

Cluster-additive functions on stable translation quivers

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Abstract. Additive functions on translation quivers have played an important role in the representation theory of finite dimensional algebras, the most prominent ones are the hammock functions introduced by S. Brenner. When dealing with cluster categories (and cluster-tilted algebras), one should look at a corresponding class of functions defined on stable translation quivers, namely the cluster-additive ones. We conjecture that the cluster-additive functions on a stable translation quiver of Dynkin type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ are non-negative linear combinations of cluster-hammock functions (with index set a tilting set). The present paper provides a first study of cluster-additive functions and gives a proof of the conjecture in the case \mathbb{A}_n .

A *translation quiver* is of the form $\Gamma = (\Gamma_0, \Gamma_1, \tau)$, where (Γ_0, Γ_1) is a locally finite quiver say with m_{xy} arrows $x \rightarrow y$, and $\tau: (\Gamma_0 \setminus \Gamma_0^p) \rightarrow \Gamma_0$ is an injective function defined on the complement of a subset $\Gamma_0^p \subseteq \Gamma_0$, such that for any pair of vertices $y, z \in \Gamma_0$, with $z \notin \Gamma_0^p$ one has $m_{\tau z, y} = m_{y, z}$. The vertices in Γ_0^p are said to be the *projective* vertices, those not in the image of τ the *injective* vertices. If there are neither projective nor injective vertices, then Γ is said to be *stable*. A typical example of a translation quiver is the Auslander-Reiten quiver of a finite-dimensional k -algebra A , where k is an algebraically closed field. Such an Auslander-Reiten quiver is equipped with an additive function on the set of vertices with values in the set of positive integers, its value at a vertex x is the length of the corresponding A -module. Here, a function $f: \Gamma_0 \rightarrow \mathbb{Z}$ is said to be *additive* provided

$$f(z) + f(\tau z) = \sum_{y \in \Gamma_0} m_{yz} f(y), \quad \text{for all } z \in \Gamma_0 \setminus \Gamma_0^p.$$

The importance of dealing with additive functions on translation quivers is well-known since a long time, of particular relevance have been the hammock functions introduced by Brenner [Br], see also [RV]; the hammock functions for the vertices of the translations quivers of the form $\Gamma = \mathbb{Z}\Delta$ with Δ a Dynkin diagram $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ have been displayed already by Gabriel [G] in 1980.

The present note is concerned with combinatorial features of cluster categories (introduced by Buan, Marsh, Reineke, Reiten, Todorov [B-T]) and cluster-tilted algebras (introduced by Buan, Marsh, Reiten [BMR]), and for simplicity we again will assume that

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we work over an algebraically closed field. The cluster categories are triangulated categories with Auslander-Reiten triangles, thus we may consider the corresponding Auslander-Reiten quivers: these are now stable translation quivers. Thus, let Γ be a stable translation quiver. Instead of looking at additive functions on Γ , we now will be interested in what we call cluster-additive functions.

We use the following notation: Any integer z can be written as $z = z^+ - z^-$ with non-negative integers z^+, z^- such that $z^+z^- = 0$ (thus $z^+ = \max\{z, 0\}$ and $z^- = \max\{-z, 0\}$). A function $f: \Gamma_0 \rightarrow \mathbb{Z}$ is said to be *cluster-additive* on Γ provided

$$f(z) + f(\tau z) = \sum_{y \in \Gamma_0} m_{yz} f(y)^+, \quad \text{for all } z \in \Gamma_0.$$

The sum of two cluster-additive functions usually is not cluster-additive. Theorem 1 will provide a criterion for such sums to be cluster-additive again: *If f, g are cluster-additive on Γ , then $f + g$ is cluster-additive if and only if f and g are compatible* (this means that $f(x)g(x) \geq 0$ for all vertices x). Theorem 2 shows that *the difference $f - g$ of cluster-additive functions f, g is cluster-additive if and only if $g \leq f$* (this means that $g(x)^+ \leq f(x)^+$ and $g(x)^- \leq f(x)^-$ for all vertices x).

The remaining parts of the paper will deal with translation quivers related to those of the form $\mathbb{Z}\Delta$ where Δ is a finite directed quiver. Recall that any locally finite directed quiver Δ gives rise to a stable translation quiver $\mathbb{Z}\Delta$ with vertex set $\Delta \times \mathbb{Z}$, with arrows $(\alpha, i): (\xi, i) \rightarrow (\eta, i)$ and $(\alpha^*, i): (\eta, i) \rightarrow (\xi, i + 1)$ for any arrow $\alpha: \xi \rightarrow \eta$ in Δ and with translation $(\xi, i) \mapsto (\xi, i - 1)$. Theorem 3 asserts that *a cluster-additive function on $\mathbb{Z}\Delta$ with Δ a finite directed quiver is uniquely determined by its values on a slice and that these values are arbitrary integers*. Thus, if Δ has n vertices, we may identify in this way the set of cluster-additive functions on Γ with the set \mathbb{Z}^n ; but note that this is just a set-theoretical bijection!

Our main interest lies in the translation quivers $\mathbb{Z}\Delta$ where Δ is a simply laced Dynkin-quiver, thus of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$, or \mathbb{E}_8 . Theorem 4 asserts that for Δ of type \mathbb{A}_n , *any cluster-additive function on $\mathbb{Z}\Delta$ is a non-negative linear combination of cluster-hammock functions* (they are introduced in section 5). We conjecture that the same assertion holds for all Dynkin cases. This would be an analog of an old theorem of Butler [Bu] which asserts that for a representation-finite algebra A , the additive functions on the Auslander-Reiten quiver of A are the linear combinations of the hammock functions.

Cluster-additive categories arise naturally in the context of cluster categories and cluster-tilted algebras (see section 10), thus one may be tempted to focus the attention to cluster-additive functions on stable translation quivers Γ such as the Auslander-Reiten quiver of a cluster category, a typical example is $\mathbb{Z}\Delta/F$ where Δ is a Dynkin quiver and $F = \tau^{-1}[1]$. It may come as a surprise that instead of looking at $\mathbb{Z}\Delta/F$, we prefer to consider cluster-additive functions on its cover $\mathbb{Z}\Delta$. After all, every cluster-additive function on $\mathbb{Z}\Delta/F$ lifts to a cluster-additive function on $\mathbb{Z}\Delta$, thus we deal with a setting which on a first sight appears to be more general. But we conjecture that all the cluster-additive functions on $\mathbb{Z}\Delta/F$ actually are F -invariant, so that we would get the shift F for free.

The experienced reader will observe that the cluster-additive functions exhibit a lot of typical features known in cluster theory (as started by Fomin and Zelevinsky and developed

further by a large number of mathematicians): that negative numbers arise only seldom, that they have to be ignored in some calculations, that there is a playing field which concerns only non-negative numbers, and if the ball leaves the field, it is bounced back immediately

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1. Preliminaries.

Let Γ be a stable translation quiver. We compare additivity and cluster-additivity and look for the image of a cluster-additive function.

(1) *A function f on Γ_0 with values in \mathbb{N}_0 is cluster-additive if and only if it is additive.*

(2) *If Γ is connected and Γ_1 is not empty, then any function $f: \Gamma_0 \rightarrow \mathbb{Z}$ which is both additive and cluster-additive takes values in \mathbb{N}_0 .*

Proof: (1) is obvious. For the proof of (2), observe that a connected stable translation quiver with at least one arrow has the property that any vertex y_0 is starting point of an arrow, say $y_0 \rightarrow z$. Now

$$f(z) + f(\tau z) = \sum_y m_{yz} f(y)^+ = \sum_y m_{yz} f(y)$$

implies that $\sum_y m_{yz}(f(y)^+ - f(y)) = 0$. However we have $f(y)^+ - f(y) \geq 0$ for all y . This shows that for $m_{yz} \neq 0$ we must have $f(y)^+ = f(y)$. Since $m_{y_0,z} \neq 0$, we see that $f(y_0)^+ = f(y_0)$. Thus $f(y_0) \geq 0$.

