injective homomorphism,

$$j : \Gamma_k(\{> y\}, \mathscr{B}^J_w) \to \Gamma_k(\{> yw_J\}, \mathscr{B}_{ww_J}),$$

defined by setting $(m_x)_{x \in (y,w] \subset \mathcal{W}^J} \mapsto (\widetilde{m_z})_{z \in (yw_J, ww_J] \subset \mathcal{W}}$, where $\widetilde{m_z} := \psi^z(\eta_x(m_x))$ if $z \in x\mathcal{W}_J$ and $\psi^z : \mathscr{B}^x_{ww_J} \to \mathscr{B}^z_{ww_J}$ denotes the isomorphism in Lemma 6.6.

We will show that such a homomorphism induces an isomorphism $(\mathscr{B}_{ww_J})^{\delta\, yw_J} \cong (\mathscr{B}^J_w)^{\delta y}$. Then, by the unicity of projective cover, the statement will follow.

Let $z \in (yw_J, ww_J]$, $z = xu$, for some $x > y \in \mathcal{W}^J$, $u \in \mathcal{W}_J$ and $u = s_1 \dots s_r$ a reduced expression with $s_i \in J$ for every i. Moreover, let $(n_v) \in \Gamma_k(\{> yw_J\}, \mathscr{B}_{ww_J})$. We prove by induction on $l(u) = r$ that there exists a section $(p_v) \in \Gamma_k(\{> yw_J\}, \mathscr{B}_{ww_J})$ such that $p_{xs_1 \dots s_i} = \psi^{xs_1 \dots s_i}(\eta_x(m_x))$ for some $m_x \in (\mathscr{B}^J_w)^x$ for any $i = 0, \dots, r$ and such that $u_{yw_J}((p_v)) = u_{yw_J}((n_v))$.

For the base step we have $r = 0$ and there is nothing to prove.

If $z = (xs_1s_2 \dots s_{r-1})s_r$ then by the inductive hypothesis there exists a section $(q_v) \in \Gamma_k(\{> yw_J\}, \mathscr{B}_{ww_J})$ and an element $m_x \in (\mathscr{B}^J_w)^x$ such that $q_{xs_1 \dots s_i} = \psi^{xs_1 \dots s_i}(\eta_x(m_x))$ and $u_y((q_v)) = u_y((n_v))$ for $i = 0, \dots, r-1$. Thus, by Lemma 6.5, the element $(p_v) \in \bigoplus_{v > yw_J} \mathscr{B}^y$ such that

$$p_{ys_1 \dots s_{r-1}s_r} = \varphi^{ys_1 \dots s_{r-1}}(p_{ys_1 \dots s_{r-1}})$$

and

$$p_{xs_1 \dots s_i} = q_{xs_1 \dots s_i} = \psi^{xs_1 \dots s_i}(\eta_x(m_x)) \quad \forall i < r$$

is a section on $\{> yw_J\}$ and verifies $u_{yw_J}((\widetilde{n_v})) = u_{yw_J}((n_v))$.

Finally, from the proof of Lemma 6.6 it follows that

$$\varphi^{ys_1 \dots s_{r-1}}(p_{ys_1 \dots s_{r-1}}) = \varphi^{ys_1 \dots s_{r-1}}(\psi^{ys_1 \dots s_{r-1}}(\eta_x(m_x)) = \psi^{xs_1 \dots s_r}(\eta_x(m_x)).$$

□

The theorem above is just the analogue of the following theorem, due to Deodhar:

Theorem 6.3. ([5])*Let $\mathcal{W}$ be a Weyl group with $\mathcal{S}$, set of simple reflections, and $J \subseteq \mathcal{S}$ such that $\mathcal{W}_J$ is finite. Let w_J be the longest element of $\mathcal{W}_J$ and $y, w \in \mathcal{W}^J$, then $P^{J,-1}_{y,w} = P_{yw_J, yw_J}$.*

Acknowledgements

I would like to thank Peter Fiebig for leading my attention to this problem and for helpful discussions, and Rocco Chirivì and Corrado De Concini for useful conversations. I would also like to thank the INdAM for funding part of my stay at the Friedrich-Alexander-Universität of Erlangen-Nürnberg. Finally, I am grateful to the Hausdorff Research Institute for Mathematics in Bonn for its hospitality during the program "On the Interaction of Representation Theory with Geometry and Combinatorics", while the last version of this paper was written.

