Semibounded representations of hermitian Lie groups

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Abstract

A unitary representation of a, possibly infinite dimensional, Lie group G is called semibounded if the corresponding operators $i\mathtt{d}\pi(x)$ from the derived representation are uniformly bounded from above on some non-empty open subset of the Lie algebra $\mathfrak{g}$ of G. A hermitian Lie group is a central extension of the identity component of the automorphism group of a hermitian Hilbert symmetric space. In the present paper we classify the irreducible semibounded unitary representations of hermitian Lie groups corresponding to infinite dimensional irreducible symmetric spaces. These groups come in three essentially different types: those corresponding to negatively curved spaces (the symmetric Hilbert domains), the unitary groups acting on the duals of Hilbert domains, such as the restricted Graßmannian, and the motion groups of flat spaces.

Keywords: infinite dimensional Lie group, unitary representation, semibounded representation, hermitian symmetric space, symmetric Hilbert domain.

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Introduction

This paper is part of a project concerned with a systematic approach to unitary representations of Banach–Lie groups in terms of conditions on spectra in the derived representation. For the derived representation to carry significant information, we have to impose a suitable smoothness condition. A unitary representation $\pi\colon G \to \mathrm{U}(\mathcal{H})$ is said to be *smooth* if the subspace $\mathcal{H}^\infty \subseteq \mathcal{H}$ of smooth vectors is dense. This is automatic for continuous representations of finite dimensional groups, but not for Banach–Lie groups ([Ne10a]). For any smooth unitary representation, the *derived representation*

$$\mathtt{d}\pi\colon \mathfrak{g} = \mathbf{L}(G) \to \mathrm{End}(\mathcal{H}^\infty), \quad \mathtt{d}\pi(x)v := \frac{d}{dt}\Big|_{t=0} \pi(\exp tx)v$$

carries significant information in the sense that the closure of the operator $\mathtt{d}\pi(x)$ coincides with the infinitesimal generator of the unitary one-parameter group $\pi(\exp tx)$. We call $(\pi, \mathcal{H})$ *semibounded* if the function

$$s_\pi\colon \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s_\pi(x) := \sup\big(\mathrm{Spec}(i\mathtt{d}\pi(x))\big)$$

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is bounded on the neighborhood of some point in $\mathfrak{g}$. Then the set W_π of all such points is an open $\mathrm{Ad}(G)$-invariant convex cone in the Lie algebra $\mathfrak{g}$. We call π *bounded* if s_π is bounded on some 0-neighborhood, i.e., $W_\pi = \mathfrak{g}$. All finite dimensional continuous unitary representations are bounded and most of the unitary representations appearing in physics are semibounded (cf. [Ca83], [Mi87, Mi89], [PS86], [SeG81], [CR87], [Se58], [Se78], [Bak07]).

For finite dimensional Lie groups, the irreducible semibounded representations are precisely the unitary highest weight representations and one has unique direct integral decompositions [Ne00, X.3/4, XI.6]. For many other classes of groups such as the Virasoro group and affine Kac–Moody groups (double extensions of loop groups with compact target groups), the irreducible highest weight representations are semibounded, but to prove the converse is more difficult and requires a thorough understanding of invariant cones in the corresponding Lie algebras as well as of convexity properties of coadjoint orbits ([Ne10c]).

Finite dimensional groups only have faithful bounded representations if their Lie algebras are compact, which is equivalent to the existence of an Ad-invariant norm on the Lie algebra. For infinite dimensional groups, the picture is much more colorful. There are many interesting bounded representations, in particular of unitary groups of C^*-algebras (cf. [BN11]) and a central result of Pickrell ([Pi88]), combinded with classification results of Kirillov, Olshanski and I. Segal ([Ol78], [Ki73], [Se57]), implies that all separable unitary representations of the unitary group $\mathrm{U}(\mathcal{H})$ of an infinite dimensional separable Hilbert space can can be classified in the same way as for their finite dimensional analogs by Schur–Weyl theory. In particular, the irreducible ones are bounded (cf. Theorem E.1).

To address classification problems one needs refined analytic tools based on recent results asserting that the space $\mathcal{H}^\infty$ of smooth vectors is a Fréchet space on which G acts smoothly ([Ne10a]). In [Ne10d] we use these facts to develop some spectral theoretic tools concerning the space of smooth vectors. This does not only lead to a complete description of semibounded representations of various interesting classes of groups such as hermitian Lie groups which are dealt with in the present paper. They also apply naturally to unitary representations of Lie supergroups generated by their odd part ([NSa10]).

Our goal is a classification of the irreducible semibounded representations and the development of tools to obtain direct integral decompositions of semibounded representations. The first part of this goal is achieved in the present paper for the class of hermitian Lie groups. These are triples (G, θ, d), where G is a connected Lie group, θ an involutive automorphism of G with the corresponding eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $d \in \mathfrak{z}(\mathfrak{k})$ (the center of $\mathfrak{k}$) an element for which $\mathrm{ad}\, d|_{\mathfrak{p}}$ is a complex structure, and $\mathfrak{p}$ carries an $e^{\mathrm{ad}\, \mathfrak{k}}$-invariant Hilbert space structure. We then write $K := (G^\theta)_0$ for the identity component of the group of θ-fixed points in G and observe that our assumptions imply that G/K is a hermitian Hilbert symmetric space. The simply connected symmetric spaces arising from this construction have been classified by W. Kaup by observing that $\mathfrak{p}$ carries a natural structure of a JH^*-triple, and these objects permit a powerful structure theory which leads to a complete classification in terms of orthogonal decompositions and simple objects (cf. [Ka81, Ka83]).

Typical examples of hermitian symmetric spaces are symmetric Hilbert domains (the negatively curved case), their duals, such as the restricted Graßmannian of a polarized Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ (the positively curved case) (cf. [PS86]), and all Hilbert spaces $\mathcal{H}$ as quotients $G = (\mathcal{H} \rtimes_\alpha K)/K$, where α is a norm-continuous unitary representation of K on $\mathcal{H}$ (the flat case). Our concept of a hermitian Lie group contains almost no restrictions on the group K, but the structure of K is crucial for the classification of semibounded representations of G. To make this more specific, we call (G, θ, d) *irreducible* if the unitary K-representation on $\mathfrak{p}$ is irreducible. In this case either $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] = \{0\}$ (the flat case) or $\mathfrak{p}$ is a simple JH^*-triple. Then we say that $\mathfrak{g}$ is *full* if $\mathrm{ad}\, \mathfrak{k} = \mathfrak{aut}(\mathfrak{p})$ is the full Lie algebra of the automorphism group $\mathrm{Aut}(\mathfrak{p})$ of the JH^*-triple $\mathfrak{p}$. One of our key observations is that the quotient Lie algebra $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no non-trivial open convex invariant cones.

The structure of this paper is as follows. Section 1 introduces the concept of a hermitian Banach–Lie group and in Section 2 we explain their connection with JH^*-triples and recall Kaup's classification of infinite dimensional simple JH^*-triples. As we shall see in the process, hermitian Lie groups G have natural central extensions $\widehat{G}$ and these central extensions often enjoy a substantially richer supply of semibounded unitary representations than the original group G. This phenomenon is also well-known for the group of diffeomorphisms of the circle (cf. [Ne10c]) and loop groups ([PS86], [Ne01b]). This motivates our detailed discussion of central extensions in Section 3. Any semibounded representation $(\pi, \mathcal{H})$ defines the open convex invariant cone $W_\pi \subseteq \mathfrak{g}$. Therefore the understanding of semibounded representations requires some information on the geometry of open invariant cones in Lie algebras. In our context we mainly need the information that for certain Lie algebras $\mathfrak{k}$, all open invariant cones in $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ are trivial. Typical examples with this property are the Lie algebras of the unitary groups of real, complex or quaternionic Hilbert spaces (Section 4). Sections 5-8 are devoted to the classification of the irreducible semibounded representations of hermitian Lie groups. The main steps in this classification are the following results on semibounded representations $(\pi, \mathcal{H})$ of G:

(1) If $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones, then we derive from the results on invariant cones developed in Section 4 that $\pi|_{Z(K)_0}$ is also semibounded.

(2) If (G, θ, d) is irreducible, then $d \in W_\pi \cup -W_\pi$. If $d \in W_\pi$, then we call $(\pi, \mathcal{H})$ a *positive energy representation.* In this case the maximal spectral value of the essentially selfadjoint operator $i\mathtt{d}\pi(d)$ is an eigenvalue and the K-representation (ρ, V) on the corresponding eigenspace is bounded and irreducible. Using the holomorphic induction techniques developed in [Ne10d] for Banach–Lie groups, it follows that $(\pi, \mathcal{H})$ is uniquely determined by (ρ, V) (Section 5). We call a bounded representation (ρ, V) of K *(holomorphically) inducible* if it corresponds as above to a unitary representation $(\pi, \mathcal{H})$ of G.

(3) If (G, θ, d) is full or G/K is flat (with $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ not containing open invariant cones), we derive an explicit characterization of the inducible bounded irreducible K-representations. We thus obtain a classification of all irreducible semibounded representations of G in terms of the corresponding K-representations (Sections 6-8). For irreducible symmetric Hilbert domains, this explicit characterization is based on the classification of unitarizable highest weight modules of locally finite

hermitian Lie algebras obtained in [NO98] (Section 7). For the c-dual spaces we obtain the surprisingly simple result that inducibility can be characterized by an easily verifiable positivity condition (Section 8).

(4) A central point in our characterization is that the group K has a normal closed subgroup K_∞ (for $K = \mathrm{U}(\mathcal{H})$ this is the group $\mathrm{U}_\infty(\mathcal{H})$) with the property that each bounded irreducible representation (ρ, V) is a tensor product of two bounded irreducible representations (ρ_0, V_0) and (ρ_1, V_1), where $\rho_0|_{K_\infty}$ is irreducible and $K_\infty \subseteq \ker \rho_1$. We show that (ρ, V) is inducible if and only if (ρ_0, V_0) has this property. Here the main point is that, even though no classification of the representations (ρ_1, V_1) is known, the representations (ρ_0, V_0) can be parameterized easily by "highest weights" (cf. Definition D.3). If V is separable, then the representation ρ_1 is trivial, which can be derived from the fact that all continuous separable unitary representations of the Banach–Lie group $\mathrm{U}(\mathcal{H})/\mathrm{U}_\infty(\mathcal{H})$ are trivial if $\mathcal{H}$ is separable (cf. [Pi88] and also Theorem E.1 below).

We collect various auxiliary results in appendices. In Appendix A we discuss operator-valued positive definite functions on Lie groups. The main result is Theorem A.7 asserting that analytic local positive definite functions extend to global ones. This generalizes the corresponding result for the scalar case in [Ne10b]. Its applications to holomorphically induced representations are developed in Appendix B, where we show that holomorphic inducibility of (ρ, V) can be characterized in terms of positive definiteness of a $B(V)$-valued function on some identity neighborhood of G. This is a key tool in Sections 7 and 8.

For the convenience of the reader we provide in Appendix C a description of various classical groups of operators on Hilbert spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Appendix D provides a complete discussion of the bounded unitary representations of the unitary group $\mathrm{U}_p(\mathcal{H})$ of an infinite dimensional real, complex or quaternionic Hilbert space for $1 < p \leq \infty$. The irreducible representations are parametrization in terms of highest weights. For $\mathbb{K} = \mathbb{C}$ this was done in [Ne98], and for $\mathbb{K} = \mathbb{R}$ and $\mathbb{H}$, these results are new but quite direct consequences of the complex case. We show in particular that all bounded representatations of these groups are direct sums of irreducible ones. If $\mathcal{H}$ is separable, this is true for any continuous unitary representation of $\mathrm{U}_\infty(\mathcal{H})$ ([Ki73]), but for $p < \infty$, the topological groups $\mathrm{U}_p(\mathcal{H})$ are not of type I (cf. [Bo80], [SV75]). In Appendix E we finally recall the special features of separable representations of unitary groups. We conclude this paper with a discussion of some open problems and some comments on variations of the concept of a hermitian Lie group.

The classification results of the present paper also contribute to the Olshanski–Pickrell program of classifying the unitary representations of automorphism groups of Hilbert symmetric spaces (cf. [Pi87, Pi88, Pi90, Pi91], [Ol78, Ol84, Ol88, Ol89, Ol90]). For symmetric spaces $M = G/K$ of finite rank, Olshanski has shown in [Ol78, Ol84] that the so-called *admissible* unitary representations of G, i.e., representations whose restriction to K is tame, lead to so-called *holomorphic* representations of the automorphism group $G^\sharp$ of a hermitian symmetric space $M^\sharp = G^\sharp/K^\sharp$ containing M as a totally real submanifold. The irreducible holomorphic representations of the automorphism group of $M^\sharp$ turn out to be highest weight representations, which correspond to the representations showing up for type $\mathrm{I}_{\mathrm{fin}}$ in our context. For spaces M of infinite rank, the classification of the admissible representations is far less complete, although Olshanski formulates in [Ol90] quite precise conjectures. These conjectures suggest that one should also try to understand the semibounded representations of mapping groups of the form $C^\infty(\mathbb{S}^1, G)$, where G is a hermitian Lie group.

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Notation and conventions: For an open or closed convex cone W in a real vector space V we write $H(W) := \{x \in V : x + W = W\}$ for the *edge of* W.

If $\mathfrak{g}$ is a real Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ its complexification, we write $\overline{z} := x - iy$ for $z = x + iy$, $x, y \in \mathfrak{g}$, and $z^* := -\overline{z}$.

For a Lie group G with Lie algebra $\mathfrak{g}$ and a topological vector space V, we associate to each $x \in \mathfrak{g}$ the left invariant differential operator on $C^\infty(G, V)$ defined by

$$(L_x f)(g) := \frac{d}{dt}\Big|_{t=0} f(g \exp(tx)) \quad \text{for} \quad x \in \mathfrak{g}.$$

By complex linear extension, we define the operators

$$L_{x+iy} := L_x + iL_y \quad \text{for} \quad z = x + iy \in \mathfrak{g}_{\mathbb{C}}, x, y \in \mathfrak{g}.$$

Accordingly, we define

$$(R_x f)(g) := \frac{d}{dt}\Big|_{t=0} f(\exp(tx)g) \quad \text{for} \quad x \in \mathfrak{g}$$

and R_z for $z \in \mathfrak{g}_{\mathbb{C}}$ by complex linear extension. For a homomorphism $\varphi \colon G \to H$ of Lie groups, we write $\mathbf{L}(\varphi) \colon \mathbf{L}(G) = \mathfrak{g} \to \mathbf{L}(H) = \mathfrak{h}$ for the derived homomorphism of Lie algebras.

For a set J, we write $|J|$ for its cardinality.

Contents

1 Hermitian Lie groups

Let G be a connected Banach–Lie group with Lie algebra $\mathfrak{g}$, endowed with an involution θ. We write $K := (G^\theta)_0$ for the identity component of the subgroup of θ-fixed points in G and

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} := \ker(\mathbf{L}(\theta) - \mathbf{1}), \quad \mathfrak{p} := \ker(\mathbf{L}(\theta) + \mathbf{1})$$

for the corresponding eigenspace decomposition of the Lie algebra. Then $\mathfrak{p} \subseteq \mathfrak{g}$ is a closed $\mathrm{Ad}(K)$-invariant subspace and we write $\mathrm{Ad}_{\mathfrak{p}} \colon K \to \mathrm{GL}(\mathfrak{p})$ for the corresponding representation of K on $\mathfrak{p}$.

Definition 1.1. (a) We call a triple $(\mathfrak{g}, \theta, d)$ consisting of a Banach–Lie algebra $\mathfrak{g}$, an involution θ of $\mathfrak{g}$ and an element $d \in \mathfrak{z}(\mathfrak{k})$ a *hermitian* Lie algebra if the following conditions are satisfied:

(H1) $\mathfrak{p}$ is a complex Hilbert space.

(H2) $\mathrm{ad}_{\mathfrak{p}} \colon \mathfrak{k} \to \mathfrak{gl}(\mathfrak{p})$ is a representation of $\mathfrak{p}$ by bounded skew-hermitian operators.

(H3) $[d,x] = ix$ for every $x \in \mathfrak{p}$.

(H4) $\ker \mathrm{ad}_{\mathfrak{p}} \subseteq \mathfrak{z}(\mathfrak{k})$.

We say that the hermitian Lie algebra $(\mathfrak{g}, \theta, d)$ is *irreducible* if, in addition, the representation of $\mathfrak{k}$ on $\mathfrak{p}$ is irreducible.

(b) A triple (G, θ, d) of a connected Lie group G, an involutive automorphism θ of G and an element $d \in \mathfrak{g}$ is called a *hermitian* Lie group if the corresponding triple $(\mathfrak{g}, \mathbf{L}(\theta), d)$ is hermitian. Since K is connected, (H2) means that

(H2') $\mathrm{Ad}_{\mathfrak{p}}$ is a unitary representation of K.

Since $\mathfrak{p}$ is a closed complement of $\mathfrak{k} = \mathbf{L}(K)$ in $\mathfrak{g}$, the quotient $M := G/K$ carries a natural Banach manifold structure for which the quotient map $q \colon G \to M$ is a submersion. Moreover, the complexification $\mathfrak{g}_{\mathbb{C}}$ decomposes into 3-eigenspaces of $\mathrm{ad}\, d$:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-, \quad \mathfrak{p}^\pm := \ker(\mathrm{ad}\, d \mp i\mathbf{1}), \tag{1}$$

and all these eigenspaces are $\mathrm{Ad}(K)$-invariant. Therefore $M = G/K$ carries a natural complex structure for which the tangent space $T_{\mathbf{1}K}(M)$ in the base point $\mathbf{1}K$ can be identified with $\mathfrak{p}^+ \cong \mathfrak{g}_{\mathbb{C}}/(\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^-)$, resp., with $\mathfrak{p}$, endowed with the complex structure defined by $\mathrm{ad}\, d$ ([Bel05, Thm. 15]). As the scalar product on $\mathfrak{p}$ is $\mathrm{Ad}(K)$-invariant, it defines a Riemannian structure on M. Endowed with these structures, M becomes a *hermitian symmetric space* (cf. [Ka83]). The fact that the triple bracket $[x,[y,z]]$ corresponds to the curvature now implies that, for $x \in \mathfrak{p}$, the operator $(\mathrm{ad}\, x)^2 \colon \mathfrak{p} \to \mathfrak{p}$ is hermitian. The symmetric space is said to be *flat* if $[\mathfrak{p},[\mathfrak{p},\mathfrak{p}]] = 0$, *positively curved* if the operators $(\mathrm{ad}\, x)^2|_{\mathfrak{p}}$, $x \in \mathfrak{p}$, are negative semidefinite, and *negatively curved* if these operators are positive semidefinite (cf. [Ne02c]).

Remark 1.2. (a) If G is simply connected, then $M = G/K$ is also simply connected because the connectedness of K implies that the natural homomorphism $\pi_1(G) \to \pi_1(G/K)$ is surjective.

(b) We neither require that $\mathfrak{k} = [\mathfrak{p},\mathfrak{p}]$ nor that $[\mathfrak{p},\mathfrak{p}]$ is dense in $\mathfrak{k}$.

(c) If $(\mathfrak{g}, \theta, d)$ is a hermitian Lie algebra, then $\theta = e^{\pi\, \mathrm{ad}\, d}$ follows immediately from the definitions. In particular, every connected Lie group G with Lie algebra $\mathfrak{g}$ carries the involution $\theta_G(g) := (\exp \pi d) g (\exp -\pi d)$ with $\mathbf{L}(\theta_G) = \theta$.

(d) From $\mathfrak{k} = \ker(\mathrm{ad}\, d) \supseteq \mathfrak{z}(\mathfrak{g})$ (H3), we derive that $\mathfrak{z}(\mathfrak{g}) = (\ker \mathrm{ad}_{\mathfrak{p}}) \cap \mathfrak{z}(\mathfrak{k})$, so that (H4) is equivalent to $\ker \mathrm{ad}_{\mathfrak{p}} = \mathfrak{z}(\mathfrak{g})$. If, in addition, $(\mathfrak{g}, \theta, d)$ is irreducible, then Schur's Lemma implies that $\mathrm{ad}_{\mathfrak{p}}(\mathfrak{z}(\mathfrak{k})) = \mathbb{R} i\mathbf{1} = \mathbb{R}\, \mathrm{ad}_{\mathfrak{p}}\, d$, which leads to

$$\mathfrak{z}(\mathfrak{k}) = \mathbb{R} d \oplus \mathfrak{z}(\mathfrak{g}). \tag{2}$$

Examples 1.3. (Flat case) (a) Suppose that $\mathcal{H}$ is a complex Hilbert space, K a connected Banach–Lie group and $\pi \colon K \to \mathrm{U}(\mathcal{H})$ a norm continuous (hence smooth) representation of K on $\mathcal{H}$ with $\ker(\mathtt{d}\pi) \subseteq \mathfrak{z}(\mathfrak{k})$. Then $G := \mathcal{H} \rtimes_\pi K$ is a symmetric Banach–Lie group with respect to the involution $\theta(v,k) := (-v,k)$. In this case $G^\theta = \{0\} \times K$, so that $G/K \cong \mathcal{H}$. If there exists an element $d \in \mathfrak{z}(\mathfrak{k})$ with $\mathtt{d}\pi(d) = i\mathbf{1}$, then (G, θ, d) is a hermitian Lie group. If the latter condition is not satisfied, we may replace K by $\mathbb{R} \times K$ and extend the given representation by $\pi(t,k) := e^{it}\pi(k)$.

(b) We can modify the construction under (a) as follows. We define the *Heisenberg group of $\mathcal{H}$* by

$$\mathrm{Heis}(\mathcal{H}) := \mathbb{R} \times \mathcal{H},$$

endowed with the multiplication

$$(t,v)(t',v') := (t + t' + \tfrac{1}{2}\omega(v,v'), v + v'), \quad \omega(v,v') := 2\,\mathrm{Im}\langle v, v'\rangle.$$

Then K acts smoothly by automorphisms on $\mathrm{Heis}(\mathcal{H})$ via $\alpha_k(t,v) = (t, \pi(k)v)$, and we obtain a Lie group $\widehat{G} := \mathrm{Heis}(\mathcal{H}) \rtimes_\alpha K$ with an involution defined by $\theta(t,v,k) := (t,-v,k)$ and $\widehat{K} := \widehat{G}^\theta = \mathbb{R} \times \{0\} \times K \cong \mathbb{R} \times K$.

For $K = \mathbb{T}\mathbf{1} \subseteq \mathrm{U}(\mathcal{H})$, we obtain in particular the *oscillator group* $\mathrm{Osc}(\mathcal{H}) = \mathrm{Heis}(\mathcal{H}) \rtimes \mathbb{T}$ of $\mathcal{H}$.

Examples 1.4. Let $\mathcal{H}$ be a complex Hilbert space which is a direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.

(a) (Positively curved case) The *restricted unitary group* $\mathrm{U}_{\mathrm{res}}(\mathcal{H}) = \mathrm{U}(\mathcal{H}) \cap \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ from Appendix C.6 is hermitian with $d := \frac{1}{2}\,\mathrm{diag}(i\mathbf{1}, -i\mathbf{1})$ and the involution

$$\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$$

whose group of fixed points is the connected group $\mathrm{U}_{\mathrm{res}}(\mathcal{H})^\theta \cong \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$. We also see that

$$\mathfrak{p} = \Big\{ \begin{pmatrix} 0 & z \\ -z^* & 0 \end{pmatrix} : z \in B_2(\mathcal{H}_-, \mathcal{H}_+) \Big\} \cong B_2(\mathcal{H}_-, \mathcal{H}_+) \cong \mathcal{H}_+ \widehat{\otimes} \mathcal{H}_-^*$$

is a complex Hilbert space, and the adjoint representation $\mathrm{Ad}_{\mathfrak{p}}(k_+, k_-)z = k_+ z k_-^{-1}$ is irreducible because it is a tensor product of two irreducible representations.

(b) (Negatively curved case) The restricted hermitian group $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ (Appendix C.6) is also hermitian with respect to the same involution and the same element $d \in \mathfrak{k}$ as in (a). We also note that $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)^\theta = \mathrm{U}_{\mathrm{res}}(\mathcal{H})^\theta \cong \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$.

Definition 1.5. (a) A *symmetric Lie algebra* is a pair $(\mathfrak{g}, \theta)$, where $\mathfrak{g}$ is a Lie algebra and $\theta \in \mathrm{Aut}(\mathfrak{g})$ (the group of topological automorphisms) with $\theta^2 = \mathrm{id}_{\mathfrak{g}}$. If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the decomposition into θ-eigenspaces, then $\mathfrak{g}^c := \mathfrak{k} + i\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$ with the involution $\theta^c(x + iy) := x - iy$, $x \in \mathfrak{k}$, $y \in \mathfrak{p}$, is called the *c-dual symmetric Lie algebra* $(\mathfrak{g}^c, \theta^c)$ of $(\mathfrak{g}, \theta)$.

(b) If $(\mathfrak{g}, \theta, d)$ is hermitian, then the c-dual Lie algebra $(\mathfrak{g}^c, \theta^c, d)$ is also hermitian, called the *c-dual hermitian Lie algebra*.

(c) For any hermitian Lie group (G, θ, d), the semidirect product $G^m := \mathfrak{p} \rtimes_{\mathrm{Ad}_{\mathfrak{p}}} K$ is called the associated *motion group*. It is hermitian with respect to the involution $\theta^m(x,k) := (-x,k)$ (Example 1.3).

Remark 1.6. Note that c-duality changes the sign of the curvature. In fact, multiplying $x \in \mathfrak{p}$ with i leads to the operator $(\mathrm{ad}\, ix)^2 = -(\mathrm{ad}\, x)^2$ on $i\mathfrak{p} \cong \mathfrak{p}$.

Example 1.7. The restricted unitary group $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+ \oplus \mathcal{H}_-)$ and the restricted pseudo-unitary group $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ from Example 1.4 are c-duals of each other:

$$\mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+ \oplus \mathcal{H}_-)^c = \mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-).$$

2 JH^*-triples and their classification

A classification of the simply connected hermitian symmetric spaces has been obtained by W. Kaup in [Ka83] (cf. Theorem 2.6 below). Kaup calls a connected Riemannian Hilbert manifold M a *symmetric hermitian manifold* (which is the same as a hermitian symmetric space) if it carries a complex structure and for each $m \in M$ there exists a biholomorphic involutive isometry s_m having m as an isolated fixed point. According to [Ka77], the category of simply connected symmetric hermitian manifolds with base point is equivalent to the category of JH^*-triples, and these are classified in [Ka83]. We briefly recall the cornerstones of Kaup's classification.

Definition 2.1. (a) A complex Hilbert space U endowed with a real trilinear map

$$\{\cdot,\cdot,\cdot\}\colon U^3 \to U, \quad (x, y, z) \mapsto (x\Box y)z := \{x, y, z\}$$

which is complex linear in x and z and antilinear in y is called a *JH^*-triple* if the following conditions are satisfied for $a, b, c, x, y, z \in U$ ([Ka81, Sect. 1]):

(JH1) $\{x, y, z\} = \{z, y, x\}$.

(JH2) $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} - \{a, \{y, x, b\}, z\} + \{a, b, \{x, y, c\}\}$.

(JH3) The operators $x\Box x$ are hermitian.

We write $\mathrm{Aut}(U)$ for the *automorphism group* of the JH^*-triple U, i.e., for the group of all unitary operators g on U satisfying $g\{x, y, z\} = \{gx, gy, gz\}$ for $x, y, z \in U$. We define the *spectral norm* on U by $\|z\|_\infty := \sqrt{\|z\Box z\|}$.

(b) For every JH^*-triple, we define the *c*-dual U^c as the same complex Hilbert space U, endowed with the triple product

$$\{x, y, z\}^c := -\{x, y, z\}.$$

This turns U^c into a JH^*-triple with $x\Box^c x = -x\Box x$.

Remark 2.2. In [Ka83, Thm. 3.9], Kaup shows that every JH^*-triple U is the orthogonal direct sum

$$U = \mathfrak{z}(U) \oplus \Big(\widehat{\bigoplus}_{j\in J} U_j\Big),$$

where the U_j, $j \in J$, are the simple triple ideals. A triple ideal is a closed subspace invariant under $U\Box U$, and the simple triple ideals are the minimal non-zero triple ideals on which the triple product is non-zero. Further,

$$\mathfrak{z}(U) := \{z \in U \colon (\forall x, y \in U)\, \{x, y, z\} = 0\},$$

the *center of* U, is flat, i.e., its triple product vanishes. Since each automorphism of U permutes the simple ideals $(U_j)_{j\in J}$ and preserves the center $\mathfrak{z}(U)$, it is easy to see that the Lie algebra of the Banach–Lie group $\mathrm{Aut}(U)$ preserves each simple ideal U_j, and therefore the identity component $\mathrm{Aut}(U)_0$ also preserves each simple ideal U_j.[1]

[1]For $X \in \mathfrak{aut}(U)$ each operator e^{tX}, $t \in \mathbb{R}$, permutes the spaces U_j. Let $P_j\colon U \to U_j$ denote the orthogonal projection onto U_j. Then, for each j, the operator $P_j e^{tX} P_j^*$ on U_j is non-zero if t is sufficiently small because it converges uniformly to $\mathbf{1}$. For any such t we then have $e^{tX}U_j \subseteq U_j$, and this implies $e^X U_j = U_j$.

Remark 2.3. (a) (From hermitian Lie groups to JH^*-triples; [Ka77]) We claim that, for each hermitian Lie algebra $(\mathfrak{g}, \theta, d)$, $\mathfrak{p}$ carries a natural structure of a JH^*-triple. First we use the embedding $\mathfrak{p} \hookrightarrow \mathfrak{g}_{\mathbb{C}}, x \mapsto x - iIx$, where $Ix = [d, x]$, to obtain an isomorphism of the complex Hilbert space $\mathfrak{p}$ with the complex abelian subalgebra $\mathfrak{p}^+ \subseteq \mathfrak{g}_{\mathbb{C}}$ from (1). Therefore it suffices to exhibit a natural JH^*-structure on $\mathfrak{p}^+$.

For $x, y, z \in \mathfrak{p}^+$, we put

$$\{x, y, z\} := [[x, \overline{y}], z] \in \mathfrak{p}^+.$$

This map is complex linear in the first and third and antilinear in the second argument. Using the grading $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^- \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^+$, the two conditions (JH1/2) are easily verified.[2] We also note that

$$x\Box x = \mathrm{ad}([x, \overline{x}]),$$

so that (H2) and $[x, \overline{x}] \in i\mathfrak{k}$ imply that $x\Box x$ is hermitian. Therefore $(\mathfrak{p}^+, \{\cdot,\cdot,\cdot\})$ is a JH^*-triple.

(b) (From JH^*-triples to hermitian Lie groups) If, conversely, U is a JH^*-triple and $\overline{U}$ denotes U, endowed with the opposite complex structure, then the Kantor–Koecher–Tits construction leads to the complex Banach–Lie algebra

$$\mathfrak{g}_{\mathbb{C}}(U) := U \oplus \mathfrak{aut}(U)_{\mathbb{C}} \oplus \overline{U},$$

endowed with the bracket

$$[(x, A, y), (x', A', y')] = (Ax' - A'x, x\Box y' - x'\Box y + [A, A'], -A^*y' + (A')^*y).$$

Since $(x\Box x)^* = x\Box x$, polarization leads to $(x\Box y)^* = y\Box x$ for $x, y \in U$. This implies that $\sigma(x, A, y) := (y, -A^*, x)$ defines an antilinear involution on $\mathfrak{g}_{\mathbb{C}}(U)$, which leads to the real form

$$\mathfrak{g} := \mathfrak{g}(U) := \mathfrak{g}_{\mathbb{C}}(U)^\sigma = \{(x, A, x) \in \mathfrak{g}_{\mathbb{C}}(U) \colon x \in U, A^* = -A\}.$$

This real form has an involution θ, defined by $\theta(x, A, x) := (-x, A, -x)$ with $\mathfrak{g}^\theta \cong \mathfrak{aut}(U)$ and $\mathfrak{g}^{-\theta} \cong U$. With the element $d \in \mathfrak{z}(\mathfrak{aut}(U))$ given by $dx = ix$, we now obtain a hermitian Lie algebra $(\mathfrak{g}, \theta, d)$ with

$$\mathfrak{p}^+ = U \times \{(0,0)\}, \quad \mathfrak{k}_{\mathbb{C}} = \mathfrak{aut}(U)_{\mathbb{C}} \quad \text{and} \quad \mathfrak{p}^- = \{(0,0)\} \times U.$$

Note that, for $x, y, z \in U \cong \mathfrak{p}^+$, we have $[[x, \overline{y}], z] = (x\Box y)z = \{x, y, z\}$. [3]

Definition 2.4. We call the irreducible hermitian Lie algebra $(\mathfrak{g}, \theta, d)$ *full* if

(F1) $[\mathfrak{p}, [\mathfrak{p}, \mathfrak{p}]] \neq 0$, i.e., $\mathfrak{p}$ is a simple JH^*-triple (Remark 2.2), and

(F2) $\mathrm{ad}_{\mathfrak{p}}(\mathfrak{k}) = \mathfrak{aut}(\mathfrak{p})$ is the Lie algebra of the automorphism group $\mathrm{Aut}(\mathfrak{p})$ of the JH^*-triple $\mathfrak{p}$.

---

[2] In [Ka83], Kaup uses the formula $\{x, y, z\} := \frac{1}{2}[[x, \overline{y}], z]$.

[3] For $A \in \mathfrak{aut}(U)$ we have $A\{x, y, z\} = \{Ax, y, z\} + \{x, Ay, z\} + \{x, y, Az\}$, so that complex linear extension leads for $A \in \mathfrak{aut}(U)_{\mathbb{C}}$ to $A\{x, y, z\} = \{Ax, y, z\} - \{x, A^*y, z\} + \{x, y, Az\}$.

Remark 2.5. (a) Suppose that $(\mathfrak{g}, \theta, d)$ is full. In view of Schur's Lemma, the irreducibility of the $\mathfrak{k}$-module $\mathfrak{p}$ implies that $\mathfrak{z}(\mathfrak{aut}(\mathfrak{p})) = \mathbb{R}i\mathbf{1} = \mathbb{R}\,\mathrm{ad}_{\mathfrak{p}}\, d \subseteq \mathrm{ad}_{\mathfrak{p}}(\mathfrak{z}(\mathfrak{k}))$. Condition (F2) also implies the converse inclusion, so that

$$\mathfrak{z}(\mathfrak{aut}(\mathfrak{p})) = \mathrm{ad}_{\mathfrak{p}}(\mathfrak{z}(\mathfrak{k})) = \mathbb{R}i\mathbf{1}.$$

In particular, we obtain

$$\mathfrak{aut}(\mathfrak{p})/\mathfrak{z}(\mathfrak{aut}(\mathfrak{p})) \cong \mathfrak{k}/\mathfrak{z}(\mathfrak{k}). \tag{3}$$

(b) If (G, θ, d) is a hermitian Lie group satisfying (F2), then $\mathfrak{z}(\mathfrak{g}) = \ker \mathrm{ad}_{\mathfrak{p}}$ (Remark 1.2(d)) leads to $\mathrm{ad}\,\mathfrak{g} \cong \mathfrak{aut}(\mathfrak{p}) \oplus \mathfrak{p} = \mathfrak{g}(\mathfrak{p})$, endowed with the Lie bracket

$$[(A, x), (A', x')] := ([A, A'] + \mathrm{ad}_{\mathfrak{p}}([x, x']), Ax' - A'x)$$

(cf. Remark 2.3(b)).

For an irreducible hermitian Lie group (G, θ, d), the connectedness of K and the irreducibility of the K-representation on $\mathfrak{p}$ implies that either $\mathfrak{p}$ is a simple JH^*-triple or flat. Kaup's classification also implies that a simple JH^*-triple either is *positive*, i.e., all operators $x \Box x$ are positive, or *negative*, i.e., all these operators are negative ([Ka83, p. 69]). Since we shall need it later on, we recall the infinite dimensional part of Kaup's classification.

Theorem 2.6. *Each simple infinite dimensional positive JH^*-triple U is isomorphic to one of the following, where $\mathcal{H}_\pm$ and $\mathcal{H}$ are complex Hilbert spaces and σ is a conjugation on $\mathcal{H}$, i.e., an antilinear isometric involution.*

(I) *The space $B_2(\mathcal{H}_-, \mathcal{H}_+)$ of Hilbert–Schmidt operators $\mathcal{H}_- \to \mathcal{H}_+$ with*

$$\langle A, B\rangle = \mathrm{tr}(AB^*) \quad \text{and} \quad \{A, B, C\} = \tfrac{1}{2}(AB^*C + CB^*A). \tag{4}$$

In this case $\|A\|_\infty = \|A\|$ is the operator norm. For $\mathcal{H}_+ \not\cong \mathcal{H}_-$, the automorphism group consists of the maps of the form $\alpha(A) = g_+ A g_-$ for $g_\pm \in \mathrm{U}(\mathcal{H}_\pm)$, and for $\mathcal{H}_+ = \mathcal{H}_- = \mathcal{H}$, we obtain an additional automorphism by $A \mapsto A^\top := \sigma A^* \sigma$.