In the case $\Gamma = \mathbb{Z}\mathbb{A}_1$ with vertices x_i ($i \in \mathbb{Z}$) such that $\tau x_i = x_{i-1}$, the additive functions are the cluster-additive functions, and these are the functions of the form $f(x_i) = (-1)^i a$, where a is a fixed integer.

(3) *Let f be cluster-additive. Let $f(z) < 0$. Then $f(\tau z) \geq -f(z) > 0$.*

Proof: By definition, $f(\tau z) + f(z)$ is a sum of positive numbers, thus non-negative.

This shows:

(4) *Any cluster-additive function with only non-positive values is the zero function.*

(5) *Let $\Gamma = \mathbb{Z}\Delta$ with Δ of Dynkin type. Any cluster-additive function on Γ with only non-negative values is the zero function.*

Proof. Let f be cluster-additive on Γ with only non-negative values. Then f is additive, but according to [HPR] any additive function on Γ with only non-negative values is the zero function.

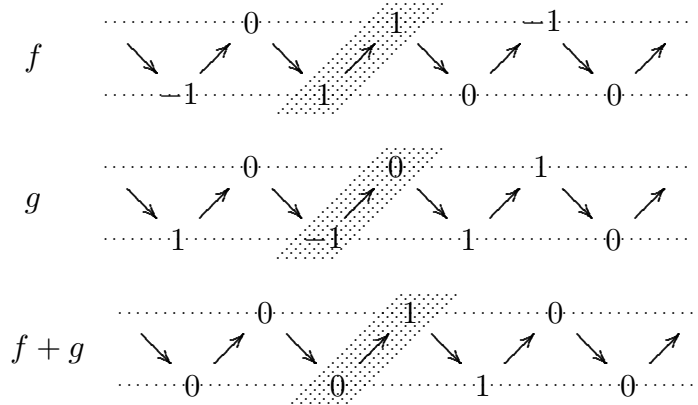
It follows from (3) and (5) that there are many stable translation quivers without non-zero cluster-additive functions. For example, *if Δ is a Dynkin quiver, then the only*

cluster-additive function f on $\Gamma = \mathbb{Z}\Delta/\tau$ is the zero function. Namely, (3) asserts that f only takes non-zero values, thus f is additive. Therefore f gives rise to an additive function on $\mathbb{Z}\Delta$ with non-negative values. According to (5) this implies that f is the zero function.

2. Sums of cluster-additive functions.

The sum of two cluster-additive functions usually will not be cluster-additive, a typical example is the following:

Example. Let $\Gamma = \mathbb{Z}A_2$.



Two cluster-additive functions f, g on Γ are said to be *compatible* provided $f(x)g(x) \geq 0$ for all vertices x . Compatibility can be characterized in many different ways (the proof is obvious):

Lemma. Let f_1, \dots, f_n be cluster-additive functions on Γ . The following conditions are equivalent:

- (i) f_1, \dots, f_n are pairwise compatible.
- (ii) If $f_i(x) < 0$ for some index i and some vertex x , then $f_j(x) \leq 0$ for $1 \leq j \leq n$.
- (iii) If $f_i(x) > 0$ for some index i and some vertex x , then $f_j(x) \geq 0$ for $1 \leq j \leq n$.
- (iv) Given a pair $i \neq j$, there is no vertex x with $f_i(x) < 0$ and $f_j(x) > 0$.

Theorem 1. Let f_1, \dots, f_a be cluster-additive functions on Γ . Then $\sum f_i$ is cluster-additive if and only if the functions are pairwise compatible.

Before we start with the proof, let us isolate a decisive property of the operator $z \mapsto z^+$.

Lemma. Let a_1, \dots, a_n be integers. Then

- (a) $(\sum_i a_i)^+ \leq \sum_i a_i^+$.
- (b) Equality holds if and only if either all a_i are non-negative or all are non-positive.

Proof: Let $a_i \geq 0$ for $1 \leq i \leq m$ and $a_i \leq 0$ for $m+1 \leq i \leq n$, let $a = \sum_{i=1}^n a_i$. Then $\sum_{i=1}^n a_i^+ = \sum_{i=1}^m a_i \geq \sum_{i=1}^n a_i = a$ and therefore $\sum_{i=1}^n a_i^+ \geq a^+$. If we have equality, and $a \geq 0$, then $0 = a - \sum_{i=1}^n a_i^+ = \sum_{i=m+1}^n (-a_i)$ shows that these $a_i = 0$, since all $-a_i$ are non-negative for $m+1 \leq i \leq n$. In this case all the a_i are non-negative. If $a \leq 0$, then

$\sum_{i=1}^m a_i = 0$ shows that these $a_i = 0$, since all a_i are non-negative for $1 \leq i \leq m$. In this case, all a_i are non-positive.

Also the converse holds: If all a_i are non-negative, then $\sum_{i=1}^n a_i^+ = \sum_{i=1}^n a_i = (\sum_{i=1}^n a_i)^+$. If all a_i are non-positive, then also $\sum_{i=1}^n a_i$ is non-positive, and $\sum_{i=1}^n a_i^+ = 0 = (\sum_{i=1}^n a_i)^+$.

Proof of Theorem 1: If Γ is of tree type \mathbb{A}_1 , then the assertion is clear. Thus we can assume that for any vertex y in Γ , there is a vertex z with $m_{yz} \neq 0$.

First let us assume that f_1, \dots, f_a are pairwise compatible and let $f = \sum_i f_i$. We claim that for all vertices y of Γ

$$(*) \quad f(y)^+ = \sum_i f_i(y)^+$$

Let \mathcal{T} be the set of vertices $x \in \Gamma_0$ such that $f_i(x) < 0$ for at least one i . If $y \in \mathcal{T}$, then $f_i(t) \leq 0$ for all $1 \leq i \leq a$, since we deal with pairwise compatible functions. It follows that $f(y) = \sum_i f_i(y) < 0$ and therefore $f(y)^+ = 0$. But also $f_i(y)^+ = 0$ for all i , this yields $(*)$ in case $y \in \mathcal{T}$.

Now assume $y \notin \mathcal{T}$. Then $f_i(y) \geq 0$ for all i , thus $f(y) = \sum_i f_i(y) \geq 0$, therefore

$$f(y)^+ = f(y) = \sum_i f_i(y) = \sum_i f_i(y)^+,$$

and we see that $(*)$ is satisfied also in this case.

Now consider some vertex z .

$$\begin{aligned} f(\tau z) + f(z) &= \sum_i f_i(\tau z) + \sum_i f_i(z) = \sum_i (f_i(\tau z) + f_i(z)) \\ &= \sum_i \left(\sum_y m_{yz} f_i(y)^+ \right) \\ &= \sum_y m_{yz} \sum_i f_i(y)^+ \\ &= \sum_y m_{yz} f(y)^+, \end{aligned}$$

where we use that all the functions f_i are cluster-additive as well as the equality $(*)$ for all y . This shows that f is cluster-additive.

Now let us assume that $f = \sum f_i$ is cluster-additive. Let z be a vertex of Γ . Then, as above, we have

$$\begin{aligned} h(\tau z) + h(z) &= \sum_i f_i(\tau z) + \sum_i f_i(z) = \sum_i (f_i(x) + f_i(z)) \\ &= \sum_i \left(\sum_y m_{yz} f_i(y)^+ \right) \\ &= \sum_y m_{yz} \sum_i f_i(y)^+, \end{aligned}$$

thus

$$\begin{aligned} 0 &= f(\tau z) + f(z) - \sum_y m_{yz} f(y)^+ \\ &= \sum_y m_{yz} \sum_i f_i(y)^+ - \sum_y m_{yz} f(y)^+ \\ &= \sum_y m_{yz} \left(\sum_i f_i(y)^+ - f(y)^+ \right). \end{aligned}$$

According to assertion (a) of the Lemma, all the brackets in the last line are non-negative, thus all the summands $m_{yz} (\sum_i f_i(y)^+ - f(y)^+)$ are non-negative. Since their sum is zero, all these summands are zero.