References

[1] H. H. Andersen, J. C. Jantzen, W. Soergel, *Representation of quantum groups at a p-th root of unity and of semisimple groups in characteristic p: Independence of p*, Astérisque **220** (1994), 1-320.
[2] T. Braden, R. MacPherson, From moment graphs to intersection cohomology, *Math. Ann.* **321** (2001), 533-551.
[3] A. Björner, F. Brenti, *Combinatorics of Coxeter Groups*, Springer, New York, 2005.
[4] F. Brenti, F. Caselli, M. Marietti, Special matchings and Kazhdan-Lusztig polynomials, *Adv. Math.*, **202** (2006), 555-601.
[5] V. Deodhar, On some geometric aspects of Bruhat orderings II. The parabolic analogue of Kazhdan-Lusztig polynomials, *J. Algebra* **111** (1987), 483-506.
[6] M. Dyer, Hecke algebras and reflections in Coxeter groups, Ph. D. Thesis, University of Sydney, 1987.
[7] P. Fiebig, Sheaves on moment graphs and a localization of Verma flags,*Adv. Math.* **217** (2008), 683-712.
[8] P. Fiebig, The combinatorics of Coxeter categories, *Trans. Amer. Math. Soc.* **360** (2008), 4211-4233.
[9] P. Fiebig, Moment graphs in representation theory and geometry, Notes (2009).
[10] P. Fiebig, Lusztig's conjecture as a moment graph problem, *Bull. London Math. Soc.* **42(6)** (2010), 957-972.
[11] P. Fiebig, The multiplicity one case of Lusztig's conjecture, Duke Math. J. 153 (2010), 551-571.
[12] P. Fiebig, Sheaves on affine Schubert varieties, modular representations and Lusztig's conjecture, *J. Amer. Math. Soc.* **24** (2011), 133-181.
[13] P. Fiebig, An upper bound on the exceptional characteristics for Lusztig's character formula, *preprint 2008*, arXiv:0811.1674.
[14] P. Fiebig, G. Williamson, Parity sheaves, moment graphs and the p-smooth locus of Schubert varieties, *preprint 2010*, arXiv:1008.0719.
[15] M. Goresky, R. Kottwitz, R. MacPherson, Equivariant cohomology, Koszul duality and the localization theorem, *Invent. Math.* **131**, No.1 (1998), 23-83.
[16] J. E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge University Press, Cambridge, 1990.
[17] D. Juteau, C. Mautner, G. Williamson, Parity sheaves. *preprint* arXiv:0906.2994, 2009.
[18] D. Kazhdan, G. Lusztig, Representation of Coxeter groups and Hecke algebras, *Invent.Math.* **53** (1979),165-184.
[19] D. Kazhdan, G. Lusztig, Schubert varieties and Poincare duality, in: Geometry of the Laplace Operator. Proc. Symp. Pure Math **36** *Amer. Math. Soc.* (1980), 185-203.
[20] D. Kazhdan, G. Lusztig, Tensor structures arising from affine Lie algebras. I-IV, *J. Amer. Math. Soc.* **6** (1993), 905–947; *J. Amer. Math. Soc.* **6** (1993), 949-1011; *J. Amer. Math. Soc.* **7** (1994), 335-381; *J. Amer. Math. Soc.* **7** (1994), 383-453.
[21] M. Kashiwara, T. Tanisaki, Kazhdan-Lusztig conjecture for affine Lie algebras with negative level, Duke Math. J. 77 (1995), no. 1, 21-62.
[22] S. Kumar, *Kac-Moody, their flag varieties and representation theory*, Progress in Mathematics, vol. 204, Birkhäuser Boston Inc., Boston, MA, 2002.

[23] G. Lusztig, Some problems in the representation theory of finite Chevalley groups, The Santa Cruz Conference on Finite Groups (Univ. California, Santa Criz, Calif., 1979) Adv. Proc. Symp. Pure Math., vol.37, Amer. Math Soc., Providence, R.I., 1980, 313-317.

Dipartimento di Matematica, Roma Tre Università degli Studi, Largo San Leonardo Murialdo 1, 00146 Roma, Italy
E-mail address: lanini@mat.uniroma3.it

Department Mathematik, Friedrich-Alexander-Universität Erlangen-Nürnberg, Bismarckstr. 1 1/2, 91054 Erlangen, Germany
E-mail address: lanini@mi.uni-erlangen.de