(II) *The space $\mathrm{Skew}_2(\mathcal{H}) := \{A \in B_2(\mathcal{H}) \colon A^\top = -A\}$ of skew-symmetric Hilbert–Schmidt operators, which is a JH^*-subtriple of $B_2(\mathcal{H})$. The automorphisms are the maps $\alpha(A) = gAg^{-1}$ for $g \in \mathrm{U}(\mathcal{H})$.*

(III) *The space $\mathrm{Sym}_2(\mathcal{H}) := \{A \in B_2(\mathcal{H}) \colon A^\top = A\}$ of symmetric Hilbert–Schmidt operators, which is another JH^*-subtriple of $B_2(\mathcal{H})$. The automorphisms are the maps $\alpha(A) = gAg^{-1}$ for $g \in \mathrm{U}(\mathcal{H})$.*

(IV) *A complex Hilbert space U with a conjugation σ and*

$$\{x, y, z\} := \langle x, y\rangle z + \langle z, y\rangle x - \langle x, \sigma(z)\rangle \sigma(y).$$

Then $\|z\|_\infty^2 = \langle z, z\rangle + \sqrt{\langle z, z\rangle^2 - |\langle z, \sigma(z)\rangle|^2}$ and the automorphisms are of the form $\alpha(v) = zg(v)$ for $z \in \mathbb{T}$ and $g \in \mathrm{O}(\mathcal{H}^\sigma)$.

The negative simple JH^*-triples are the c-duals of the positive ones and have the same automorphism group.

Proof. The classification can be found in [Ka83, Thm. 3.9]; see also [Ka81, p. 474] for the type IV case.

For a description of the automorphism groups of simple JH^*-triples, we refer to [Ka97, pp. 199/200] and [Ka75, p. 91]. To see that Kaup's list also describes the automorphism groups of the associated JH^*-triples, we note that any automorphism of U preserves the spectral norm on U, hence extends to the bidual U_∞'' of the completion U_∞ of U with respect to the spectral norm (see also [Ne01a, Thm. V.11]). Conversely, Kaup's list shows that all automorphisms of U_∞'' preserve the subspace U on which they induce JH^*-triple automorphisms. It is clear that $\mathrm{Aut}(U^c) = \mathrm{Aut}(U)$. □

Automorphisms of symmetric Hilbert domains

Remark 2.7. For every positive JH^*-triple U, the set

$$\mathcal{D} := \{x \in U \colon \|x\|_\infty < 1\}$$

is a *symmetric Hilbert domain*, i.e., for each point $m \in \mathcal{D}$, there exists an involutive biholomorphic map s_m having m as an isolated fixed point ([Ka83, p. 71]). Since this is clear for $m = 0$ with $s_0(x) = -x$, the main point is that the group $\mathrm{Aut}(\mathcal{D})$ of biholomorphic maps acts transitively on $\mathcal{D}$. This group is a Banach–Lie group and $\theta(g) := s_0 g s_0$ defines an involution on $\mathrm{Aut}(\mathcal{D})$ with $\mathrm{Aut}(\mathcal{D})^\theta = \mathrm{Aut}(U)$ and the corresponding decomposition

$$\mathfrak{aut}(\mathcal{D}) := \mathbf{L}(\mathrm{Aut}(\mathcal{D})) \cong \mathfrak{aut}(U) \oplus U, \quad \mathfrak{aut}(\mathcal{D})^\theta = \mathfrak{aut}(U), \quad \mathfrak{p} = U.$$

It is easy to see that $(\mathrm{Aut}(\mathcal{D})_0, \theta, i\mathbf{1})$ is a hermitian Lie group, and if U is simple, then it is full. We also derive that $\mathfrak{aut}(\mathcal{D}) \cong \mathfrak{g}(U)$ (Remark 2.3(b)).

We now describe the automorphism groups of infinite dimensional irreducible symmetric Hilbert domains which correspond to the four families of infinite dimensional positive simple JH^*-triples.

Type I: The group $G := \mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ (cf. Example 1.4(b)) acts transitively on

$$\mathcal{D} := \{Z \in B_2(\mathcal{H}_-, \mathcal{H}_+) \colon \|Z\| < 1\}$$

by fractional linear transformations

$$gZ = (aZ + b)(cZ + d)^{-1} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(cf. [NO98]). The stabilizer of 0 in G is $K := \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$, so that $\mathcal{D} \cong G/K$. With Theorem 2.6 we see that $\mathrm{Aut}(\mathcal{D})_0 \cong \mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)/\mathbb{T}\mathbf{1}$, so that $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ is a central $\mathbb{T}$-extension of $\mathrm{Aut}(\mathcal{D})_0$ which is easily seen to be a hermitian Lie group with respect to $d := \frac{i}{2}\,\mathrm{diag}(\mathbf{1}, -\mathbf{1})$. From Theorem 2.6 it also follows that $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ is full.

Type II: For a conjugation σ on the complex Hilbert space $\mathcal{H}$, the subgroup $G := \mathrm{O}^*_{\mathrm{res}}(\mathcal{H}) \subseteq \mathrm{U}_{\mathrm{res}}(\mathcal{H},\mathcal{H})$ (cf. Appendix C.6) acts transitively on

$$\mathcal{D} = \{Z \in B_2(\mathcal{H})\colon Z^\top = -Z, \|Z\| < 1\}.$$

The stabilizer of 0 is $K := \left\{\begin{pmatrix} g & 0 \\ 0 & g^{-\top}\end{pmatrix} : g \in \mathrm{U}(\mathcal{H})\right\} \cong \mathrm{U}(\mathcal{H})$, so that $G/K \cong \mathcal{D}$. In this case $\mathrm{Aut}(\mathcal{D})_0 \cong \mathrm{O}^*_{\mathrm{res}}(\mathcal{H})/\{\pm\mathbf{1}\}$ and $\mathrm{O}^*_{\mathrm{res}}(\mathcal{H})$ is a hermitian Lie group with respect to $d := \frac{i}{2}\operatorname{diag}(\mathbf{1}, -\mathbf{1})$.
Type III: Likewise, the subgroup $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}) \subseteq \mathrm{U}_{\mathrm{res}}(\mathcal{H},\mathcal{H})$ (cf. Appendix C.6) acts transitively on

$$\mathcal{D} = \{Z \in B_2(\mathcal{H})\colon Z^\top = Z, \|Z\| < 1\}.$$

The stabilizer K of 0 is isomorphic to $\mathrm{U}(\mathcal{H})$, $\mathrm{Aut}(\mathcal{D})_0 \cong \mathrm{Sp}_{\mathrm{res}}(\mathcal{H})/\{\pm\mathbf{1}\}$, and $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H})$ is a hermitian Lie group with respect to $d := \frac{i}{2}\operatorname{diag}(\mathbf{1}, -\mathbf{1})$.
Type IV: We consider the orthogonal group $G := \mathrm{O}(\mathbb{R}^2, \mathcal{H}_\mathbb{R})_0$ of the indefinite quadratic form

$$q(x, v) := \|x\|^2 - \|v\|^2$$

on $\mathbb{R}^2 \oplus \mathcal{H}_\mathbb{R}$, where $\mathcal{H}_\mathbb{R}$ is an infinite dimensional real Hilbert space. Then

$$d := \left(\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}, 0\right) \in \mathfrak{g} := \mathfrak{o}(\mathbb{R}^2 \oplus \mathcal{H}_\mathbb{R})$$

satisfies $(\operatorname{ad} d)^3 + \operatorname{ad} d = 0$, so that $\theta := e^{\pi \operatorname{ad} d}$ defines an involution on G with

$$K := G^\theta = \mathrm{SO}_2(\mathbb{R}) \times \mathrm{O}(\mathcal{H}_\mathbb{R}).$$

Elements of the Lie algebra $\mathfrak{g} = \mathfrak{o}(\mathbb{R}^2, \mathfrak{o}(\mathcal{H}_\mathbb{R}))$ have a block structure

$$x = \begin{pmatrix} a & b^\top \\ b & c\end{pmatrix}, \quad \text{with} \quad b = (b_1, b_2) \in \mathcal{H}_\mathbb{R}^2 \cong (\mathcal{H}_\mathbb{R})_\mathbb{C}.$$

From

$$\left[\begin{pmatrix} a & 0 \\ 0 & c\end{pmatrix}, \begin{pmatrix} 0 & b^\top \\ b & 0\end{pmatrix}\right] = \begin{pmatrix} 0 & (cb - ba)^\top \\ cb - ba & 0\end{pmatrix}$$

we derive that $a \in \mathfrak{so}_2(\mathbb{R})$ acts on $\mathfrak{p} \cong \mathcal{H}_\mathbb{R}^2$ by $b \mapsto -ba$. Therefore $\operatorname{ad} d$ defines a complex structure on $\mathfrak{p} \cong \mathcal{H}_\mathbb{R}^2$, acting by

$$I(v, w) = -(v, w)\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} = (-w, v).$$

This leads to a natural identification of $\mathfrak{p}$ with $(\mathcal{H}_\mathbb{R})_\mathbb{C}$, where $\mathfrak{o}(\mathcal{H}_\mathbb{R}) \subseteq \mathfrak{k}$ acts in the natural way by complex linear operators. Since the representation of $\mathfrak{o}(\mathcal{H}_\mathbb{R})$ on $\mathcal{H}_\mathbb{R}$ is absolutely irreducible, i.e., its commutant is $\mathbb{R}\mathbf{1}$, the representation of $\mathfrak{k}$ on the complex Hilbert space $\mathfrak{p}$ is irreducible. We therefore obtain the hermitian Lie group (G, θ, d).

Putting $\mathcal{H} := (\mathcal{H}_\mathbb{R})_\mathbb{C} \cong \mathfrak{p}$, we then obtain $\mathfrak{p}_\mathbb{C} \cong \mathcal{H}^2$ with

$$\mathfrak{p}^\pm \cong \{b \mp iIb\colon b \in \mathcal{H}\} = \{(b, \mp ib)\colon b \in \mathcal{H}\}.$$

In this realization, an easy matrix calculation leads with the isomorphism $\mathcal{H} \to \mathfrak{p}^+, x \mapsto (x, -ix)$ for $x, y, z \in \mathfrak{p}^+ \cong \mathcal{H}$ to

$$[[x, \overline{y}], z] = 2(\langle x, y\rangle z + \langle z, y\rangle x - \langle x, \overline{z}\rangle \overline{y}).$$

Therefore the associated JH^*-triple is of type IV (Theorem 2.6).

3 Central extensions of hermitian Lie groups

In Section 6-8 below we show that for certain hermitian Lie groups G there are natural central extensions $\widehat{G}$ with a substantially richer semibounded unitary representation theory. This motivates our detailed discussion of central extensions in the present section. Its goal is to show that for every simple JH^*-triple U, the Lie algebra $\mathfrak{g}(U)$ has a universal central extension $\widehat{\mathfrak{g}}(U)$ whose center is at most 2-dimensional and which is *integrable* in the sense that it is the Lie algebra of a simply connected Banach–Lie group $\widehat{G}(U)$. For these central Lie group extensions we give rather direct constructions based on [Ne02].

Definition 3.1. (a) Let $\mathfrak{z}$ be a topological vector space and $\mathfrak{g}$ be a topological Lie algebra. A *continuous $\mathfrak{z}$-valued 2-cocycle* is a continuous skew-symmetric function $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ with

$$\omega([x, y], z) + \omega([y, z], x) + \omega([z, x], y) = 0 \quad \text{for} \quad x, y, z \in \mathfrak{g}.$$

It is called a *coboundary* if there exists a continuous linear map $\alpha\colon \mathfrak{g} \to \mathfrak{z}$ with $\omega(x, y) = \alpha([x, y])$ for all $x, y \in \mathfrak{g}$. We write $Z^2_c(\mathfrak{g}, \mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles, $B^2_c(\mathfrak{g}, \mathfrak{z})$ for the subspace of coboundaries defined by continuous linear maps, and define the *second continuous Lie algebra cohomology space*

$$H^2_c(\mathfrak{g}, \mathfrak{z}) := Z^2_c(\mathfrak{g}, \mathfrak{z})/B^2_c(\mathfrak{g}, \mathfrak{z}).$$

(b) If ω is a continuous $\mathfrak{z}$-valued cocycle on $\mathfrak{g}$, then we write $\mathfrak{z} \oplus_\omega \mathfrak{g}$ for the topological Lie algebra whose underlying topological vector space is the product space $\mathfrak{z} \times \mathfrak{g}$, and whose bracket is defined by

$$[(z, x), (z', x')] = \big(\omega(x, x'), [x, x']\big).$$

Then $q\colon \mathfrak{z} \oplus_\omega \mathfrak{g} \to \mathfrak{g}, (z, x) \mapsto x$ is a central extension and $\sigma\colon \mathfrak{g} \to \mathfrak{z} \oplus_\omega \mathfrak{g}, x \mapsto (0, x)$ is a continuous linear section of q.

(c) A *morphism of central extensions*

$$\mathfrak{z}_1 \xrightarrow{i_1} \widehat{\mathfrak{g}}_1 \xrightarrow{q_1} \mathfrak{g}, \qquad \mathfrak{z}_2 \xrightarrow{i_2} \widehat{\mathfrak{g}}_2 \xrightarrow{q_2} \mathfrak{g}$$

is a continuous homomorphism $\varphi\colon \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ with $q_2 \circ \varphi = q_1$ and $\varphi \circ i_1 = i_2 \circ \psi$ for a continuous linear map $\psi\colon \mathfrak{z}_1 \to \mathfrak{z}_2$. For $\mathfrak{z}_1 = \mathfrak{z}_2$ and $\psi = \mathrm{id}_{\mathfrak{z}_1}$, the morphism φ is called an *equivalence*. The space $H^2_c(\mathfrak{g}, \mathfrak{z})$ classifies equivalence classes of topologically split central extensions of $\mathfrak{g}$ by $\mathfrak{z}$ (cf. [Ne02b, Rem. 1.2]).

(d) A central extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \to \mathfrak{g}$ is called *universal* if, for every other central extension $\mathfrak{z}_1 \hookrightarrow \widehat{\mathfrak{g}}_1 \to \mathfrak{g}$, there exists a unique morphism $\varphi\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}_1$ of central extensions. This condition readily implies that universal central extensions are unique up to isomorphism. According to [Ne02, Prop. I.13], a perfect Banach–Lie algebra [4] for which $H^2_c(\mathfrak{g}, \mathbb{R})$ is finite dimensional has a universal central extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ satisfying $\mathfrak{z} \cong H^2_c(\mathfrak{g}, \mathbb{R})^*$.

(e) We say that $\mathfrak{g}$ is *centrally closed* if $H^2_c(\mathfrak{g}, \mathbb{R}) = \{0\}$. From [Ne02b, Lemma 1.11, Cor. 1.14] it follows that a perfect central extension $q\colon \widehat{\mathfrak{g}} \to \mathfrak{g}$ with finite dimensional kernel which is centrally closed is universal.

Definition 3.2. (a) Let $(\mathfrak{g}, \theta, d)$ be a hermitian Lie algebra, write $\langle\cdot,\cdot\rangle$ for the scalar product on $\mathfrak{p}$ and let $\omega_{\mathfrak{p}}(x, y) := 2\,\mathrm{Im}\langle x, y\rangle$ be the corresponding symplectic form on $\mathfrak{p}$. We extend $\omega_{\mathfrak{p}}$ to a skew-symmetric form $\omega_{\mathfrak{p}}$ on $\mathfrak{g}$ satisfying $i_x\omega_{\mathfrak{p}} = 0$ for every $x \in \mathfrak{k}$. We claim that $\omega_{\mathfrak{p}}$ is a 2-cocycle, i.e., an element of $Z^2(\mathfrak{g}, \mathbb{R})$. As $\omega_{\mathfrak{p}}$ is $\mathfrak{k}$-invariant, $i_x\mathsf{d}\omega_{\mathfrak{p}} = -\mathsf{d}(i_x\omega_{\mathfrak{p}}) = 0$ for $x \in \mathfrak{k}$. Hence it remains to see that

$$(\mathsf{d}\omega_{\mathfrak{p}})(x, y, z) = -\sum_{\text{cyc.}} \omega_{\mathfrak{p}}([x, y], z)$$

vanishes for $x, y, z \in \mathfrak{p}$, but this follows from $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$.

We call $\omega_{\mathfrak{p}}$ the *canonical cocycle* of the hermitian Lie algebra $(\mathfrak{g}, \mathbf{L}(\theta), d)$.

(b) Let $(\mathfrak{g}, \theta, d)$ be a hermitian Lie algebra and $\omega_{\mathfrak{p}} \in Z^2(\mathfrak{g}, \mathbb{R})$ be its canonical 2-cocycle (Definition 3.2(a)). We write $\widehat{\mathfrak{g}} = \mathbb{R} \oplus_{\omega_{\mathfrak{p}}} \mathfrak{g}$ for the corresponding central extension with the bracket

$$[(z, x), (z', x')] := (\omega_{\mathfrak{p}}(x, x'), [x, x']).$$

Since $\omega_{\mathfrak{p}}$ is θ-invariant, $\widehat{\theta}(z, x) := (z, \mathbf{L}(\theta)x)$ defines an involution on the Banach–Lie algebra $\widehat{\mathfrak{g}}$ with eigenspace decomposition

$$\widehat{\mathfrak{k}} = \mathbb{R} \oplus \mathfrak{k} \quad \text{and} \quad \widehat{\mathfrak{p}} = \mathfrak{p}.$$

We thus obtain a hermitian Lie algebra $(\widehat{\mathfrak{g}}, \widehat{\theta}, d)$. In Theorem 3.9 below we shall see how to obtain a corresponding Lie group.

Lemma 3.3. *Let $(\mathfrak{g}, \theta, d)$ be irreducible hermitian and $\omega \in Z^2_c(\mathfrak{g}, \mathbb{R})$ be a continuous 2-cocycle. Then the following assertions hold:*

(i) *The class $[\omega] \in H^2_c(\mathfrak{g}, \mathbb{R})$ can be represented by a cocycle vanishing on $\mathfrak{k} \times \mathfrak{p}$.*

(ii) *If ω vanishes on $\mathfrak{k} \times \mathfrak{k}$, then $[\omega] \in \mathbb{R}[\omega_{\mathfrak{p}}]$, where $\omega_{\mathfrak{p}}$ is the canonical cocycle.*

Proof. (i) In view of [Ne10e, Thm. 9.1], the group $e^{\mathbb{R}\,\mathrm{ad}\,d} = e^{[0,2\pi]\,\mathrm{ad}\,d}$ acts trivially on $H^2_c(\mathfrak{g}, \mathbb{R})$, so that

$$\widetilde{\omega}(x, y) := \frac{1}{2\pi}\int_0^{2\pi} \omega(e^{t\,\mathrm{ad}\,d}x, e^{t\,\mathrm{ad}\,d}y)\,dt$$

[4] We call a Lie algebra $\mathfrak{g}$ *perfect* if $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, i.e., if it is spanned by all brackets. A topological Lie algebra $\mathfrak{g}$ is called *topologically perfect* if the subspace $[\mathfrak{g}, \mathfrak{g}]$ is dense in $\mathfrak{g}$.

is a 2-cocycle with the same cohomology class as ω and which coincides with ω on $\mathfrak{k} \times \mathfrak{k}$. Since $\widetilde{\omega}$ is $\operatorname{ad} d$-invariant, we obtain for $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$:

$$0 = \widetilde{\omega}([d,x],y) + \widetilde{\omega}(x,[d,y]) = \widetilde{\omega}(x,[d,y]),$$

so that $[d,\mathfrak{p}] = \mathfrak{p}$ leads to $\widetilde{\omega}(\mathfrak{k},\mathfrak{p}) = \{0\}$.

Suppose that ω, and hence also $\widetilde{\omega}$, vanishes on $\mathfrak{k} \times \mathfrak{k}$. From the Cartan formula we further obtain for $x \in \mathfrak{k}$ the relation $\mathcal{L}_x\widetilde{\omega} = \mathsf{d}(i_x\widetilde{\omega}) = 0$, so that $\widetilde{\omega}|_{\mathfrak{p}\times\mathfrak{p}}$ is a $\mathfrak{k}$-invariant alternating form, hence extends to a $\mathfrak{k}$-invariant skew-hermitian form

$$\widetilde{\omega}_{\mathbb{C}} \colon \mathfrak{p} \times \mathfrak{p} \to \mathbb{C}, \quad \widetilde{\omega}_{\mathbb{C}}(x,y) := \widetilde{\omega}(x,y) + i\widetilde{\omega}(-Ix,y), \quad Ix = [d,x].$$

This form can be represented by a skew-hermitian operator $A = -A^* \in B(\mathfrak{p})$ via

$$\widetilde{\omega}_{\mathbb{C}}(x,y) = \langle Ax, y\rangle \quad \text{for} \quad x,y \in \mathfrak{p}.$$

Now the $\mathfrak{k}$-invariance of $\widetilde{\omega}_{\mathbb{C}}$ implies that A commutes with $\operatorname{ad}\mathfrak{k}$, so that the irreducibility of the $\mathfrak{k}$-module $\mathfrak{p}$ leads to $A \in i\mathbb{R}\mathbf{1}$. We conclude that there exists a $\lambda \in \mathbb{R}$ with

$$\widetilde{\omega}(x,y) = \operatorname{Re}\langle \lambda i x, y\rangle = -\lambda \operatorname{Im}\langle x,y\rangle = -\frac{\lambda}{2}\omega_{\mathfrak{p}}(x,y),$$

so that $\widetilde{\omega} = -\frac{\lambda}{2}\omega_{\mathfrak{p}}$. □

Remark 3.4. (a) If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a direct sum of topological Lie algebras, where $H^2_c(\mathfrak{g}_j,\mathbb{R}) = \{0\}$ for $j = 1,2$ and $\mathfrak{g}_1$ is topologically perfect, then $H^2_c(\mathfrak{g},\mathbb{R}) = \{0\}$.

In fact, if $\omega \in Z^2_c(\mathfrak{g},\mathbb{R})$ is a cocycle, then the triviality of the cohomology of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ implies that we may subtract a coboundary, so that we can assume that ω vanishes on $\mathfrak{g}_j \times \mathfrak{g}_j$ for $j = 1,2$. The cocycle property further implies that

$$\omega([x_1,x_2],y) = 0 \quad \text{for} \quad x_1,x_2 \in \mathfrak{g}_1, y \in \mathfrak{g}_2.$$

Since $\mathfrak{g}_1$ is topologically perfect, we obtain $\omega = 0$.

(b) [Ne02, Prop. I.5]: If $\mathcal{H}$ is a complex Hilbert space of any dimension, then the second continuous cohomology $H^2_c(\mathfrak{gl}(\mathcal{H}),\mathbb{C})$ of the complex Banach–Lie algebra $\mathfrak{gl}(\mathcal{H})$ vanishes. This implies in particular that it also vanishes for all its real forms:

$$H^2_c(\mathfrak{u}(\mathcal{H}),\mathbb{R}) = \{0\} \quad \text{and} \quad H^2_c(\mathfrak{u}(\mathcal{H}_+,\mathcal{H}_-),\mathbb{R}) = \{0\}.$$

We now apply all this to the Lie algebras $\mathfrak{k}$ corresponding to the hermitian operator groups from Section 2.

Example 3.5. Type I: (a) For $\mathfrak{g} = \mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)$, we have $\mathfrak{k} = \mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)$. If at least one space $\mathcal{H}_\pm$ is infinite dimensional, then Remark 3.4(b) implies that $H^2_c(\mathfrak{u}(\mathcal{H}_\pm),\mathbb{R}) = \{0\}$, and since $\mathfrak{u}(\mathcal{H}_+)$ or $\mathfrak{u}(\mathcal{H}_-)$ is perfect ([Ne02, Lemma I.3]), (a) leads to $H^2_c(\mathfrak{k},\mathbb{R}) = \{0\}$.

Therefore every class $[\omega] \in H^2_c(\mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-),\mathbb{R})$ can be represented by a cocycle vanishing on $\mathfrak{k} \times \mathfrak{k}$, so that Lemma 3.3 shows that $H^2_c(\mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-),\mathbb{R}) = \mathbb{R}[\omega_{\mathfrak{p}}]$. If not both $\mathcal{H}_\pm$ are infinite dimensional, then $\mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-) = \mathfrak{u}(\mathcal{H}_+,\mathcal{H}_-)$, so that

Remark 3.4(b) implies that its second cohomology is trivial. In this case we therefore have $H^2_c(\mathfrak{u}_{\rm res}(\mathcal{H}_+, \mathcal{H}_-), \mathbb{R}) = 0$.

If both spaces $\mathcal{H}_\pm$ are infinite dimensional, then the cocycle $\omega_{\mathfrak{p}}$ leads to a central extension $\widehat{\mathfrak{g}} = \widehat{\mathfrak{u}}_{\rm res}(\mathcal{H}_+, \mathcal{H}_-)$ of $\mathfrak{g} = \mathfrak{u}_{\rm res}(\mathcal{H}_+, \mathcal{H}_-)$ with $\widehat{\mathfrak{k}} \cong \mathbb{R} \oplus \mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)$, so that $\dim \mathfrak{z}(\widehat{\mathfrak{k}}) = 3$ and $\dim \mathfrak{z}(\widehat{\mathfrak{g}}) = 2$.

We claim that the Lie algebra $\widehat{\mathfrak{g}}$ is centrally closed. To this end, we first recall the central $\mathbb{R}$-extension of $\mathfrak{g}$ described in [Ne02] whose cocycle is defined in terms of the (2×2)-block matrix structure of the elements $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ of $\mathfrak{g}$ by

$$\omega(x, y) := \mathrm{tr}([x, y]_{11}) = \mathrm{tr}(x_{12}y_{21} - y_{12}x_{21}).$$

(cf. [Ne02, Rem. IV.5]). Identifying a Hilbert–Schmidt operator $z \in B_2(\mathcal{H}_-, \mathcal{H}_+)$ with the corresponding matrix

$$\widetilde{z} := \begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \in \mathfrak{p},$$

this leads to

$$\omega(\widetilde{z}, \widetilde{w}) = \mathrm{tr}(zw^* - wz^*) = 2i \,\mathrm{Im}\, \mathrm{tr}(zw^*).$$

This means that $\omega = i\omega_{\mathfrak{p}}$ and thus [Ne02, Prop. IV.8] implies that $\widehat{\mathfrak{g}}$ is perfect and centrally closed, hence a universal central extension (Definition 3.1(e)).

To obtain a (simply connected) Lie group $\widehat{G} = \widehat{\mathrm{U}}_{\rm res}(\mathcal{H}_+, \mathcal{H}_-)$ with Lie algebra $\widehat{\mathfrak{g}}$, we first consider the Banach–Lie algebra

$$\mathfrak{g}_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{g} \colon a \in B_1(\mathcal{H}_+), d \in B_1(\mathcal{H}_-) \right\}$$

(cf. Appendix C.5). Then

$$\sigma \colon \mathfrak{g}_1 \to \widehat{\mathfrak{g}}, \quad \sigma(x) := (-i \,\mathrm{tr}\, x_{11}, x)$$

is a homomorphism of Lie algebras because the linear functional $\alpha(x) := -i\,\mathrm{tr}(x_{11})$ on $\mathfrak{g}_1$ satisfies $\alpha([x, y]) = -i\omega(x, y) = \omega_{\mathfrak{p}}(x, y)$ for $x, y \in \mathfrak{g}_1$. As $\mathrm{tr} \colon B_1(\mathcal{H}_+) \to \mathbb{C}$ vanishes on $[B(\mathcal{H}_+), B_1(\mathcal{H}_+)]$, the homomorphism σ can be combined with the inclusion $\mathfrak{k} \hookrightarrow \widehat{\mathfrak{g}}, x \mapsto (0, x)$ to a surjective homomorphism,

$$\mathfrak{g}_1 \rtimes \mathfrak{k} \to \widehat{\mathfrak{g}}, \quad (x, y) \mapsto (-i\,\mathrm{tr}(x_{11}), x + y).$$

Passing from $\mathfrak{g}_1$ to

$$\mathfrak{sg}_1 := \ker(\mathrm{tr}\,|_{\mathfrak{g}_1}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{g}_1 \colon \ \mathrm{tr}(a) + \mathrm{tr}(d) = 0 \right\},$$

we also obtain a surjective homomorphism

$$\gamma_{\mathfrak{g}} \colon \mathfrak{sg}_1 \rtimes \mathfrak{k} \to \widehat{\mathfrak{g}}, \quad (x, y) \mapsto (-i\,\mathrm{tr}(x_{11}), x + y).$$

(b) For the positive JH^*-triple structure on $\mathfrak{p} = B_2(\mathcal{H}_-, \mathcal{H}_+)$, the situation on the group level looks as follows. The subgroup $K = \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-) \subseteq \mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ is simply connected by Kuiper's Theorem, and the group

$$G_1 := \{g \in G \colon \|\mathbf{1} - g_{11}\|_1 < \infty, \|\mathbf{1} - g_{22}\|_1 < \infty\} \tag{5}$$

is a Banach–Lie group with Lie algebra $\mathfrak{g}_1$ and a polar decomposition $G_1 = K_1 \exp(\mathfrak{p})$, where

$$K_1 := K \cap G_1 \cong \mathrm{U}_1(\mathcal{H}_+) \times \mathrm{U}_1(\mathcal{H}_-)$$

(cf. Remark C.11). For the subgroup $SG_1 := \ker(\det \colon G_1 \to \mathbb{T})$ we have

$$SK_1 := K \cap SG_1 \cong \{(k_1, k_2) \in \mathrm{U}_1(\mathcal{H}_+) \times \mathrm{U}_1(\mathcal{H}_-) \colon \det(k_1)\det(k_2) = 1\}.$$

Since $\widetilde{\mathrm{U}}_1(\mathcal{H}_\pm) \cong \mathbb{R} \ltimes \mathrm{SU}(\mathcal{H}_\pm)$ holds for the universal covering group of $\mathrm{U}_1(\mathcal{H}_\pm)$, we have the universal covering group

$$\widetilde{SK}_1 \cong \mathbb{R} \ltimes (\mathrm{SU}(\mathcal{H}_+) \times \mathrm{SU}(\mathcal{H}_-)),$$

which, according to the polar decomposition (cf. Remark C.11), is contained in the universal covering group $\widetilde{SG}_1$ of SG_1. If $\widehat{G}$ is a connected Lie group with Lie algebra $\widehat{\mathfrak{g}}$, the homomorphism $\gamma_{\mathfrak{g}}$ integrates to a surjective homomorphism of Lie groups

$$\gamma_G \colon \widetilde{SG}_1 \rtimes K \to \widehat{G}$$

which restricts to

$$\gamma_G \colon \widetilde{SK}_1 \rtimes K \to \widehat{K} \cong \mathbb{R} \times K, \quad ((t, k_1, k_2), (u_1, u_2)) \mapsto (t, k_1 u_1, k_2 u_2)$$

(cf. [Ne02, Sect. IV], where this is used to construct $\widehat{G}$ as a quotient group). The kernel of γ_G is isomorphic to $\mathrm{SU}(\mathcal{H}_+) \times \mathrm{SU}(\mathcal{H}_-)$.

(c) For the negative JH^*-triple structure on $\mathfrak{p} = B_2(\mathcal{H}_-, \mathcal{H}_+)$ and $G = \mathrm{U}_{\mathrm{res}}(\mathcal{H})$, we define G_1 and SG_1 as in (5). Then the inclusion $\mathrm{SU}(\mathcal{H}) \to SG_1$ is a weak homotopy equivalence, and in particular SG_1 is simply connected (use [Ne02, Prop. III.2(c)] and the polar decomposition). We thus obtain a surjective homomorphism of Lie groups $\gamma_G \colon SG_1 \rtimes K \to \widehat{G}$ which restricts to

$$\gamma_G \colon SK_1 \rtimes K \to \widehat{K} \cong \mathbb{T} \times K, \quad ((t, k_1, k_2), (u_1, u_2)) \mapsto (t, k_1 u_1, k_2 u_2).$$

(d) We also note that the positive JH^*-triple structure $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ on $U_+ := B_2(\mathcal{H}_-, \mathcal{H}_+)$ leads to the Lie algebra

$$\mathfrak{g}(U_+) \cong \mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)/i\mathbb{R}\mathbf{1} \cong \mathrm{ad}(\mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)),$$

and if both $\mathcal{H}_\pm$ are infinite dimensional, then we obtain the non-trivial central extension

$$\widehat{\mathfrak{g}}(U_+) \cong \widehat{\mathfrak{u}}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-).$$

Accordingly, the negative JH^*-triple structure $\{x,y,z\} = -\frac{1}{2}(xy^*z + zy^*x)$ on $U_- := B_2(\mathcal{H}_-, , \mathcal{H}_+)$ leads to

$$\mathfrak{g}(U_-) \cong \mathfrak{u}_{\rm res}(\mathcal{H})/i\mathbb{R}\mathbf{1} \cong \mathrm{ad}(\mathfrak{u}_{\rm res}(\mathcal{H})),$$

and if both $\mathcal{H}_\pm$ are infinite dimensional, we have the non-trivial central extension

$$\widehat{\mathfrak{g}}(U_-) \cong \widehat{\mathfrak{u}}_{\rm res}(\mathcal{H}).$$

Example 3.6. Type II/III: (a) For $\mathfrak{k} = \mathfrak{u}(\mathcal{H})$ and $\dim \mathcal{H} = \infty$, Remark 3.4(b) implies $H^2_c(\mathfrak{k}, \mathbb{R}) = \{0\}$, and we obtain as above with Lemma 3.3 that $H^2_c(\mathfrak{sp}_{\rm res}(\mathcal{H}), \mathbb{R}) = \mathbb{R}[\omega_\mathfrak{p}]$ and likewise that $H^2_c(\mathfrak{o}^*_{\rm res}(\mathcal{H}), \mathbb{R}) = \mathbb{R}[\omega_\mathfrak{p}]$. According to [Ne02, Prop. I.11], $[\omega_\mathfrak{p}] \neq 0$ in both cases, and [Ne02, Prop. IV.8] implies that $\widehat{\mathfrak{g}}$ is centrally closed. Note that $\dim \mathfrak{z}(\widehat{\mathfrak{k}}) = 2$ and $\dim \mathfrak{z}(\widehat{\mathfrak{g}}) = 1$.

(b) On the group level, we have for the positive JH^*-triple structure in a similar fashion as for type I a Lie algebra $\mathfrak{g}_1 = \mathfrak{k}_1 \oplus \mathfrak{p}$ with $\mathfrak{k}_1 \cong \mathfrak{u}_1(\mathcal{H})$ and a surjective homomorphism

$$\gamma \colon \mathfrak{g}_1 \rtimes \mathfrak{k} \to \widehat{\mathfrak{g}}, \quad (x,y) \mapsto (-i\,\mathrm{tr}(x_{11}), x+y).$$

Again, $K = \mathrm{U}(\mathcal{H})$ is simply connected and the group

$$G_1 := \{g \in G \colon \|\mathbf{1} - g_{11}\|_1 < \infty, \|\mathbf{1} - g_{22}\|_1 < \infty\} \tag{6}$$

is a Banach–Lie group with Lie algebra $\mathfrak{g}_1$ and a polar decomposition $G_1 = K_1 \exp(\mathfrak{p})$, where $K_1 := K \cap G_1 \cong \mathrm{U}_1(\mathcal{H})$. From the polar decomposition we obtain in particular $\pi_1(G_1) \cong \pi_1(K_1) \cong \mathbb{Z}$. Hence the universal covering group $\widetilde{G}_1$ contains $\widetilde{K}_1 \cong \widetilde{\mathrm{U}}_1(\mathcal{H}) \cong \mathbb{R} \ltimes \mathrm{SU}(\mathcal{H})$. Now γ integrates to a surjective homomorphism of Lie groups $\gamma_G \colon \widetilde{G}_1 \rtimes K \to \widehat{G}$ which restricts to

$$\gamma_G \colon \widetilde{K}_1 \rtimes K \to \widehat{K} \cong \mathbb{R} \times K, \quad ((t,k),u) \mapsto (t, ku)$$

(cf. [Ne02, Sect. IV]). The kernel of γ_G is isomorphic to $\mathrm{SU}(\mathcal{H})$.