It follows that for any y we have

$$\sum_i f_i(y)^+ = f(y)^+$$

(since there is z with $m_{yz} \neq 0$). According to the assertion (b) of the Lemma, we conclude that all the values $f_i(y)$ for $1 \leq i \leq a$ are non-negative or all are non-positive. But this means that the functions f_1, \dots, f_a are compatible.

3. Subtraction.

Let us introduce the following *partial ordering* on the set of cluster-additive functions on Γ . If f, g are cluster-additive functions on Γ , we write $g \leq f$ provided $g(x)^+ \leq f(x)^+$ as well as $g(x)^- \leq f(x)^-$ for all vertices x of Γ .

Theorem 2. *Let f, g be cluster-additive functions on Γ . Then $f - g$ is cluster-additive if and only if $g \leq f$.*

Proof. First, let us assume that $g \leq f$. We claim that

$$(f - g)(x)^+ = f(x)^+ - g(x)^+$$

for all vertices x . Namely, if $g(x) > 0$, then $g(x) = g(x)^+ \leq f(x)^+$, and therefore $f(x) = f(x)^+$, thus $g(x) \leq f(x)$ and therefore $(f - g)(x) = f(x) - g(x) \geq 0$, thus

$$(f - g)(x)^+ = (f - g)(x) = f(x) - g(x) = f(x)^+ - g(x)^+.$$

Also, if $g(x) < 0$, then $g(x)^+ = 0$, and $0 < -g(x) = g(x)^- \leq f(x)^-$, thus $f(x)^+ = 0$. Also, $f(x)^- = -f(x)$ and therefore $g(x) \geq f(x)$, thus $(f - g)(x) = f(x) - g(x) \leq 0$. It follows that

$$(f - g)(x)^+ = 0 = f(x)^+ - g(x)^+.$$

Finally, if $g(x) = 0$, then also $g(x)^+ = 0$ and

$$(f - g)(x)^+ = f(x)^+ = f(x)^+ - g(x)^+.$$

Let z be a vertex of Γ , then

$$\begin{aligned} (f - g)(\tau z) + (f - g)(z) &= f(\tau z) - f(z) + g(\tau z) - g(z) \\ &= \sum_y m_{yz} f(y)^+ - \sum_y m_{yz} g(y)^+ \\ &= \sum_y m_{yz} (f - g)(y)^+. \end{aligned}$$

This shows that $f - g$ is cluster-additive.

Conversely, assume that $f - g$ is cluster-additive. Since the sum $f = (f - g) + g$ of the cluster-additive functions f, g is cluster-additive, we know by Theorem 1 that $f - g$ and g are compatible functions, thus $(f - g)(x)g(x) \geq 0$ for all vertices x , thus $f(x)g(x) \geq g(x)g(x)$ for all x . If $g(x) > 0$, then this implies that $f(x) \geq g(x) > 0$, thus $f(x)^+ \geq g(x)^+$ and $f(x)^- = g(x)^-$. If $g(x) < 0$, then $f(x) \leq g(x) < 0$, therefore $g(x)^- = -g(x) \leq -f(x) = f(x)^-$ and $g(x)^+ = 0 = f(x)^+$. Of course, if $g(x) = 0$, then $g(x)^+ = 0 \leq f(x)^+$ and $g(x)^- = 0 \leq f(x)^-$. This shows that $g \leq f$.

4. The restriction of cluster-additive functions to a slice.

We consider now cluster-additive functions on a translation quiver $\Gamma = \mathbb{Z}\Delta$, where Δ is a finite directed quiver. The subset $\Delta_0 \times \{0\}$ is called a slice of Γ (all the slices are obtained by considering the subsets $\eta(\Delta'_0 \times \{0\})$, where $\eta: \mathbb{Z}\Delta' \rightarrow \mathbb{Z}\Delta$ is an isomorphism of translation quivers).

Theorem 3. *Let Δ be a finite directed quiver. Any function $f: \Delta_0 \times \{0\} \rightarrow \mathbb{Z}$ can be extended uniquely to a cluster-additive function on $\mathbb{Z}\Delta$.*

This may be reformulated as follows:

Corollary. *The restriction furnishes a bijection between the set of cluster-additive functions on $\mathbb{Z}\Delta$ and the functions $f: \Delta_0 \times \{0\} \rightarrow \mathbb{Z}$.*

Proof of Lemma: Let x be a source in Δ . Then f is defined for $(\xi, 0)$ and all its direct successors, thus we use the defining property of a cluster-additive function in order to define $f(\xi, 1)$. Inductively we define in this way $f(\eta, j)$ for all vertices η of Δ and all $j > 0$. The dual procedure yields the values $f(\eta, j)$ for $j < 0$.

Remark 1. Note that we need here that Δ is finite. For example, if Δ is the linearly ordered quiver of type \mathbb{A}_∞ , then any function $f: \mathbb{Z}\Delta \rightarrow \mathbb{N}_0$ which is constant on the slices $\Delta \times \{i\}$ for all $i \in \mathbb{Z}$, is additive, thus cluster-additive and of course not determined by the value taken on one of these slices.

We also may look at $\Gamma = \mathbb{Z}\Delta$ with Δ a locally finite (but not necessarily finite) directed quiver. A slice S of Γ may be said to be *generating* provided we obtain all vertices from S of Γ using reflections at sinks and at sources. If Δ a finite, then any slice is generating, but in general not. If $\Gamma = \mathbb{Z}\mathbb{A}_\infty$, then a slice S is generating if and only if no arrow in S belongs to an infinite path. The corollary can be generalized as follows: *Let S be a generating slice. Then the restriction function $f \mapsto f|_S$ is bijective.*

Remark 2. The extension of a function $f: \Delta_0 \times \{i\} \rightarrow \mathbb{Z}$ to a cluster-additive function on Γ can be achieved by using what one may call cluster-reflections. Given a locally finite quiver Δ and a vertex x of Δ , which is a sink or a source, then the *cluster-reflection* σ_x maps any function $f: \Delta_0 \rightarrow \mathbb{Z}$ to the function $\sigma_x f$ with $(\sigma_x f)(y) = f(y)$ for $y \neq x$ and $(\sigma_x f)(x) = -f(x) + \sum_y m_{xy} f(y)^+$ and $\sigma_x f$ should be considered as a function on $(\sigma_x \Delta)_0$, where $\sigma_x \Delta$ is obtained from Δ by changing the orientation of all the arrows involving x (thus replacing a source by a sink and vice versa). Starting with a source x of $\Delta = \Delta \times \{i\}$, then we may identify $\sigma_x \Delta$ with the slice obtained by deleting x and adding $\tau^{-1}x$; given

a function $f: \Delta \rightarrow \mathbb{Z}$, and looking for its cluster-addition extension, then we have to use $\sigma_x f$ on the slice $\sigma_x \Delta$.

Altogether we see that the restrictions of a cluster-additive function on Γ to the various slices of Γ are obtained from each other by a sequence of cluster-reflections.

5. Cluster-hammock functions.

Here we introduce some basic cluster-additive functions. As before, we consider a translation quiver $\Gamma = \mathbb{Z}\Delta$, where Δ is a finite directed quiver.

Recall the definition of the *left hammock function* h'_p for a vertex p of Γ (and that left hammock functions with finite support are called *hammock functions*). First, $h'_p(p) = 1$. Second, if z is not a successor of p , then $h'_p(z) = 0$. Third, assume that $h'_p(y)$ is defined for all proper predecessors y of z ; if there is an arrow $y \rightarrow z$ with $h'_p(y) > 0$, then

$$h'_p(z) = -h'_p(\tau z) + \sum_y m_{yz} h'_p(y),$$

otherwise $h'_p(z) = 0$.

It is well-known that all the values $h'_p(z)$ are non-negative; the support of h'_p will be denoted by H_p . If Δ is a Dynkin quiver (thus of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$, or \mathbb{E}_8), then H_p is finite and there is a unique vertex νp with $h_p(\nu p) \neq 0$ such that any vertex y with $h_p(y) \neq 0$ is a predecessor of νp ; the map $\nu: \Gamma_0 \rightarrow \Gamma_0$ is called the *Nakayama shift* (see [G], where also typical hammock functions are displayed; but note that in contrast to the definition given in this paper, but also in [B] and [RV], Gabriel extends the function $h'_p|_{H_p}$ to an additive function on all of $\mathbb{Z}\Delta$). The shift $\nu\tau^{-1}$ is usually denoted by $[1]$, the shift $\nu\tau^{-2}$ by F .

We insert here that $\mathbb{Z}\Delta$ may be interpreted as the Auslander-Reiten quiver of the derived category $D^b(\text{mod } A)$, where A is the path algebra of the opposite quiver of Δ , see [H]. Given an indecomposable A -module X , we denote by $[X]$ the corresponding vertex in $\mathbb{Z}\Delta$. In this interpretation, $[1]$ corresponds to the shift functor of the derived category and F to the functor $[1]\tau^{-1}$ (also denoted by F) which is used in order to define the corresponding cluster category, see [B-T].

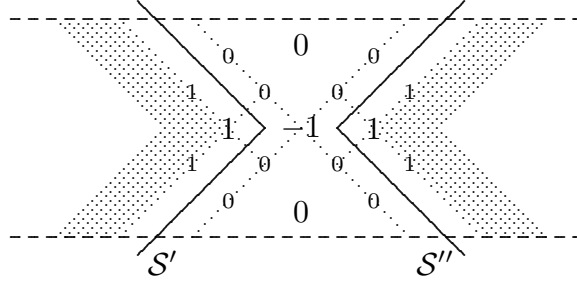
If Δ is connected and not one of these Dynkin diagrams, then the support H_p of h'_p is infinite, for any vertex p of $\mathbb{Z}\Delta$.

For any vertex x of Γ , we now define a cluster-additive function h_x as follows: Let \mathcal{S} be any slice containing x , let $h_x(x) = -1$ and $h_x(y) = 0$ for $y \neq x$ in \mathcal{S} .

According to Theorem 3, we know that h_x extends in a unique way to a cluster-additive function h_x on Γ and this extension is independent of the choice of \mathcal{S} . We call h_x the *cluster-hammock function* for the vertex x .

Proof of the independency: There is a slice \mathcal{S}' with x the unique sink of \mathcal{S}' and a slice \mathcal{S}'' with x the unique source of \mathcal{S}'' , and all other slices containing x are obtained from \mathcal{S}' or also \mathcal{S}'' by reflections at sinks or sources different from x . The corresponding cluster-reflections σ_y do not change the value 0.

Note that the proof shows that $h_x(y) = 0$ for all vertices $y \neq x$ which belong to the convex hull of \mathcal{S}' and \mathcal{S}'' .

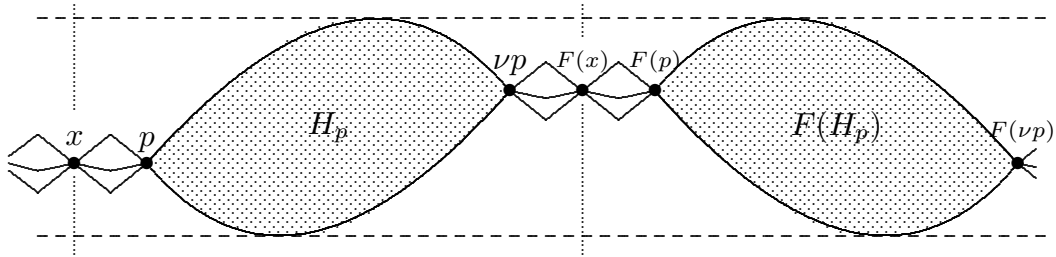


Lemma. Let $\Gamma = \mathbb{Z}\Delta$ with Δ a Dynkin quiver, then h_x is F -invariant. The support of h_x consists of

- the F -orbit of x and h_x takes the value -1 on these vertices, as well as
- the F -shifts of the hammock $H_{\tau^{-1}x}$ and here h_x takes positive values, namely

$$h_x(y) = h'_{\tau^{-1}x}(y) \quad \text{for } y \in H_{\tau^{-1}x}.$$

Here is a schematic illustration, where we write $p = \tau^{-1}x$ (the vertical dotted lines mark a fundamental domain for the action of F):



We have mentioned that h_x is F -invariant: $h_x(y) = h_x(Fy)$ for all $y \in \Gamma$. But this means also that $h_x = h_{Fx}$. Thus, when dealing with a set of cluster-hammock functions, we may restrict to look at those indexed by elements in some fixed fundamental domain for F .

Let us mention a property of the hammock functions h'_p (and of $h_{\tau p}$) which will be used in the next section. If there is a sectional path from p to a vertex y , then $h'_p(y) \geq 1$ (or better: in this case, $h'_p(y)$ is the number of sectional paths from p to y).

We call a subset \mathcal{T} of $\mathbb{Z}\Delta$ *confined* provided there is a slice \mathcal{S} such that \mathcal{T} is contained in the convex hull of \mathcal{S} and $\tau\mathcal{S}[1]$; note that this is the Auslander-Reiten quiver $\Gamma(A)$ of a hereditary algebra A of type Δ , with \mathcal{S} the indecomposable projective A -modules, and $\tau\mathcal{S}[1]$ the indecomposable injective A -modules.

We call a subset \mathcal{T} of Γ a *tilting set* provided we can identify Γ as a translation quiver with $D^b(\text{mod } A)$ for some hereditary algebra A such that \mathcal{T} are just the positions of the indecomposable direct summands of a tilting A -module. Subsets of tilting sets are called *partial tilting sets*.

Lemma. Let X, Y be indecomposable A -modules. Then $\text{Ext}^1(X, Y) = 0$ if and only if $h_{[Y]}([X]) = 0$.

Proof: There is the Auslander-Reiten formula $\text{Ext}^1(X, Y) \simeq D \text{Hom}(\tau^{-1}Y, X)$ and $\dim \text{Hom}(\tau^{-1}Y, X) = h'_{[\tau^{-1}Y]}([X]) = h_{[Y]}([X])$.

Corollary. *A subset \mathcal{T} of Γ is partial tilting if and only if \mathcal{T} is confined and $h_x(x') = 0$ for all pairs $x \neq x'$ in \mathcal{T} .*

6. Non-negative linear combinations of cluster-hammock functions.

Again, we deal with a translation quiver $\Gamma = \mathbb{Z}\Delta$, where Δ is a finite directed quiver.

Proposition 1. *Consider a set h_1, \dots, h_n of cluster-hammock functions. These functions are pairwise compatible if and only if there is a tilting set \mathcal{T} such that any h_i is of the form h_x with $x \in \mathcal{T}$.*

Proof: Let \mathcal{T} be a tilting set and $x, x' \in \mathcal{T}$. We have to show that $h_x, h_{x'}$ are compatible. This is clear if $h_x = h_{x'}$. Thus assume that $h_x \neq h_{x'}$, thus x and x' do not belong to the same F -orbit of Γ_0 . Let $h_{x'}(y) < 0$, then y belongs to the F -orbit of x' , thus $h_x(y) = h_x(x') = 0$. This shows that $h_x(y)h_{x'}(y) = 0$. Similarly, we see: if $h_x(y) < 0$, then $h_x(y)h_{x'}(y) = 0$. For the remaining vertices y we have both $h_x(y) \geq 0$ and $h_{x'}(y) \geq 0$, thus also $h_x(y)h_{x'}(y) \geq 0$.

Conversely, assume that the functions h_1, \dots, h_n are pairwise compatible. First, we show that for $h_i \neq h_j$, and $h_j = h_y$ for some vertex y , then $h_i(y) = 0$. Namely, $h_i(y)h_y(y) = h_i(y)h_j(y) \geq 0$, and $h_y(y) = -1$ shows that $h_i(y) \leq 0$. But $h_i(y) < 0$ would imply that $h_i = h_y$, a contradiction. Thus $h_i(y) = 0$.