(c) For the negative JH^*-triples of type II/III we write

$$\beta_\pm((x,y),(x',y')) := \langle x, \sigma(y')\rangle \mp \langle x', \sigma(y)\rangle$$

and obtain

$$G = \mathrm{O}(\mathcal{H}^2, \beta_+) \cap \mathrm{U}_{\rm res}(\mathcal{H}^2), \quad \text{resp.,} \quad \mathrm{Sp}(\mathcal{H}^2, \beta_-) \cap \mathrm{U}_{\rm res}(\mathcal{H}^2).$$

In both cases $K = (G^\theta)_0 = \{\mathrm{diag}(g, g^{-\top}) \colon g \in \mathrm{U}(\mathcal{H})\} \cong \mathrm{U}(\mathcal{H})$. We define G_1 as in (5) and obtain $K_1 := G_1 \cap G \cong \mathrm{U}_1(\mathcal{H})$. From the polar decomposition and [Ne02, Prop. III.15], we derive that the inclusion $G_1 \to \mathrm{O}_2(\mathcal{H}^2, \beta_+)$, resp., $G_1 \to \mathrm{Sp}_2(\mathcal{H}^2, \beta_-)$ are weak homotopy equivalences. This shows in particular that $\pi_1(G_1) \cong \mathbb{Z}/2\mathbb{Z}$ for type II and that G_1 is simply connected for type III ([Ne02, Thm. II.14]). In both cases we have a surjective homomorphism of Lie groups $\gamma_G \colon \widetilde{G}_1 \rtimes K \to \widehat{G}$, but the two types lead to different pictures for the subgroup $\widehat{K}_1 = \langle \exp_{\widetilde{G}_1} \mathfrak{k}_1 \rangle$. For type III we have $\widetilde{G}_1 = G_1$ and $\widehat{K}_1 = K_1$, but for type II, the group $\widehat{K}_1$ is the unique 2-fold covering of $K_1 \cong \mathbb{T} \ltimes \mathrm{SU}(\mathcal{H})$, i.e., $\widehat{K}_1 \cong \mathbb{T} \ltimes \mathrm{SU}(\mathcal{H})$ with the universal covering map $q_{G_1} \colon \widetilde{G}_1 \to G_1$ satisfying $q_{G_1}(z,k) = (z^2,k) \in \mathbb{T} \ltimes \mathrm{SU}(\mathcal{H}) \cong K_1$. Put differently, the projection $\sqrt{\det} \colon \widehat{K}_1 \to \mathbb{T}, (z,k) \mapsto z$ is a square root of the pullback of the determinant to $\widehat{K}_1$ which satisfies $\mathbf{L}(\sqrt{\det}) = \frac{1}{2}\,\mathrm{tr}$.

Example 3.7. Type IV: Since $\mathfrak{g}_{\mathbb{C}} = \mathfrak{o}(\mathbb{R}^2, \mathcal{H}_{\mathbb{R}})_{\mathbb{C}} \cong \mathfrak{o}(\mathbb{C}^2 \oplus \mathcal{H})$, with $\mathcal{H} = (\mathcal{H}_{\mathbb{R}})_{\mathbb{C}}$, is the full orthogonal Lie algebra of the complex Hilbert space $\mathbb{C}^2 \oplus \mathcal{H}$, all its central extensions are trivial ([Ne02, Prop. I.7]).

Remark 3.8. (a) From the preceding discussion, we conclude that if $\mathfrak{g}$ is one of the hermitian Lie algebras

$$\mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-), \quad \mathfrak{sp}_{\mathrm{res}}(\mathcal{H}), \quad \mathfrak{o}^*_{\mathrm{res}}(\mathcal{H}) \quad \text{or} \quad \mathfrak{o}(\mathbb{R}^2, \mathcal{H}_{\mathbb{R}}),$$

the canonical central extension $\widehat{\mathfrak{g}}$ is either trivial (type I with $\mathcal{H}_+$ or $\mathcal{H}_-$ finite dimensional, and type IV), or $\widehat{\mathfrak{g}}$ is the universal central extension of $\mathfrak{g}$ (cf. Definition 3.1(d)). The same holds for the c-dual Lie algebras

$$\mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+ \oplus \mathcal{H}_-), \quad \mathfrak{sp}_{\mathrm{res}}(\mathcal{H}^2_{\mathbb{H}}), \quad \mathfrak{o}_{\mathrm{res}}(\mathcal{H}^{\mathbb{R}}) \quad \text{and} \quad \mathfrak{o}(\mathbb{R}^2 \oplus \mathcal{H}_{\mathbb{R}})$$

(cf. Appendix C.3).

(b) If $(\mathfrak{g}, \theta, d)$ is a full hermitian Lie algebra, then $\mathrm{ad}\,\mathfrak{g} \cong \mathfrak{g}(\mathfrak{p})$ holds for the corresponding JH^*-triple $\mathfrak{p}$. If $\widehat{\mathfrak{g}}(\mathfrak{p})$ is the universal central extension of $\mathfrak{g}(\mathfrak{p})$ from (a), then the universal property (cf. Definition 3.1) implies the existence of a morphism $\alpha \colon \widehat{\mathfrak{g}}(\mathfrak{p}) \to \mathfrak{g}$ of central extensions of $\mathfrak{g}(\mathfrak{p})$. Then α maps the finite dimensional center $\mathfrak{z}(\widehat{\mathfrak{g}}(\mathfrak{p}))$ into $\mathfrak{z}(\mathfrak{g})$, and with any decomposition $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z} \oplus \alpha(\mathfrak{z}(\widehat{\mathfrak{g}}(\mathfrak{p})))$, we obtain a direct decomposition $\mathfrak{g} \cong \mathfrak{z} \oplus \alpha(\widehat{\mathfrak{g}}(\mathfrak{p}))$, where $\alpha(\widehat{\mathfrak{g}}(\mathfrak{p})) = [\mathfrak{g}, \mathfrak{g}]$ is the commutator algebra.

For the classification of irreducible semibounded representations it therefore suffices to consider the simply connected Lie group $\widehat{G}(\mathfrak{p})$ with Lie algebra $\widehat{\mathfrak{g}}(\mathfrak{p})$. This is the approach we shall follow in Sections 7 and 8. The existence of $\widehat{G}(\mathfrak{p})$ is a consequence of the following theorem.

Theorem 3.9. *If (G, θ, d) is full, then the Lie algebra $\widehat{\mathfrak{g}} = \mathbb{R} \oplus_{\omega_{\mathfrak{p}}} \mathfrak{g}$ is integrable, i.e., there exists a Lie group $\widehat{G}$ with Lie algebra $\widehat{\mathfrak{g}}$.*

Moreover, $\widehat{\mathfrak{g}}(\mathfrak{p})_{\mathbb{C}}$ is integrable for every simple JH^*-triple $\mathfrak{p}$.

Proof. The Lie algebra $\widehat{\mathfrak{g}}$ is clearly integrable if $[\omega_{\mathfrak{p}}] = 0$, because this implies that $\widehat{\mathfrak{g}} \cong \mathbb{R} \oplus \mathfrak{g}$.

Since the adjoint representation yields a morphism $\mathrm{Ad} \colon G \to G(\mathfrak{p})$ of the simply connected Lie groups with the Lie algebras $\mathfrak{g}$, resp., $\mathfrak{g}(\mathfrak{p})$, it suffices to show that $\mathfrak{g}^\sharp := \mathbb{R} \oplus_{\omega_{\mathfrak{p}}} \mathfrak{g}(\mathfrak{p})$ is integrable. Then the pullback of the corresponding central Lie group extension $G^\sharp$ of $G(\mathfrak{p})$ by Ad is a central extension of G with Lie algebra $\widehat{\mathfrak{g}}$.

We recall that the cocycle $\omega_{\mathfrak{p}}$ is non-trivial for $\mathfrak{g}(\mathfrak{p})$ only if the JH^*-triple $\mathfrak{p}$ is of type I with $\dim \mathcal{H}_\pm = \infty$ or of type II or III (Remark 3.8). We also know from [Ne02, Def. IV.4, Prop. IV.8] that the universal central extension $\widehat{\mathfrak{g}}(\mathfrak{p})$ of $\mathfrak{g}(\mathfrak{p})$ and its complexification are integrable. For Type II/III, the assertion follows from $\widehat{\mathfrak{g}}(\mathfrak{p}) \cong \mathfrak{g}^\sharp$.

For type I with either $\mathcal{H}_+$ or $\mathcal{H}_-$ finite dimensional, we have for $\mathfrak{p}$ positive $\widehat{G}(\mathfrak{p}) \cong \widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-) = \widetilde{\mathrm{U}}(\mathcal{H}_+,\mathcal{H}_-)$ with $\widehat{G}(\mathfrak{p})_{\mathbb{C}} \cong \widetilde{\mathrm{GL}}(\mathcal{H}_+ \oplus \mathcal{H}_-)$, and for type IV we have for $\mathfrak{p}$ positive $\widehat{G}(\mathfrak{p}) \cong \widetilde{\mathrm{O}}(\mathbb{R}^2, \mathcal{H}_{\mathbb{R}})$ with $\widehat{G}(\mathfrak{p})_{\mathbb{C}} \cong \widetilde{\mathrm{O}}(\mathbb{C}^2 \oplus \mathcal{H})$ (Example 3.7).

For Type I with $\mathcal{H}_\pm$ both infinite dimensional and $\mathfrak{p}$ positive, the identity component of the center of the simply connected Lie group $\widehat{G}(\mathfrak{p})$ with Lie algebra $\widehat{\mathfrak{g}}(\mathfrak{p})$ is isomorphic to

$$\mathbb{R} \times \mathbb{T} \cong \mathbb{R} \times \{(t\mathbf{1}, t^{-1}\mathbf{1}) \colon t \in \mathbb{T}\} \subseteq \mathbb{R} \times \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-),$$

where factorization of the circle subgroup yields a group with Lie algebra $\mathbb{R} \oplus_{\omega_{\mathfrak{p}}} \mathfrak{g}(\mathfrak{p})$. If $\mathfrak{p}$ is negative, then

$$Z(\widehat{G}(\mathfrak{p}))_0 \cong \mathbb{T}^2 \cong \mathbb{T} \times \{(t\mathbf{1}, t^{-1}\mathbf{1}) \colon t \in \mathbb{T}\} \subseteq \mathbb{T} \times \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-),$$

where factorization of the second $\mathbb{T}$-factor yields a group with Lie algebra $\mathbb{R} \oplus_{\omega_{\mathfrak{p}}} \mathfrak{g}(\mathfrak{p})$ (cf. the discussion of the corresponding groups in Remark 2.7). The construction of the complex Lie group $\widehat{\mathrm{GL}}_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-) \cong \widehat{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}_+,\mathcal{H}_-)_{\mathbb{C}}$ can be found in [Ne02, Def. IV.4]. □

Remark 3.10. (a) If $[\mathfrak{p},\mathfrak{p}] = \{0\}$, then the assertion of the preceding theorem is true for trivial reasons. In this case $\mathfrak{g} \cong \mathfrak{p} \rtimes \mathfrak{k}$ leads to $\widehat{\mathfrak{g}} \cong \widehat{\mathfrak{p}} \rtimes \mathfrak{k}$, where $\widehat{\mathfrak{p}} = \mathbb{R} \oplus_{\omega_{\mathfrak{p}}} \mathfrak{p}$ is a Heisenberg algebra (cf. Example 1.3(b)). Therefore $\widehat{G} := \mathrm{Heis}(\mathfrak{p}) \rtimes K$ is a corresponding Lie group.

(b) The Lie algebra cocycle $\omega_{\mathfrak{p}}$ defines a G-invariant non-degenerate closed 2-form Ω on M, for which $\Omega_{\mathbf{1}K}$ corresponds to $\omega_{\mathfrak{p}}$ under the natural identification $T_{\mathbf{1}K} \cong \mathfrak{p}$. Here the closedness of this 2-form follows from the closedness of its pullback $q^*\Omega$ to G under the quotient map $G \to G/K = M$ which is a left invariant 2-form on G with $(q^*\Omega)_{\mathbf{1}} = \omega_{\mathfrak{p}}$ (cf. [CE48, Thm. 10.1]).

On the central extension $\widehat{\mathfrak{g}} = \mathbb{R} \oplus_{\omega_{\mathfrak{p}}} \mathfrak{g}$, the coadjoint action of $\widehat{G}$ factors through an action of G, and the linear functional $\lambda(t,x) := t$ satisfies $G_\lambda = K$, so that its orbit $\mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda$ is isomorphic to G/K. Therefore the canonical action of $\widehat{G}$ on the symplectic Kähler manifold (M,Ω) is hamiltonian.

Remark 3.11. Suppose that $\mathfrak{p}$ is a simple infinite dimensional JH^*-triple and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \widehat{\mathfrak{g}}(\mathfrak{p})$ be the universal central extension of $\mathfrak{g}(\mathfrak{p})$ (Remark 3.8). We now exhibit an inclusion of a Banach-Lie algebra $\mathfrak{k}_1 \hookrightarrow \mathfrak{k}$ such that $\mathfrak{g}_1 := \mathfrak{k}_1 \oplus \mathfrak{p}$ carries a natural Banach–Lie algebra structure with a continuous inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$ such that the summation map $\mathfrak{g}_1 \rtimes \mathfrak{k} \to \mathfrak{g}, (x,y) \mapsto x+y$ is a quotient homomorphism. We have the following cases:

(I_∞) $\mathfrak{p} = B_2(\mathcal{H}_-,\mathcal{H}_+)$ with both $\mathcal{H}_\pm$ infinite dimensional,

$$\mathfrak{k} = \mathbb{R} \oplus \mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-), \qquad \mathfrak{k}_1 := \mathfrak{u}_1(\mathcal{H}_+) \oplus \mathfrak{u}_1(\mathcal{H}_-),$$

and $\mathfrak{g}_1 := \mathfrak{k}_1 \oplus \mathfrak{p} \subseteq \mathfrak{gl}(\mathcal{H})$ with the inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}, x \mapsto (-i\,\mathrm{tr}\,x_{11}, x)$.

($\mathrm{I}_{\mathrm{fin}}$) $\mathfrak{p} = B_2(\mathcal{H}_-,\mathcal{H}_+)$ with $\mathcal{H}_+$ or $\mathcal{H}_-$ finite dimensional. Then $\mathfrak{k} = \mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)$, $\mathfrak{k}_1 := \mathfrak{u}_1(\mathcal{H}_+) \oplus \mathfrak{u}_1(\mathcal{H}_-)$ and $\mathfrak{g}_1 := \mathfrak{k}_1 \oplus \mathfrak{p} \subseteq \mathfrak{gl}(\mathcal{H})$ with the canonical inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g} \subseteq \mathfrak{g}(\mathcal{H})$.

(II) $\mathfrak{p} = \mathrm{Skew}_2(\mathcal{H}) = \{A \in B_2(\mathcal{H})\colon A^\top = -A\}$. Then $\mathfrak{k} \cong \mathbb{R} \oplus \mathfrak{u}(\mathcal{H})$, $\mathfrak{k}_1 := \mathfrak{u}_1(\mathcal{H})$ and $\mathfrak{g}_1 := \mathfrak{k}_1 \oplus \mathfrak{p} \subseteq \mathfrak{gl}(\mathcal{H}^2)$ with the inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$ given by $x \mapsto (-i \operatorname{tr} x_{11}, x)$.

(III) $\mathfrak{p} = \mathrm{Sym}_2(\mathcal{H}) = \{A \in B_2(\mathcal{H})\colon A^\top = A\}$. Then $\mathfrak{k} \cong \mathbb{R} \oplus \mathfrak{u}(\mathcal{H})$, $\mathfrak{k}_1 := \mathfrak{u}_1(\mathcal{H})$ and $\mathfrak{g}_1 := \mathfrak{k}_1 \oplus \mathfrak{p} \subseteq \mathfrak{gl}(\mathcal{H}^2)$ with the inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$ given by $x \mapsto (-i \operatorname{tr} x_{11}, x)$.

(IV) $\mathfrak{p} = \mathcal{H}_\mathbb{R}^2 \cong \mathbb{R}^2 \otimes \mathcal{H}_\mathbb{R}$ with a real Hilbert space $\mathcal{H}_\mathbb{R}$. Then $\mathfrak{k} = \mathfrak{so}_2(\mathbb{R}) \oplus \mathfrak{o}(\mathcal{H}_\mathbb{R})$, $\mathfrak{k}_1 := \mathfrak{so}_2(\mathbb{R}) \oplus \mathfrak{o}_1(\mathcal{H}_\mathbb{R})$ and $\mathfrak{g}_1 := \mathfrak{k}_1 \oplus \mathfrak{p} \subseteq \mathfrak{gl}(\mathbb{R}^2 \oplus \mathcal{H}_\mathbb{R})$ with the canonical inclusion $\mathfrak{g}_1 \hookrightarrow \mathfrak{g}$.

4 Open invariant cones in hermitian Lie algebras

If each open convex invariant cone W in a Banach–Lie algebra $\mathfrak{g}$ is trivial, then this holds in particular for the cone W_π of any semibounded unitary representation. As a consequence, every semibounded representation is bounded. Therefore we study in this section criteria for the triviality of open invariant cones in Banach–Lie algebras.

4.1 Lie algebras with no open invariant cones

Definition 4.1. We say that the Banach–Lie algebra $\mathfrak{k}$ has *no open invariant cones* if each non-empty open invariant convex cone $W \subseteq \mathfrak{k}$ coincides with $\mathfrak{k}$.

Lemma 4.2. (a) *If $\mathfrak{n} \trianglelefteq \mathfrak{k}$ is a closed ideal, then $\mathfrak{k}/\mathfrak{n}$ has no open invariant cones if and only if all open invariant cones in $\mathfrak{k}$ intersect $\mathfrak{n}$.*

(b) *If $\mathfrak{n} \trianglelefteq \mathfrak{k}$ is a closed ideal such that neither $\mathfrak{n}$ nor $\mathfrak{k}/\mathfrak{n}$ have open invariant cones, then $\mathfrak{k}$ has no open invariant cones.*

(c) *If $\mathfrak{k} = \bigoplus_{i=1}^n \mathfrak{k}_i$ is a direct sum of Lie algebras with no open invariant cones, then $\mathfrak{k}$ has no open invariant cones.*

Proof. (a) Let $q\colon \mathfrak{k} \to \mathfrak{k}/\mathfrak{n}$ denote the quotient map. If $W \subseteq \mathfrak{k}$ is an open invariant cone intersecting $\mathfrak{n}$ trivially, then $W + \mathfrak{n} = q^{-1}(q(W))$ is a proper open invariant cone in $\mathfrak{k}$, hence $q(W)$ is a proper open invariant cone in $\mathfrak{k}/\mathfrak{n}$.

Now suppose that $\mathfrak{k}/\mathfrak{n}$ has no open invariant cones and let $W \subseteq \mathfrak{k}$ be an open invariant cone. Since $q(W)$ is an open invariant cone in $\mathfrak{k}/\mathfrak{n}$, it contains 0, which means that $W \cap \mathfrak{n} \neq \emptyset$.

(b) Let $\emptyset \neq W \subseteq \mathfrak{k}$ be an open invariant cone. From (a) we derive that $W \cap \mathfrak{n}$ is a non-empty open invariant cone in $\mathfrak{n}$. As $\mathfrak{n}$ contains no open invariant cones, we obtain $0 \in W \cap \mathfrak{n} \subseteq W$, and thus $W = \mathfrak{k}$.

(c) follows from (b) by induction. □

Example 4.3. If K is a connected Lie group with compact Lie algebra, then $\mathrm{Ad}(K)$ is compact, so that averaging with respect to Haar measure implies that every open invariant cone $\emptyset \neq W \subseteq \mathfrak{k}$ intersects $\mathfrak{z}(\mathfrak{k})$. In particular, $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ has no open invariant cones because it is semisimple.

Below we shall see that there is an interesting family of infinite dimensional Banach–Lie algebras behaving very much like compact ones. An important point of the triviality of all open invariant cones is that this property permits us to draw conclusions concerning boundedness of unitary representations, such as the following. For the definition of boundedness and semiboundedness of a unitary representation, we refer to the introduction.

Proposition 4.4. *Let K be a Lie group for which $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ has no open invariant cone. If (ρ, V) is a semibounded unitary representation of K for which $\rho|_{Z(K)_0}$ is bounded, then (ρ, V) is bounded. In particular, every irreducible semibounded representation of K is bounded.*

Proof. Since ρ is semibounded, the open cone $W_\rho \subseteq \mathfrak{k}$ is non-trivial, and Lemma 4.2(a) implies that $W_\rho \cap \mathfrak{z}(\mathfrak{k}) \neq \emptyset$. The assumption that $\rho|_{Z(K)_0}$ is bounded further implies that $\mathfrak{z}(\mathfrak{k}) + W_\rho = W_\rho$, which leads to $0 \in W_\rho$, hence to $W_\rho = \mathfrak{k}$, i.e., ρ is bounded.

If, in addition, ρ is irreducible, then $\rho(Z(K)) \subseteq \mathbb{T}\mathbf{1}$ follows from Schur's Lemma, and the preceding argument implies that ρ is bounded. □

4.2 Open invariant cones in unitary Lie algebras

In this section we show that, for any real, complex or quaternionic Hilbert space $\mathcal{H}$, the Lie algebra $\mathfrak{u}(\mathcal{H})/\mathfrak{z}(\mathfrak{u}(\mathcal{H}))$ has no open invariant cones. Note that the center $\mathfrak{z}(\mathfrak{u}(\mathcal{H}))$ is only non-trivial for complex Hilbert spaces, where it is one-dimensional, and for 2-dimensional real Hilbert spaces.

Lemma 4.5. *Let J be a set and S_J be the group of all permutations of J acting canonically on $\ell^\infty(J, \mathbb{R})$. Then every S_J-invariant non-empty open convex cone in $\ell^\infty(J, \mathbb{R})$ contains a constant function.*

Proof. **Step 1:** If J is finite, then S_J is finite, so that the assertion follows by averaging. We may therefore assume that J is infinite. Let $\emptyset \neq W \subseteq \ell^\infty(J, \mathbb{R})$ be an open S_J-invariant convex cone. Then W contains a constant function if and only if $0 \in W + \mathbb{R}\mathbf{1}$, where $\mathbf{1}$ stands for the constant function 1. We may therefore assume that $W = W + \mathbb{R}\mathbf{1}$ and show that $0 \in W$.

Step 2: Since the functions x with only finitely many values are dense in $\ell^\infty(J, \mathbb{R})$, the cone W contains such an element. Let $J = J_0 \cup \cdots \cup J_N$ be the corresponding decomposition of J for which $x|_{J_k}$ is constant x_k for $k = 1, \ldots, N$. We write this as

$$x = \sum_{k=0}^{N} x_k \chi_{J_k},$$

where χ_{J_k} is the characteristic function of J_k. Since J is infinite, there exists a k with $|J_k| = |J|$. We may w.l.o.g. assume that $k = 0$ and observe that $y := x - x_0\mathbf{1} \in W$.

Fix $k > 0$ and put $J_k' := J \setminus (J_0 \cup J_k)$. As $|J_0| = |J_0 \times \mathbb{N}|$, there exist pairwise disjoint subsets $J_k^n \subseteq J_0$, $n \in \mathbb{N}$, and involutions $\sigma_n \in S_J$ with

$$\sigma_n(J_k') = J_k^n, \quad \sigma_n(J_k^n) = J_k' \quad \text{and} \quad \sigma_n|_{J_k \cup J_0 \setminus J_k^n} = \mathrm{id}\,.$$

Then $y^M := \frac{1}{M}\sum_{n=1}^{M} y \circ \sigma_n$ coincides with y on J_k, and on $J_0 \cup J_k'$ it is bounded by $\frac{1}{M}\|y\|_\infty$. Therefore $y_k \chi_{J_k} = \lim_{M\to\infty} y^M \in \overline{W}$ holds for $k = 1, \ldots, N$.

Step 3: Let $M \subseteq J$ be any subset with $|J \setminus M| = |J|$ and observe that there exists a subset $M' \subseteq J_0$ with the same cardinality for which $J_0 \setminus M'$ still has the same cardinality as J. Since W is open, there exists an $\varepsilon > 0$ with $z := y \pm \varepsilon\chi_{M'} \in W$ and $\varepsilon < |y_j|$ whenever $y_j \neq 0$. Then the argument from Step 2 applies to z instead of y, which leads to $\pm\varepsilon\chi_{M'} \in \overline{W}$. We conclude that $\chi_{M'} \in H(\overline{W}) = H(W)$. As

$$(|M|, |J \setminus M|) = (|M'|, |J \setminus M'|),$$

there exists a permutation $\sigma \in S_J$ with $\sigma(M) = M'$, so that we also obtain $\chi_M \in H(W)$.

Step 4: From $|J \setminus J_k| \geq |J_0| = |J|$, we further derive that $\chi_{J_k} \in H(W)$ for $k = 1, \ldots, N$, and this eventually leads to $0 = y - \sum_{k=1}^{N} y_k \chi_{J_k} \in W$. □

Theorem 4.6. *If $\mathcal{H}$ is a complex Hilbert space, then the Banach–Lie algebra $\mathfrak{pu}(\mathcal{H}) = \mathfrak{u}(\mathcal{H})/\mathbb{R}i\mathbf{1}$ has no non-trivial open invariant cones, i.e., each non-empty open invariant cone in $\mathfrak{u}(\mathcal{H})$ intersects $\mathbb{R}i\mathbf{1}$.*

Proof. If $\dim \mathcal{H} < \infty$, then the assertion follows from the fact that $\mathfrak{pu}(\mathcal{H})$ is a compact Lie algebra with trivial center. We may therefore assume that $\dim \mathcal{H} = \infty$ and that $W \subseteq \mathfrak{u}(\mathcal{H})$ is an open invariant convex cone.

Since the elements with finite spectrum are dense in $\mathfrak{u}(\mathcal{H})$ by the Spectral Theorem, W contains such an element X. Let $(e_j)_{j \in J}$ be an orthonormal basis of $\mathcal{H}$ consisting of eigenvectors for X. Then the space $\mathfrak{t}$ of all diagonal operators in $\mathfrak{u}(\mathcal{H})$ with respect to (e_j) is isomorphic to $\ell^\infty(J, i\mathbb{R})$, and since any permutation σ of the basis vectors corresponds to an element $u_\sigma \in \mathrm{U}(\mathcal{H})$, the invariance of W implies that the cone $W \cap \mathfrak{t} \subseteq \mathfrak{t} \cong \ell^\infty(J, i\mathbb{R})$ is invariant under the action of S_J. It is non-empty because it contains X. Therefore Lemma 4.5 implies that $W \cap i\mathbb{R}\mathbf{1} \neq \emptyset$. □

Theorem 4.7. *If $\mathcal{H}_\mathbb{R}$ is a real Hilbert space with $\dim \mathcal{H}_\mathbb{R} > 2$, then the Banach–Lie algebra $\mathfrak{o}(\mathcal{H}_\mathbb{R})$ has no non-trivial open invariant cones.*

Proof. If $2 < n := \dim \mathcal{H}_\mathbb{R} < \infty$, then $\mathfrak{o}(\mathcal{H}_\mathbb{R}) \cong \mathfrak{so}_n(\mathbb{R})$ is a compact semisimple Lie algebra. Hence every open invariant convex cone in $\mathfrak{o}(\mathcal{H}_\mathbb{R})$ contains the only fixed point 0 of the adjoint action, hence cannot be proper.

We now assume that $\dim \mathcal{H}_\mathbb{R} = \infty$ and choose an orthogonal complex structure I on $\mathcal{H}_\mathbb{R}$ (cf. Example C.6). This complex structure defines on $\mathcal{H}_\mathbb{R}$ the structure of a complex Hilbert space $\mathcal{H} := (\mathcal{H}_\mathbb{R}, I)$.

Let $W \subseteq \mathfrak{o}(\mathcal{H}_\mathbb{R})$ be a non-empty open convex invariant cone and $p \colon \mathfrak{o}(\mathcal{H}_\mathbb{R}) \to \mathfrak{u}(\mathcal{H})$ be the fixed point projection for the circle group $e^{\mathbb{R}\,\mathrm{ad}\, I}$. Then $p(W) = W \cap \mathfrak{u}(\mathcal{H})$ is a non-empty open invariant cone in $\mathfrak{u}(\mathcal{H})$ ([Ne10c, Prop. 2.11]), so that Theorem 4.6 implies that $\lambda I \in W$ holds for some $\lambda \in \mathbb{R}$.

Let $J \colon \mathcal{H} \to \mathcal{H}$ be an anticonjugation (cf. Appendix C.1). Then $J \in \mathfrak{o}(\mathcal{H}_\mathbb{R})$ is a second complex structure anticommuting with I. This leads to $[J, I] = JI - IJ = 2JI$, and hence to $e^{\frac{\pi}{2}\,\mathrm{ad}\, J} I = e^{\pi J} I = -I$. This shows that $\pm I \in W$, which leads to $0 \in W$, and finally to $W = \mathfrak{o}(\mathcal{H}_\mathbb{R})$. □

Theorem 4.8. *If $\mathcal{H}$ is a quaternionic Hilbert space, then the Banach–Lie algebra $\mathfrak{sp}(\mathcal{H}) = \mathfrak{u}_\mathbb{H}(\mathcal{H})$ has no non-trivial open invariant cones.*

Proof. If $\dim \mathcal{H} < \infty$, then $\mathfrak{sp}(\mathcal{H})$ is a compact simple Lie algebra, so that every open invariant cone contains 0, hence cannot be proper.

Let $\mathcal{H}^{\mathbb{C}}$ denote the complex Hilbert space underlying $\mathcal{H}$ and note that $\mathcal{H}^{\mathbb{C}}$ can be written as $\mathcal{K}^2$, where $\mathcal{K}$ is a complex Hilbert space and the quaternionic structure on $\mathcal{H}^{\mathbb{C}}$ is given in terms of a conjugation τ on $\mathcal{K}$ by the anticonjugation $\sigma(v, w) := (-\tau w, \tau v)$ on $\mathcal{K}^2$. Then

$$\mathfrak{sp}(\mathcal{H}) = \{x \in \mathfrak{u}(\mathcal{K}^2) \colon \sigma x = x\sigma\}$$

contains the operator $d(v, w) := (iv, -iw)$ whose centralizer is

$$\{x \in \mathfrak{sp}(\mathcal{H}) \subseteq \mathfrak{gl}(\mathcal{K}^2) \colon x_{12} = x_{21} = 0\} = \{\mathrm{diag}(x, -x^\top) \colon x \in \mathfrak{u}(\mathcal{K})\} \cong \mathfrak{u}(\mathcal{K}).$$

Let $W \subseteq \mathfrak{sp}(\mathcal{H})$ be a non-empty open convex invariant cone and $p \colon \mathfrak{sp}(\mathcal{H}) \to \mathfrak{u}(\mathcal{K})$ be the fixed point projection for the circle group $e^{\mathbb{R}\,\mathrm{ad}\,d}$. Then $p(W) = W \cap \mathfrak{u}(\mathcal{H}, I)$ is a non-empty open invariant cone in $\mathfrak{u}(\mathcal{K})$ ([Ne10c, Prop. 2.11]), so that Theorem 4.6 implies that $\lambda d \in W$ holds for some $\lambda \in \mathbb{R}$.

The Lie algebra $\mathfrak{sp}(\mathcal{H})$ also contains the operator $K(v, w) := (-w, v)$ because it commutes with σ. Hence W is also invariant under $e^{\mathbb{R}\,\mathrm{ad}\,K}$. From $Kd = -dK$ it follows that

$$[K, [K, d]] = -2[K, dK] = -2[K, d]K = 4dK^2 = -4d,$$

so that $e^{\frac{\pi}{2}\,\mathrm{ad}\,K} d = -d$ leads to $0 \in W$, and finally to $W = \mathfrak{sp}(\mathcal{H})$. □

4.3 Applications to hermitian Lie algebras

Lemma 4.9. *Let $\mathfrak{k}$ be a Banach–Lie algebra for which $\mathfrak{k}/\mathfrak{z}(\mathfrak{k}) = \oplus_{j=1}^n \mathfrak{k}_j$, where each $\mathfrak{k}_j$ is either compact semisimple or isomorphic to $\mathfrak{u}(\mathcal{H})/\mathfrak{z}(\mathfrak{u}(\mathcal{H}))$ for a real, complex or quaternionic Hilbert space $\mathcal{H}$. Then every non-empty open invariant convex cone $W \subseteq \mathfrak{k}$ intersects $\mathfrak{z}(\mathfrak{k})$.*

Proof. In view of Theorems 4.6 and 4.7, 4.8 and Example 4.3, Lemma 4.2(c) implies that $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones. Hence the assertion follows from Lemma 4.2(a). □

Lemma 4.10. *If $(\mathfrak{g}, \theta, d)$ is a full irreducible hermitian Lie algebra, then $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones.*

Proof. In view of Lemma 4.9, we only have to show that $\mathfrak{k}/\mathfrak{z}(\mathfrak{k}) \cong \mathfrak{aut}(\mathfrak{p})/\mathfrak{z}(\mathfrak{aut}(\mathfrak{p}))$ (cf. (3)) has the required structure. This is an easy consequence of Theorem 2.6:

Type I: $\mathfrak{aut}(\mathfrak{p}) \cong (\mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-))/\mathbb{R}i(\mathbf{1}, \mathbf{1})$ implies $\mathfrak{k}/\mathfrak{z}(\mathfrak{k}) \cong \mathfrak{pu}(\mathcal{H}_+) \oplus \mathfrak{pu}(\mathcal{H}_-)$.

Type II/III: $\mathfrak{k}/\mathfrak{z}(\mathfrak{k}) \cong \mathfrak{pu}(\mathcal{H})$.

Type IV: $\mathfrak{k}/\mathfrak{z}(\mathfrak{k}) \cong \mathfrak{o}(\mathcal{H}_{\mathbb{R}})$. □

Proposition 4.11. *If $(\mathfrak{g}, \theta, d)$ is a hermitian Lie algebra for which $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones, then every open invariant cone $\emptyset \neq W \subseteq \mathfrak{g}$ intersects $\mathfrak{z}(\mathfrak{k})$.*

Proof. Let $p_{\mathfrak{k}} \colon \mathfrak{g} \to \mathfrak{k}$ denote the projection along $\mathfrak{p}$. This is the fixed point projection with respect to the action of the circle group $e^{\mathbb{R} \operatorname{ad} d} \subseteq \operatorname{Aut}(\mathfrak{g})$, so that

$$p_{\mathfrak{k}}(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{t \operatorname{ad} d} x \, dt$$

implies that $p_{\mathfrak{k}}(W) = W \cap \mathfrak{k}$ (cf. [Ne10c, Prop. 2.11]). We conclude that $W \cap \mathfrak{k}$ is a non-empty open invariant cone in $\mathfrak{k}$ which intersects $\mathfrak{z}(\mathfrak{k})$ by Lemma 4.2(a). $\square$

5 Semiboundedness and positive energy

In this section we start with our analysis of semibounded unitary representations of hermitian Lie groups. If (G, θ, d) is irreducible, we first show that each irreducible semibounded representation $(\pi, \mathcal{H})$ satisfies $d \in W_\pi \cup -W_\pi$. If $d \in W_\pi$, then we call $(\pi, \mathcal{H})$ a *positive energy representation.* In this case the maximal spectral value of the essentially selfadjoint operator $i\mathtt{d}\pi(d)$ is an eigenvalue and the K-representation (ρ, V) on the corresponding eigenspace is bounded and irreducible. Since $(\pi, \mathcal{H})$ is uniquely determined by (ρ, V), we call a bounded representation (ρ, V) of K *(holomorphically) inducible* if it corresponds in this way to a unitary representation $(\pi, \mathcal{H})$ of G. The classification of the holomorphically inducible representations (ρ, V) is carried out for flat, negatively and positively curved spaces G/K in Sections 6-8.

5.1 From semiboundedness to positive energy

From the introduction we recall the concept of a semibounded unitary representation and the corresponding open invariant cone W_π.

Definition 5.1. Let (G, θ, d) be a connected hermitian Banach–Lie group.

(a) A unitary representation $(\pi, \mathcal{H})$ of a hermitian Lie group G is called a *positive energy representation* if $\sup \operatorname{Spec}(i\mathtt{d}\pi(d)) < \infty$, i.e., if the infinitesimal generator $-i\mathtt{d}\pi(d)$ of the one-parameter group $\pi(\exp td)$ is bounded from below. Note that this requirement is weaker than the condition $d \in W_\pi$.

(b) We recall that $\mathfrak{k}_{\mathbb{C}} + \mathfrak{p}^-$ is a closed subalgebra of $\mathfrak{g}_{\mathbb{C}}$ defining a complex structure on the homogeneous space $M := G/K$. Let (ρ, V) be a bounded representation of K and define a bounded Lie algebra representation $\beta \colon \mathfrak{p}^+ \rtimes \mathfrak{k}_{\mathbb{C}} \to B(V)$ by $\beta(\mathfrak{p}^+) = \{0\}$ and $\beta|_{\mathfrak{k}} = \mathtt{d}\rho$ (cf. (1)). Then the associated Hilbert bundle $\mathbb{V} = G \times_K V$ over G/K carries the structure of a holomorphic bundle on which G acts by holomorphic bundle automorphisms ([Ne10c, Thm. I.6]). A unitary representation $(\pi, \mathcal{H})$ of G is said to be *holomorphically induced from* (ρ, V) if there exists a G-equivariant realization $\Psi \colon \mathcal{H} \to \Gamma(\mathbb{V})$ as a Hilbert space of holomorphic sections such that the evaluation $\operatorname{ev}_{\mathbf{1}K} \colon \mathcal{H} \to V = \mathbb{V}_{\mathbf{1}K}$ defines an isometric embedding $\operatorname{ev}_{\mathbf{1}K}^* \colon V \hookrightarrow \mathcal{H}$. If a unitary representation $(\pi, \mathcal{H})$ holomorphically induced from (ρ, V) exists, then it is uniquely determined ([Ne10d, Def. 2.10]) and we call (ρ, V) *(holomorphically) inducible.*

This concept of inducibility involves a choice of sign. Replacing d by $-d$ changes the complex structure on G/K and exchanges $\mathfrak{p}^+$ with $\mathfrak{p}^-$.