Now, let $h_1 = h_x$ for some $x \in \Gamma_0$. Let \mathcal{S} be the slice in Γ such that $\tau^{-2}F^{-1}x$ is the unique source. Let $\mathcal{S}' = \mathcal{S}[1]$, this is the slice with unique source $\tau^{-1}x$. Clearly, the convex hull \mathcal{F} of \mathcal{S} and \mathcal{S}' is a fundamental domain for F , thus $h_i = h_{x_i}$ for some $x_i \in \mathcal{F}$. Since $\tau^{-1}x$ is the unique source of \mathcal{S}' , we see that $h_x(z) > 0$ for all $z \in \mathcal{S}'$. Assume that some x_j belongs to \mathcal{S}' , then $x \neq x_j$, thus $h_x \neq h_{x_j}$ (since x, x_j belong to the fundamental domain \mathcal{F} of F), but then we know that $h_x(x_j) = 0$, a contradiction. In this way, we see that all the vertices x_i belong to the convex hull of \mathcal{S} and $\tau\mathcal{S}' = \tau\mathcal{S}[1]$, thus the set $\mathcal{T} = \{x_1, \dots, x_n\}$ is confined. Since also $h_x(x') = 0$ for $x \neq x'$ in \mathcal{T} , we see that \mathcal{T} is a tilting set.

Corollary. *A linear combination $h = \sum_{x \in \mathcal{T}} n_x h_x$ with positive integers n_x is cluster-additive if and only if \mathcal{T} is a partial cluster-tilting set.*

Proof. This is a direct consequence of Theorem 1 and Proposition 1.

Proposition 2. *Let $f = \sum_{x \in \mathcal{T}} n_x h_x$ for some tilting set \mathcal{T} and $n_x \in \mathbb{N}_0$, then $f(x) = -n_x$ for $x \in \mathcal{T}$ and $f(y) \geq 0$ provided the intersection of \mathcal{T} and the F -orbit of y is empty. Thus*

$$f = \sum_{x \in \mathcal{T}} n_x h_x = - \sum_{x \in \mathcal{T}} f(x) h_x = \sum_{x \in \mathcal{T}} f(x)^- h_x = \sum_{x \in \Gamma^0} f(x)^- h_x.$$

where Γ^0 is the convex hull of some slice \mathcal{S} and $\tau\mathcal{S}[1]$.

Conjecture. Let $\Gamma = \mathbb{Z}\Delta$ where Δ is one of the Dynkin diagrams $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ and let f be cluster-additive on Γ . Then f is a non-negative linear combination of cluster-hammock functions (and therefore of the form

$$\sum_{x \in \mathcal{T}} n_x h_x$$

for a tilting set \mathcal{T} and integers $n_x \in \mathbb{N}_0$, for all $x \in \mathcal{T}$).

If this conjecture is true, then any cluster-additive function satisfies the following properties:

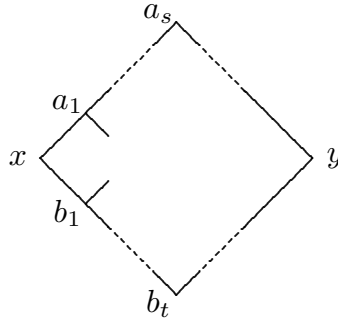
- (a) f is F -invariant.
- (b) $\{x \in \Gamma_0 \mid f(x) < 0\}$ is the union of the F -orbits of a partial tilting set.
- (c) There is a partial tilting set \mathcal{T} with

$$f = \sum_{x \in \mathcal{T}} f(x)^- h_x$$

A proof of the conjecture in the case \mathbb{A}_n will be given in section 9. We also note that it is not difficult to exhibit explicitly all the cluster-additive functions on $\mathbb{Z}\Delta$, where Δ is a quiver of type \mathbb{D}_4 , thus verifying the conjecture also in this case.

7. The rectangle rule.

Lemma. Let f be cluster-additive on the following translation quiver with $s \geq 1, t \geq 1$:



Then for $y = y(s, t)$ with $f(x) \leq 0$, we have

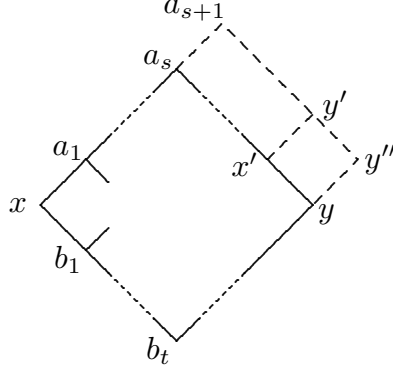
$$f(y) = f(x)^- + \sum_{1 \leq i \leq s-1} f(a_i)^- + f(a_s)^+ + \sum_{1 \leq j \leq t-1} f(b_j)^- + f(b_t)^+.$$

In particular, $f(y) \geq f(x)^- \geq 0$.

Proof, by induction on s and t .

If $s = t = 1$, then $f(y) = f(a_1)^+ + f(b_1)^+ - f(x)$.

Now assume that we know the formula for some s, t . Let us increase s by 1, thus we deal with



For $t = 1$, we have $x' = a_s$ and $y' = a_{s+1}$, otherwise $x' = y(s, t - 1)$ and $y' = y(s + 1, t - 1)$.

Now, consider first the case $t = 1$. Then (since $f(y) \geq 0$):

$$\begin{aligned}
f(y'') &= f(y')^+ + f(y) - f(x') \\
&= f(a_{s+1})^+ + f(x)^- + \sum_{1 \leq i \leq s-1} f(a_i)^- + f(a_s)^+ + \sum_{1 \leq j \leq t-1} f(b_j)^- + f(b_t)^+ - f(a_s) \\
&= f(a_{s+1})^+ + f(x)^- + \sum_{1 \leq i \leq s} f(a_i)^- + \sum_{1 \leq j \leq t-1} f(b_j)^- + f(b_t)^+
\end{aligned}$$

where the last equality comes from $f(a_s)^+ - f(a_s) = f(a_s)^-$.

Second, let $t \geq 2$. Then both $f(y) \geq 0$, $f(y') \geq 0$, thus

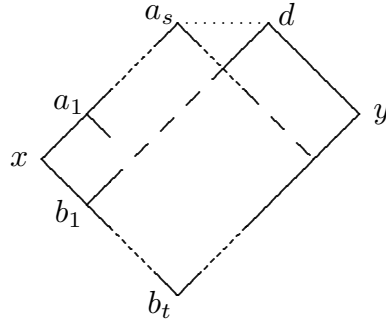
$$\begin{aligned}
f(y'') &= f(y') + f(y) - f(x') \\
&= f(x)^- + \sum_{1 \leq i \leq s} f(a_i)^- + f(a_{s+1})^+ + \sum_{1 \leq j \leq t-2} f(b_j)^- + f(b_{t-1})^+ \\
&\quad + f(x)^- + \sum_{1 \leq i \leq s-1} f(a_i)^- + f(a_s)^+ + \sum_{1 \leq j \leq t-1} f(b_j)^- + f(b_t)^+ \\
&\quad - f(x)^- - \sum_{1 \leq i \leq s-1} f(a_i)^- - f(a_s)^+ - \sum_{1 \leq j \leq t-2} f(b_j)^- - f(b_{t-1})^+ \\
&= f(x)^- + \sum_{1 \leq i \leq s} f(a_i)^- + f(a_{s+1})^+ + \sum_{1 \leq j \leq t-1} f(b_j)^- + f(b_t)^+,
\end{aligned}$$

as we want.

By symmetry, the same argument works, if we increase t instead of s . This completes the proof.

Extended version. Let f be cluster-additive on the following translation quiver with

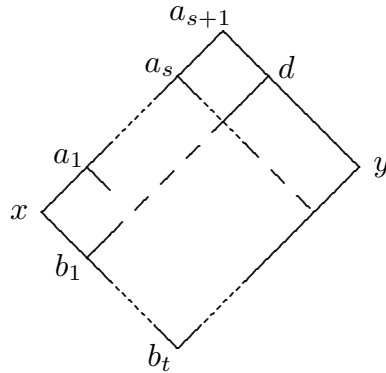
$s \geq 1, t \geq 1$:



Then for $y = y(s + 1, t)$ with $f(x) \leq 0$, we have

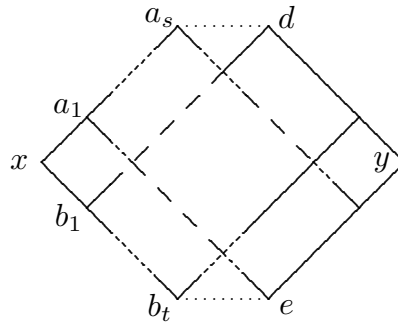
$$f(y) = f(x)^- + \sum_{1 \leq i \leq s-1} f(a_i)^- + \sum_{1 \leq j \leq t-1} f(b_j)^- + f(b_t)^+.$$

Proof: We add a vertex a_{s+1} and arrows $a_s \rightarrow a_{s+1}$ and $a_{s+1} \rightarrow d$, so that we obtain a rectangle. Also, we extend f to be defined on the rectangle by setting $f(a_s) = 0$. Then the extended function satisfies the cluster-additivity condition on all the meshes of the rectangle and we can apply the lemma.



There is also a corresponding double extended version for dealing with $\mathbb{Z}\Delta$ where Δ is of type \mathbb{A}_{s+t+1} .

Double extended version. Let f be cluster-additive on the following translation quiver with $s \geq 1, t \geq 1$:



Then for $y = y(s + 1, t + 1)$ with $f(x) \leq 0$, we have

$$f(y) = f(x)^- + \sum_{1 \leq i \leq s-1} f(a_i)^- + \sum_{1 \leq j \leq t-1} f(b_j)^-.$$

8. Wings.

Let $s \geq 0, t \geq 1$, let y be a wing vertex of rank $s + t + 1$, say with sectional paths

$$p[1] \rightarrow p[2] \rightarrow \cdots \rightarrow p[s + t + 1] = y, \quad y = [s + t + 1]q \rightarrow \cdots \rightarrow [2]q \rightarrow q[1].$$

Lemma. Assume that

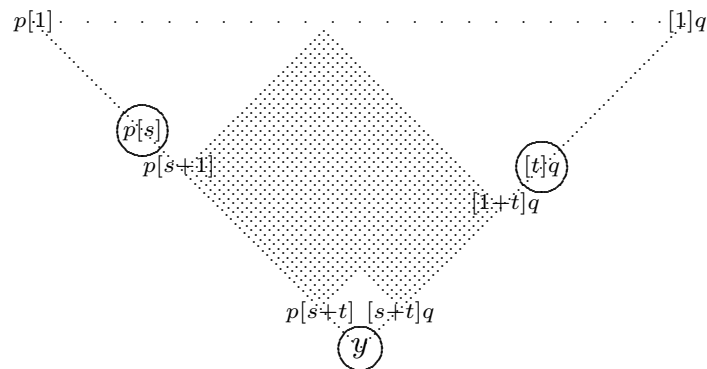
$$f(p[s]) \leq 0, \quad f(p[s + i]) \geq 0, \text{ for } 1 \leq i \leq t, \quad f(p[s + t + 1]) \leq 0.$$

Then

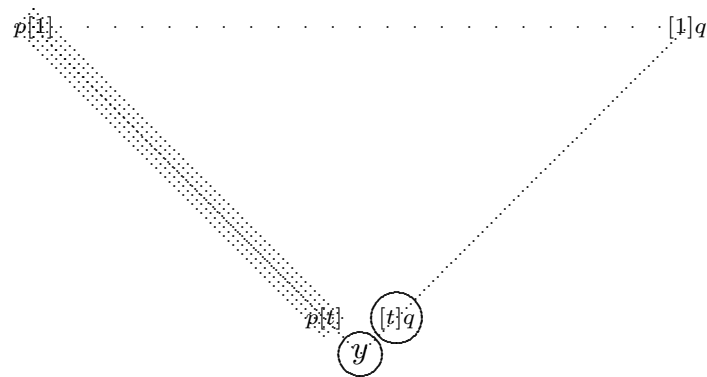
$$f([t]q) = \min_{1 \leq i \leq t} f(p[s + i]).$$

Also, f is non-negative on all vertices between $p[s + 1]$ and $[1 + t]q$ different from y .

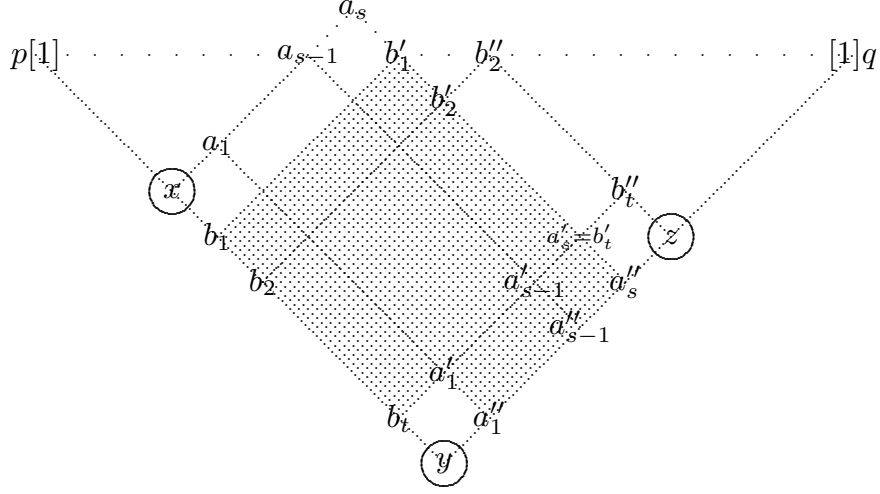
Here is a sketch which exhibits the vertices in question in case $s \geq 1$:



The case $s = 0$ looks as follows:



Proof: Let us use the following labels for the relevant vertices of the wing



In particular

$$x = p[s], \quad z = [t]q, \quad b_i = p[s + i], \quad a_i = \tau^{-i}p[s - i].$$

Note that we have added the vertex a_s to the wing, (with additional arrows $a_{s-1} \rightarrow a_s$ and $a_s \rightarrow b'_1$) and we put $f(a_s) = 0$, as in the proof of the extended rectangle rule.

Using the new labels, the assumptions read:

$$f(x) \leq 0, \quad f(b_i) \geq 0, \quad \text{for } 1 \leq i \leq t, \quad f(y) \leq 0$$

and the assertion is that f is non-negative on the shaded area (the vertices between b_1 and a''_s different from y) and that

$$f(z) = -\min(f(b_i) \mid 1 \leq i \leq t).$$

The rectangle rule asserts that f is bounded below by $f(x)^-$ on the rectangle between $\tau^{-1}x$ and $a'_s = b'_t$. By assumption, f is non-negative on the vertices b_1, \dots, b_t . Thus, concerning the non-negativity assertion, it remains to show that f is non-negative on the vertices a''_1, \dots, a''_s .

The rectangle rule asserts that

$$f(a'_i) = f(x)^- + \sum_{j=1}^{i-1} f(a_j)^- + f(a_i)^+ + \sum_{j=1}^{t-1} f(b_j)^- + f(b_t)^+.$$

Since $f(y) \leq 0$ and $f(a'_1) \geq 0$, we have $f(a''_1) = f(a'_1) - f(b_t) \geq 0$. Assume by induction that we know that $f(a''_i) = f(a'_i) - f(b_t) \geq 0$, then we get

$$\begin{aligned} f(a''_{i+1}) + f(a'_i) &= f(a''_i)^+ + f(a'_{i+1})^+ \\ &= f(a''_i) + f(a'_{i+1}) \\ &= f(a'_i) - f(b_t) + f(a'_{i+1}) \end{aligned}$$

and therefore

$$f(a''_{i+1}) = f(a'_{i+1}) - f(b_t).$$

By the rectangle rule for a'_{i+1} we see that $f(a'_{i+1}) - f(b_t) \geq 0$ provided $i + 1 \leq s$.