Theorem 5.2. *Let $(\pi, \mathcal{H})$ be a semibounded representation of the hermitian Lie group G for which $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones. Then the following assertions hold:*

(i) *$\pi|_{Z(K)_0}$ is semibounded and $W_\pi \cap \mathfrak{z}(\mathfrak{k}) \neq \emptyset$.*

(ii) *π is a direct sum of representations which are bounded on $Z(G)$.*

(iii) *If π is bounded on $Z(G)_0$ and (G, θ, d) is irreducible, then $d \in W_\pi \cup -W_\pi$. If $d \in W_\pi$, then π is a positive energy representation, and if $d \in -W_\pi$, then the dual representation π^* is.*

(iv) *If (G, θ, d) is irreducible and π is irreducible, then $d \in W_\pi \cup -W_\pi$.*

Proof. (i) The representation $\zeta := \pi|_{Z(K)_0}$ satisfies $W_\zeta \supseteq W_\pi \cap \mathfrak{z}(\mathfrak{k})$, and Proposition 4.11 implies that the latter open cone is non-empty.

(ii) From (i) it follows that $\pi|_{Z(K)_0}$ is semibounded, hence given by a regular spectral measure on the dual space $\mathfrak{z}(\mathfrak{k})'$ ([Ne09, Thm. 4.1]). Now the regularity of the spectral measure implies (ii).

(iii) If π is bounded on $Z(G)_0$, then $\mathfrak{z}(\mathfrak{g}) + W_\pi = W_\pi$, so that

$$\emptyset \neq W_\pi \cap \mathfrak{z}(\mathfrak{k}) = \mathfrak{z}(\mathfrak{g}) \oplus (W_\pi \cap \mathbb{R}d)$$

follows from $\mathfrak{z}(\mathfrak{k}) = \mathbb{R}d \oplus \mathfrak{z}(\mathfrak{g})$ (Remark 1.2(d)). This proves (iii). If $d \in W_\pi$, then π is a positive energy representation and otherwise π^* with $W_{\pi^*} = -W_\pi$ has this property.

(iv) If π is irreducible, then $\pi(Z(G)) \subseteq \mathbb{T}\mathbf{1}$ follows from Schur's Lemma. In particular, $\pi|_{Z(G)}$ is bounded, so that (iii) applies. □

Lemma 5.3. *Suppose that (G, θ, d) is a hermitian Lie group and that $(\pi, \mathcal{H})$ is a smooth unitary representation of G. Then the following assertions hold:*

(a) *$\mathcal{H}$ decomposes into $Z(K)_0$-eigenspaces, which coincide with the $\exp(\mathbb{R}d)$-eigenspaces*

$$\mathcal{H}_\lambda = \{v \in \mathcal{H} \colon (\forall t \in \mathbb{R})\, \pi(\exp td)v = e^{it\lambda}v\}, \quad \lambda \in \mathbb{R}.$$

(b) *The corresponding projections $P_\lambda \colon \mathcal{H} \to \mathcal{H}_\lambda$ preserve the space $\mathcal{H}^\infty$ of smooth vectors. In particular, $\mathcal{H}^\infty \cap \mathcal{H}_\lambda \supseteq P_\lambda(\mathcal{H}^\infty)$ is dense in $\mathcal{H}_\lambda$ for every λ.*

(c) *If $\lambda := \sup \mathrm{Spec}(i\mathtt{d}\pi(d)) < \infty$, i.e., $(\pi, \mathcal{H})$ is a positive energy representation, then $\{0\} \neq \mathcal{H}^\infty \cap \mathcal{H}_\lambda \subseteq (\mathcal{H}^\infty)^{\mathfrak{p}^-}$.*

Proof. (a) Since π is irreducible, $\pi(Z(G)) \subseteq \mathbb{T}\mathbf{1}$ follows from Schur's Lemma. We consider the representation of the abelian group

$$Z(K)_0 = \exp(\mathfrak{z}(\mathfrak{k})) = Z(G)_0 \exp(\mathbb{R}d)$$

on $\mathcal{H}$ (Remark 2.5). From $\pi(Z(G)) \subseteq \mathbb{T}\mathbf{1}$ and $\exp(2\pi d) \in Z(G)$, we derive that

$$\pi(Z(K)_0) = \pi(Z(G)_0)\pi(\exp([0, 2\pi]d)) \subseteq \mathbb{T}\pi(\exp([0, 2\pi]d))$$

is relatively compact in the strong operator topology, so that $\mathcal{H}$ is an orthogonal direct sum of $Z(K)_0$-eigenspaces which coincide with the $\exp(\mathbb{R}d)$-eigenspaces. Suppose that $\pi(\exp 2\pi d) = e^{2\pi i\mu_0}\mathbf{1}$, so that $\mu - \mu_0 \in \mathbb{Z}$ whenever $\mathcal{H}_\mu \neq \{0\}$. Then

$$P_\mu v := \frac{1}{2\pi}\int_0^{2\pi} e^{-it\mu}\pi(\exp td)v\,dt$$

is the orthogonal projection onto $\mathcal{H}_\mu$.

(b) All these projections preserve $\mathcal{H}^\infty$ ([Ne10d, Prop. 3.4(i)]), so that $P_\mu(\mathcal{H}^\infty) = \mathcal{H}^\infty \cap \mathcal{H}_\mu$ is dense in $\mathcal{H}_\mu = P_\mu(\mathcal{H})$.

(c) If $\lambda := \sup(\mathrm{Spec}(i\mathtt{d}\pi(d)))$ is finite and $v \in \mathcal{H}_\lambda$ any smooth vector, then $\mathtt{d}\pi(\mathfrak{p}^-)v \in \mathcal{H}_{\lambda+1} = \{0\}$. □

Theorem 5.4. *Suppose that (G, θ, d) is a hermitian Lie group for which $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones. For any irreducible semibounded positive energy representation $(\pi, \mathcal{H})$ of G, the K-representation ρ on $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$ is bounded and irreducible, and $(\pi, \mathcal{H})$ is holomorphically induced from (ρ, V).*

Proof. The smoothness of ρ follows from the density of $V \cap \mathcal{H}^\infty$ in V. Since π is semibounded, the corresponding cone W_π is non-empty, and we know from Proposition 4.11 that $W_\pi \cap \mathfrak{z}(\mathfrak{k}) \neq \emptyset$. We conclude that the open cone $W_\rho \supseteq W_\pi \cap \mathfrak{k}$ also intersects $\mathfrak{z}(\mathfrak{k})$.

From Lemma 5.3(c) we know that $V_\lambda := V \cap \mathcal{H}_\lambda \neq \{0\}$. Since $V^\infty := V \cap \mathcal{H}^\infty$ is dense in V and P_λ-invariant, the subspace $V_\lambda^\infty = P_\lambda(V^\infty)$ is dense in V_λ. The representation ρ_λ of K on V_λ satisfies $\rho_\lambda(Z(K)_0) \subseteq \mathbb{T}\mathbf{1}$, which implies that $\mathfrak{z}(\mathfrak{k}) + W_{\rho_\lambda} = W_{\rho_\lambda}$. As the open invariant cone W_{ρ_λ} in $\mathfrak{k}$ also intersects $\mathfrak{z}(\mathfrak{k})$ (Lemma 4.9), we obtain $W_{\rho_\lambda} = \mathfrak{k}$, i.e., ρ_λ is bounded.

Now we apply [Ne10d, Thm. 2.17] to the closed subspace $V_\lambda \subseteq \mathcal{H}$ and the representation of $\mathfrak{p}^+ \rtimes \mathfrak{k}_\mathbb{C}$ on V_λ defined by $\beta(\mathfrak{p}^+) = \{0\}$. We derive in particular that $(\pi, \mathcal{H})$ is holomorphically induced from $(\rho_\lambda, V_\lambda)$ and hence that the irreducibility of π implies the irreducibility of ρ_λ ([Ne10d, Cor. 2.14]).

We obtain in particular $V_\lambda \subseteq \mathcal{H}^\omega$, so that $U(\mathfrak{g})_\mathbb{C}V_\lambda^\infty$ spans a dense subspace of $\mathcal{H}$. Next we use the PBW Theorem to see that

$$U(\mathfrak{g}_\mathbb{C})V_\lambda^\infty = U(\mathfrak{p}^+)U(\mathfrak{k}_\mathbb{C})U(\mathfrak{p}^-)V_\lambda^\infty = U(\mathfrak{p}^+)V_\lambda^\infty,$$

and the eigenvalues of $-i\mathtt{d}\pi(d)$ on this space are contained in $\lambda + \mathbb{N}_0$. In particular, λ is the minimal eigenvalue of $-i\mathtt{d}\pi(d)$ on $\mathcal{H}$. Since the same argument applies to all other eigenvalues of $-i\mathtt{d}\pi(d)$ in V, they are equal, so that $V = V_\lambda$. □

Remark 5.5. In view of Theorem 5.4, every irreducible semibounded representation $(\pi, \mathcal{H})$ of G is determined by the bounded irreducible K-representation (ρ, V). According to [Ne10d, Cor. 2.16], two such G-representations $(\pi_j, \mathcal{H}_j)$, $j = 1, 2$, are equivalent if and only if the corresponding K-representations (ρ_j, V_j) are equivalent. Therefore the classification of the separable semibounded irreducible G-representations means to determine the inducible bounded irreducible representations (ρ, V) of K.

The following theorem follows from [Ne10d, Thm. 3.7, Thm. 3.14]. In particular, it provides a criterion for semiboundedness of a G-representation in terms of boundedness of a K-representation plus a positive energy condition.

Theorem 5.6. *If $(\pi, \mathcal{H})$ is a smooth positive energy representation for which the K-representation ρ on $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$ is bounded, then it is holomorphically induced from (ρ, V) and it is semibounded with $d \in W_\pi$.*

Proposition 5.7. *If the irreducible hermitian Lie group (G, θ, d) has a bounded unitary representation with $\mathtt{d}\pi(\mathfrak{p}) \neq \{0\}$, then the JH^*-triple $\mathfrak{p}$ is negative.*

Proof. Let $(\pi, \mathcal{H})$ be a bounded unitary representation with $\mathtt{d}\pi(\mathfrak{p}) \neq \{0\}$. Then $\ker \mathtt{d}\pi \trianglelefteq \mathfrak{g}$ is a closed ideal, so that the simplicity of the JH^*-triple $\mathfrak{p}$ implies that $\mathfrak{p} \cap \ker \mathtt{d}\pi = \{0\}$, hence that $\ker \mathtt{d}\pi \subseteq \ker \mathrm{ad}_{\mathfrak{p}} = \mathfrak{z}(\mathfrak{g})$. We may therefore assume that $\mathtt{d}\pi$ is faithful, so that $\|x\| := \|\mathtt{d}\pi(x)\|$ defines an invariant norm on $\mathfrak{g}$.

Since (G, θ, d) is irreducible, either $\mathfrak{p}$ is flat, positive or negative. If $\mathfrak{p}$ is flat, then $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}]$ is a 2-step nilpotent Lie subalgebra and $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] + \mathbb{R}d$ is solvable and locally finite, i.e., every finite subset generates a finite dimensional subalgebra. For a finite dimensional Lie algebra, the existence of an invariant norm implies that it is compact. If it is also solvable, it must be abelian, but $\mathfrak{p} + [\mathfrak{p}, \mathfrak{p}] + \mathbb{R}d$ is non-abelian. Therefore $\mathfrak{p}$ is not flat.

If $\mathfrak{p}$ is positive, then [Ka83, Lemma 3.3] implies the existence of an element $z \in \mathfrak{p}$ which is a tripotent for the Jordan triple structure, i.e., $\{z, z, z\} = z$. Then $\mathfrak{g}_z := \mathbb{R}z + \mathbb{R}[d, z] + \mathbb{R}[z, [d, z]]$ is a θ-invariant subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. This contradicts the existence of an invariant norm. Therefore $\mathfrak{p}$ is negative. □

Remark 5.8. If $(\pi, \mathcal{H})$ is a semibounded representation with $d \in W_\pi$, then the boundedness of π is equivalent to the boundedness of the operator $\mathtt{d}\pi(d)$. In fact, the boundedness of $\mathtt{d}\pi(d)$ implies that $d \in W_\pi \cap -\overline{W_\pi}$, hence that $W_\pi \cap -W_\pi \neq \emptyset$. But this implies that $0 \in W_\pi$, i.e., $W_\pi = \mathfrak{g}$.

5.2 Necessary conditions for inducibility

In this subsection we discuss some necessary conditions for inducibility of a bounded representation (ρ, V) of K. To evaluate these conditions in concrete cases, we also express them in terms of root decompositions.

Definition 5.9. (a) Let $C_{\mathfrak{p}} \subseteq \mathfrak{k}$ be the closed convex cone generated by the elements of the form $i[z^*, z]$, $z \in \mathfrak{p}^+$.

(b) For a convex cone $C \subseteq \mathfrak{k}$, a smooth unitary representation (ρ, V) of K is said to be *C-dissipative* if $i\mathtt{d}\rho(y) \leq 0$ for $y \in C$.

Remark 5.10. (a) Since its set of generators is $\mathrm{Ad}(K)$-invariant, the cone $C_{\mathfrak{p}} \subseteq \mathfrak{k}$ is $\mathrm{Ad}(K)$-invariant. In general it does not have any interior points. For the hermitian Lie algebras $\mathfrak{g} = \mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$, $\mathfrak{sp}_{\mathrm{res}}(\mathcal{H})$ and $\mathfrak{o}^*_{\mathrm{res}}(\mathcal{H})$, the cone $C_{\mathfrak{p}}$ lies in the subspace of compact operators in $\mathfrak{k}$, so that it never has interior points with respect to the operator norm.

(b) Passing from $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ to the c-dual symmetric Lie algebra $\mathfrak{g}^c = \mathfrak{k}+i\mathfrak{p} \subseteq \mathfrak{g}_{\mathbb{C}}$ does not change the subspaces $\mathfrak{p}^{\pm} \subseteq \mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^c_{\mathbb{C}}$, but for $z \in \mathfrak{p}_{\mathbb{C}}$ the corresponding element z^* changes sign. This leads to $C_{i\mathfrak{p}} = -C_{\mathfrak{p}}$.

(c) The elements in $\mathfrak{p}^+$ can also be written as $x - iIx$, $x \in \mathfrak{p}$, where $I = \mathrm{ad}_{\mathfrak{p}} d$ is the canonical complex structure on $\mathfrak{p}$. We then have

$$i[(x - iIx)^*, x - iIx] = i[-x - iIx, x - iIx] = 2[Ix, x].$$

Lemma 5.11. *If the K-representation (ρ, V) is inducible, then*

$$\mathtt{d}\rho([z^*, z]) \geq 0 \quad \textit{for} \quad z \in \mathfrak{p}^+, \tag{7}$$

i.e., (ρ, V) is $C_{\mathfrak{p}}$-dissipative.

Proof. If the G-representation $(\pi, \mathcal{H}_V)$ is holomorphically induced from (ρ, V), then we have for $v \in (\mathcal{H}^\infty)^{\mathfrak{p}^-}$ and $z \in \mathfrak{p}^+$ the relation

$$\langle \mathtt{d}\rho([z^*, z])v, v\rangle = \langle [\mathtt{d}\pi(z^*), \mathtt{d}\pi(z)]v, v\rangle = \langle \mathtt{d}\pi(z^*)\mathtt{d}\pi(z)v, v\rangle = \|\mathtt{d}\pi(z)v\|^2 \geq 0.$$

Since $(\mathcal{H}^\infty)^{\mathfrak{p}^-}$ is dense in V and $\mathtt{d}\rho([z^*, z])$ is bounded, this proves (7). $\square$

Example 5.12. Consider the full hermitian Lie algebra $\mathfrak{g} = \mathfrak{u}_{\mathrm{res}}(\mathcal{H})$, where the involution is obtained from a decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ and the element

$$d := \frac{i}{2}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}.$$

Then $\mathfrak{k} = \mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)$ and $\mathfrak{p}^+ = \begin{pmatrix} 0 & B_2(\mathcal{H}_-, \mathcal{H}_+) \\ 0 & 0 \end{pmatrix}$. For $z \in B_2(\mathcal{H}_-, \mathcal{H}_+)$ we have

$$\left[\begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ z^* & 0 \end{pmatrix}\right] = \begin{pmatrix} zz^* & 0 \\ 0 & -z^*z \end{pmatrix}.$$

Therefore

$$C_{\mathfrak{p}} \subseteq \{(a, d) \in \mathfrak{k} \colon i \cdot a \geq 0, -i \cdot d \geq 0\}.$$

(b) For the c-dual Lie algebra $\mathfrak{g}^c = \mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$, we find that

$$C_{\mathfrak{p}} \subseteq \{(a, d) \in \mathfrak{k} \colon -ia \geq 0, id \geq 0\}.$$

The most effective way to draw further consequences from the necessary condition Lemma 5.11 is to use root decompositions (see Appendix C.2 for detailed definitions).

Lemma 5.13. *Suppose that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is hermitian, $\mathfrak{t} \subseteq \mathfrak{k}$ is maximal abelian in $\mathfrak{g}$, and $\mathfrak{g}^\alpha_{\mathbb{C}} = \mathbb{C}x_\alpha$ with $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cong \mathfrak{sl}_2(\mathbb{C})$. Then the following assertions hold:*

(i) *If $x_\alpha \in \mathfrak{k}_{\mathbb{C}}$, then α is compact, i.e., $\alpha \in \Delta_c$.*

(ii) *If $\mathfrak{p}$ is a negative/positive JH^*-triple and $x_\alpha \in \mathfrak{p}_{\mathbb{C}}$, then α is compact/non-compact.*

(iii) *If (β, V) is a $*$-representation of $\mathfrak{g}_{\mathbb{C}}$ on a pre-Hilbert space and $v_\lambda \in V$ an eigenvector of $\mathfrak{g}_{\mathbb{C}}^{-\alpha} + \mathfrak{t}_{\mathbb{C}}$, then*

$$\lambda(\check{\alpha}) \begin{cases} \leq 0 & \text{for } \alpha \in \Delta_c \\ \geq 0 & \text{for } \alpha \in \Delta_{nc}. \end{cases}$$

Proof. (i) If $x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha \subseteq \mathfrak{k}_{\mathbb{C}}$, then the real Lie algebra $\mathfrak{g}(\alpha) := \mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g} \subseteq \mathfrak{k}$ has a non-trivial bounded representation $\mathrm{ad}_{\mathfrak{p}}$, hence is compact. Now Lemma C.2 implies that $\alpha \in \Delta_c$.

(ii) The 3-dimensional Lie algebra $\mathfrak{g}(\alpha) = \mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g}$ is hermitian with $d_\alpha := -\frac{i}{2}\check{\alpha}$ and $\mathfrak{p}^+ = \mathbb{C}x_\alpha$. Now

$$\{x_\alpha, x_\alpha, x_\alpha\} = [[x_\alpha, \overline{x_\alpha}], x_\alpha] = -[[x_\alpha, x_\alpha^*], x_\alpha] = \alpha([x_\alpha^*, x_\alpha])x_\alpha.$$

Therefore the JH^*-triple $\mathbb{C}x_\alpha$ is positive if $\mathfrak{g}(\alpha) \cong \mathfrak{su}_{1,1}(\mathbb{C})$ and negative if $\mathfrak{g}(\alpha) \cong \mathfrak{su}_2(\mathbb{C})$ (Lemma C.2).

(iii) As in the proof of Lemma 5.11, we obtain for a unit vector v_λ of weight λ and $x_\alpha \in \mathfrak{g}_{\mathbb{C}}^\alpha$:

$$\lambda([x_\alpha^*, x_\alpha]) = \langle \beta([x_\alpha^*, x_\alpha])v, v\rangle = \langle \beta(x_\alpha)^*\beta(x_\alpha)v, v\rangle = \|\beta(x_\alpha)v\|^2 \geq 0.$$

Now (iii) follows from (24) in Definition C.3. □

Example 5.14. (Example 5.12 continued) For the Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, we choose an orthonormal basis $(e_j)_{j\in J}$ such that $J = J_+ \cup J_-$, where $(e_j)_{j\in J_\pm}$ is an orthonormal basis of $\mathcal{H}_\pm$.

(a) For the Lie algebra $\mathfrak{g} = \mathfrak{u}_{\mathrm{res}}(\mathcal{H})$, the subalgebra

$$\mathfrak{t} := \{X \in \mathfrak{g} \colon (\forall j \in J)\, Xe_j \in \mathbb{C}e_j\} \cong \ell^\infty(J, i\mathbb{R})$$

is elliptic and maximal abelian with root system

$$\Delta = \Delta_c = \{\varepsilon_j - \varepsilon_k \colon j \neq k \in J\}$$

(cf. Example C.4). From $\Delta_p^+ = \{\varepsilon_j - \varepsilon_k \colon j \in J_+, k \in J_-\}$ (Example C.7), we therefore get $E_{jj} - E_{kk} \in iC_{\mathfrak{p}}$ for $j \in J_+, k \in J_-$. For any weight $\mu = (\mu_j)_{j\in J}$ of a representation (ρ, V) of K, the relation $\mathtt{d}\rho([z^*, z]) \geq 0$ for each $z \in \mathfrak{p}^+$ now implies

$$\mu_j \leq \mu_k \quad \text{for} \quad j \in J_+, k \in J_-, \quad \text{i.e.,} \quad \sup(\mu_+) \leq \inf(\mu_-) \quad \text{for} \quad \mu_\pm := \mu|_{J_\pm}. \tag{8}$$

(b) For the c-dual Lie algebra $\mathfrak{g}^c = \mathfrak{u}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$, the sets Δ_k and $\Delta_p^\pm$ are the same, but we have $E_{jk}^* = -E_{kj}$ for $j \in J_+, k \in J_-$, so that $\Delta_c = \Delta_k$ and $\Delta_{nc} = \Delta_p$. For any weight $\mu = (\mu_j)_{j\in J}$ of a representation (ρ, V) of K, the relation $\mathtt{d}\rho([z^*, z]) \geq 0$ therefore implies

$$\mu_j \geq \mu_k \quad \text{for} \quad j \in J_+, k \in J_-, \quad \text{i.e.,} \quad \inf(\mu_+) \geq \sup(\mu_-). \tag{9}$$

5.3 The Reduction Theorem

Suppose that (G, θ, d) is a simply connected full hermitian Lie group for which K is a direct product of a finite dimensional Lie group with compact Lie algebra and groups isomorphic to full unitary groups $\mathrm{U}(\mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional real, complex or quaternionic Hilbert space. We write $K_\infty \trianglelefteq K$ for the normal integral subgroup obtained by replacing each $\mathrm{U}(\mathcal{H})$-factor by $\mathrm{U}_\infty(\mathcal{H})$. In Theorem 5.17 below we show that bounded irreducible representations (ρ, V) of K are tensor products of two bounded irreducible representations (ρ_0, V_0) and (ρ_1, V_1), where $\rho_0|_{K_\infty}$ is irreducible and $K_\infty \subseteq \ker \rho_1$. In this section we see how this can be used to derive a factorization of irreducible semibounded representations, resp., to reduce the inducibility problem to the case $\rho = \rho_0$.

We start with a general lemma on unitary representations of discrete groups.

Lemma 5.15. (Factorization Lemma) *Let G be a group, $N \trianglelefteq G$ be a normal subgroup and $(\pi, \mathcal{H})$ be an irreducible unitary representation of G such that the restriction $\pi|_N$ contains an irreducible representation $(\pi_0, \mathcal{H}_0)$ whose equivalence class is G-invariant. Let*

$$\widehat{G} := \{(g, u) \in G \times \mathrm{U}(\mathcal{H}_0) \colon (\forall n \in N)\ \pi_0(gng^{-1}) = u\pi_0(n)u^*\}$$

denote the corresponding central $\mathbb{T}$-extension of G with kernel $Z := \{\mathbf{1}\} \times \mathbb{T}\mathbf{1}$ and note that $\sigma \colon N \to \widehat{G}, n \mapsto (n, \pi_0(n))$ defines a group homomorphism. Then

(a) $\widehat{\pi}_0(g, u) := u$ *defines an irreducible representation $(\widehat{\pi}_0, \mathcal{H}_0)$ of $\widehat{G}$ for which $\widehat{\pi}_0 \circ \sigma = \pi_0$ is irreducible, and there exists*

(b) *an irreducible representation $(\widehat{\pi}_1, \mathcal{H}_1)$ of $\widehat{G}$ with $\sigma(N) \subseteq \ker \widehat{\pi}_1$ and $\widehat{\pi}_1(\mathbf{1}, t) = t^{-1}\mathbf{1}$ for $t \in \mathbb{T}$,*

such that $q^*\pi \cong \widehat{\pi}_0 \otimes \widehat{\pi}_1$ holds for the projection map $q \colon \widehat{G} \to G$.

Proof. Our assumption implies that G preserves the π_0-isotypic subspace for $\pi|_N$, so that the irreducibility of π implies that $\pi|_N$ is of the form $\pi_0 \otimes \pi_1$, where $(\pi_0, \mathcal{H}_0)$ is irreducible and $(\pi_1, \mathcal{H}_1)$ is trivial. Since the class $[\pi_0] \in \widehat{N}$ is G-invariant, the map $\widehat{G} \to G, (g, u) \mapsto g$ is surjective with kernel $\{\mathbf{1}\} \times \mathbb{T}\mathbf{1}$, hence a central extension. Clearly, $(\widehat{\pi}_0, \mathcal{H}_0)$ is a unitary representation of $\widehat{G}$ whose restriction to $\sigma(N)$ is irreducible. Since the operators $(\widehat{\pi}_0(g, u) \otimes \mathbf{1})^{-1}\pi(g)$ commute with $\pi(N)$, they are of the form $\mathbf{1} \otimes \widehat{\pi}_1(g, u)$ for some unitary operator $\widehat{\pi}_1(g, u) \in \mathrm{U}(\mathcal{H}_1)$. By definition, $\widehat{\pi}_1(n, \pi_0(n)) = \mathbf{1}$ for $n \in N$. That $\widehat{\pi}_1$ is a representation follows from

$$\begin{aligned}
&\mathbf{1} \otimes \widehat{\pi}_1((g_1, u_1)(g_2, u_2)) = \big(\widehat{\pi}_0(g_1g_2, u_1u_2) \otimes \mathbf{1}\big)^{-1}\pi(g_1g_2) \\
&= (u_2^{-1}u_1^{-1} \otimes \mathbf{1})\pi(g_1)\pi(g_2) = (u_2^{-1} \otimes \mathbf{1})(\mathbf{1} \otimes \widehat{\pi}_1(g_1, u_1))\pi(g_2) \\
&= (\mathbf{1} \otimes \widehat{\pi}_1(g_1, u_1))(u_2^{-1} \otimes \mathbf{1})\pi(g_2) = (\mathbf{1} \otimes \widehat{\pi}_1(g_1, u_1))(\mathbf{1} \otimes \widehat{\pi}_1(g_2, u_2)).
\end{aligned}$$

□

Lemma 5.16. *Let K_j, $j = 1, \ldots, n$, be Lie groups for which all bounded unitary representations are direct sums of irreducible ones. Then the same holds for the product group $K := \prod_{j=1}^n K_j$ and each bounded irreducible representation of K is a tensor product of bounded irreducible representations of the K_j.*

Proof. We argue by induction, so that it suffices to consider the case $n = 2$. Let $(\pi, \mathcal{H})$ be a bounded unitary representation of $K = K_1 \times K_2$. Since $\pi|_{K_1}$ is a direct sum of irreducible representations, we obtain a decomposition of $\mathcal{H}$ into isotypical subspaces $\mathcal{H}_j$ which are also invariant under K_2. Hence we may assume that $\pi|_{K_1}$ is isotypical, so that $\mathcal{H} \cong \mathcal{H}_1 \otimes \mathcal{H}_2$ and $\pi(k_1, k_2) = \pi_1(k_1) \otimes \pi_2(k_2)$, where $(\pi_1, \mathcal{H}_1)$ is an irreducible representation of K_1. As $(\pi_2, \mathcal{H}_2)$ also is a direct sum of irreducible representations, the assertion follows from the fact that tensor products of irreducible representations of K_1, resp., K_2, define irreducible representations of $K_1 \times K_2$. $\square$

Theorem 5.17. *Suppose that $K = \prod_{j=0}^n K_j$ is a product of connected Lie groups, where K_0 has a compact Lie algebra and the groups K_j, $j > 0$, are isomorphic to a full unitary group $\mathrm{U}(\mathcal{H}_j)$ of an infinite dimensional Hilbert space $\mathcal{H}_j$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Accordingly, we define $K_{j,\infty}$ as $\mathrm{U}_\infty(\mathcal{H}_j)$. Then*

$$K_\infty := K_0 \times \prod_{j=1}^n K_{j,\infty}$$

is a normal Lie subgroup and each irreducible bounded representation (ρ, V) of K is a tensor product of two bounded irreducible representations (ρ_0, V_0) and (ρ_1, V_1), where $\rho_0|_{K_\infty}$ is irreducible and $K_\infty \subseteq \ker \rho_1$.

Proof. Each bounded representation of K_j extends to a holomorphic representation of its universal complexification $K_{j,\mathbb{C}}$. For the groups of the form $\mathrm{U}_\infty(\mathcal{H}_j)$ where $\mathcal{H}_j$ is complex, it therefore follows from [Ne98, Thm. III.14] that all bounded representations are direct sums of irreducible ones, and if $\mathcal{H}_j$ is a Hilbert spaces over $\mathbb{K} = \mathbb{R}, \mathbb{H}$, this is a consequence of Theorems D.5 and D.6. Since K_0 has a compact Lie algebra, it is a product $K_0 = Z_0 \times K_0'$, where K_0' is compact semisimple and Z_0 is abelian. Since every continuous unitary representation of K_0' is a direct sum of irreducible ones and $\rho(Z_0) \subseteq \mathbb{T}\mathbf{1}$ by Schur's Lemma, Lemma 5.16 implies that the restriction $\rho|_{K_\infty}$ is a direct sum of irreducible representations which in turn are tensor products of irreducible ones.

Since each bounded irreducible representation of $\mathrm{U}_\infty(\mathcal{H})$ extends to $\mathrm{U}(\mathcal{H})$ (cf. Definition D.3 and Theorems D.5 and D.6), the same holds for the bounded irreducible representations of K_∞. Hence the assertion follows from Lemma 5.15, which applies with the trivial central extension $\widehat{K} \cong \mathbb{T} \times K$ of K. $\square$

Let (G, θ, d) be a simply connected full irreducible hermitian Lie group for which $K = (G^\theta)_0$ satisfies the assumption of Theorem 5.17. Then $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}_\infty = \mathbf{L}(K_\infty)$ follows from the concrete description of $\mathfrak{p}$ in all types (cf. Remark 2.7). Therefore each bounded irreducible representation (ρ, V) of K is a tensor product of two bounded irreducible representations (ρ_0, V_0) and (ρ_1, V_1), where $\rho_0|_{K_\infty}$ is irreducible and $K_\infty \subseteq \ker \rho_1$.

From $[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}_\infty$ we derive the existence of a homomorphism $p \colon G \to K/K_\infty$, so that, if $\overline{\rho}_1 \colon K/K_\infty \to \mathrm{U}(\mathcal{H}_1)$ denotes the induced representation of the quotient group, then $\pi_1 := \overline{\rho}_1 \circ p$ is a bounded irreducible representation of G. If (ρ_0, V_0) is extendable to a semibounded representation $(\pi_0, \mathcal{H}_0)$ of G, then

$$\pi := \pi_0 \otimes \pi_1, \qquad \mathcal{H} := \mathcal{H}_0 \otimes V_1$$

defines a semibounded unitary representation of G on $\mathcal{H}$ which is holomorphically induced from the irreducible representation (ρ, V), hence irreducible.

Theorem 5.18. *If (G, θ, d) is a simply connected full irreducible hermitian Lie group for which $K = (G^\theta)_0$ satisfies the assumptions of Theorem 5.17, then the representation (ρ, V) is holomorphically inducible if and only if (ρ_0, V_0) has this property.*

Proof. We know from Theorem B.1 below that (ρ, V) is holomorphically inducible if and only if the corresponding function f_ρ on a $\mathbf{1}$-neighborhood of G, defined by

$$f_\rho(\exp x) = F_\rho(x) \quad \text{and} \quad F_\rho(x_+ * x_0 * x_-) = e^{\mathtt{d}\rho(x_0)}, \quad x_\pm \in \mathfrak{p}^\pm, x_0 \in \mathfrak{k}_\mathbb{C},$$

is positive definite on a $\mathbf{1}$-neighborhood in G. From the factorization of ρ, we obtain a factorization

$$f_\rho(g) = f_{\rho_0}(g) \otimes f_{\rho_1}(g) = f_{\rho_0}(g) \otimes \pi_1(g),$$

which leads to

$$f_\rho(gh^{-1}) = f_{\rho_0}(gh^{-1}) \otimes \pi_1(gh^{-1}) = (\mathbf{1} \otimes \pi_1(g))(f_{\rho_0}(gh^{-1}) \otimes \mathbf{1})(\mathbf{1} \otimes \pi_1(h))^*.$$

Therefore the function f_ρ is positive definite if and only if f_{ρ_0} is positive definite (cf. Remark A.2), and this shows that (ρ, V) is inducible if and only if (ρ_0, V_0) has this property. □

Remark 5.19. In view of Theorem 5.18, the classification problem for irreducible semibounded representations of G reduces to the classification of the inducible irreducible bounded representations for which $\rho|_{K_\infty}$ is irreducible and the disjoint problem of the classification of irreducible bounded representations of the group K/K_∞. The latter contains in particular the classification problem for irreducible bounded representations of groups like $\mathrm{U}(\mathcal{H})/\mathrm{U}_\infty(\mathcal{H})$, which is the identity component of the unitary group of the C^*-algebra $B(\mathcal{H})/K(\mathcal{H})$. For a construction procedure for irreducible unitary representations of unitary groups of C^*-algebras, we refer to [BN11].

5.4 Triple decompositions in complex groups

This brief subsection sets the stage for a uniform treatment of the inducibility problem for bounded K-representations (ρ, V) for groups related to positive and negative JH^*-triples.

Lemma 5.20. *Let (G, θ, d) be a hermitian Lie group, G_c be a complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$, $P^\pm := \exp \mathfrak{p}^\pm$, and $K_c := \langle \exp \mathfrak{k}_\mathbb{C} \rangle$ be the Lie subgroup of G_c corresponding to $\mathfrak{k}_\mathbb{C}$. Then the multiplication map*

$$\mu \colon P^+ \times K_c \times P^- \to G_c, \quad (p_+, k, p_-) \mapsto p_+ k p_-$$

is biholomorphic onto an open subset of G_c and the subgroups $P^\pm$ are simply connected.

Proof. The existence of the Lie subgroups $P^\pm$ and K_c of G_c follows from the fact the Lie algebras $\mathfrak{p}^\pm$ and $\mathfrak{k}_\mathbb{C}$ are closed ([Mais62]). That the subgroups $P^\pm$ are simply connected follows from

$$\mathrm{Ad}(\exp z)d = e^{\mathrm{ad}\, z}d = d + [z,d] = d \mp iz \quad \text{for} \quad z \in \mathfrak{p}^\pm,$$

which implies that $\exp|_{\mathfrak{p}^\pm}$ is injective. It also shows that, for $p_\pm \in P^\pm$, the relation $\mathrm{Ad}(p_\pm)d = d$ implies $p_\pm = \mathbf{1}$.