It remains to calculate the value $f(z)$.

Using induction on i , we show that

$$f(b''_i) = f(b_i) - \min(f(b_j) \mid 1 \leq j < i)$$

for $i \geq 2$.

The rectangle rule for b'_i yields

$$\begin{aligned} f(b'_i) &= f(x)^- + \sum_{j=1}^{s-1} f(a_j)^- + f(a_s)^+ + \sum_{j=1}^{i-1} f(b_j)^- + f(b_i)^+ \\ &= f(x)^- + \sum_{j=1}^{s-1} f(a_j)^- + f(b_i), \end{aligned}$$

since $f(a_s) = 0$ and all $f(b_j) \geq 0$. Similarly, for b'_{i+1} we get:

$$f(b'_{i+1}) = f(x)^- + \sum_{j=1}^{s-1} f(a_j)^- + f(b_{i+1}),$$

thus

$$f(b'_{i+1}) - f(b'_i) = f(b_{i+1}) - f(b_i).$$

For $i = 2$, we have

$$f(b''_2) = f(b'_2) - f(b'_1),$$

since $f(b'_2) \geq 0$, thus

$$f(b''_2) = f(b'_2) - f(b'_1) = f(b_2) - f(b_1) = f(b_2) - \min(f(b_j) \mid 1 \leq j < 2),$$

as we have claimed.

Similarly, we have for all $i \geq 2$

$$\begin{aligned} f(b''_{i+1}) &= f(b''_i)^+ + f(b'_{i+1}) - f(b'_i) \\ &= f(b''_i)^+ + f(b_{i+1}) - f(b_i). \end{aligned}$$

By induction, we may assume that

$$f(b''_i) = f(b_i) - \min(f(b_j) \mid 1 \leq j < i),$$

and we have to distinguish two cases:

First, assume that $f(b_i'') \leq 0$. Then $f(b_i'')^+ = 0$ and $f(b_i) \leq \min(f(b_j) \mid 1 \leq j < i)$, so that $\min(f(b_j) \mid 1 \leq j \leq i) = f(b_i)$. Thus

$$\begin{aligned} f(b_{i+1}'') &= f(b_i'')^+ + f(b_{i+1}) - f(b_i) \\ &= 0 + f(b_{i+1}) - \min(f(b_j) \mid 1 \leq j \leq i), \end{aligned}$$

as we want to show.

In the second case, $f(b_i'') \geq 0$, thus $f(b_i'')^+ = f(b_i'')$ and $f(b_i) \geq \min(f(b_j) \mid 1 \leq j < i)$, so that $\min(f(b_j) \mid 1 \leq j \leq i) = \min(f(b_j) \mid 1 \leq j < i)$. Thus

$$\begin{aligned} f(b_{i+1}'') &= f(b_i'')^+ + f(b_{i+1}) - f(b_i) \\ &= f(b_i) - \min(f(b_j) \mid 1 \leq j < i) + f(b_{i+1}) - f(b_i) \\ &= \min(f(b_j) \mid 1 \leq j \leq i) + f(b_{i+1}). \end{aligned}$$

Thus we see that

$$f(b_t'')^+ = f(b_t) - \min(f(b_i) \mid 1 \leq i \leq t).$$

On the other hand, the calculations in the first part of the proof had shown that $f(a_s'') \geq 0$ and that

$$f(a_s'') - f(a_s') = -f(b_t).$$

It follows that

$$\begin{aligned} f(z) &= f(b_t'')^+ + f(a_s'') - f(a_s') = f(b_t'')^+ + f(a_s'')^+ - f(a_s') \\ &= f(b_t'')^+ - f(b_t) = f(b_t) - \min(f(b_i) \mid 1 \leq i \leq t) - f(b_t) \\ &= -\min(f(b_i) \mid 1 \leq i \leq t) \end{aligned}$$

This completes the proof.

9. The case $\Gamma = \mathbb{Z}\mathbb{A}_n$

Consider now the case $\Gamma = \mathbb{Z}\Delta$ with Δ of type \mathbb{A}_n .

Theorem 4. *Let $\Gamma = \mathbb{Z}\Delta$ with Δ of type \mathbb{A}_n . Then any cluster-additive function on Γ is a non-negative linear combination of cluster-hammock functions.*

If $n = 1$, then any cluster-additive function on Γ is a non-negative multiple of one of the two cluster-hammock functions. Thus, we can assume that $n \geq 2$.

Let f be a cluster-additive function on Γ .

(1) *If z is a vertex of Γ with $f(z) \leq 0$, then there is a vertex $z' \neq z$ with $f(z') \leq 0$ and a sectional path from z to z' or from z' to z .*

Proof: Since $n \geq 2$, there is an arrows $a_1 \rightarrow a_0 = z$. Choose m maximal such that there exists a sectional path

$$a_m \rightarrow \cdots \rightarrow a_1 \rightarrow a_0 = z.$$

If $f(a_i) \leq 0$ for some $1 \leq i \leq m$, then let $z' = a_i$. Otherwise we consider the wing with corners

$$p[1] = a_m, \quad z, \quad q[1] = \tau^{-m} a_m.$$

The wing lemma (with $s = 0$) asserts that $f(z) \leq 0$ (even $f(z) < 0$) for $z' = \tau^{-1} a_1$.

(2) If $f(z) < 0$ for some vertex z , then $f(z) = f(Fz)$ and

$$f(z)^- h_z \leq f.$$

Proof: According to (1), there is a vertex $y = z'$ with $f(y) \leq 0$ and a sectional path from z to y or from y to z . Up to duality, we can assume that there is a sectional path from y to z (otherwise we consider instead of Γ the opposite translation quiver). Also, we can assume that we choose y such that the path from y to x is of smallest possible length (thus f is positive on all the vertices between y and z). Consider the wing with corners

$$p[1], \quad y, \quad [1]q,$$

thus there are sectional paths

$$p[1] \rightarrow p[2] \rightarrow \cdots \rightarrow p[m] = y, \quad y = [m]q \rightarrow \cdots \rightarrow [2]q \rightarrow q[1]$$

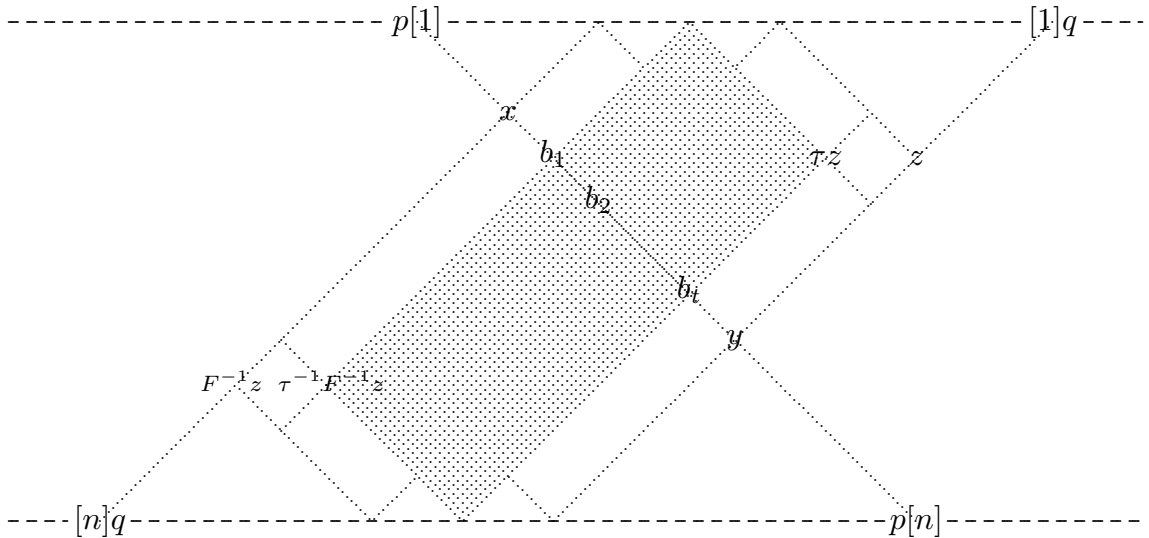
and z is one of the vertices $[j]q$ with $1 \leq j < m$. Let $s \geq 0$ be maximal with $f(p[s]) \leq 0$ and $t = m - s - 1$. We claim that $t \geq 1$ and that $z = [t]q$.