The map μ is an orbit map for the holomorphic action of the direct product group $P^+ \times (K_c \rtimes P^-)^{\mathrm{op}}$ on G_c by $(g,h).x := gxh$. Since its differential in $(\mathbf{1},\mathbf{1},\mathbf{1})$ is the summation map, hence invertible, it follows from the complex Inverse Function Theorem that μ is a covering map onto an open subset of G_c. Therefore it remains to show that the fiber of $\mathbf{1}$ is trivial. If $p_+kp_- = \mathbf{1}$, then $p_+ = p_-^{-1}k^{-1}$ implies that

$$\mathrm{Ad}(p_+)d - d = \mathrm{Ad}(p_-^{-1}k^{-1})d - d = \mathrm{Ad}(p_-^{-1})d - d \in \mathfrak{p}^+ \cap \mathfrak{p}^- = \{0\},$$

so that $p_+ = p_- = \mathbf{1}$. It follows that μ is injective, hence biholomorphic onto its image. □

Now we consider a simply connected hermitian Lie group (G,θ,d) for which there exists a simply connected Lie group $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}_\mathbb{C}$. This assumption is satisfied if G is the simply connected Lie group with Lie algebra $\mathfrak{g} = \widehat{\mathfrak{g}}(\mathfrak{p})$ (Theorem 3.9). Then there exists a homomorphism $\eta_G\colon G \to G_\mathbb{C}$ integrating the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{g}_\mathbb{C}$. Since $\mathbf{L}(\eta_G)$ is injective, the kernel of η_G is discrete. This in turn implies that $\eta_G|_K\colon K \to G_\mathbb{C}$ has a discrete kernel, so that the Complexification Theorem in [GN03, Thm. IV.7] implies the existence of a universal complexification $\eta_K\colon K \to K_\mathbb{C}$ of K. Let $K_c := \langle \exp \mathfrak{k}_\mathbb{C}\rangle \subseteq G_\mathbb{C}$ denote the integral subgroup corresponding to $\mathfrak{k}_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$ and $q_c\colon K_\mathbb{C} \to K_c$ be the natural map whose existence follows from the universal property of $K_\mathbb{C}$. We then obtain a covering map

$$\widetilde{\mu}\colon P^+ \times K_\mathbb{C} \times P^- \to \Omega := P^+K_cP^- \subseteq G_\mathbb{C}, \quad (p^+,k,p^-) \mapsto p^+q_c(k)p^-.$$

Let $U \subseteq \mathfrak{p}$ be a K-invariant open convex symmetric 0-neighborhood for which the map $U \times K \to \exp(U)K, (x,k) \mapsto \exp x \cdot k$ is a diffeomorphism onto an open subset $U_G = \exp(U)K$ of G contained in $\eta_G^{-1}(\Omega)$. The K-invariance and the symmetry of U imply that

$$U_G = \exp(U)K = K \exp U = K \exp(-U) = U_G^{-1}.$$

Let

$$\widetilde{\eta}_G\colon U_G \to P^+ \times K_\mathbb{C} \times P^-$$

be the unique continuous lift of $\eta_G|_{U_G}$ with $\widetilde{\eta}_G(\mathbf{1}) = (\mathbf{1},\mathbf{1},\mathbf{1})$ and $\widetilde{\mu} \circ \widetilde{\eta}_G = \eta_G|_{U_G}$. Writing $x = (x_+,x_0,x_-)$ for the elements of $P^+ \times K_\mathbb{C} \times P^-$, we thus obtain an analytic map

$$\kappa\colon U_G \to K_\mathbb{C}, \quad g \mapsto \widetilde{\eta}_G(g)_0.$$

From the uniqueness of lifts and the fact that K normalizes $P^\pm$, we derive that κ is K-biequivariant, i.e.,

$$\kappa(k_1gk_2) = k_1\kappa(g)k_2 \quad \text{for} \quad g \in G, k_1,k_2 \in K.$$

For a bounded representation (ρ, V) of K and its holomorphic extension

$$\rho_{\mathbb{C}} \colon K_{\mathbb{C}} \to \mathrm{GL}(V) \quad \text{with} \quad \rho_{\mathbb{C}} \circ \eta_K = \rho,$$

we now obtain the analytic function

$$f_\rho := \rho_{\mathbb{C}} \circ \kappa \colon U_G \to \mathrm{GL}(V) \tag{10}$$

which coincides with the corresponding function in Theorem B.1 in an identity neighborhood. Now let $W \subseteq U$ be an open symmetric K-invariant 0-neighborhood for which $W_G \subseteq U_G$ satisfies $W_G W_G^{-1} = W_G W_G \subseteq U_G$. We write $M_W := W_G/K$ for the corresponding open subset of $M = G/K$. From the equivariance property

$$f_\rho(k_1 g k_2) = \rho(k_1) f_\rho(g) \rho(k_2) \quad \text{for} \quad g \in G, k_1, k_2 \in K,$$

it follows that f_ρ defines a kernel function

$$F_\rho \colon M_W \times M_W \to B(V), \quad (gK, hK) \mapsto (f_\rho(h)^{-1})^* f_\rho(h^{-1}g) f_\rho(g)^{-1}.$$

The kernel F_ρ is hermitian in the sense that

$$F_\rho(z, w)^* = F_\rho(w, z) \quad \text{for} \quad z, w \in M_W.$$

Moreover, for each $h \in G$, the function

$$G \to B(V), \quad g \mapsto (f_\rho(h)^{-1})^* f_\rho(h^{-1}g) f_\rho(g)^{-1}$$

is annihilated by the left invariant differential operators L_w, $w \in \mathfrak{p}^-$, because f_ρ has this property, which in turn is a consequence of the construction of κ. Therefore F_ρ is holomorphic in the first argument ([Ne10d, Def. 1.3]), and since it is hermitian, it is antiholomorphic in the second argument, i.e., holomorphic as a function on the complex manifold $M_W \times \overline{M_W}$ because we know already that it is smooth. Here $\overline{M_W}$ denotes the complex manifold M_W, endowed with the opposite complex structure.

Proposition 5.21. *Let (G, θ, d) be a hermitian Lie group for which $\mathfrak{g}_{\mathbb{C}}$ is integrable and $M_W = W_G/K$ be as above. Then the following are equivalent for a bounded unitary representation (ρ, V) of K:*

(i) *The representation (ρ, V) is holomorphically inducible.*

(ii) *The function f_ρ is positive definite on $W_G W_G^{-1}$.*

(iii) *The holomorphic kernel F_ρ on $M_W \times \overline{M_W}$ is positive definite.*

Proof. Since f_ρ is analytic, the Extension Theorem A.7 implies that it is positive definite if and only if it is positive definite on some identity neighborhood. Therefore the equivalence of (i) and (ii) follows from Theorem B.1.

Further, Remark A.2 implies that the kernel F_ρ is positive definite if and only if f_ρ is a positive definite function, so that (ii) and (iii) are also equivalent. □

Remark 5.22. Suppose that $\mathfrak{p}$ is a simple JH^*-triple and $\mathfrak{g} = \widehat{\mathfrak{g}}(\mathfrak{p})$, so that $\mathfrak{g}_{\mathbb{C}}$ is integrable (Theorem 3.9). As we have seen in Remark 3.11, the simply connected group G with Lie algebra $\widehat{\mathfrak{g}}(\mathfrak{p})$ is a quotient of a semidirect product $G_1 \rtimes K \to G$, where G_1 is the simply connected covering of a group of operators. Then f_ρ is positive definite on $W_G W_G^{-1}$ if and only if its pullback $\widetilde{f}_\rho$ to $G_1 \rtimes K$ is positive definite, but this function satisfies

$$\begin{aligned}\widetilde{f}_\rho((g_2,k_2)^{-1}(g_1,k_1)) &= \rho(k_2)^{-1}\widetilde{f}_\rho((g_2^{-1},\mathbf{1})(g_1,\mathbf{1}))\rho(k_1)\\ &= \rho(k_2)^*\widetilde{f}_\rho((g_2^{-1}g_1,\mathbf{1}))\rho(k_1),\end{aligned}$$

so that $\widetilde{f}_\rho$ is positive definite if and only if the restriction $\widetilde{f}_\rho|_{G_1}$ is positive definite (Remark A.2). For $K_1 := (G_1^\theta)_0$, the function $\widetilde{f}_\rho$ only depends on the representation $\widetilde{\rho}\colon K_1 \to \mathrm{U}(V)$ obtained by pulling ρ back by the canonical map $K_1 \to K$.

6 Motion groups and their central extensions

In this section we obtain a complete classification of the irreducible semibounded representations of the hermitian motion group $G = \mathfrak{p} \rtimes_\alpha K$ and of its canonical central extension $\widehat{G} = \mathrm{Heis}(\mathfrak{p}) \rtimes_\alpha K$ (Example 1.3). For $G = \mathfrak{p} \rtimes K$, Theorem 6.1 asserts that all semibounded unitary representation are trivial on $\mathfrak{p}$, hence factor through (semibounded) representations of K. If, in addition, $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no invariant cones, then Proposition 4.4 shows that all semibounded irreducible representations of K are bounded. For the central extension $\widehat{G}$, Theorem 6.2 asserts that a bounded irreducible representation (ρ, V) of $\widehat{K} = \mathbb{R} \times K$ with $\rho(t,\mathbf{1}) = e^{ict}\mathbf{1}$ on the center is inducible if and only if $c \geq 0$.

Theorem 6.1. *If $[\mathfrak{p},\mathfrak{p}] = \{0\}$ and $G = \mathfrak{p} \rtimes K$, then all semibounded unitary representations $(\pi, \mathcal{H})$ factor through K, i.e., $\mathfrak{p} \subseteq \ker \mathtt{d}\pi$.*

Proof. Let $(\pi, \mathcal{H})$ be a semibounded unitary representation of G and $\widehat{G} := G \times \mathbb{R}$. We consider the representation $\widehat{\pi}$ of $\widehat{G}$ defined by $\widehat{\pi}(g,t) := e^{it}\pi(g)$. Since $W_\pi \neq \emptyset$, the open cone $C_{\widehat{\pi}}$ of all elements x of $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ for which $s_{\widehat{\pi}} < 0$ holds on a neighborhood of x is non-empty. For the abelian ideal $\mathfrak{p}$ of $\widehat{\mathfrak{g}}$, we thus obtain from [Ne10c, Lemma 7.6] that

$$C_{\widehat{\pi}} + \mathfrak{p} = C_{\widehat{\pi}} + [d,\mathfrak{p}] \subseteq C_{\widehat{\pi}} + [\mathfrak{p},\widehat{\mathfrak{g}}] = C_{\widehat{\pi}},$$

and hence that $\mathfrak{p} \subseteq H(C_{\widehat{\pi}}) = \ker \mathtt{d}\pi$. □

Now we turn to the central extension

$$\widehat{G} = \mathrm{Heis}(\mathfrak{p}) \rtimes K \cong \mathfrak{p} \oplus (\mathbb{R} \oplus K)$$

with Lie algebra $\widehat{\mathfrak{g}} \cong \mathbb{R} \oplus_{\omega_\mathfrak{p}} \mathfrak{g}$, where $\omega_\mathfrak{p}(x,y) = 2\,\mathrm{Im}\langle x,y\rangle$. We assume that $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones. From Theorem 5.4 we know that every irreducible semibounded representation $(\pi, \mathcal{H})$ of positive energy is holomorphically induced from the bounded $\widehat{K}$-representation on $V = \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$. We now turn to the question which bounded $\widehat{K}$-representations (ρ, V) are holomorphically inducible.

In view of Remark 5.10(c), the cone $C_{\mathfrak{p}}$ is generated by the elements of the form $[Ix, x]$, $x \in \mathfrak{p}$. From

$$[Ix, x] = 2\,\mathrm{Im}\langle Ix, x\rangle = 2\|x\|^2$$

it now follows that

$$C_{\mathfrak{p}} = \mathbb{R}^+ \times \{0\} \subseteq \mathbb{R} \times \mathfrak{k},$$

which leads to the necessary condition

$$-i\mathtt{d}\rho(1,0) \geq 0. \tag{11}$$

Any irreducible representation (ρ, V) of $\widehat{K} \cong \mathbb{R} \times K$ is of the form $\rho(t,k) = e^{ict}\rho_1(k)$, where (ρ_1, V) is an irreducible K-representation. Now condition (11) means that $c \geq 0$.

For $c = 0$, we obtain the unique extension to G by $\pi(t,v,k) := \rho(t,k) = \rho_1(k)$ satisfying $\mathfrak{p} \subseteq \ker \pi$. To deal with the cases $c > 0$, we have to introduce the canonical representation of the Heisenberg group on the Fock space $S(\mathfrak{p}) = \widehat{\bigoplus}_{n\geq 0} S^n(\mathfrak{p})$ (cf. [Ot95]). On the dense subspace $S(\mathfrak{p})_0 = \sum_{n=0}^{\infty} S^n(\mathfrak{p})$ of $S(\mathfrak{p})$, we have for each $v \in \mathfrak{p}$ the *creation operator*

$$a^*(v)(v_1 \vee \cdots \vee v_n) := v \vee v_1 \vee \cdots \vee v_n.$$

This operator has an adjoint $a(v)$ on $S(\mathfrak{p})_0$, given by

$$a(v)\Omega = 0, \quad a(v)(v_1 \vee \cdots \vee v_n) = \sum_{j=0}^{n} \langle v_j, v\rangle v_1 \vee \cdots \vee \widehat{v_j} \vee \cdots \vee v_n,$$

where $\widehat{v_j}$ means omitting the factor v_j. One easily verifies that these operators satisfy the canonical commutation relations (CCR):

$$[a(v), a(w)] = 0, \quad [a(v), a^*(w)] = \langle w, v\rangle \mathbf{1}. \tag{12}$$

For each $v \in \mathfrak{p}$, the operator $a(v) + a^*(v)$ on $S(\mathfrak{p})_0$ is essentially self-adjoint and

$$W(v) := e^{\frac{i}{\sqrt{2}}\overline{a(v)+a^*(v)}} \in \mathrm{U}(S(\mathfrak{p}))$$

is a unitary operator. These operators satisfy the *Weyl relations*

$$W(v)W(w) = W(v+w)e^{\frac{i}{2}\,\mathrm{Im}\langle v,w\rangle} \quad \text{for} \quad v, w \in \mathfrak{p}$$

(cf. [Ne10c]). For the Heisenberg group $\mathrm{Heis}(\mathfrak{p}) := \mathbb{R} \times \mathfrak{p}$ with the multiplication

$$(t,v)(t',v') := (t + t' + \mathrm{Im}\langle v, v'\rangle, v + v')$$

we thus obtain by $W(t,v) := e^{it/2}W(v)$ a unitary representation on $S(\mathfrak{p})$, called the *Fock representation* which is actually smooth (cf. [Ne10c, Sect. 9.1]). Using the natural representations

$$S^n(U)(v_1 \vee \cdots \vee v_n) := Uv_1 \vee \cdots \vee Uv_n$$

of the unitary group $\mathrm{U}(\mathfrak{p})$ on the symmetric powers $S^n(\mathfrak{p})$, and combining them to a unitary representation $(S, S(\mathfrak{p}))$, we obtain a smooth representation $(\pi_s, S(\mathfrak{p}))$, defined by

$$\pi_s(t, v, k) := W(t, v)S(k) = e^{it/2}W(v)S(k)$$

of $\widehat{G} = \mathrm{Heis}(\mathfrak{p}) \rtimes K$ on $S(\mathfrak{p})$ with $\Omega \in S(\mathfrak{p})^\infty$. Then $\mathtt{d}\pi_s(\mathfrak{p}^+) = a^*(\mathfrak{p})$ and $\mathtt{d}\pi_s(\mathfrak{p}^-) = a(\mathfrak{p})$ lead to $(S(\mathfrak{p})^\infty)^{\mathfrak{p}^-} = \mathbb{C}\Omega$, and it follows from [Ne10d, Thm. 2.17] that $(\pi_s, S(\mathfrak{p}))$ is holomorphically induced from the character $\rho_s \colon \widehat{K} = \mathbb{R} \times K \to \mathbb{T}$, given by $\rho_s(t, k) = e^{it/2}$. Composing with the automorphisms of $\mathrm{Heis}(\mathcal{H})$ given by $\mu_h(t, v) := (h^2 t, hv)$, $h > 0$, we see that all characters $\rho_c(t, k) := e^{ict}\mathbf{1}$ of $\widehat{K}$ are holomorphically inducible. We write $(\pi_c, S(\mathfrak{p}))$ for the corresponding unitary representation of $\widehat{G}$.

With $\pi_1(t, v, k) := \rho_1(k)$ we now put

$$\pi := \pi_c \otimes \pi_1 \quad \text{on} \quad \mathcal{H} := S(\mathfrak{p}) \otimes V_1$$

an observe that this unitary representation is holomorphically induced from $\rho = \rho_c \otimes \rho_1$. Its semiboundedness follows from Theorem 5.6. This completes the proof of the following theorem:

Theorem 6.2. *A bounded irreducible unitary representation (ρ, V) of $\widehat{K} = \mathbb{R} \times K$ with $\rho(t, \mathbf{1}) = e^{ict}\mathbf{1}$ is holomorphically inducible to a semibounded positive energy representation of $\widehat{G} = \mathrm{Heis}(\mathfrak{p}) \rtimes K$ if and only if $c \geq 0$.*

Remark 6.3. If $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones, then Theorem 5.2(iii) implies that every semibounded unitary representation π of $\widehat{G}$ satisfies $d \in W_\pi \cup -W_\pi$. If $d \in W_\pi$, then it is of positive energy and if, in addition, π is irreducible, Theorem 5.4 implies that it is holomorphically induced from a bounded representation of $\widehat{K}$, hence covered by Theorem 6.2.

Example 6.4. (a) Theorem 6.2 applies in particular to the oscillator group $\widehat{G} = \mathrm{Osc}(\mathcal{H}) = \mathrm{Heis}(\mathcal{H}) \rtimes \mathbb{T}$ of a complex Hilbert space $\mathcal{H}$ from Example 1.3(b). In this case $G = \mathcal{H} \rtimes \mathbb{T}$ is the minimal hermitian Lie group with $\mathfrak{p} = \mathcal{H}$.

In the physics literature the unitary representations of $\mathrm{Osc}(\mathcal{H})$ are known as the representations of the CCR with a number operator. It is known that the positive energy representations of $\mathrm{Osc}(\mathcal{H})$ are direct sums of Fock representations (cf. [Ch68]). Therefore our Theorem 6.2 generalizes Chaiken's uniqueness result on the irreducible positive energy representations of $\mathrm{Osc}(\mathcal{H})$.

(b) The maximal hermitian group (G, θ, d) with $\mathfrak{p} = \mathcal{H}$ is $G = \mathcal{H} \rtimes \mathrm{U}(\mathcal{H})$. For this group, Theorem 6.2 provides a complete classification of the semibounded positive energy representations in terms of bounded irreducible representations (ρ, V) of $\mathrm{U}(\mathcal{H})$. Those representations for which $\rho|_{\mathrm{U}_\infty(\mathcal{H})}$ is irreducible are described in Appendix D (cf. Definition D.3). According to Theorem E.1, all separable continuous irreducible representations of $\mathrm{U}(\mathcal{H})$ have this property.

7 Hermitian groups with positive JH^*-triples

In this section we consider the case where $M = G/K$ is an infinite dimensional irreducible symmetric Hilbert domain, i.e., $\mathfrak{p}$ is a positive simple infinite dimensional

JH^*-triple of type I-IV, and G is a simply connected Lie group with Lie algebra $\mathfrak{g} = \widehat{\mathfrak{g}}(\mathfrak{p})$. In view of [Ne02c, Prop. 3.15, Thm. 5.1], the polar map

$$K \times \mathfrak{p} \to G, \quad (k, x) \mapsto k \exp x \tag{13}$$

is a diffeomorphism because the positivity of the JH^*-triple $\mathfrak{p}$ implies that G/K is a symmetric space with seminegative curvature. Let $G_{\mathbb{C}}$ denote the simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ whose existence follows from Theorem 3.9.

Proposition 7.1. *Let (G, θ, d) be a simply connected hermitian Lie group for which $\mathfrak{p}$ is a positive JH^*-triple and $G_{\mathbb{C}}$ be a simply connected group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Then the canonical morphism $\eta_G \colon G \to G_{\mathbb{C}}$ is a covering of a closed subgroup $\eta_G(G)$ of $G_{\mathbb{C}}$ which is contained in the open subset $P^+K_cP^-$, where $P^\pm = \exp \mathfrak{p}^\pm$ and $K_c := \langle \exp \mathfrak{k}_{\mathbb{C}} \rangle$.*

Proof. In view of $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{k}$, $\eta_G(G) \subseteq P^+K_cP^-$ follows from the corresponding assertion for the group $\mathrm{Ad}(G) \subseteq \mathrm{Aut}(\mathfrak{p})_0$, for which it follows from the fact that the G-orbit of the base point in the complex homogeneous space $G_{\mathbb{C}}/K_cP^-$ corresponds to the domain

$$\mathcal{D} = \{z \in \mathfrak{p}^+ \colon \|z\|_\infty < 1\} \subseteq \mathfrak{p}^+ \cong P^+$$

(via the exponential map of P^+) and the open embedding $P^+ \hookrightarrow G_{\mathbb{C}}/(K_cP^-)$ (cf. [Ka83]). For domains of type I-III, this also follows from the discussion in [NO98, Sect. III]. □

We can now refine some of the results developed in the context of Proposition 5.21. Since G has a diffeomorphic polar map (see (13)), Proposition 7.1 shows that we may put $U_G = W_G = G$. We thus obtain an analytic map

$$\kappa \colon G \to K_{\mathbb{C}}, \quad g \mapsto \widetilde{\eta}_G(g)_0$$

defined on the whole group G. Let $\overline{\kappa}(g) := \overline{\kappa(g)}$, where $k \mapsto \overline{k}$ denotes complex conjugation of $K_{\mathbb{C}}$ with respect to K. This function plays the role of κ if we exchange P^+ and P^-, which is the context of [NO98, Sect. II]. We now obtain on $M = G/K$ a well-defined function

$$gK \mapsto (\kappa(g)^{-1})^*\kappa(g)^{-1} = \overline{\kappa}(g)\overline{\kappa}(g)^*.$$

According to [NO98, Lemma II.3], the canonical holomorphic kernel function $Q_M^c \colon M \times \overline{M} \to K_c$ satisfies

$$Q_c(gK, gK) = \eta_K(\overline{\kappa}(g)\overline{\kappa}(g)^*) \quad \text{for} \quad g \in G.$$

Since $M = G/K \cong \mathfrak{p}$ according to the polar decomposition, the space M is simply connected. Hence the canonical lift

$$Q_M \colon M \times \overline{M} \to K_{\mathbb{C}}$$

of Q_c to the covering group $K_{\mathbb{C}}$ of K_c determined by $Q_M(\mathbf{1}K, \mathbf{1}K) = \mathbf{1}$ satisfies

$$Q_M(gK, gK) = \overline{\kappa}(g)\overline{\kappa}(g)^* \quad \text{for} \quad g \in G.$$

For a bounded representation (ρ, V) of K and its holomorphic extension $\rho_{\mathbb{C}} \colon K_{\mathbb{C}} \to \mathrm{GL}(V)$, we recall from (10) in Section 5 the analytic function $f_\rho := \rho_{\mathbb{C}} \circ \kappa \colon G \to \mathrm{GL}(V)$ and the corresponding holomorphic kernel function

$$F_\rho \colon M \times \overline{M} \to B(V), \quad (gK, hK) \mapsto (f_\rho(h)^{-1})^* f_\rho(h^{-1}g) f_\rho(g)^{-1}.$$

On the diagonal $\Delta_M \subseteq M \times \overline{M}$ we have

$$\begin{aligned} F_\rho(gK, gK) &= (f_\rho(g)^{-1})^* f_\rho(g^{-1}g) f_\rho(g)^{-1} = (f_\rho(g)^{-1})^* f_\rho(g)^{-1} \\ &= \rho_{\mathbb{C}}((\kappa(g)^{-1})^* \kappa(g)^{-1}) = \rho_{\mathbb{C}}(Q_M(gK, gK)). \end{aligned}$$

Therefore F_ρ coincides with the kernel $Q_\rho := \rho_{\mathbb{C}} \circ Q_M$, because both kernels are holomorphic on $M \times \overline{M}$ and the diagonal $\Delta_M \subseteq M \times M$ is totally real.

Proposition 5.21 now specializes as follows:

Proposition 7.2. *Let (G, θ, d) be a hermitian Lie group for which $\mathfrak{p}$ is a positive simple JH^*-triple and $\mathfrak{g}_{\mathbb{C}}$ is integrable. Then the following are equivalent for a bounded unitary representation (ρ, V) of K:*

(i) *The representation (ρ, V) is holomorphically inducible.*

(ii) *The function $f_\rho \colon G \to B(V)$ is positive definite.*

(iii) *The kernel $Q_\rho \colon M \times M \to B(V)$ is positive definite.*

From now on we assume that $\mathfrak{g} = \widehat{\mathfrak{g}}(\mathfrak{p})$ is the universal central extension of $\mathfrak{g}(\mathfrak{p})$ (Remark 3.8) and recall from the discussion in the proof of Theorem 3.9 that in all cases the corresponding simply connected Lie group G has the property that $K := (G^\theta)_0$ has the product structure required in Theorem 5.17. In Remark 5.22 we have seen that f_ρ is positive definite if and only if the corresponding function $\widetilde{f}_\rho \colon G_1 \to B(V)$ is positive definite, which only depends on the representation $\widetilde{\rho} \colon K_1 \to \mathrm{U}(V)$ obtained by pulling ρ back via the canonical map $K_1 \to K$.

As $\mathfrak{p}^\pm \subseteq (\mathfrak{g}_1)_{\mathbb{C}}$, the kernel Q_ρ coincides with the corresponding kernel that we obtain from the triple decomposition of the open subset $P^+ K_{1,c} P^-$ of $(G_1)_{\mathbb{C}}$. Therefore $\widetilde{f}_\rho$ is positive definite if and only if the kernel $Q_\rho = Q_{\widetilde{\rho}}$ is positive definite on M. Since the positive definite kernels $Q_{\widetilde{\rho}}$ have been classified in [NO98, Thm. IV.1] in terms of unitarity of related highest weight modules, and all bounded irreducible representations $(\widetilde{\rho}, V)$ of K_1 can be described as highest weight representations, we can now derive a classification of the inducible irreducible bounded representations in terms of conditions on the highest weight of ρ_0 (cf. Theorem 5.18). In the following theorem, we evaluate the characterizations from [NO98] case by case for the four types of hermitian groups. Here we use the classification of bounded irreducible representations of $\mathrm{U}(\mathcal{H})$ whose restriction to $\mathrm{U}_\infty(\mathcal{H})$ is irreducible in terms of highest weights (cf. Definition D.3).

Theorem 7.3. (Classification Theorem) *For an irreducible bounded representation (ρ, V) of K whose restriction to K_∞ is irreducible, we obtain the following characterization of the inducible representations with respect to the element $d = \frac{i}{2}\,\mathrm{diag}(\mathbf{1}, -\mathbf{1})$:*

(I_∞) *Let $\mathcal{H}_\pm$ both be infinite dimensional and $\widehat{G} = \widehat{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ with $\widehat{K} \cong \mathbb{R} \times \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$. Then $\rho(t, k_1, k_2) = e^{ict}\pi_{\lambda_+}(k_1) \otimes \pi_{\lambda_-}(k_2)$ is inducible if and only if $\lambda_+ \geq 0 \geq \lambda_-$ and*

$$c \in |\operatorname{supp}(\lambda_+)| + |\operatorname{supp}(\lambda_-)| + \mathbb{N}_0.$$

($\mathrm{I}_{\mathrm{fin}}$) *Let $\dim \mathcal{H}_- < \infty$, $G = \widetilde{\mathrm{U}}(\mathcal{H}_+, \mathcal{H}_-)$ and $K \cong \mathrm{U}(\mathcal{H}_+) \times \widetilde{\mathrm{U}}(\mathcal{H}_-)$. Then $\rho(k_1, k_2) = \pi_{\lambda_+}(k_1) \otimes \pi_{\lambda_-}(k_2)$ is inducible if and only if $\lambda_+ \geq 0 \geq \lambda_-$ and*

$$c := -\max(\lambda_-) \in \{a, a+1, \ldots, b\} \cup]b, \infty[\subseteq \mathbb{Z}$$

for $a := |\operatorname{supp}(\lambda_+)| + |\operatorname{supp}(\lambda_- + c)|$ and $b := a - 1 + |J_- \setminus \operatorname{supp}(\lambda_- + c)|$.

(II) *For $\widehat{G} = \widehat{\mathrm{O}}^*_{\mathrm{res}}(\mathcal{H})$ and $\widehat{K} = \mathbb{R} \times \mathrm{U}(\mathcal{H})$, the representation $\rho(t, k) = e^{ict}\pi_\lambda(k)$ is inducible if and only if*

$$\lambda \geq 0 \quad \text{and} \quad c \in |\operatorname{supp}(\lambda)| + \mathbb{N}_0.$$

(III) *For $\widehat{G} = \widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H})$ and $\widehat{K} \cong \mathbb{R} \times \mathrm{U}(\mathcal{H})$, the representation $\rho(t, k) = e^{ict}\pi_\lambda(k)$ is inducible if and only if*

$$\lambda \geq 0 \quad \text{and} \quad 2c \in |\operatorname{supp}(\lambda)| + |\{j \in J \colon \lambda_j > 1\}| + \mathbb{N}_0.$$

Proof. (I_∞) From Example 3.5 we obtain the structure of $\widehat{K}$ and that $\widehat{G}$ is a quotient of a semidirect product via the homomorphism $\gamma_G \colon \widetilde{SG_1} \rtimes K \to \widehat{G}$ integrating

$$\gamma_{\mathfrak{g}} \colon \mathfrak{sg}_1 \rtimes \mathfrak{k} \to \widehat{\mathfrak{g}}, \quad (x, y) \mapsto (-i \operatorname{tr}(x_{11}), x + y).$$

Let (ρ, V) be a bounded irreducible representation of $\widehat{K}$ for which the restriction to $\widehat{K}_\infty = \mathbb{R} \times \mathrm{U}_\infty(\mathcal{H}_+) \times \mathrm{U}_\infty(\mathcal{H}_-)$ is irreducible. Then

$$\rho(t, k_1, k_2) = e^{ict}\pi_{\lambda_+}(k_1) \otimes \pi_{\lambda_-}(k_2),$$

where $\lambda_\pm \colon J_\pm \to \mathbb{Z}$ are finitely supported functions, corresponding to the highest weights and $(e_j)_{j \in J_\pm}$ in $\mathcal{H}_\pm$ are orthonormal bases (cf. Definition D.3). On

$$\mathfrak{sk}_1 \cong \{(x, y) \in \mathfrak{u}_1(\mathcal{H}_+) \times \mathfrak{u}_1(\mathcal{H}_-) \colon \operatorname{tr}(x) + \operatorname{tr}(y) = 0\}$$

this leads to a representation with highest weight $\lambda = (\lambda_+ + c, \lambda_-) = (\lambda_+, \lambda_- - c)$, where we consider λ as a function $J = J_+ \dot\cup J_- \to \mathbb{R}$. In fact, for a diagonal matrix $x = \operatorname{diag}((x_j)_{j \in J})$ and a λ-weight vector v_λ, we have

$$\begin{aligned} &\rho(\exp(-i \operatorname{tr} x_{11}, x))v_\lambda = \rho(-i \operatorname{tr} x_{11}, \exp x)v_\lambda \\ &= e^{c \sum_{j \in J_+} x_j} e^{\sum_{j \in J_+} (\lambda_+)_j x_j + \sum_{j \in J_-} (\lambda_-)_j x_j} v_\lambda = e^{\sum_{j \in J} \lambda_j x_j} v_\lambda. \end{aligned}$$

From Lemma 5.13 and Example 5.14(b) we obtain the necessary condition $c + \min(\lambda_+) \geq \max(\lambda_-)$. Since $\lambda_\pm$ have finite support, this implies that $c \geq 0$.

To derive the classification from [NO98, Prop. I.7], we put $J_1 := J_-$, $J_2 := J_+$ and

$$M := \min(\lambda_+ + c) = c + \min(\lambda_+) \geq m := \max(\lambda_-).$$

According to loc. cit., a necessary condition for inducibility is that

$$q'' := |\operatorname{supp}(\lambda_+ + c - M)| \quad \text{and} \quad p'' := |\operatorname{supp}(\lambda_- - m)|$$

are both finite. This implies that $M = c$ and $m = 0$, so that

$$a := p'' + q'' = |\operatorname{supp}(\lambda_+)| + |\operatorname{supp}(\lambda_-)|.$$

Since $J_\pm$ are both infinite,

$$b := a - 1 + \min\{|J_+ \setminus \operatorname{supp}(\lambda_+)|, |J_- \setminus \operatorname{supp}(\lambda_-)|\} = \infty,$$

and loc. cit. shows that inducibility is equivalent to $c = M - m \in a + \mathbb{N}_0$.

($\mathrm{I}_{\mathrm{fin}}$) With the same group SG_1 as for the previous case, we obtain a surjective homomorphism $\gamma_G \colon \widetilde{S}G_1 \rtimes K \to G$ integrating the summation homomorphism $\gamma_{\mathfrak{g}} \colon \mathfrak{sg}_1 \rtimes \mathfrak{k} \to \mathfrak{g}, (x,y) \mapsto x + y$. Now $K_\infty = \mathrm{U}_\infty(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ and (ρ, V) has the form

$$\rho(k_1, k_2) = \pi_{\lambda_+}(k_1) \otimes \pi_{\lambda_-}(k_2).$$

On $\mathfrak{sk}_1$ this leads to a representation with highest weight $\lambda = (\lambda_+, \lambda_-) \colon J \to \mathbb{R}$. From Lemma 5.13 and Example 5.14(b) we obtain the necessary condition

$$M := \min(\lambda_+) \geq m := \max(\lambda_-)$$

(see also [NO98, Prop. I.5(ii)]). To derive the classification from [NO98, Prop. I.7], we put $J_1 := J_-$ and $J_2 := J_+$. With the finiteness of

$$q'' := |\operatorname{supp}(\lambda_+ - M)| \quad \text{and} \quad p'' := |\operatorname{supp}(\lambda_- - m)|,$$

we derive that $M = 0$ and hence that $\lambda_+ \geq 0 \geq \lambda_-$, so that

$$a := p'' + q'' = |\operatorname{supp}(\lambda_+)| + |\operatorname{supp}(\lambda_- - m)|.$$

Now

$$b := a - 1 + \min\{|J_+ \setminus \operatorname{supp}(\lambda_+)|, |J_- \setminus \operatorname{supp}(\lambda_-)|\} = a - 1 + |J_- \setminus \operatorname{supp}(\lambda_-)| < \infty$$

and loc. cit. shows that inducibility is equivalent to

$$c := -m \in \{a, a+1, \ldots, b\} \cup]b, \infty[.$$

(II) From Example 3.6 we recall that $\widehat{K} \cong \mathbb{R} \times \mathrm{U}(\mathcal{H}) = \mathbb{R} \times K$, and that $\widehat{G}$ is obtained as a quotient of a semidirect product via the homomorphism $\gamma_G \colon \widetilde{G}_1 \rtimes K \to \widehat{G}$ integrating

$$\gamma_{\mathfrak{g}} \colon \mathfrak{g}_1 \rtimes \mathfrak{k} \to \widehat{\mathfrak{g}}, \quad (x,y) \mapsto (-i\operatorname{tr}(x_{11}), x + y).$$

Let (ρ, V) be a bounded irreducible representation of $\widehat{K}$ for which the restriction to $\widehat{K}_\infty = \mathbb{R} \times \mathrm{U}_\infty(\mathcal{H})$ is irreducible. Then $\rho(t,k) = e^{ict}\pi_\lambda(k)$, where $\lambda\colon J \to \mathbb{Z}$ is finitely supported. On $\mathfrak{k}_1 \cong \mathfrak{u}_1(\mathcal{H})$, this leads to a representation with highest weight $\lambda + c$. From Lemma 5.13 and the description of the coroots for $\Delta_p^+ \subseteq \Delta_{nc}$ in Example C.9, we derive the necessary condition

$$2c + \lambda_j + \lambda_k \geq 0 \quad \text{for} \quad j \neq k \in J$$

(see also [NO98]). As λ has finite support, we obtain $c \geq 0$.

Comparing our Δ_p^+ with the positive system used in [NO98], it follows that we have to apply [NO98, Prop. I.11] to $-\lambda - c$. For

$$M := \max(-c - \lambda) = -\min(\lambda) - c \quad \text{we put} \quad p' := |J \setminus \lambda^{-1}(-M - c)|.$$

Then p' has to be finite, so that $M = -c \leq 0$, $\lambda \geq 0$ and $p' = |\operatorname{supp}(\lambda)|$. According to [NO98, Prop. I.11], inducibility is equivalent to $c \in |\operatorname{supp}(\lambda)| + \mathbb{N}_0$.

(III) Here we have a similar situation as for type II. Now $\rho(t,k) = e^{ict}\pi_\lambda(k)$ leads to a representation with highest weight $\lambda + c$ of $\mathfrak{k}_1 \cong \mathfrak{u}_1(\mathcal{H})$ and Lemma 5.13 with Example C.8 lead to the necessary condition

$$2c + \lambda_j + \lambda_k \geq 0 \quad \text{for} \quad j, k \in J,$$

which is equivalent to $c + \lambda \geq 0$. As λ has finite support, this implies $c \geq 0$.

For $M := \max(-c - \lambda) = -c - \min(\lambda)$ we put

$$q' := |\{j \in J\colon \lambda_j > \min(\lambda)\}| \quad \text{and} \quad r' := |\{j \in J\colon \lambda_j > \min(\lambda) + 1\}|.$$

Then r' and q' have to be finite, so that $M = -c$, $\min \lambda = 0$ and $q' = |\operatorname{supp}(\lambda)|$. According to [NO98, Prop. I.9], inducibility is equivalent to $2c \in q' + r' + \mathbb{N}_0$. □

Remark 7.4. (a) For the types I_∞, II and III, the number c is called the *central charge* of the representation. The preceding theorem shows that $c = 0$ implies $\lambda = 0$, hence that the irreducible representation ρ is trivial. In particular, the groups $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$, $\mathrm{O}^*_{\mathrm{res}}(\mathcal{H})$ and $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H})$ have no non-trivial irreducible semibounded representation $(\pi, \mathcal{H})$ with $\mathfrak{p} \not\subseteq \ker \mathtt{d}\pi$. All irreducible semibounded representations of these groups are pullbacks of bounded representations of K/K_∞ by the canonical homomorphism $G \to K/K_\infty$.

For $K = \mathrm{U}(\mathcal{H})$, the quotient $K/K_\infty \cong \mathrm{U}(\mathcal{H})/\mathrm{U}_\infty(\mathcal{H})$ is the identity component of the unitary group of the C^*-algebra $B(\mathcal{H})/K(\mathcal{H})$, hence has enough bounded unitary representations to separate the points. However, these representations all live on non-separable spaces (see [Pi88] and Theorem E.1 below).

(b) The representations for which V is one dimensional are called *of scalar type.* For the groups $\widehat{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ and $\widehat{\mathrm{O}}^*_{\mathrm{res}}(\mathcal{H})$ they are parameterized by $c \in \mathbb{N}_0$, and for $\widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H})$ we have $c \in \frac{1}{2}\mathbb{N}_0$.

For $c = \frac{1}{2}$ we obtain in particular the metaplectic representation of $\widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H})$ on the even Fock space $S^{\mathrm{even}}(\mathcal{H})$ and for $\rho(t,k) = e^{it/2}k$ on $V = \mathcal{H}$ we obtain the metaplectic representation on the odd Fock space $S^{\mathrm{odd}}(\mathcal{H})$ (cf. [Ot95], [Ne10c, Sect. 9.1], [NO98, Sect. IV]).

(c) For type $\mathrm{I}_{\mathrm{fin}}$ ($\dim \mathcal{H}_- < \infty$), the scalar type condition means that $\lambda_+ = 0$ and $\lambda_- = -c$, so that $a = 0$ and $b = |J_-| - 1$ lead to the condition

$$c \in \{0, 1, 2, |J_-| - 1\} \quad \text{or} \quad c > |J_-| - 1.$$

For type I_{fin} and $\sum_j (\lambda_-)_j = -\sum_j (\lambda_+)_j$, the representation $\rho = \pi_{\lambda_+} \otimes \pi_{\lambda_-}$ vanishes on

$$Z(\widetilde{G})_0 = \{(e^{-it}, t) \colon t \in \mathbb{R}\} \subseteq K \cong \mathrm{U}(\mathcal{H}_+) \times (\mathbb{R} \ltimes \mathrm{SU}(\mathcal{H}_-)),$$

so that we also obtain non-trivial inducible representations vanishing on the center. These lead to semibounded representations of $\mathrm{U}(\mathcal{H}_+, \mathcal{H}_-)/\mathbb{T}\mathbf{1}$.

For the domains of type IV, the situation is quite trivial, as the following theorem shows.

Theorem 7.5. *The universal covering group* $\widetilde{\mathrm{O}}(\mathbb{R}^2, \mathcal{H}_\mathbb{R})$ *of* $\mathrm{O}(\mathbb{R}^2, \mathcal{H}_\mathbb{R})$ *has no semibounded unitary representation* $(\pi, \mathcal{H}_0)$ *with* $\mathfrak{p} \not\subseteq \ker(\mathtt{d}\pi)$.

Proof. Put $G := \mathrm{O}(\mathbb{R}^2, \mathcal{H}_\mathbb{R})$ and write $\widetilde{G}$ for its universal covering group so that $K \cong \mathrm{SO}_2(\mathbb{R}) \times \mathrm{O}(\mathcal{H}_\mathbb{R})$ and $\widetilde{K} \cong \mathbb{R} \times \mathrm{O}(\mathcal{H}_\mathbb{R}) \subseteq \widetilde{G}$.

Let (ρ, V) be an irreducible holomorphically inducible bounded representation of $\widetilde{K}$. Then ρ is of the form $\rho(t, k) = e^{ict}\rho_0(k)$, where (ρ_0, V) is an irreducible representation of $\mathrm{O}(\mathcal{H}_\mathbb{R})$. In view of Theorem 5.17, it suffices to show that, if $\rho|_{\mathrm{O}_\infty(\mathcal{H}_\mathbb{R})}$ is irreducible and non-trivial, then (ρ, V) is not inducible.

According to Theorem D.5, every irreducible bounded representation (ρ, V) of $\mathrm{O}_\infty(\mathcal{H}_\mathbb{R})$ is a highest weight representation (ρ_λ, V) with finitely supported highest weight. Now [NO98, Sect. 1, Thm. IV.1] implies that the kernel Q_{ρ_λ} on G/K is not positive definite for $\lambda \neq 0$, so that (ρ_λ, V) is not inducible. □

Remark 7.6. The results of the present section imply in particular that the unitary representations of the groups G_1 constructed in [NO98] all extend to unitary representations of the full hermitian groups $\widehat{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$, $\widehat{\mathrm{O}}^*_{\mathrm{res}}(\mathcal{H})$ and $\widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H})$.

8 Hermitian groups with negative JH^*-triples

In this section we consider the case where $\mathfrak{p}$ is an infinite dimensional negative simple JH^*-triple, i.e., G/K is c-dual to a symmetric Hilbert domain of type I-IV, i.e., a hermitian symmetric space of "compact type" in the sense of Kaup (cf. [Ka81, Ka83]). We assume that G is a simply connected Lie group with Lie algebra $\mathfrak{g} = \widehat{\mathfrak{g}}(\mathfrak{p})$ (Remark 3.8) and recall that this implies that K has the product structure required in Theorem 5.17.

Theorem 8.1. *Suppose that G is full and simply connected and that $\mathfrak{p}$ is an infinite dimensional negative JH^*-triple. Then the following are equivalent for a bounded unitary irreducible representation (ρ, V) of K:*

(i) *(ρ, V) is holomorphically inducible.*

(ii) $\mathtt{d}\rho([z^*, z]) \geq 0$ *for* $z \in \mathfrak{p}^+$ *(ρ is anti-dominant).*

Proof. (i) $\Rightarrow$ (ii) follows from Lemma 5.11.

(ii) $\Rightarrow$ (i): In view of Theorem 5.18, we may w.l.o.g. assume that $\rho|_{K_\infty}$ is irreducible, i.e., $\rho = \rho_0$. We have seen in Remark 3.11 that, for all types, we have a surjective submersion $G_1 \rtimes K \to G$. In view of Remark 5.22, it suffices to show that the function f_ρ is positive definite in a 0-neighborhood of the group G_1 if (ii) is satisfied. To verify this, we consider all types separately, where for type I-III we take $d = \frac{i}{2}\operatorname{diag}(\mathbf{1}, -\mathbf{1})$.

(I_∞) Here $G = \widehat{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}_+ \oplus \mathcal{H}_-)$ and $K = \mathbb{T} \times \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ (Example 3.5(c)), so that ρ has the form $\rho(z, k_1, k_2) = z^c \rho_{\lambda_+}(k_1) \otimes \rho_{\lambda_-}(k_2)$, where $c \in \mathbb{Z}$ and $\lambda_\pm \colon J_\pm \to \mathbb{Z}$ are finitely supported functions, corresponding to the highest weights, and $(e_j)_{j \in J_\pm}$ in $\mathcal{H}_\pm$ are orthonormal bases. From

$$G_1 = \mathrm{SU}_{1,2}(\mathcal{H}_+ \oplus \mathcal{H}_-), \quad K_1 = S(\mathrm{U}_1(\mathcal{H}_+) \times \mathrm{U}_1(\mathcal{H}_-)) \cong \mathbb{T} \rtimes (\mathrm{SU}(\mathcal{H}_+) \times \mathrm{SU}(\mathcal{H}_-))$$

(cf. Example 3.5 and Appendix C.7), we obtain on K_1 a representation with the bounded highest weight $\lambda = (c + \lambda_+, \lambda_-)$. Condition (ii) now leads with Lemma 5.13 and (8) in Example 5.14(a) to

$$c + \max(\lambda_+) \leq \min(\lambda_-). \tag{14}$$

Hence there exists a linear order $\preceq$ on $J = J_+ \dot\cup J_-$ with $J_+ \prec J_-$ for which $\lambda \colon J \to \mathbb{Z}$ is increasing. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ denote the corresponding bounded highest weight representation of $\mathrm{SU}(\mathcal{H})$ whose existence follows from [Ne98, Prop. III.7].

In view of (14), any highest weight vector v_λ generates the K_1-representation $(\rho_\lambda, V_\lambda)$ of highest weight λ annihilated by $\widetilde{\mathfrak{p}}^- := \mathfrak{p}^- \cap \mathfrak{gl}_1(\mathcal{H})$. For the orthogonal projection $p_V \colon \mathcal{H}_\lambda \to V_\lambda$ we therefore find

$$f_\rho(g) = p_V \pi_\lambda(g) p_V \in B(V_\lambda)$$

because this relation holds for $g \in K_1 \subseteq \mathrm{SU}(\mathcal{H})$ and both sides are annihilated by the differential operators L_z, $z \in \widetilde{\mathfrak{p}}^-$. We conclude that the function f_ρ is positive definite on the dense subset $W_G W_G^{-1} \cap \mathrm{SU}(\mathcal{H})$ of G_1, so that its continuity implies that it is positive definite.

(I_{fin}) Here we have $G = \mathrm{U}(\mathcal{H}_+ \oplus \mathcal{H}_-)$ and $K = \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$, so that ρ has the form $\rho(k_1, k_2) = \rho_{\lambda_+}(k_1) \otimes \rho_{\lambda_-}(k_2)$ and $\lambda = (\lambda_+, \lambda_-)$ has finite support. In view of Lemma 5.13 and (8) in Example 5.14(b), condition (ii) here means that

$$\max(\lambda_+) \leq \min(\lambda_-). \tag{15}$$

Hence there exists a linear order $\preceq$ on $J = J_+ \dot\cup J_-$ with $J_+ \prec J_-$ for which λ is increasing. Let $(\pi_\lambda, \mathcal{H}_\lambda)$ denote the corresponding bounded highest weight representation of G (cf. Appendix D). As a consequence of (15), any highest weight vector v_λ generates the K-representation $(\rho_\lambda, V_\lambda)$ of highest weight λ, and the subspace V_λ is annihilated by $\mathfrak{p}^-$. As above, it now follows that

$$f_\rho(g) = p_V \pi_\lambda(g) p_V \quad \text{for} \quad g \in G,$$

and hence that ρ is inducible.

(II) Here $G = \widehat{\mathrm{O}}_{\mathrm{res}}(\mathcal{H}_{\mathbb{R}})$ (cf. Example C.9 and Appendix C.6) and $K \cong \mathbb{T} \times \mathrm{U}(\mathcal{H})$ (cf. Example 3.6), so that ρ has the form $\rho(z,k) = z^c \rho_\mu(k)$ with $c \in \mathbb{Z}$ and highest weight $\mu\colon J \to \mathbb{Z}$. From $G_1 = \widetilde{\mathrm{O}}_{1,2}(\mathcal{H}_{\mathbb{R}})$ and the fact that $K_1 \cong \widehat{\mathrm{U}}_1(\mathcal{H})$ is the unique 2-fold covering of $\mathrm{U}_1(\mathcal{H}) \cong \mathbb{T} \ltimes \mathrm{SU}(\mathcal{H})$, we obtain on K_1 the representation with the bounded highest weight $\lambda := c/2 + \mu$ (cf. Example 3.6). With Lemma 5.13 and Example C.9 condition (ii) now translates into

$$\lambda_j + \lambda_k = c + \mu_j + \mu_k \leq 0 \quad \text{for} \quad j \neq k \in J. \tag{16}$$

As μ is integral with finite support, we get $c \leq 0$. We pick a linear order on J for which $\lambda\colon J \to \mathbb{Z}$ is increasing. From [Ne98, Sect. VII] we obtain a bounded highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $\widetilde{\mathrm{O}}_1(\mathcal{H}_{\mathbb{R}})$, resp., a holomorphic representation of its universal complexification $\widetilde{\mathrm{O}}_1(\mathcal{H}_{\mathbb{R}})_{\mathbb{C}} \cong \mathrm{O}_1(\mathcal{H}^2, \beta)$ (cf. Appendix C.1). The remaining argument is similar as for type I_∞ and uses the density of $\widetilde{\mathrm{O}}_1(\mathcal{H}_{\mathbb{R}})$ in $G_1 = \widetilde{\mathrm{O}}_{1,2}(\mathcal{H}_{\mathbb{R}})$.

(III) Here $G = \widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H}_{\mathbb{H}}) \subseteq G_{\mathbb{C}} \cong \widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H}^2, \omega)$ (cf. Example C.8), and the argument is quite similar to type III. Condition (ii) translates into

$$\lambda_j = c + \mu_j \leq 0 \quad \text{for} \quad j \in J,$$

$c \leq 0$, and $\lambda_j \in \frac{1}{2}\mathbb{Z}$ for every $j \in J$. Since λ is bounded, we can argue with a holomorphic highest weight representation of the group $\mathrm{Sp}_1(\mathcal{H}^2, \omega)$ whose existence follows from [Ne98, Sect. VI].

(IV) Here $G = \mathrm{O}(\mathbb{R}^2 \oplus \mathcal{H}_{\mathbb{R}})$ is a full orthogonal group and

$$K = \mathrm{SO}_2(\mathbb{R}) \times \mathrm{O}(\mathcal{H}_{\mathbb{R}}) \cong \mathbb{T} \times \mathrm{O}(\mathcal{H}_{\mathbb{R}}),$$

so that ρ has the form $\rho(z,k) = z^c \rho_\mu(k)$ with $c \in \mathbb{Z}$ and a bounded highest weight representation ρ_μ of $\mathrm{O}(\mathcal{H}_{\mathbb{R}})$. Let J parameterize an orthonormal basis of $\mathbb{C} \oplus (\mathcal{H}_{\mathbb{R}}, I)$, where I is a complex structure on $\mathcal{H}_{\mathbb{R}}$, such that $j_0 \in J$ corresponds to the basis element $e_{j_0} := (1, 0)$. Then $\lambda_{j_0} := c$ and $\lambda|_{J \setminus \{j_0\}} := \mu$ defines a function $\lambda\colon J \to \mathbb{Z}$. With Lemma 5.13 and Example C.10, we see that condition (ii) means that

$$c \pm \mu_j \geq 0 \quad \text{for} \quad j \neq j_0, \tag{17}$$

i.e., $c \geq 0$ and $|\mu_j| \leq c$ (cf. [NO98, Sect. I]). From [Ne98, Sect. VII] we now obtain a bounded highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of G. As a consequence of (17), any highest weight vector v_λ generates the K-representation $(\rho_\lambda, V_\lambda)$ of highest weight λ, and the subspace V_λ is annihilated by $\mathfrak{p}^-$. With similar arguments as for type $\mathrm{I}_{\mathrm{fin}}$, it now follows that

$$f_\rho(g) = p_V \pi_\lambda(g) p_V \in B(V_\lambda)$$

is positive definite, so that ρ is inducible. □

Remark 8.2. Suppose that, in the preceding proof, for one of the groups

$$\widehat{\mathrm{U}}_{\mathrm{res}}(\mathcal{H}), \quad \widehat{\mathrm{O}}_{\mathrm{res}}(\mathcal{H}_{\mathbb{R}}), \quad \widehat{\mathrm{Sp}}_{\mathrm{res}}(\mathcal{H}_{\mathbb{H}})$$

we have $c = 0$. Then the corresponding highest weight λ is finitely supported and we derive from [Ne98, Sects. V-VII] that the corresponding unitary highest weight representation of the corresponding non-extended group

$$\mathrm{U}_{\mathrm{res}}(\mathcal{H}), \quad \mathrm{O}_{\mathrm{res}}(\mathcal{H}_{\mathbb{R}}), \quad \mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_{\mathbb{H}})$$

extends to a bounded representation of the corresponding full group

$$\mathrm{U}(\mathcal{H}), \quad \mathrm{O}(\mathcal{H}_{\mathbb{R}}), \quad \mathrm{Sp}(\mathcal{H}_{\mathbb{H}}).$$

In particular, the corresponding highest weight representation is bounded.

Suppose, conversely, that $c \neq 0$. We want to show that the corresponding highest weight representation π_λ is unbounded. To this end, it suffices to show that for the corresponding set $\mathcal{P}_\lambda$ of weights, the set of all values $\mathcal{P}_\lambda(d)$ is infinite.

For type I_∞ we have $\lambda = (c + \lambda_+, \lambda_-)\colon J \to \mathbb{Z}$ and the group $\mathcal{W} = S_{(J)}$ of all finite permutations of J acts on the weight set $\mathcal{P}_\lambda$ with respect to $\mathfrak{t}_1 \subseteq \mathfrak{k}_1$. Let $F \subseteq J_+$ be a finite subset not contained in the support of λ_+ and $w \in \mathcal{W}$ be an involution with $w(F_+) \subseteq J_- \setminus \mathrm{supp}(\lambda_-)$ and $w(j) = j$ for $j \notin F_+ \cup w(F_+)$. Then

$$\lambda - w\lambda = c \sum_{j \in F_+} (\varepsilon_j - \varepsilon_{w(j)})$$

has on $-i \cdot d$ the value $c|F_+|$, which can be arbitrarily large.

For type II we have $\lambda = c/2 + \mu$ and the weight set is invariant under the Weyl group $\mathcal{W}$ which contains in particular sign changes $s(h_j) = (\varepsilon_j h_j)$, where $\varepsilon_j \in \{\pm 1\}$, and $F = \{j \in J\colon \varepsilon_j = -1\}$ can be any finite subset of J with an even number of elements (cf. Example C.6). For $F \cap \mathrm{supp}(\mu) = \emptyset$, we then obtain

$$(\lambda - w\lambda)(-i \cdot d) = c|F|,$$

which can be arbitrarily large.

For type III we have $\lambda = c + \mu$ and a similar argument applies.

Example 8.3. (Finite type I) If one of the spaces $\mathcal{H}_+$ or $\mathcal{H}_-$ in $\mathcal{H} = \mathcal{H}_+ + \mathcal{H}_-$ is finite dimensional, then $G = \mathrm{U}_{\mathrm{res}}(\mathcal{H}) = \mathrm{U}(\mathcal{H})$ is the full unitary group. Since every open invariant cone in $\mathfrak{u}(\mathcal{H})$ intersects the center (Theorem 4.6), all irreducible semibounded representations of G are bounded (Proposition 4.4).

According to Theorem 5.17, every bounded irreducible representation (ρ, V) of $\mathrm{U}(\mathcal{H})$ is a tensor product of a representation (ρ_0, V_0) whose restriction to $\mathrm{U}_\infty(\mathcal{H})$ is irreducible and of a representation (ρ_1, V_1) with $\mathrm{U}_\infty(\mathcal{H}) \subseteq \ker \rho_1$. The representations ρ_0 are easily classified by their highest weights, as described in Appendix D but for the representations ρ_1 there is no concrete classification available (cf. Remark 5.19).

Example 8.4. (Infinite type I) Consider the group $\widehat{G} = \widehat{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ with $\dim \mathcal{H}_+ = \dim \mathcal{H}_- = \infty$. For representations which are trivial on the center, i.e., $c = 0$, we obtain for $\lambda = (\lambda_+, \lambda_-)$ the necessary condition $\lambda_+ \leq 0 \leq \lambda_-$, which leads to a representation of $K \cong \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ on a subspace

$$\mathcal{H}_{\lambda_+} \otimes \mathcal{H}_{\lambda_-} \subseteq (\mathcal{H}_+^*)^{\otimes n} \otimes \mathcal{H}_-^{\otimes m} \subseteq (\mathcal{H}^*)^{\otimes n} \otimes \mathcal{H}^{\otimes m}$$

(cf. Appendix D).

On the latter space we have a unitary representation of $\mathrm{U}(\mathcal{H})$. We claim that it contains a subrepresentation which is holomorphically induced from the representation of $\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$ on $\mathcal{H}_{\lambda_+} \otimes \mathcal{H}_{\lambda_-}$. In fact, the natural action of $\mathfrak{p}^- \cong B_2(\mathcal{H}_+, \mathcal{H}_-)$ annihilates the subspace $\mathcal{H}_{\lambda_+} \subseteq (\mathcal{H}_+^*)^{\otimes n} \subseteq (\mathcal{H}^*)^{\otimes n}$ and likewise, it annihilates the subspace $\mathcal{H}_{\lambda_-} \subseteq \mathcal{H}_-^{\otimes m} \subseteq \mathcal{H}^{\otimes m}$. Now the assertion follows from [Ne10d, Cor. 3.9].

A Analytic operator-valued positive definite functions

In this appendix we discuss operator-valued positive definite functions on Lie groups. The main result is Theorem A.7, asserting that local analytic positive definite functions extend to global ones. This generalizes the corresponding result for the scalar case in [Ne10b].

Definition A.1. Let X be a set and $\mathcal{K}$ be a Hilbert space.

(a) A function $Q\colon X \times X \to B(\mathcal{K})$ is called a *$B(\mathcal{K})$-valued kernel.* It is said to be *hermitian* if $Q(z,w)^* = Q(w,z)$ holds for all $z, w \in X$.

(b) A hermitian $B(\mathcal{K})$-valued kernel K on X is said to be *positive definite* if for every finite sequence $(x_1, v_1), \ldots, (x_n, v_n)$ in $X \times \mathcal{K}$ we have

$$\sum_{j,k=1}^{n} \langle Q(x_j, x_k)v_k, v_j\rangle \geq 0.$$

(c) If $(S, *)$ is an involutive semigroup, then a function $\varphi\colon S \to B(\mathcal{K})$ is called *positive definite* if the kernel $Q_\varphi(s,t) := \varphi(st^*)$ is positive definite.

(d) Positive definite kernels can be characterized as those for which there exists a Hilbert space $\mathcal{H}$ and a function $\gamma\colon X \to B(\mathcal{H}, \mathcal{K})$ such that

$$Q(x,y) = \gamma(x)\gamma(y)^* \quad \text{for} \quad x, y \in X$$

(cf. [Ne00, Thm. I.1.4]). Here one may assume that the vectors $\gamma(x)^*v$, $x \in X, v \in \mathcal{K}$, span a dense subspace of $\mathcal{H}$. Then the pair $(\gamma, \mathcal{H})$ is called a *realization of K*. The map $\Phi\colon \mathcal{H} \to \mathcal{K}^X, \Phi(v)(x) := \gamma(x)v$, then realizes $\mathcal{H}$ as a Hilbert subspace of $\mathcal{K}^X$ with continuous point evaluations $\mathrm{ev}_x\colon \mathcal{H} \to \mathcal{K}$. It is the unique Hilbert subspace in $\mathcal{K}^X$ with this property for which $Q(x,y) = \mathrm{ev}_x \mathrm{ev}_y^*$ for $x, y \in X$. We write $\mathcal{H}_Q \subseteq \mathcal{K}^X$ for this subspace and call it the *reproducing kernel Hilbert space with kernel Q*.

Remark A.2. Let $Q\colon X \times X \to B(\mathcal{K})$ and $f\colon X \to \mathrm{GL}(\mathcal{K})$ be functions. It is obvious that the kernel Q is positive definite if and only if the modified kernel

$$(x,y) \mapsto f(x)Q(x,y)f(y)^*$$

is positive definite.

Theorem A.3. *Let M be an analytic Fréchet manifold and $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces. Then a function $\gamma\colon M \to B(\mathcal{H}, \mathcal{K})$ is analytic if and only if the kernel $Q(x,y) := \gamma(x)\gamma(y)^* \in B(\mathcal{K})$ is analytic on an open neighborhood of the diagonal in $M \times M$.*

Proof. Since the map $B(\mathcal{H},\mathcal{K}) \times B(\mathcal{H},\mathcal{K}) \to B(\mathcal{K}), (A,B) \mapsto AB^*$ is continuous and real bilinear, it is real analytic. Therefore the analyticity of γ implies that Q is analytic.

Conversely, we assume that Q is analytic. On the analytic Fréchet manifold $X := M \times \mathcal{K}$ we obtain the positive definite kernel

$$\widetilde{Q}((m,v),(m',v')) := \langle Q(m,m')v', v\rangle = \langle \gamma(m)\gamma(m')^*v', v\rangle = \langle \gamma(m')^*v', \gamma(m)^*v\rangle$$

which is analytic by assumption. Therefore [Ne10b, Thm. 5.1] implies that the map

$$\Gamma\colon M \times \mathcal{K} \to \mathcal{H}, \quad (m,v) \mapsto \gamma(m)^*v$$

is analytic.

Since the assertion we have to prove is local, we may w.l.o.g. assume that M is an open subset of a real Fréchet space V. Pick $x_0 \in M$. Then the definition of analyticity implies the existence of an open neighborhood U_V of x_0 in the complexification $V_{\mathbb{C}}$ with $U_V \cap V \subseteq M$ and an open 0-neighborhood $U_{\mathcal{K}} \subseteq \mathcal{K}$ such that Γ extends to a holomorphic function

$$\Gamma_{\mathbb{C}}\colon U_V \times U_{\mathcal{K}} \to \mathcal{H}.$$

Then $\Gamma_{\mathbb{C}}(m,v)$ is linear in the second argument, hence of the form $\Gamma_{\mathbb{C}}(m,v) = \gamma_{\mathbb{C}}(m)^*v$, where $\gamma_{\mathbb{C}}(m)^* \in B(\mathcal{K},\mathcal{H})$. From $\|\gamma(m)\|^2 = \|Q(m,m)\|$ we further derive that γ is locally bounded, so that we may w.l.o.g. assume that $\gamma_{\mathbb{C}}$ is bounded. Now [Ne00, Prop. I.1.9] implies that $\gamma_{\mathbb{C}}\colon U_V \to B(\mathcal{K})$ is holomorphic, and hence that γ is real analytic in a neighborhood of x_0. ☐

Definition A.4. Let $\mathcal{K}$ be a Hilbert space, G be a group, and $U \subseteq G$ be a subset. A function $\varphi\colon UU^{-1} \to B(\mathcal{K})$ is said to be *positive definite* if the kernel

$$Q_\varphi\colon U \times U \to B(\mathcal{K}), \quad (x,y) \mapsto \varphi(xy^{-1})$$

is positive definite.

Definition A.5. A Lie group G is said to be *locally exponential* if it has an exponential function $\exp\colon \mathfrak{g} = \mathbf{L}(G) \to G$ for which there is an open 0-neighborhood U in $\mathbf{L}(G)$ mapped diffeomorphically onto an open subset of G. If, in addition, G is analytic and the exponential function is an analytic local diffeomorphism in 0, then G is called a *BCH–Lie group*. Then the Lie algebra $\mathbf{L}(G)$ is a *BCH–Lie algebra*, i.e., there exists an open 0-neighborhood $U \subseteq \mathfrak{g}$ such that, for $x,y \in U$, the BCH series

$$x * y = x + y + \frac{1}{2}[x,y] + \cdots$$

converges and defines an analytic function $U \times U \to \mathfrak{g}, (x,y) \mapsto x * y$. The class of BCH–Lie groups contains in particular all Banach–Lie groups ([Ne06, Prop. IV.1.2]).

Definition A.6. Let $(\rho, \mathcal{D})$ be a $*$-representation of $U_{\mathbb{C}}(\mathfrak{g})$ on the pre-Hilbert space $\mathcal{D}$, i.e.,

$$\langle \rho(D)v, w\rangle = \langle v, \rho(D^*)w\rangle \quad \text{for} \quad D \in U_{\mathbb{C}}(\mathfrak{g}), v, w \in \mathcal{D}.$$

We call a subset $E \subseteq \mathcal{D}$ *equianalytic* if there exists a 0-neighborhood $U \subseteq \mathfrak{g}$ such that

$$\sum_{n=0}^{\infty} \frac{\|\rho(x)^n v\|}{n!} < \infty \quad \text{for} \quad v \in E, x \in U.$$

This implies in particular that each $v \in E$ is an analytic vector for every $\rho(x)$, $x \in \mathfrak{g}$.

Theorem A.7. (Extension of local positive definite analytic functions) *Let G be a simply connected Fréchet–BCH–Lie group, $V \subseteq G$ an open connected $\mathbf{1}$-neighborhood, $\mathcal{K}$ be a Hilbert space and $\varphi\colon VV^{-1} \to B(\mathcal{K})$ be an analytic positive definite function. Then there exists a unique analytic positive definite function $\widetilde{\varphi}\colon G \to B(\mathcal{K})$ extending φ.*

Proof. The uniqueness of $\widetilde{\varphi}$ follows from the connectedness of G and the uniqueness of analytic continuation.

Step 1: To show that $\widetilde{\varphi}$ exists, we consider the reproducing kernel Hilbert space $\mathcal{H}_Q \subseteq \mathcal{K}^V$ defined by the kernel Q via $f(g) = Q_g f$ for $g \in V$ and $Q(h, g) = Q_h Q_g^*$ for $g, h \in V$ (Definition A.1(d)). Then the analyticity of the function

$$V \times V \to B(\mathcal{K}), \quad (g, h) \mapsto Q(g, h) = \varphi(gh^{-1})$$

implies that the map $\eta\colon V \to B(\mathcal{H}_Q, \mathcal{K}), g \mapsto Q_g$ is analytic (Theorem A.3). Here we use that G is Fréchet. Hence all functions in $\mathcal{H}_Q$ are analytic, so that we obtain for each $x \in \mathfrak{g}$ an operator

$$L_x\colon \mathcal{H}_Q \to C^\omega(V, \mathcal{K}), \quad (L_x f)(g) := \frac{d}{dt}\Big|_{t=0} f(g \exp(tx)) = \frac{d}{dt}\Big|_{t=0} Q_{g\exp(tx)} f, \quad g \in V.$$

For $v \in \mathcal{K}$, we then have

$$\begin{aligned}(L_x Q_h^* v)(g) &= \frac{d}{dt}\Big|_{t=0} Q_{g\exp(tx)} Q_h^* v = \frac{d}{dt}\Big|_{t=0} \varphi(g\exp(tx)h^{-1})v \\ &= \frac{d}{dt}\Big|_{t=0} Q_g Q^*_{h\exp(-tx)} v = \frac{d}{dt}\Big|_{t=0} (Q^*_{h\exp(-tx)} v)(g)\end{aligned}$$

for $g, h \in V, x \in \mathfrak{g}$, which means that

$$L_x Q_h^* = \frac{d}{dt}\Big|_{t=0} Q^*_{h\exp(-tx)} \in B(\mathcal{K}, \mathcal{H}_Q).$$

Iterating this argument, we see by induction that, for $x_1, \ldots, x_n \in \mathfrak{g}$,

$$L_{x_1} \cdots L_{x_n} Q_h^* = \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\Big|_{t_1=\ldots=t_n=0} Q^*_{h\exp(-t_n x_n)\cdots\exp(-t_1 x_1)} \tag{18}$$

defines a bounded operator $\mathcal{K} \to \mathcal{H}_Q$.

Step 2: For an open subset $W \subseteq V$, we thus obtain the subspace

$$\mathcal{D}(W) := \operatorname{span}\{L_{x_1} \cdots L_{x_n} Q_h^* v\colon v \in \mathcal{K}, h \in W, x_1, \ldots, x_n \in \mathfrak{g}, n \in \mathbb{N}_0\}$$

of $\mathcal{H}_Q$ and operators

$$\rho(x) := L_x|_{\mathcal{D}} \colon \mathcal{D}(V) \to \mathcal{D}(V), \quad x \in \mathfrak{g},$$

defining a representation $\rho\colon \mathfrak{g} \to \mathrm{End}(\mathcal{D}(V))$.[5] Here we use that $\mathcal{D}(V) \subseteq C^\omega(V, \mathcal{K})$ and the fact that $\mathfrak{g}$ acts by left invariant vector fields on this space. Next we observe that

$$\begin{aligned}
&\langle L_{x_1} \cdots L_{x_n} Q_h^* v, Q_g^* w\rangle \\
&= \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\Big|_{t_1=\ldots=t_n=0} \langle Q_g Q^*_{h\exp(-t_n x_n)\cdots\exp(-t_1 x_1)} v, w\rangle \\
&= \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\Big|_{t_1=\ldots=t_n=0} \langle \varphi(g\exp(t_1 x_1)\cdots\exp(t_n x_n)h^{-1})v, w\rangle \\
&= \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\Big|_{t_1=\ldots=t_n=0} \langle Q_h^* v, Q^*_{g\exp(t_1 x_1)\cdots\exp(t_n x_n)} w\rangle \\
&= \langle Q_h^* v, (-L_{x_n})\cdots(-L_{x_1}) Q_g^* w\rangle.
\end{aligned}$$

Therefore $\rho(x) \subseteq -\rho(x)^*$, and thus ρ extends to a $*$-representation of $U_{\mathbb{C}}(\mathfrak{g})$ on $\mathcal{D}(V)$.

Step 3: Since η is analytic, we derive from (18) for each $g \in V$ and sufficiently small $x \in \mathfrak{g}$, the relation

$$Q^*_{g\exp(-x)} = \sum_{n=0}^\infty \frac{1}{n!}\rho(x)^n Q_g^*. \tag{19}$$

Since derivatives of analytic functions are also analytic, using (18) again implies that $\mathcal{D}(V)$ consists of analytic vectors for the representation ρ of $\mathfrak{g}$.

Step 4: Let $W_G \subseteq V$ be an open $\mathbf{1}$-neighborhood and $W_{\mathfrak{g}} \subseteq \mathfrak{g}$ an open balanced 0-neighborhood with $W_G \exp(W_{\mathfrak{g}}) \subseteq V$. Next we show that $\mathcal{D}(W_G)$ is equianalytic and spans a dense subspace of $\mathcal{H}_Q$.

Since the map $W_G \times W_{\mathfrak{g}} \to B(\mathcal{K}, \mathcal{H}_Q), (g,x) \mapsto Q^*_{g\exp(x)}$ is analytic, [Ne10b, Lemma 7.2] shows that, after shrinking W_G and $W_{\mathfrak{g}}$, we may assume that

$$Q^*_{g\exp(-x)} = \sum_{n=0}^\infty \frac{1}{n!}\rho(x)^n Q_g^* \quad \text{for} \quad g \in W_G, x \in W_{\mathfrak{g}}. \tag{20}$$

This implies that $\mathcal{D}(W_G)$ is equianalytic. To see that $\mathcal{D}(W_G)$ is dense, we use the analyticity of η to see that $\eta(V)^*\mathcal{K} \subseteq \overline{\mathcal{D}(W_G)}$, which implies that $\mathcal{D}(W_G)$ is dense in $\mathcal{H}_Q$.

Applying [Ne10b, Thm. 6.8] to the $*$-representation $(\rho, \mathcal{D}(W_G))$, we now obtain a continuous unitary representation $(\pi, \mathcal{H}_Q)$ of G with $\pi(\exp(x)) = e^{\overline{\rho(x)}}$ for every $x \in \mathfrak{g}$. Then

$$\widetilde{\varphi}(g) := Q_{\mathbf{1}}\pi(g)Q_{\mathbf{1}}^*$$

[5] In [Ne10b, Thm. 7.3] we prove a version of the present theorem for the special case $\mathcal{K} = \mathbb{C}$. In loc.cit. we claim that $\mathcal{H}_Q^0$ is invariant under the Lie algebra $\mathfrak{g}$, but this need not be the case. The argument given here, where $\mathcal{H}_Q^0$ is enlarged to $\mathcal{D}(V)$ corrects this point.

is a positive definite $B(\mathcal{K})$-valued function on G, and for $x \in W_{\mathfrak{g}}$ we obtain from (20)

$$\widetilde{\varphi}(\exp x)v = Q_{\mathbf{1}}\pi(\exp x)Q_{\mathbf{1}}^* v = Q_{\mathbf{1}} \sum_{n=0}^{\infty} \frac{\rho(x)^n}{n!} Q_{\mathbf{1}}^* v = Q_{\mathbf{1}} Q_{-\exp x}^* v = \varphi(\exp x)v.$$

Since the kernel

$$(g,h) \mapsto \widetilde{\varphi}(gh^{-1}) = Q_{\mathbf{1}}\pi(gh^{-1})Q_{\mathbf{1}}^* = (Q_{\mathbf{1}}\pi(g))(Q_{\mathbf{1}}\pi(h))^*$$

is analytic in a neighborhood of the diagonal, Theorem A.3 implies that the map

$$G \to B(\mathcal{H}_Q, \mathcal{K}), \quad g \mapsto Q_{\mathbf{1}}\pi(g)$$

is analytic, and this in turn implies that $\widetilde{\varphi}$ is analytic. As $\widetilde{\varphi}$ coincides with φ in a $\mathbf{1}$-neighborhood, the analyticity of φ and $\widetilde{\varphi}$, together with the connectedness of V lead to $\widetilde{\varphi}|_V = \varphi$. □

B Applications to holomorphically induced representations

Let $\mathfrak{g}$ be a Banach–Lie algebra and $d \in \mathfrak{g}$ be an elliptic element, i.e., $e^{\mathbb{R}\,\mathrm{ad}\,d}$ is bounded. We say that d satisfies the *splitting condition* if 0 is an isolated spectral value of $\mathrm{ad}\,d$ (cf. [Ne10d]). With $\mathfrak{h} := \ker \mathrm{ad}\,d$, we then obtain an $\mathrm{ad}\,d$-invariant direct sum decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{p}^-$, where the spectrum of $\mp i\,\mathrm{ad}\,d$ on $\mathfrak{p}^{\pm}$ is contained in $]0,\infty[$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We consider a bounded representation (ρ, V) of $H = \langle \exp \mathfrak{h} \rangle \subseteq G$ and want to obtain criteria for the holomorphic inducibility of (ρ, V).

We recall the closed subalgebra $\mathfrak{q} = \mathfrak{p}^+ \rtimes \mathfrak{h}_{\mathbb{C}} \subseteq \mathfrak{g}_{\mathbb{C}}$. Let $U_{\pm} \subseteq \mathfrak{p}^{\pm}$ and $U_0 \subseteq \mathfrak{h}_{\mathbb{C}}$ be open convex 0-neighborhoods for which the BCH-multiplication map

$$U_+ \times U_0 \times U_- \to \mathfrak{g}_{\mathbb{C}}, \quad (x_+, x_0, x_-) \mapsto x_+ * x_0 * x_-$$

is biholomorphic onto an open subset U of $\mathfrak{g}_{\mathbb{C}}$. We then define a holomorphic map

$$F \colon U \to B(V), \quad F(x_+ * x_0 * x_-) := e^{\mathtt{d}\rho(x_0)}.$$

Theorem B.1. *The following are equivalent:*

(i) (ρ, V) *is holomorphically inducible.*

(ii) $f_\rho(\exp x) := F(x)$ *defines a positive definite analytic function on a* $\mathbf{1}$*-neighborhood of* G.

Proof. (i) ⇒ (ii): Let $(\pi, \mathcal{H})$ be the unitary representation of G obtained by holomorphic induction from (ρ, V). We identify V with the corresponding closed subspace of $\mathcal{H}$ and write $p_V \colon \mathcal{H} \to V$ for the corresponding orthogonal projection. For $v \in V \subseteq (\mathcal{H}^\omega)^{\mathfrak{p}^-}$ ([Ne10d, Rem. 2.18]) we let $f_\rho^v \colon U_v \to \mathcal{H}$ be a holomorphic map on an open convex 0-neighborhood $U_v \subseteq U$ satisfying $f_\rho^v(x) = \pi(\exp x)v$ for $x \in U_v \cap \mathfrak{g}$.

Then $\mathtt{d}\pi(\mathfrak{p}^-)v = \{0\}$ implies that $L_z f^v_\rho = 0$ for $z \in \mathfrak{p}^-$. For $w \in V$ and $z \in \mathfrak{p}^+$, we also obtain

$$\langle (R_z f^v_\rho)(x), w\rangle = \langle \mathtt{d}\pi(z) f^v_\rho(x), w\rangle = \langle f^v_\rho(x), \mathtt{d}\pi(z^*)w\rangle = 0.$$

This proves that $R_z(p_V \circ f_v) = 0$. We conclude that, for $x_\pm$ and x_0 sufficiently close to 0, we have

$$p_V f^v_\rho(x_+ * x_0 * x_-) = f\rho^v(x_0) = e^{\mathtt{d}\rho(x_0)}v = F(x_+ * x_0 * x_-)v.$$

Therefore $p_V \circ f^v_\rho$ extends holomorphically to U and

$$\langle \pi(\exp x)v, w\rangle = \langle F(x)v, w\rangle \quad \text{for} \quad x \in U_v \cap \mathfrak{g}, v, w \in V.$$

We conclude that $F(x) = p_V \pi(\exp x) p_V$ holds for x sufficiently close to 0, and hence that $f_\rho(\exp x) = p_V \pi(\exp x) p_V$ defines a positive definite function on a $\mathbf{1}$-neighborhood of the real Lie group G.

(ii) $\Rightarrow$ (i): From Theorem A.7 it follows that some restriction of f_ρ to a possibly smaller $\mathbf{1}$-neighborhood in G extends to a global analytic positive definite function φ. Then the vector-valued GNS construction yields a unitary representation of G on the corresponding reproducing kernel Hilbert space $\mathcal{H}_\varphi \subseteq V^G$ for which all the elements of $\mathcal{H}^0_\varphi = \operatorname{span}(\varphi(G)V)$ are analytic vectors. In particular, $V \subseteq \mathcal{H}^\omega_\varphi$ consists of smooth vectors, and the definition of f_ρ implies that $\mathtt{d}\pi(\mathfrak{p}^-)V = \{0\}$. Therefore [Ne10d, Thm. 2.17] implies that the representation $(\pi, \mathcal{H}_\varphi)$ is holomorphically induced from (ρ, V). □

C Classical groups of operators

In this appendix we review the zoo of classical groups of operators on Hilbert spaces showing up in this paper.

C.1 Unitary and general linear groups

For a Hilbert space $\mathcal{H}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we write $\mathrm{GL}(\mathcal{H}) = \mathrm{GL}_\mathbb{K}(\mathcal{H})$ for the group of $\mathbb{K}$-linear topological isomorphisms of $\mathcal{H}$, which is the unit group of the real Banach algebra $B(\mathcal{H})$ of bounded $\mathbb{K}$-linear operators on $\mathcal{H}$. It contains the subgroup

$$\mathrm{U}_\mathbb{K}(\mathcal{H}) := \mathrm{U}(\mathcal{H}) := \{g \in \mathrm{GL}_\mathbb{K}(\mathcal{H}) \colon g^* = g^{-1}\} \tag{21}$$

of unitary operators with Lie algebra $\mathfrak{u}(\mathcal{H}) = \{x \in \mathfrak{gl}(\mathcal{H}) \colon x^* = -x\}$. If $\mathcal{H}$ is real, then we also write $\mathrm{O}(\mathcal{H}) := \mathrm{U}_\mathbb{R}(\mathcal{H})$, and if $\mathcal{H}$ is quaternionic, we also write $\mathrm{Sp}(\mathcal{H}) := \mathrm{U}_\mathbb{H}(\mathcal{H})$ for the corresponding unitary groups.

In many situations it is convenient to describe real Hilbert spaces as pairs $(\mathcal{H}, \sigma)$, where $\mathcal{H}$ is a complex Hilbert space and $\sigma \colon \mathcal{H} \to \mathcal{H}$ is a *conjugation*, i.e., an antilinear isometry with $\sigma^2 = \mathrm{id}_\mathcal{H}$. Then $\mathcal{H}^\sigma := \{v \in \mathcal{H} \colon \sigma(v) = v\}$ is a real Hilbert space and, conversely, every real Hilbert space $\mathcal{H}$ arises this way by the canonical conjugation σ on $\mathcal{H}_\mathbb{C}$ with $(\mathcal{H}_\mathbb{C})^\sigma = \mathcal{H}$. Then

$$\mathrm{O}(\mathcal{H}) \cong \{g \in \mathrm{U}(\mathcal{H}_\mathbb{C}) \colon g\sigma = \sigma g\} \quad \text{and} \quad \mathrm{GL}_\mathbb{R}(\mathcal{H}) \cong \{g \in \mathrm{GL}(\mathcal{H}_\mathbb{C}) \colon g\sigma = \sigma g\}. \tag{22}$$

For $x \in B(\mathcal{H}_\mathbb{C})$ we put $x^\top := \sigma x^* \sigma$ and obtain

$$\mathrm{O}(\mathcal{H}) = \{g \in \mathrm{U}(\mathcal{H}_\mathbb{C}) \colon g^\top = g^{-1}\}, \quad \mathfrak{o}(\mathcal{H}) = \{x \in \mathfrak{u}(\mathcal{H}_\mathbb{C}) \colon x^\top = -x\}.$$

For the complex symmetric bilinear form $\beta(x,y) := \langle x, \sigma y\rangle$ on $\mathcal{H}_\mathbb{C}$, the orthogonal group is

$$\mathrm{O}(\mathcal{H}_\mathbb{C}, \beta) = \{g \in \mathrm{GL}(\mathcal{H}_\mathbb{C}) \colon g^\top = g^{-1}\}$$

with Lie algebra $\mathfrak{o}(\mathcal{H}_\mathbb{C}, \beta) \cong \mathfrak{o}(\mathcal{H})_\mathbb{C}$.

In the following we write $\mathrm{Herm}(\mathcal{H}) := \{x \in B(\mathcal{H}) \colon x^* = x\}$,

$$\mathrm{Sym}(\mathcal{H}) := \{x \in B(\mathcal{H}) \colon x^\top = x\} \quad \text{and} \quad \mathrm{Skew}(\mathcal{H}) := \{x \in B(\mathcal{H}) \colon x^\top = -x\}.$$

A quaternionic Hilbert space $\mathcal{H}$ can be considered as a complex Hilbert space $\mathcal{H}^\mathbb{C}$ (the underlying complex Hilbert space), endowed with an *anticonjugation* σ, i.e., σ is an antilinear isometry with $\sigma^2 = -\mathbf{1}$. Then

$$\mathrm{Sp}(\mathcal{H}) = \{g \in \mathrm{U}(\mathcal{H}^\mathbb{C}) \colon g\sigma = \sigma g\} \quad \text{and} \quad \mathrm{GL}_\mathbb{H}(\mathcal{H}) = \{g \in \mathrm{GL}(\mathcal{H}^\mathbb{C}) \colon g\sigma = \sigma g\}. \quad (23)$$

For $x \in B(\mathcal{H}^\mathbb{C})$ we put $x^\sharp := \sigma x^* \sigma$ and obtain

$$\mathrm{Sp}(\mathcal{H}) = \{g \in \mathrm{U}(\mathcal{H}^\mathbb{C}) \colon g^\sharp = g^{-1}\}, \quad \mathfrak{sp}(\mathcal{H}) = \{x \in \mathfrak{u}(\mathcal{H}^\mathbb{C}) \colon x^\sharp = -x\}.$$

For the complex skew-symmetric bilinear form $\omega(x,y) := \langle x, \sigma y\rangle$ on $\mathcal{H}^\mathbb{C}$, the symplectic group is

$$\mathrm{Sp}(\mathcal{H}^\mathbb{C}, \omega) = \{g \in \mathrm{GL}(\mathcal{H}^\mathbb{C}) \colon g^\sharp = g^{-1}\}$$

with Lie algebra $\mathfrak{sp}(\mathcal{H}^\mathbb{C}, \omega) = \mathfrak{sp}(\mathcal{H})_\mathbb{C}$. In [Ne02] we also use the notation $\mathrm{GL}(\mathcal{H}_\mathbb{C}, \sigma) = \mathrm{O}(\mathcal{H}_\mathbb{C}, \beta)$ in the real case and $\mathrm{GL}(\mathcal{H}^\mathbb{C}, \sigma) = \mathrm{Sp}(\mathcal{H}^\mathbb{C}, \omega)$ in the quaternionic case.

C.2 Root systems of classical groups

Definition C.1. (a) Let $\mathfrak{g}$ be a real Lie algebra and $\mathfrak{g}_\mathbb{C}$ be its complexification. If $\sigma \colon \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ denotes the complex conjugation with respect to $\mathfrak{g}$, we write $x^* := -\sigma(x)$ for $x \in \mathfrak{g}_\mathbb{C}$, so that $\mathfrak{g} = \{x \in \mathfrak{g}_\mathbb{C} \colon x^* = -x\}$. Let $\mathfrak{t} \subseteq \mathfrak{g}$ be a maximal abelian subalgebra and $\mathfrak{h} := \mathfrak{t}_\mathbb{C} \subseteq \mathfrak{g}_\mathbb{C}$ be its complexification. For a linear functional $\alpha \in \mathfrak{h}^*$,

$$\mathfrak{g}_\mathbb{C}^\alpha = \{x \in \mathfrak{g}_\mathbb{C} \colon (\forall h \in \mathfrak{h})\ [h,x] = \alpha(h)x\}$$

is called the corresponding *root space*, and

$$\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \colon \mathfrak{g}_\mathbb{C}^\alpha \neq \{0\}\}$$

is the *root system* of the pair $(\mathfrak{g}_\mathbb{C}, \mathfrak{h})$. We then have $\mathfrak{g}_\mathbb{C}^0 = \mathfrak{h}$ and $[\mathfrak{g}_\mathbb{C}^\alpha, \mathfrak{g}_\mathbb{C}^\beta] \subseteq \mathfrak{g}_\mathbb{C}^{\alpha+\beta}$, hence in particular $[\mathfrak{g}_\mathbb{C}^\alpha, \mathfrak{g}_\mathbb{C}^{-\alpha}] \subseteq \mathfrak{h}$.

(b) If $\mathfrak{g}$ is a Banach–Lie algebra, then we say that $\mathfrak{t}$ is *elliptic* if the group $e^{\mathrm{ad}\,\mathfrak{t}}$ is bounded in $B(\mathfrak{g})$. We then have

(I1) $\alpha(\mathfrak{t}) \subseteq i\mathbb{R}$ for $\alpha \in \Delta$, and therefore

(I2) $\sigma(\mathfrak{g}_{\mathbb{C}}^{\alpha}) = \mathfrak{g}_{\mathbb{C}}^{-\alpha}$ for $\alpha \in \Delta$.

Lemma C.2. *Suppose that $\mathfrak{t} \subseteq \mathfrak{g}$ is elliptic. For $0 \neq x_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$, the subalgebra*

$$\mathfrak{g}_{\mathbb{C}}(x_\alpha) := \mathrm{span}_{\mathbb{C}}\{x_\alpha, x_\alpha^*, [x_\alpha, x_\alpha^*]\}$$

is σ-invariant and of one of the following types:

(A) *The abelian type: $[x_\alpha, x_\alpha^*] = 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ is two dimensional abelian.*

(N) *The nilpotent type: $[x_\alpha, x_\alpha^*] \neq 0$ and $\alpha([x_\alpha, x_\alpha^*]) = 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ is a three dimensional Heisenberg algebra.*

(S) *The simple type: $\alpha([x_\alpha, x_\alpha^*]) \neq 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cong \mathfrak{sl}_2(\mathbb{C})$. In this case we distinguish two cases:*

(CS) *$\alpha([x_\alpha, x_\alpha^*]) > 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g} \cong \mathfrak{su}_2(\mathbb{C})$, and*

(NS) *$\alpha([x_\alpha, x_\alpha^*]) < 0$, i.e., $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g} \cong \mathfrak{su}_{1,1}(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{R})$.*

Proof. (cf. [Ne10c, App. C]) First we note that, in view of $x_\alpha^* \in \mathfrak{g}_{\mathbb{C}}^{-\alpha}$, [Ne98, Lemma I.2] applies, and we see that $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ is of one of the three types (A), (N) and (S). We note that $\alpha([x_\alpha, x_\alpha^*]) \in \mathbb{R}$ because of (I2) and $[x_\alpha, x_\alpha^*] \in i\mathfrak{t}$. Now it is easy to check that $\mathfrak{g}_{\mathbb{C}}(x_\alpha) \cap \mathfrak{g}$ is of type (CS), resp., (NS), according to the sign of this number. □

Definition C.3. Assume that $\mathfrak{g}_{\mathbb{C}}^{\alpha} = \mathbb{C}x_\alpha$ is one dimensional and that $\mathfrak{g}_{\mathbb{C}}(x_\alpha)$ is of type (S). Then there exists a unique element $\check{\alpha} \in \mathfrak{h} \cap [\mathfrak{g}_{\mathbb{C}}^{\alpha}, \mathfrak{g}_{\mathbb{C}}^{-\alpha}]$ with $\alpha(\check{\alpha}) = 2$. It is called the *coroot of* α. The root $\alpha \in \Delta$ is said to be *compact* if for $0 \neq x_\alpha \in \mathfrak{g}_{\mathbb{C}}^{\alpha}$ we have $\alpha([x_\alpha, x_\alpha^*]) > 0$ and *non-compact* otherwise. We write Δ_c for the set of compact and Δ_{nc} for the set of non-compact roots. Lemma C.2 implies that

$$\check{\alpha} \in \mathbb{R}^+[x_\alpha, x_\alpha^*] \quad \text{for} \quad \alpha \in \Delta_c \quad \text{and} \quad \check{\alpha} \in \mathbb{R}^+[x_\alpha^*, x_\alpha] \quad \text{for} \quad \alpha \in \Delta_{nc}. \tag{24}$$

The Weyl group $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h})$ is the subgroup generated by all reflections

$$r_\alpha(x) := x - \alpha(x)\check{\alpha}.$$

It acts on the dual space by the adjoint maps

$$r_\alpha^*(\beta) := \beta - \beta(\check{\alpha})\alpha.$$

We now describe the relevant root data for the three types of unitary Lie algebras over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

Example C.4. (Root data of unitary Lie algebras) Let $\mathcal{H}$ be a complex Hilbert space with orthonormal basis $(e_j)_{j \in J}$ and $\mathfrak{t} \subseteq \mathfrak{g} := \mathfrak{u}(\mathcal{H})$ be the subalgebra of all diagonal operators with respect to the e_j. Then $\mathfrak{t}$ is elliptic and maximal abelian, $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \cong \ell^\infty(J, \mathbb{C})$, and the set of roots of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(\mathcal{H})$ with respect to $\mathfrak{h}$ is given by

$$\Delta = \{\varepsilon_j - \varepsilon_k \colon j \neq k \in J\}.$$

Here the operator $E_{jk}e_m := \delta_{km}e_j$ is an $\mathfrak{h}$-eigenvector in $\mathfrak{gl}(\mathcal{H})$ generating the corresponding eigenspace and $\varepsilon_j(\mathrm{diag}(h_k)_{k\in J}) = h_j$. From $E_{jk}^* = E_{kj}$ it follows that

$$(\varepsilon_j - \varepsilon_k)\check{} = E_{jj} - E_{kk} = [E_{jk}, E_{kj}] = [E_{jk}, E_{jk}^*],$$

so that $\Delta = \Delta_c$, i.e., all roots are compact.

The Weyl group $\mathcal{W}$ is isomorphic to the group $S_{(J)}$ of finite permutations of J, acting in the canonical way on $\mathfrak{h}$. It is generated by the reflections $r_{jk} := r_{\varepsilon_j - \varepsilon_k}$ corresponding to the transpositions of $j \neq k \in J$.

Example C.5. (Root data of symplectic Lie algebras) For a complex Hilbert space $\mathcal{H}$ with a conjugation σ, we consider the quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}} := \mathcal{H}^2$, where the quaternionic structure is defined by the anticonjugation $\widetilde{\sigma}(v, w) := (\sigma w, -\sigma v)$. Then $\mathfrak{g} := \mathfrak{sp}(\mathcal{H}_{\mathbb{H}}) = \{x \in \mathfrak{u}(\mathcal{H}^2) : \widetilde{\sigma}x = x\widetilde{\sigma}\}$ and

$$\mathfrak{sp}(\mathcal{H}_{\mathbb{H}})_{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \in B(\mathcal{H}^2) : B^\top = B, C^\top = C \right\}.$$

Let $(e_j)_{j\in J}$ be an orthonormal basis of $\mathcal{H}$ and $\mathfrak{t} \subseteq \mathfrak{g} \subseteq \mathfrak{u}(\mathcal{H}^2)$ be the subalgebra of all diagonal operators with respect to the basis elements $(e_j, 0)$ and $(0, e_k)$ of $\mathcal{H}^2$. Then $\mathfrak{t}$ is elliptic and maximal abelian in $\mathfrak{g}$. Moreover, $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \cong \ell^\infty(J, \mathbb{C})$, consists of diagonal operators of the form $h = \mathrm{diag}((h_j), (-h_j))$, and the set of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}$ is given by

$$\Delta = \{\pm 2\varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k) : j \neq k, j, k \in J\},$$

where $\varepsilon_j(h) = h_j$. If we write $E_j \in \mathfrak{h}$ for the element defined by $\varepsilon_k(E_j) = \delta_{jk}$, then the coroots are given by

$$(\varepsilon_j \pm \varepsilon_k)\check{} = E_j \pm E_k \quad \text{for} \quad j \neq k \quad \text{and} \quad (2\varepsilon_j)\check{} = E_j. \tag{25}$$

Here the roots $\varepsilon_j - \varepsilon_k$ correspond to block diagonal operators, the roots $\varepsilon_j + \varepsilon_k$ to strictly upper triangular operators, and the roots $-\varepsilon_j - \varepsilon_k$ to strictly lower triangular operators. Again, all roots are compact, and the Weyl group $\mathcal{W}$ is isomorphic to the group $N \rtimes S_{(J)}$, where $N \cong \{\pm 1\}^{(J)}$ is the group of finite sign changes on $\ell^\infty(J, \mathbb{R})$. In fact, the reflection $r_{\varepsilon_j - \varepsilon_k}$ acts as a transposition and the reflection $r_{2\varepsilon_j}$ changes the sign of the jth component.

Example C.6. (Root data of orthogonal Lie algebras) Let $\mathcal{H}_{\mathbb{R}}$ be an infinite dimensional real Hilbert space and $(e_j)_{j\in\widetilde{J}}$ be an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$. Since $\widetilde{J}$ is infinite, it contains a subset J for which there exists an involution $\eta : \widetilde{J} \to \widetilde{J}$ with $\eta(J) = \widetilde{J} \setminus J$ and $\widetilde{J} = J \dot{\cup} \eta(J)$. Then

$$Ie_j := \begin{cases} e_{\eta(j)} & \text{for } j \in J \\ -e_{\eta(j)} & \text{for } j \in \eta(J) \end{cases}$$

defines an orthogonal complex structure on $\mathcal{H}_{\mathbb{R}}$. This complex structure defines on $\mathcal{H}_{\mathbb{R}}$ the structure of a complex Hilbert space $\mathcal{H} := (\mathcal{H}_{\mathbb{R}}, I)$. We write σ for the conjugation on $\mathcal{H}$ defined by $\sigma(e_j) = e_j$ for $j \in J$.

Then $\iota\colon \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}, v \mapsto \frac{1}{\sqrt{2}}(v, \sigma(v))$ is real linear and isometric. Since its image is a totally real subspace, ι extends to a unitary isomorphism $\iota_{\mathbb{C}}\colon \mathcal{H}_{\mathbb{C}} \to \mathcal{H}^2$. Let $\beta\colon \mathcal{H}^2 \to \mathbb{C}$ denote the complex bilinear extension of the scalar product of $\mathcal{H}_{\mathbb{R}}$, so that $\mathfrak{o}(\mathcal{H}_{\mathbb{R}})_{\mathbb{C}} \cong \mathfrak{o}(\mathcal{H}^2, \beta)$. It is given by

$$\beta((x,y),(x',y')) = \beta(x,y') + \beta(x',y) = \langle x, \sigma(y')\rangle + \langle x', \sigma(y)\rangle$$

because the right hand side is complex bilinear and has the correct restriction to $\iota(\mathcal{H})$. This implies that

$$\begin{aligned}\mathfrak{o}(\mathcal{H}_{\mathbb{R}})_{\mathbb{C}} \cong \mathfrak{o}(\mathcal{H}^2,\beta) &= \left\{X \in \mathfrak{gl}(\mathcal{H}^2)\colon X^\top \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} X = 0\right\} \\ &= \left\{\begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \in B(\mathcal{H}^2)\colon B^\top = -B, C^\top = -C\right\}.\end{aligned}$$

For the conjugation $\widetilde{\sigma}(v,w) = (\sigma(w), \sigma(v))$ with respect to $\iota(\mathcal{H})$, we have

$$\mathfrak{g} := \mathfrak{o}(\mathcal{H}_{\mathbb{R}}) \cong \{x \in \mathfrak{u}(\mathcal{H}^2)\colon \widetilde{\sigma}x = x\widetilde{\sigma}\}.$$

Now $(e_j)_{j\in J}$ is an orthonormal basis of the complex Hilbert space $\mathcal{H}$ and the subalgebra $\mathfrak{t} \subseteq \mathfrak{g} \subseteq \mathfrak{u}(\mathcal{H}^2)$ of all diagonal operators with respect to the basis elements $(e_j, 0)$ and $(0, e_k)$ of $\mathcal{H}^2$ is elliptic and maximal abelian in $\mathfrak{g}$. Again, $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \cong \ell^\infty(J, \mathbb{C})$ consists of diagonal operators of the form $h = \mathrm{diag}((h_j), (-h_j))$, and the set of roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}$ is given by

$$\Delta = \{\pm(\varepsilon_j \pm \varepsilon_k)\colon j \neq k, j,k \in J\}.$$

As in Example C.5, all roots are compact, but since $2\varepsilon_j$ is not a root and the reflection $r_{\varepsilon_j+\varepsilon_k}$ changes the sign of the j- and the k-component, the Weyl group $\mathcal{W}$ is isomorphic to the group $N_{\mathrm{even}} \rtimes S_{(J)}$, where N_{even} is the group of finite even sign changes.

C.3 c-duality and complexification of $\mathfrak{gl}_{\mathbb{K}}(\mathcal{H})$

In the preceding discussion we have seen 3 types of unitary groups $\mathrm{U}_{\mathbb{K}}(\mathcal{H})$: $\mathrm{O}(\mathcal{H})$, $\mathrm{U}(\mathcal{H})$ and $\mathrm{Sp}(\mathcal{H})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathbb{H}$, and their complexifications $\mathrm{O}(\mathcal{H}_{\mathbb{C}}, \beta)$, $\mathrm{GL}(\mathcal{H})$ and $\mathrm{Sp}(\mathcal{H}^{\mathbb{C}}, \omega)$. We have also seen the subgroups $\mathrm{GL}_{\mathbb{K}}(\mathcal{H}) = \mathrm{U}_{\mathbb{K}}(\mathcal{H}) \exp(\mathrm{Herm}_{\mathbb{K}}(\mathcal{H}))$ of $\mathrm{U}_{\mathbb{K}}(\mathcal{H})_{\mathbb{C}}$ which are symmetric Lie groups with respect to the involution $\theta(g) = (g^*)^{-1}$ and the corresponding decomposition $\mathfrak{gl}_{\mathbb{K}}(\mathcal{H}) = \mathfrak{u}_{\mathbb{K}}(\mathcal{H}) \oplus \mathrm{Herm}_{\mathbb{K}}(\mathcal{H})$ of the Lie algebra. The corresponding c-dual symmetric Lie algebras correspond to unitary groups:

($\mathbb{R}$) $\mathfrak{gl}_{\mathbb{R}}(\mathcal{H})^c = \mathfrak{o}(\mathcal{H}) \oplus i\,\mathrm{Sym}(\mathcal{H}) \cong \mathfrak{u}(\mathcal{H}_{\mathbb{C}})$.

($\mathbb{C}$) $\mathfrak{gl}_{\mathbb{C}}(\mathcal{H})^c = \mathfrak{u}(\mathcal{H}) \oplus i\,\mathrm{Herm}(\mathcal{H}) \cong \mathfrak{u}(\mathcal{H})^2$.

($\mathbb{H}$) $\mathfrak{gl}_{\mathbb{H}}(\mathcal{H})^c = \mathfrak{sp}(\mathcal{H}) \oplus i\,\mathrm{Herm}_{\mathbb{H}}(\mathcal{H}) \cong \mathfrak{u}(\mathcal{H}^{\mathbb{C}})$.

The complexifications of these Lie algebras are

$$\mathfrak{gl}_{\mathbb{R}}(\mathcal{H})_{\mathbb{C}} \cong \mathfrak{gl}(\mathcal{H}_{\mathbb{C}}), \quad \mathfrak{gl}_{\mathbb{C}}(\mathcal{H})_{\mathbb{C}} \cong \mathfrak{gl}(\mathcal{H})^2 \quad \text{and} \quad \mathfrak{gl}_{\mathbb{H}}(\mathcal{H})_{\mathbb{C}} \cong \mathfrak{gl}(\mathcal{H}^{\mathbb{C}}).$$

C.4 Variants of hermitian groups

We shall also need the following variant of the hermitian groups.

Example C.7. If $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is an orthogonal decomposition of the complex Hilbert space $\mathcal{H}$, then $h((z_+, z_-), (w_+, w_-)) := \langle z_+, w_+\rangle - \langle z_-, w_-\rangle$ defines a hermitian form on $\mathcal{H}$. We write

$$\mathrm{U}(\mathcal{H}_+, \mathcal{H}_-) := \mathrm{U}(\mathcal{H}, h) \subseteq \mathrm{GL}(\mathcal{H}) \tag{26}$$

for the corresponding group of complex linear h-isometries. Its Lie algebra $\mathfrak{u}(\mathcal{H}_+, \mathcal{H}_-)$ is a real form of $\mathfrak{gl}(\mathcal{H})$.

Let $J := J_+ \dot\cup J_-$ be such that $(e_j)_{j \in J_\pm}$ is an ONB of $\mathcal{H}_\pm$. We have seen in Example C.4, that the root system Δ of $\mathfrak{u}(\mathcal{H}_+, \mathcal{H}_-)_\mathbb{C} \cong \mathfrak{gl}(\mathcal{H}) \cong \mathfrak{u}(\mathcal{H})_\mathbb{C}$ with respect to the subalgebra $\mathfrak{h} \cong \ell^\infty(J, \mathbb{C})$ of diagonal matrices with respect to the e_j is given by

$$\Delta = \{\varepsilon_j - \varepsilon_k \colon j \neq k \in J\}$$

(cf. Example C.4). For $d := \frac{i}{2}\operatorname{diag}(\mathbf{1}, -\mathbf{1})$, we obtain the compact roots

$$\Delta_c = \Delta_k := \{\alpha \in \Delta \colon \alpha(d) = 0\} = \{\varepsilon_j - \varepsilon_k \colon j \neq k \in J_+; j \neq k \in J_-\}$$

corresponding to the complexification $\mathfrak{gl}(\mathcal{H}_+) \oplus \mathfrak{gl}(\mathcal{H}_-)$ of the centralizer

$$\mathfrak{u}(\mathcal{H}_+, \mathcal{H}_-) \cap \mathfrak{u}(\mathcal{H}) \cong \mathfrak{u}(\mathcal{H}_+) \oplus \mathfrak{u}(\mathcal{H}_-)$$

of d in $\mathfrak{u}(\mathcal{H}_+, \mathcal{H}_-)$, and the non-compact roots $\Delta_{nc} = \Delta_p^+ \cup \Delta_p^-$, where

$$\Delta_p^\pm := \{\alpha \in \Delta \colon \alpha(-i \cdot d) = \pm 1\} = \pm\{\varepsilon_j - \varepsilon_k \colon j \in J_+, k \in J_-\} \tag{27}$$

correspond to the $\pm i$-eigenspaces of $\operatorname{ad} d$ in $\mathfrak{gl}(\mathcal{H})$ (cf. Example 5.14(b)).

Example C.8. For a complex Hilbert space $\mathcal{H}$ with a conjugation σ, we define the corresponding *symplectic group* by

$$\mathrm{Sp}(\mathcal{H}) := \left\{ g \in \mathrm{U}(\mathcal{H}, \mathcal{H}) \colon g^\top \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} g = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \right\} \tag{28}$$

with the Lie algebra

$$\mathfrak{sp}(\mathcal{H}) = \left\{ \begin{pmatrix} A & B \\ B^* & -A^\top \end{pmatrix} \in B(\mathcal{H}^2) \colon A^* = -A, B^\top = B \right\}$$

and note that

$$\mathfrak{sp}(\mathcal{H})_\mathbb{C} = \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \in B(\mathcal{H}^2) \colon B^\top = B, C^\top = C \right\}.$$

The anticonjugation $\widetilde{\sigma}(v, w) := (\sigma w, -\sigma v)$ defines a quaternionic structure on $\mathcal{H}^2$. We write $\mathcal{H}_\mathbb{H}$ for the so obtained quaternionic Hilbert space. The skew-symmetric complex bilinear form on $\mathcal{H}^2$ defined by $\widetilde{\sigma}$ is given by

$$\omega((v, w), (v', w')) = \langle (v, w), (-\sigma w', \sigma v')\rangle = \langle w, \sigma v'\rangle - \langle v, \sigma w'\rangle$$

and $\mathfrak{sp}(\mathcal{H})_{\mathbb{C}} \cong \mathfrak{sp}(\mathcal{H}^2, \omega) \cong \mathfrak{sp}(\mathcal{H}_{\mathbb{H}})_{\mathbb{C}}$ (cf. Example C.5).

Let $(e_j)_{j \in J}$ be an ONB of $\mathcal{H}$. In Example C.5 we have seen that the root system Δ of $\mathfrak{sp}(\mathcal{H}^2, \omega)$ with respect to the subalgebra $\mathfrak{h} \cong \ell^\infty(J, \mathbb{C})$ of diagonal matrices is

$$\Delta = \{\pm 2\varepsilon_j, \pm(\varepsilon_j \pm \varepsilon_k) \colon j \neq k, j, k \in J\},$$

For $d := \frac{i}{2} \operatorname{diag}(\mathbf{1}, -\mathbf{1}) \in \mathfrak{h}$, we obtain the compact roots

$$\Delta_c = \Delta_k := \{\alpha \in \Delta \colon \alpha(d) = 0\} = \{\varepsilon_j - \varepsilon_k \colon j \neq k \in J\}$$

corresponding to the complexification of $\mathfrak{sp}(\mathcal{H}) \cap \mathfrak{u}(\mathcal{H}^2) \cong \mathfrak{u}(\mathcal{H})$, and the non-compact roots

$$\Delta_p^{\pm} := \{\alpha \in \Delta \colon \alpha(-i \cdot d) = \pm 1\} = \pm\{\varepsilon_j + \varepsilon_k \colon j, k \in J\}.$$

Example C.9. In a similar fashion, we define for a complex Hilbert space $\mathcal{H}$ with a conjugation σ

$$\mathrm{O}^*(\mathcal{H}) := \left\{ g \in \mathrm{U}(\mathcal{H}, \mathcal{H}) \colon g^\top \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} g = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \right\} \tag{29}$$

with the Lie algebra

$$\mathfrak{o}^*(\mathcal{H}) = \left\{ \begin{pmatrix} A & B \\ B^* & -A^\top \end{pmatrix} \in B(\mathcal{H}^2) \colon A^* = -A, B^\top = -B \right\}$$

and

$$\mathfrak{o}^*(\mathcal{H})_{\mathbb{C}} = \left\{ \begin{pmatrix} A & B \\ C & -A^\top \end{pmatrix} \in B(\mathcal{H}^2) \colon B^\top = -B, C^\top = -C \right\}.$$

The conjugation $(v, w) \mapsto (\sigma w, \sigma v)$ defines a symmetric complex bilinear form

$$\beta((v, w), (v', w')) = \langle (v, w), (\sigma w', \sigma v') \rangle = \langle v, \sigma w' \rangle + \langle w, \sigma v' \rangle$$

on $\mathcal{H}^2$ with $\mathfrak{o}^*(\mathcal{H})_{\mathbb{C}} \cong \mathfrak{o}(\mathcal{H}^2, \beta)$.

Let $(e_j)_{j \in J}$ be an ONB of $\mathcal{H}$. According to Example C.6, the root system Δ of $\mathfrak{o}(\mathcal{H}^2, \beta)$ with respect to the subalgebra $\mathfrak{h} \cong \ell^\infty(J, \mathbb{C})$ of diagonal matrices is given by

$$\Delta = \{\pm \varepsilon_j \pm \varepsilon_k \colon j \neq k \in J\},$$

For $d := \frac{i}{2} \operatorname{diag}(\mathbf{1}, -\mathbf{1}) \in \mathfrak{h}$, we obtain the compact roots

$$\Delta_c = \Delta_k := \{\alpha \in \Delta \colon \alpha(d) = 0\} = \{\varepsilon_j - \varepsilon_k \colon j \neq k \in J\}$$

corresponding to the complexification of $\mathfrak{o}^*(\mathcal{H}) \cap \mathfrak{u}(\mathcal{H}^2) \cong \mathfrak{u}(\mathcal{H})$ and the non-compact roots

$$\Delta_p^{\pm} := \{\alpha \in \Delta \colon \alpha(-i \cdot d) = \pm 1\} = \pm\{\varepsilon_j + \varepsilon_k \colon j \neq k \in J\}.$$

Example C.10. For a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, we consider the pseudounitary group $\mathrm{O}(\mathbb{R}^2, \mathcal{H}_{\mathbb{R}})$ of the indefinite quadratic form

$$q(x, v) := \|x\|^2 - \|v\|^2 = \langle Q(x, v), (x, v) \rangle, \quad Q = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},$$

on $\mathbb{R}^2 \oplus \mathcal{H}_\mathbb{R}$. Let $\mathcal{H} := (\mathbb{R}^2 \oplus \mathcal{H}_\mathbb{R}, I)$ be the complex Hilbert space, where I is a complex structure on $\mathbb{R}^2 \oplus \mathcal{H}_\mathbb{R}$ given by $I(x, y) = (-y, x)$ on $\mathbb{R}^2$ and an by an orthogonal complex structure on $\mathcal{H}_\mathbb{R}$. Let $(e_j)_{j \in J}$ be an orthonormal basis of this complex Hilbert space and $j_0 \in J$ with $e_{j_0} = (1, 0) \in \mathbb{R}^2$ and define a conjugation σ on $\mathcal{H}$ by $\sigma(e_j) = e_j$ for $j \in J$.

We realize $\mathrm{O}(\mathbb{R}^2, \mathcal{H}_\mathbb{R})$ as a subgroup of $\mathrm{O}(\mathcal{H}^2, \beta)$, where β is the complex bilinear extension of β to $\mathcal{H}^2 \cong \mathcal{H}_\mathbb{C}$ as follows. The conjugation $(v, w) \mapsto (\sigma w, \sigma v)$ defines a symmetric complex bilinear form

$$\beta((v, w), (v', w')) = \langle (Qv, Qw), (\sigma w', \sigma v') \rangle = \langle Qv, \sigma w' \rangle + \langle Qw, \sigma v' \rangle$$

on $\mathcal{H}^2$ with $\mathfrak{o}(\mathbb{R}^2, \mathcal{H}_\mathbb{R})_\mathbb{C} \cong \mathfrak{o}(\mathcal{H}^2, \beta)$. Then the corresponding complex orthogonal group is

$$\mathrm{O}(\mathcal{H}^2, \beta) = \left\{ g \in \mathrm{GL}(\mathcal{H}^2) \colon g^\top \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} g = \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} \right\} \tag{30}$$

with the Lie algebra

$$\begin{aligned} \mathfrak{o}(\mathcal{H}^2, \beta) &= \left\{ X \in B(\mathcal{H}^2) \colon X^\top \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} + \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix} X = 0 \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in B(\mathcal{H}^2) \colon B^\top = -QBQ, C^\top = -QCQ, D = -QA^\top Q \right\}. \end{aligned}$$

The subalgebra $\mathfrak{h} \cong \ell^\infty(J, \mathbb{C})$ of diagonal matrices $\mathrm{diag}((h_j), (-h_j)) \in \mathfrak{gl}(\mathcal{H}^2)$ is maximal abelian in $\mathfrak{o}(\mathcal{H}^2, \beta)$ and the root system Δ of $\mathfrak{o}(\mathcal{H}^2, \beta)$ with respect to $\mathfrak{h}$ is given by

$$\Delta = \{\pm\varepsilon_j \pm \varepsilon_k \colon j \neq k \in J\}.$$

Then

$$d := \mathrm{diag}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right) \in \mathfrak{o}(\mathbb{R}^2, \mathcal{H}_\mathbb{R})$$

corresponds to $h = \mathrm{diag}(h_j) \in \mathfrak{h}$ with $h_j = \delta_{j, j_0} i$, so that

$$\Delta_k := \{\alpha \in \Delta \colon \alpha(d) = 0\} = \{\pm\varepsilon_j \pm \varepsilon_k \colon j \neq k \in J \setminus \{j_0\}\}$$

and

$$\Delta_p^\pm := \{\alpha \in \Delta \colon \alpha(-i \cdot d) = \pm 1\} = \{\pm\varepsilon_{j_0} + \varepsilon_j, \pm\varepsilon_{j_0} - \varepsilon_j \colon j_0 \neq j \in J\}.$$

C.5 Groups related to Schatten classes

For the algebra $B(\mathcal{H})$ of bounded operators on the $\mathbb{K}$-Hilbert space $\mathcal{H}$, the ideal of compact operators is denoted $K(\mathcal{H}) = B_\infty(\mathcal{H})$, and for $1 \leq p < \infty$, we write

$$B_p(\mathcal{H}) := \{A \in K(\mathcal{H}) \colon \mathrm{tr}((A^*A)^{p/2}) < \infty\}$$

for the *Schatten ideals*. In particular, $B_2(\mathcal{H})$ is the space of *Hilbert–Schmidt operators* and $B_1(\mathcal{H})$ the space of *trace class operators*. For two Hilbert spaces $\mathcal{H}_\pm$, we put

$$B_2(\mathcal{H}_-, \mathcal{H}_+) := \{A \in B(\mathcal{H}_-, \mathcal{H}_+) \colon \mathrm{tr}(A^*A) < \infty\}.$$

For $1 \leq p \leq \infty$, we obtain Lie groups

$$\mathrm{GL}_p(\mathcal{H}) := \mathrm{GL}(\mathcal{H}) \cap (\mathbf{1} + B_p(\mathcal{H})) \quad \text{and} \quad \mathrm{U}_p(\mathcal{H}) := \mathrm{U}(\mathcal{H}) \cap \mathrm{GL}_p(\mathcal{H})$$

with the Lie algebras

$$\mathfrak{gl}_p(\mathcal{H}) := B_p(\mathcal{H}) \quad \text{and} \quad \mathfrak{u}_p(\mathcal{H}) := \mathfrak{u}(\mathcal{H}) \cap \mathfrak{gl}_p(\mathcal{H}).$$

For $\mathbb{K} = \mathbb{C}$, we have a *determinant homomorphism*

$$\det\colon \mathrm{GL}_1(\mathcal{H}) \to \mathbb{C}^\times \quad \text{with} \quad \det(\mathrm{U}_1(\mathcal{H})) = \mathbb{T}.$$

With $\mathrm{SU}(\mathcal{H}) := \ker(\det|_{\mathrm{U}_1(\mathcal{H})})$, we then obtain

$$\mathrm{U}_1(\mathcal{H}) \cong \mathbb{T} \ltimes \mathrm{SU}(\mathcal{H}) \quad \text{and} \quad \widetilde{\mathrm{U}}_1(\mathcal{H}) \cong \mathbb{R} \ltimes \mathrm{SU}(\mathcal{H}) \tag{31}$$

for the simply connected covering group (cf. [Ne04, Prop. IV.21]).

C.6 Restricted groups

Let $\mathcal{H}$ be a complex Hilbert space which is a direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. Then

$$B_{\mathrm{res}}(\mathcal{H}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in B(\mathcal{H})\colon b \in B_2(\mathcal{H}_-, \mathcal{H}_+), c \in B_2(\mathcal{H}_+, \mathcal{H}_-) \right\}$$

is a complex Banach-$*$ algebra. Its unit group is

$$\mathrm{GL}_{\mathrm{res}}(\mathcal{H}) = \mathrm{GL}(\mathcal{H}) \cap B_{\mathrm{res}}(\mathcal{H}).$$

Intersecting with $\mathrm{GL}_{\mathrm{res}}(\mathcal{H})$, we obtain various *restricted classical groups*:

$\mathrm{U}_{\mathrm{res}}(\mathcal{H}) := \mathrm{U}(\mathcal{H}) \cap \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ (the *restricted unitary group*).

$\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-) := \mathrm{U}(\mathcal{H}_+, \mathcal{H}_-) \cap \mathrm{GL}_{\mathrm{res}}(\mathcal{H})$ (the *restricted pseudo-unitary group*).

$\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}) := \mathrm{Sp}(\mathcal{H}) \cap \mathrm{GL}_{\mathrm{res}}(\mathcal{H} \oplus \mathcal{H})$ (the *restricted symplectic group*).

$\mathrm{O}^*_{\mathrm{res}}(\mathcal{H}) := \mathrm{O}^*(\mathcal{H}) \cap \mathrm{GL}_{\mathrm{res}}(\mathcal{H} \oplus \mathcal{H})$.

The unitary group in c-duality with $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$ is $\mathrm{U}_{\mathrm{res}}(\mathcal{H}_+ \oplus \mathcal{H}_-)$ (cf. Example 1.7). The groups $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H})$ and $\mathrm{O}^*_{\mathrm{res}}(\mathcal{H})$ also have corresponding c-dual unitary groups which can be realized as follows. Let $\mathcal{H}_{\mathbb{H}}$ denote $\mathcal{H}^2$, endowed with its canonical quaternionic structure given by $\widetilde{\sigma}(v, w) = (-\sigma w, \sigma v)$, so that $\mathfrak{gl}_{\mathbb{H}}(\mathcal{H}_{\mathbb{H}})_{\mathbb{C}} \cong \mathfrak{gl}(\mathcal{H}^2)$ (cf. Appendix C.3) and $\mathfrak{sp}(\mathcal{H}_{\mathbb{H}})_{\mathbb{C}} \cong \mathfrak{sp}(\mathcal{H}^2, \omega)$. Then

$$\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_{\mathbb{H}}) := \mathrm{Sp}(\mathcal{H}_{\mathbb{H}}) \cap \mathrm{GL}_{\mathrm{res}}(\mathcal{H} \oplus \mathcal{H})$$

is a group whose Lie algebra $\mathfrak{sp}_{\mathrm{res}}(\mathcal{H}_{\mathbb{H}})$ is c-dual to $\mathfrak{sp}_{\mathrm{res}}(\mathcal{H})$.

For a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ with complex structure I and the corresponding complex Hilbert space $\mathcal{H} = (\mathcal{H}_{\mathbb{R}}, I)$, the realization of $\mathrm{O}(\mathcal{H}_{\mathbb{R}})$ as $\mathrm{U}(\mathcal{H}^2) \cap \mathrm{O}(\mathcal{H}^2, \beta)$ leads to the *restricted unitary group*

$$\mathrm{O}_{\mathrm{res}}(\mathcal{H}_{\mathbb{R}}) := \mathrm{O}(\mathcal{H}_{\mathbb{R}}) \cap \mathrm{GL}_{\mathrm{res}}(\mathcal{H} \oplus \mathcal{H})$$

of those orthogonal operators $g = g_0 + g_1$ on $\mathcal{H}_{\mathbb{R}}$ for which the antilinear part g_1 with respect to I is Hilbert–Schmidt and the subgroup of I-linear elements in $\mathrm{O}_{\mathrm{res}}(\mathcal{H}_{\mathbb{R}})$ is the unitary group $\mathrm{U}(\mathcal{H})$ of the complex Hilbert space $\mathcal{H}$.

C.7 Doubly restricted groups

If $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is an orthogonal decomposition of the complex Hilbert space $\mathcal{H}$, then we write operators on $\mathcal{H}$ as (2×2)-matrices. The subspace

$$B_{1,2}(\mathcal{H}) := \left\{ x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in B(\mathcal{H}) \colon \|x_{11}\|_1, \|x_{12}\|_2, \|x_{21}\|_2, \|x_{22}\|_1 < \infty \right\} \quad (32)$$

is a Banach algebra with the associated group

$$\mathrm{GL}_{1,2}(\mathcal{H}) := \{ g \in \mathrm{GL}(\mathcal{H}) \colon \mathbf{1} - g \in B_{1,2}(\mathcal{H}) \}. \quad (33)$$

For a subgroup $G \subseteq \mathrm{GL}(\mathcal{H})$, the corresponding *doubly restricted group* is now defined as $G_{1,2} := G \cap \mathrm{GL}_{1,2}(\mathcal{H})$. We shall need these groups for $G = \mathrm{U}_{\mathrm{res}}(\mathcal{H}_+, \mathcal{H}_-)$, $\mathrm{O}^*_{\mathrm{res}}(\mathcal{H})$, $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H})$, and also for the corresponding unitary c-dual groups $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, $\mathrm{O}_{\mathrm{res}}(\mathcal{H}_{\mathbb{R}})$, $\mathrm{Sp}_{\mathrm{res}}(\mathcal{H}_{\mathbb{H}})$.

C.8 Polar decomposition

Remark C.11. In [Ne02, Thm. II.6, Prop. III.8] it is shown that for all groups G discussed in this appendix, the Lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{with} \quad \mathfrak{k} = \{x \in \mathfrak{g} \colon x^* = -x\} \quad \text{and} \quad \mathfrak{p} = \{x \in \mathfrak{g} \colon x^* = x\},$$

and that the corresponding polar map $K \times \mathfrak{p} \to G, (k, x) \mapsto k \exp x$ is a diffeomorphism (see also the appendix in [Ne04]). For the doubly restricted group $G_{1,2}$, see in particular [NO98, Lemma III.6].

D Bounded representations of $\mathrm{U}_p(\mathcal{H})$, $1 < p \leq \infty$

In this section we completely describe the bounded representations of the unitary groups $G := \mathrm{U}_p(\mathcal{H})_0$, $1 < p \leq \infty$, where $\mathcal{H}$ is an infinite dimensional Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. For $\mathbb{K} = \mathbb{C}$, we have shown in [Ne98, Thm. III.14] that all bounded unitary representations of G, resp., all holomorphic $*$-representations of the complexified group $G_{\mathbb{C}} = \mathrm{GL}_p(\mathcal{H})$ are direct sums of irreducible ones and that the irreducible ones are classified by their "highest weights". In this section we explain this classification in some detail and extend it to real and quaternionic groups.

To this end, we work with the root systems Δ from Examples C.4, C.5 and C.6, where $\mathfrak{h} = \mathfrak{t}_{\mathbb{C}} \cong \ell^p(J, \mathbb{C}) \subseteq \mathfrak{g}_{\mathbb{C}}$ now stands for the Lie algebra of diagonal operators in $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{t} = \mathfrak{g} \cap \mathfrak{h}$. The root data does not depend on the parameter p.

Definition D.1. A continuous linear functional $\beta \in \mathfrak{h}'$ is called a *weight* if $\beta(\check{\alpha}) \in \mathbb{Z}$ holds for each $\alpha \in \Delta$. We write $\mathcal{P}$ for the additive group of weights and $\mathcal{Q} \subseteq \mathcal{P}$ for the subgroup generated by Δ. Note that the Weyl group $\mathcal{W}$ acts naturally on $\mathcal{P}$ (cf. [Ne98], [Ne04]).

Remark D.2. (a) For $q \in [1,\infty[$ defined by $\frac{1}{q}+\frac{1}{p}=1$, we have $\mathfrak{h}' \cong \ell^q(J,\mathbb{C})$, so that we consider elements of this space as functions $\beta\colon J \to \mathbb{C}$. Since the root system Δ contains all roots of the form $\varepsilon_j - \varepsilon_k$, $j \neq k \in J$, it follows from [Ne98, Prop. III.10] that each weight $\beta \in \mathcal{P}$ is finitely supported with $\beta(J) \subseteq \mathbb{Z}$. Conversely, the description of the coroots in (25) implies for $\mathbb{K} = \mathbb{H}$ and $\mathbb{R}$ (Examples C.5 and C.6) that any finitely supported $\mathbb{Z}$-valued function on J is a weight, i.e., $\mathcal{P} \cong \mathbb{Z}^{(J)}$ is a free group over J.

(b) The weight group $\mathcal{P}$ can be identified with the character group of the Banach–Lie group $T = \exp \mathfrak{t}$ by assigning to λ the character given by $\chi_\lambda(t) := \prod_{j\in J} t_j^{\lambda_j}$.

Definition D.3. From [Ne98, Prop. III.10, Sects. VI, VIII] we know that, for each $\lambda \in \mathcal{P}$, there exists a unique bounded unitary representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $\mathrm{U}_p(\mathcal{H})$ whose weight set is given by

$$\mathcal{P}_\lambda = \mathrm{conv}(\mathcal{W}\lambda) \cap (\lambda + \mathcal{Q}). \tag{34}$$

From [Ne98, Thm. III.15, Sects. VI,VII], we even know that these representation extend to bounded representations of the full unitary group $\mathrm{U}(\mathcal{H})$. The uniqueness of the extension to $\mathrm{U}_\infty(\mathcal{H})$ follows from the density of $\mathrm{U}_p(\mathcal{H})$ in $\mathrm{U}_\infty(\mathcal{H})$, and the uniqueness of the extension to $\mathrm{U}(\mathcal{H})$ follows from the perfectness of the Lie algebra $\mathfrak{u}(\mathcal{H})$ ([Ne02, Lemma I.3]) because any two extensions $\mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H}_\lambda)$ differ by a continuous homomorphism

$$\mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H}_\lambda) \cap \pi_\lambda(\mathrm{U}_p(\mathcal{H}))' = \mathbb{T}\mathbf{1}$$

by Schur's Lemma. We also write π_λ for the unique extension to $\mathrm{U}(\mathcal{H})$.

Remark D.4. (a) Since the representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of $\mathrm{U}_\infty(\mathcal{H})$ is uniquely determined by (34), and the right hand side only depends on the Weyl group orbit, which in turn coincides with the set of extreme points of its convex hull ([Ne98, Lemma I.19]), it follows that $\pi_\mu \cong \pi_\lambda$ if and only if $\mu \in \mathcal{W}\lambda$. Hence the equivalence classes of these representations are parameterized by the orbit space $\mathcal{P}/\mathcal{W}$.

(b) For $\mathbb{K} = \mathbb{C}$, we write every $\lambda \in \mathcal{P}$ as $\lambda = \lambda_+ - \lambda_-$ for $\lambda_\pm := \max(\pm\lambda, 0)$. With [BN11, Thm. 2.2] we then obtain the factorization

$$\pi_\lambda \cong \pi_{\lambda_+} \otimes \pi_{\lambda_-}^*$$

(see also [Ki73] for the case where $\mathcal{H}$ is separable).

The first part of the following theorem is a variation of [Ne98, Thm. III.14] which is the corresponding result for the groups $\mathrm{U}_p(\mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space. From [Ne98] we know that this result does not extend to $p = 1$.

Theorem D.5. *Let $\mathcal{H}_\mathbb{R}$ be an infinite dimensional real Hilbert space. For $1 < p \leq \infty$, every bounded unitary representation of the simply connected covering group $\widetilde{\mathrm{O}}_p(\mathcal{H}_\mathbb{R})_0$ of $\mathrm{O}_p(\mathcal{H}_\mathbb{R})_0$ is a direct sum of bounded irreducible representations. The bounded irreducible representations are highest weight representations (ρ_λ, V) with finitely supported highest weights $\lambda\colon J \to \mathbb{Z}$. In particular, they have a unique extension to the full orthogonal group $\mathrm{O}(\mathcal{H}_\mathbb{R})$.*

Proof. Let $G := \widetilde{\mathrm{O}}_p(\mathcal{H}_\mathbb{R})_0$. Passing to the derived representation of the complexified Lie algebra $\mathfrak{g}_\mathbb{C} = \mathfrak{o}_p(\mathcal{H}_\mathbb{R})_\mathbb{C} \cong \mathfrak{o}_p(\mathcal{H}^2, \beta)$, we obtain a $*$-representation (ρ, V) of $\mathfrak{g}_\mathbb{C}$. Applying [Ne98, Lemma III.13] to $\mathfrak{h} \subseteq \mathfrak{g}_\mathbb{C}$, it follows that each bounded $*$-representation (ρ, V) of $\mathfrak{g}_\mathbb{C}$ is a direct sum of $\mathfrak{h}$-weight spaces and that the weight set $\mathcal{P}_V$ of V is contained in $\mathcal{P} \cong \mathbb{Z}^{(J)}$. For $\mu \in \mathbb{Z}^{(J)}$, we have

$$\|\mu\|_1 = \sum_j |\mu_j| \leq \sum_j |\mu_j|^q = \|\mu\|_q^q,$$

so that the boundedness of ρ implies the boundedness of the set $\mathcal{P}_V \subseteq \mathfrak{h}'$ of weights of (ρ, V) as a subset of $\ell^1(J, \mathbb{C}) \subseteq \ell^\infty(J, \mathbb{C})'$.

Let $\mathfrak{gl}_p(\mathcal{H}) \hookrightarrow \mathfrak{o}_p(\mathcal{H}^2, \beta), x \mapsto \mathrm{diag}(x, -x^\top)$ denote the canonical embedding and $d := \frac{i}{2}\,\mathrm{diag}(\mathbf{1}, -\mathbf{1})$. Then [Ne98, Thm. III.14] implies that the restriction of ρ to $\mathfrak{gl}_p(\mathcal{H})$ is a direct sum of irreducible representations. From the decomposition into weight spaces for $\mathfrak{h} \cong \ell^p(J, \mathbb{C})$ and the boundedness of the weight set $\mathcal{P}_V$ in ℓ^1, we obtain an extension of ρ from $\mathfrak{h}$ to a bounded representation of $\mathfrak{h}_\infty := \ell^\infty(J, \mathbb{C})$, hence to a bounded representation $(\widehat{\rho}, V)$ of the semidirect sum $\mathfrak{g}_\mathbb{C} \rtimes \mathfrak{h}_\infty$. Since the operator $-i\widehat{\rho}(d)$ is bounded with integral eigenvalues, it has only finitely many eigenspaces, and all its eigenspaces are $\mathfrak{gl}_p(\mathcal{H})$-invariant.

Let (ρ_0, V_0) be an irreducible $\mathfrak{gl}_p(\mathcal{H})$-subrepresentation of the eigenspace corresponding to the maximal eigenvalue and $\mathfrak{p}^\pm \subseteq \mathfrak{g}_\mathbb{C}$ denote the $\pm i$-eigenspaces of $\mathrm{ad}\, d$. Then $\widehat{\rho}(\mathfrak{p}^+)V_0 = \{0\}$. Next we observe that the representation of $\mathfrak{g}$ on the closed subspace $\widehat{V_0}$ generated by V_0 is also $\widehat{\rho}(d)$-invariant, hence a representation of the real Lie algebra $\mathfrak{g} + \mathbb{R}d$. We conclude with [Ne10d, Thm. 2.17] that it is holomorphically induced from the irreducible representation (ρ_0, V_0) of $\mathfrak{gl}_p(\mathcal{H}) + \mathbb{C}d$, hence irreducible by [Ne10d, Cor 2.14]. This shows that every bounded representation (ρ, V) of $\mathfrak{g}$ on a non-zero Hilbert space contains an irreducible one, and therefore Zorn's Lemma implies that ρ is a direct sum of irreducible representations.

Now we suppose that (ρ, V) is irreducible and that V_0 is chosen as above. For each weight $\mu\colon J \to \mathbb{Z}$ in $\mathcal{P}_V$ we have $\mu(-i \cdot d) = \sum_{j \in J} \mu_j$. Since $\mathcal{P}_V$ is invariant under the Weyl group $\mathcal{W}$ which contains all finite sign changes (Example C.5), the maximality of $\mu(-i \cdot d)$ among $(\mathcal{W}\mu)(-i \cdot d)$ implies that each weight $\mu \in \mathcal{P}_{V_0}$ satisfies $\mu_j \in \mathbb{N}_0$ for each $j \in J$. Since the representation (ρ_0, V_0) of $\mathfrak{gl}(\mathcal{H})$ is a representation with some highest weight λ ([Ne98, Thm. III.14]), it follows from [Ne98, Prop. VII.2] that (ρ, V) contains the representation with the highest weight λ, hence is a highest weight representations because it is irreducible. □

Almost the same arguments as for the orthogonal groups apply to the unitary groups $\mathrm{Sp}_p(\mathcal{H}_\mathbb{H})$ of a quaternionic Hilbert space:

Theorem D.6. *Let $\mathcal{H}_\mathbb{H} = \mathcal{H}^2$ be the quaternionic Hilbert space canonically associated to the complex Hilbert space $\mathcal{H}$ and a conjugation σ on $\mathcal{H}$. For $1 < p \leq \infty$, every bounded unitary representation of $\mathrm{Sp}_p(\mathcal{H}_\mathbb{H})$ is a direct sum of bounded irreducible representations. The bounded irreducible representations are highest weight representations (ρ_λ, V) with finitely supported highest weights $\lambda\colon J \to \mathbb{Z}$. In particular, they have a unique extension to the full symplectic group $\mathrm{Sp}(\mathcal{H}_\mathbb{H})$.*

For a separable Hilbert space, the representations discussed in this section coincide with those representations of $\mathrm{U}_\infty(\mathcal{H})$ extending to strongly continuous representations on the full unitary groups $\mathrm{U}(\mathcal{H})$. Their restrictions to the direct limit groups $\mathrm{U}_\infty(\mathbb{K}) := \varinjlim \mathrm{U}_n(\mathbb{K})$ are precisely the tame representations ([Ol90, Sect. 3], [Ol78]).

E Separable representations of infinite dimensional unitary groups

Theorem E.1. (Kirillov, Olshanski, Pickrell) *For a separable continuous unitary representation π of the unitary group $\mathrm{U}(\mathcal{H})$ of a separable infinite dimensional Hilbert space $\mathcal{H}$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, the following assertions hold:*

(a) *π is also continuous with respect to the strong operator topology.*

(b) *π is a direct sum of irreducible representations.*

(c) *All irreducible representations occur in a tensor product $\mathcal{H}_{\mathbb{C}}^{\otimes n}$ for some $n \in \mathbb{N}_0$. In particular they are bounded.*

(d) *π extends uniquely to a continuous representation of the group $\mathrm{U}(\mathcal{H})^\sharp$ with the same commutant, where*

$$\mathrm{U}(\mathcal{H})^\sharp \cong \begin{cases} \mathrm{U}(\mathcal{H}_{\mathbb{C}}) & \text{for } \mathbb{K} = \mathbb{R}, \\ \mathrm{U}(\mathcal{H})^2 & \text{for } \mathbb{K} = \mathbb{C}, \\ \mathrm{U}(\mathcal{H}^{\mathbb{C}}) & \text{for } \mathbb{K} = \mathbb{H}, \end{cases}$$

is the group with Lie algebra $\mathbf{L}(\mathrm{U}(\mathcal{H})^\sharp) = \mathfrak{u}(\mathcal{H}) + i\,\mathrm{Herm}(\mathcal{H})$. Here $\mathcal{H}^{\mathbb{C}}$ denotes the complex Hilbert space underlying the quaternionic Hilbert space $\mathcal{H}$, and, for $\mathbb{K} = \mathbb{C}$, the inclusion $\eta\colon \mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H})^2$ is the diagonal embedding.

Proof. (a) is due to Pickrell ([Pi88]).

(b) and (c) are claimed by Kirillov in [Ki73] but detailed proofs have been provided by Olshanski in [Ol78] (see also [Ol90]). Note that, for $\mathbb{K} = \mathbb{C}$, $\mathcal{H}_{\mathbb{C}} \cong \mathcal{H} \oplus \mathcal{H}^*$, so that one could also say that the irreducible representations occur in some $\mathcal{H}^{\otimes n} \otimes (\mathcal{H}^*)^{\otimes m}$.

(d) Here the main idea is that (a) and (c) implies that the representation extends to a representation of the full contraction semigroup $C(\mathcal{H})$ in which $\mathrm{U}(\mathcal{H})$ is strongly dense. Then one applies a holomorphic extension argument which yields a representation of the c-dual group of $\mathrm{GL}(\mathcal{H})$, and the compatibility with the embedding $\eta\colon \mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H})^\sharp$ shows that we actually obtain a representation of $\mathrm{U}(\mathcal{H})^\sharp$. □

Corollary E.2. *Let K be a quotient of a product $K_1 \times \cdots \times K_n$, where each K_j is compact or a quotient of some group $\mathrm{U}(\mathcal{H})$, where $\mathcal{H}$ is a separable $\mathbb{K}$-Hilbert space. Then every separable continuous unitary representation π of K is a direct sum of irreducible representations which are bounded.*

The preceding corollary means that the separable representation theory of K very much resembles the representation theory of a compact group.

F Perspectives and open problems

F.1 Positive energy without semiboundedness

Problem F.1. To clarify the precise relation between semiboundedness and the positive energy condition, one has to answer the following question: Is every irreducible positive energy representation of a full hermitian Lie group G semibounded? From Theorem 5.2(iv) we know that the converse is true.

Here is what we know: Let $(\pi, \mathcal{H})$ be an irreducible positive energy representation of the hermitian Lie group (G, θ, d). From [Ne10d, Prop. 3.6] we derive that $\mathcal{H}$ is generated by the closed subspace $V := \overline{(\mathcal{H}^\infty)^{\mathfrak{p}^-}}$. Now the problem is to show that the K-representation ρ on V is bounded because then Theorem 5.6 applies. From Lemma 5.3 we know that $\mathcal{H}$ decomposes into eigenspaces of $\mathtt{d}\pi(d)$.

If we assume, in addition, that $\mathcal{H}$ and hence also V are separable, then Corollary E.2 applies to the K-representation on V, so that we obtain a bounded irreducible K-representation $W \subseteq V$, but a priori we do not know if W can be chosen in such a way that $W \cap \mathcal{H}^\infty$ is dense in W. Hence we cannot apply the tools from [Ne10d]. What is missing at this point are tools to decompose the K-representation on the Fréchet space $\mathcal{H}^\infty$ (cf. [Ne10a]) that would lead to the existence of an irreducible subrepresentation. In a certain sense we are asking for a Peter–Weyl theory for groups such as $\mathrm{U}(\mathcal{H})$.

Problem F.2. Classify semibounded representations of Fréchet–Lie groups such as $\mathrm{Diff}(\mathbb{S}^1)$, the Virasoro group and affine Kac–Moody groups. This requires in particular to extend the tools developed in [Ne10d] to strongly continuous automorphism groups $(\alpha_t)_{t\in\mathbb{R}}$ on Banach–Lie groups, where the infinitesimal generator $\alpha'(0)$ is unbounded. One also has to develop suitable direct integral techniques which permit to identify smooth vectors.

Problem F.3. Use the Lücher-Mack Theorem [MN11] to determine which irreducible representations we obtain by restriction from hermitian groups to automorphism groups of real forms $M_\mathbb{R}$ of hermitian symmetric spaces M. It seems that many of the irreducible semibounded representations remain irreducible when restricted to such subgroups.

On the level of the group K, typical examples where restrictions to rather small subgroups remain irreducible arise in the situation of Theorem E.1(d). In this context all highest weight representation of the group $\mathrm{U}(\mathcal{H})^\sharp$ with non-negative highest weight remain irreducible when restricted to the subgroup $\mathrm{U}(\mathcal{H})$.

Problem F.4. It is also of some interest to identify the irreducible semibounded representations for hermitian groups (G, θ, d) which are not irreducible. Here the main issue is to understand the situation where $\mathfrak{p}$ contains infinitely many simple JH^*-ideals.

F.2 Other restricted hermitian groups

Example F.5. For smooth manifolds M with $\dim M > 1$, the group $C^\infty(M, K)$, K a compact Lie group, has natural homomorphisms into the group

$$G := \mathrm{U}_{\mathrm{res,p}}(\mathcal{H}) := \Big\{ g \in \mathrm{U}(\mathcal{H})\colon g_{12} \in B_p(\mathcal{H}_-, \mathcal{H}_+), g_{21} \in B_p(\mathcal{H}_+, \mathcal{H}_-) \Big\}$$

for some $p > 2$ (cf. [Mi87, Mi89], [Pi89]). The Lie algebra of this group contains the element $d := \frac{i}{2}\operatorname{diag}(\mathbf{1}, -\mathbf{1})$ for which $\theta := e^{\pi \operatorname{ad} d}$ defines an involution and we can also ask for its semibounded unitary representations. Note that (G, θ, d) is not a hermitian Lie group in the sense of Definition 1.1 because $\mathfrak{p} \cong B_p(\mathcal{H}_-, \mathcal{H}_+)$ carries for $p > 2$ no Hilbert space structure invariant under the adjoint action of the group $K = G^\theta \cong \mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-)$.

First we claim that $H^2_c(\mathfrak{g}, \mathbb{R}) = \{0\}$, i.e., that all central extensions of $\mathfrak{g}$ are trivial. In fact, with the same arguments as in the proof of Lemma 3.3, we see that every continuous cocycle $\omega\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is equivalent to one vanishing on $\mathfrak{k} \times \mathfrak{g}$. Then ω defines on $\mathfrak{p} \times \mathfrak{p}$ an $\mathrm{Ad}(K)$-invariant skew-symmetric bilinear form, and Lemma 3.3 implies that its restriction to the dense subspace $B_2(\mathcal{H}_-, \mathcal{H}_+)$ is a multiplies of the canonical cocycle $\operatorname{Im} \operatorname{tr}(xy^*)$. Since this cocycle is not continuous with respect to $\|\cdot\|_p$, we obtain $\omega = 0$, and therefore $H^2_c(\mathfrak{g}, \mathbb{R}) = \{0\}$.

Next we observe that every semibounded unitary representation π of G restricts to a semibounded representation of the subgroup $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$, whose central charge c vanishes (Theorem 8.1). If $\mathtt{d}\pi$ is non-zero on $\mathfrak{p}$, then Corollary E.2 further implies that π extends to a bounded highest weight representation $(\pi_\lambda, \mathcal{H}_\lambda)$ of the full unitary group $\mathrm{U}(\mathcal{H})$. Therefore the group $\mathrm{U}_{\mathrm{res,p}}(\mathcal{H})$ has no unbounded semibounded irreducible unitary representations. If $\mathcal{H}$ is separable, then Pickrell show in [Pi90, Prop. 7.1] that all separable continuous unitary representations of $\mathrm{U}_{\mathrm{res,p}}(\mathcal{H})$ extend uniquely to strongly continuous representations of the full unitary group $\mathrm{U}(\mathcal{H})$, hence are direct sums of bounded representation.

Problem F.6. For the Schatten norms, we have the estimate

$$\|AB\|_p \leq \|A\|_{p_1} \|B\|_{p_2} \quad \text{for} \quad \frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$$

([GGK00, Th. IV.11.2]). This leads in particular to

$$\|AB\|_{p/2} \leq \|A\|_p \|B\|_p.$$

For each $p \geq 2$ we thus obtain for $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ a Banach algebra

$$B_{p/2,p}(\mathcal{H}) := \Big\{ x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in B(\mathcal{H})\colon \|x_{11}\|_{p/2}, \|x_{12}\|_p, \|x_{21}\|_p, \|x_{22}\|_{p/2} < \infty \Big\} \tag{35}$$

and a corresponding unitary group

$$\mathrm{U}_{p/2,p}(\mathcal{H}) := \mathrm{U}(\mathcal{H}) \cap (\mathbf{1} + B_{p/2,p}(\mathcal{H})). \tag{36}$$

Accordingly, we obtain for each $p \geq 2$ a variant of a hermitian Lie algebra by

$$\mathfrak{g} := \mathbb{R}d + \mathfrak{u}_{p/2,p}(\mathcal{H}) \cong \mathfrak{u}_{p/2,p}(\mathcal{H}) \rtimes_{\mathrm{ad}\, d} \mathbb{R} \quad \text{for} \quad d = \frac{i}{2}\,\mathrm{diag}(\mathbf{1}, -\mathbf{1})$$

and a corresponding group $G := \mathrm{U}_{p/2,p}(\mathcal{H}) \rtimes \mathbb{R}$. Then

$$\mathfrak{k} = \mathfrak{z}_{\mathfrak{g}}(d) \cong \mathfrak{u}_{p/2}(\mathcal{H}_+) \oplus \mathfrak{u}_{p/2}(\mathcal{H}_-) \oplus \mathbb{R}.$$

With a similar argument as in Lemma 3.3, we see that each cohomology class in $H^2_c(\mathfrak{g}, \mathbb{R})$ contains a d-invariant cocycle ω. Then $\omega(\mathfrak{k}, \mathfrak{p}) = \{0\}$, and a similar argument as in Example F.5 implies that it vanishes on $\mathfrak{p} \times \mathfrak{p}$. However, for $p > 2$, the Lie algebra $\mathfrak{u}_{p/2}(\mathcal{H}_+)$ has non-trivial cocycles which can be written as $\omega_D(x, y) := \mathrm{tr}([x, y]D)$ for $D \in B(\mathcal{H}_+)$ if $p \leq 4$, and for $D \in B_q(\mathcal{H}_+)$ if $p > 4$ and $\frac{1}{q} + \frac{4}{p} = 1$ ([Ne03, Prop. III.19]).

Suppose that $p > 2$. One can show that the Lie algebra $\mathfrak{k}$ also has the property that $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains no open invariant cones. In particular, all irreducible semibounded representations of G are holomorphically induced from bounded representations of $K = (G^\theta)_0$ (Theorem 5.4). For $p > 4$, one can apply the results from Appendix D to the groups $\mathrm{U}_{p/4}(\mathcal{H}_\pm)$ to show that, for all bounded unitary representations of central extensions of the groups $\mathrm{U}_{p/2}(\mathcal{H}_\pm)$, the center acts trivially, which leaves only representations extending to highest weight representations of the full unitary groups $\mathrm{U}(\mathcal{H}_\pm)$. From that one concludes that the central extensions of $\mathrm{U}_{p/2,p}(\mathcal{H})$ do not lead to new semibounded representations, and that all semibounded representations extend to the larger group $\mathrm{U}_{\mathrm{res,p}}(\mathcal{H})$, hence even to $\mathrm{U}(\mathcal{H})$ (cf. Example F.5).

For $2 < p \leq 4$ the preceding method does not work. It is an interesting problem whether in this case the central extensions of $\mathrm{U}_{p/2}(\mathcal{H}_\pm)$ have bounded irreducible representations which are non-trivial on the center. We do not expect that this is the case.

We also note that, since $\mathrm{U}_{p/2}(\mathcal{H})$ is contained in $\mathrm{U}_{p/2,p}(\mathcal{H})$, the structure of the bounded representations of G follows from [Ne98, Thm. III.14]. They all extend to highest weight representations of $\mathrm{U}(\mathcal{H})$ (Definition D.3).

For $p = 2$, the group $\mathrm{U}_{p/2}(\mathcal{H}_\pm) = \mathrm{U}_1(\mathcal{H}_\pm)$ has no non-trivial central extensions, but it has a bounded representation theory which is not of type I (cf. [Ne98]). Furthermore $\mathfrak{k}/\mathfrak{z}(\mathfrak{k})$ contains non-trivial open invariant cones. Therefore one can expect that in this case G has a rich but also more complicated variety of semibounded unitary representations.

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