First of all, for $t = 0$, the rectangle rule would imply that $f([j]q) \geq 0$ for $1 \leq j < m$, but z is of the form $[j]q$ and $f(z) < 0$.

This means that we can use the wing lemma, it asserts that $f([t]q) = \min(f(p[s+i]) \mid 1 \leq i \leq t)$ and that $f([j]q) \geq 0$ for $1+t \leq j \leq s+t$. Since $f(z) < 0$, with z of the form $[j]q$ and $j \leq s+t$, it follows that $j \leq t$. On the other hand, we know that f is positive on all vertices between y and z , thus we see that $j = t$.

Let $x = p[s]$, $b_i = p[s+i]$, and $z = [t]q$ and note that we have $f(x) \leq 0$, $f(y) \leq 0$, and $f(b_i) \geq 1$ for $1 \leq i \leq t$. This yields the upper wing in the following picture, namely the wing with corners

$$p[1], \quad y, \quad [1]q.$$



According to the wing lemma, we know that

$$f(z) = -\min(f(b_i) \mid 1 \leq i \leq t) < 0$$

But starting with x and y , we may also look at the wing with corners

$$[n]q, x, p[n],$$

and use the dual argument: the dual of the wing lemma concerns the vertex $F^{-1}z$ (as well as the vertices between $F^{-1}z$ and x), it yields

$$f(F^{-1}z) = -\min(f(b_i) \mid 1 \leq i \leq t).$$

This shows that

$$f(z) = f(F^{-1}z).$$

Also, the rectangle rule for $F^{-1}z$ (or the dual rectangle rule for z) assert that f is bounded from below by $f(z)^- = -f(z)$ on the rectangle starting with $\tau^{-1}F^{-1}z$ and ending with τz (the shaded area).

Using induction as well as duality, we see that $f(F^a z) = f(z)$ for all $a \in \mathbb{Z}$. Also, it follows that

$$f(z)^- h_z \leq f.$$

Proof of Theorem 4. Choose some slice \mathcal{S} . Given a function g on the set of vertices of Γ , we write

$$|g|_{\mathcal{S}} = \sum_{x \in \mathcal{S}} |g(x)|.$$

thus $|g|_{\mathcal{S}} = 0$ if and only if $g(x) = 0$ for all $x \in \mathcal{S}$. In case g is cluster-additive, we know from section 1 that $|g|_{\mathcal{S}} = 0$ if and only if g is the zero function.

We want to show any cluster-additive function f on Γ is a non-negative linear combination of cluster-hammock functions. We use induction on $|f|_{\mathcal{S}}$. If $|f|_{\mathcal{S}} = 0$, then f is the zero function.

Now assume that $|f|_{\mathcal{S}} > 0$. According to the assertion (5) in section 1, there is some vertex z with $f(z) < 0$.

According to (2), we know that $h_z \leq f(z)^- h_z \leq f$. We see by Theorem 2 that $f - h_z$ is cluster-additive again, and $|f - h_z|_{\mathcal{S}} < |f|_{\mathcal{S}}$. Thus, by induction, $f - h_z$ is a non-negative linear combination of cluster-hammock functions and then also $f = (f - h_z) + h_z$ is a non-negative linear combination of cluster-hammock functions. This completes the proof.

10. Cluster-tilted algebras.

Let A be a finite-dimensional hereditary k -algebra (k an algebraically closed field). Let T be a tilting A -module, \mathcal{T} the set of isomorphism classes of indecomposable direct summands of T , and $F\mathcal{T}$ the union of the F -orbits which contain elements of \mathcal{T} . Let B

be the opposite endomorphism ring of T in the cluster category $\mathcal{C} = D^b(\text{mod } A)/F$ (see [B-T]), thus B is a cluster-tilted algebra.

Define a function d_T on the Auslander-Reiten quiver Γ of $D^b(\text{mod } A)$ as follows: Consider the projection

$$D^b(\text{mod } A) \longrightarrow D^b(\text{mod } A)/F = \mathcal{C}_A \longrightarrow \mathcal{C}_A/\langle T \rangle = \text{mod } B,$$

and denote it by π .

Let y be a vertex of Γ , thus the isomorphism class of an indecomposable object in $D^b(\text{mod } A)$. If y is not in $F\mathcal{T}$, then $\pi(y)$ is the isomorphism class of an indecomposable B -module and we denote by $d_T(y)$ its k -dimension. On the other hand, if the F -orbit of y contains an element x of \mathcal{T} , and $x = [X]$, where X is an indecomposable direct summand of T , then let $d_T(x) = n_x$ be the Krull-Remak-Schmidt multiplicity of X in T , note that this is also the k -dimension of the corresponding simple B -module S_x . In this way we obtain a function

$$d_T: \Gamma_0 \rightarrow \mathbb{Z}$$

which obviously is F -invariant.

Of course, instead of looking at the k -dimension of the B -modules, one may also consider their length. In this way, one similarly defines the function

$$l_T: \Gamma_0 \rightarrow \mathbb{Z}$$

with $l_T(y)$ the length of $\pi(y)$ in case y is not in $F\mathcal{T}$, and with $l_T(y) = -1$ otherwise. If the tilting module T is multiplicity-free, then $l_T = d_T$. For a general tilting module T , let T' be multiplicity-free with the same indecomposable direct summands as T , then $l_T = d_{T'}$.

Lemma. *The function d_T on Γ is cluster-additive and we have*

$$d_T = \sum_{x \in \mathcal{T}} n_x h_x.$$

Proof: Let us consider the mesh of $\mathbb{Z}\Delta$ ending in z , say with arrows $y_i \rightarrow z$, $1 \leq i \leq s$. We assume that the vertices y_{r+1}, \dots, y_s belong to $F\mathcal{T}$, and y_1, \dots, y_r not.

First, consider the case that neither z nor τz belong to $F\mathcal{T}$, thus we may consider the Auslander-Reiten sequence ending in Z . By the assumption on the y_i , we see that the Auslander-Reiten sequence has the form

$$0 \rightarrow \tau Z \rightarrow \bigoplus_{i=1}^r Y_i^{m_i} \rightarrow Z \rightarrow 0,$$

with indecomposable B -modules Z and Y_i such that $[Z] = z$, $[Y_i] = y_i$ and where $m_i = m_{y_i, z}$. It follows that

$$\begin{aligned} d_T(z) + d_T(\tau z) &= \dim Z + \dim \tau Z \\ &= \sum_{i=1}^r m_i \dim Y_i = \sum_{i=1}^r m_{y_i, z} d_T(y_i) \\ &= \sum_{i=1}^s m_{y_i, z} d_T(y_i)^+, \end{aligned}$$

since $d_T(y_i) < 0$ for $r + 1 \leq i \leq s$.

Next, let τz belong to $F\mathcal{T}$, thus z is a projective vertex, say $z = [Z]$ for some indecomposable projective B -module Z . By the assumption on the y_i , the radical of Z has the form $\text{rad } Z = \bigoplus_{i=1}^r Y_i^{m_i}$ and $Z/\text{rad } Z$ has dimension n_z . This means

$$\begin{aligned} d_T(z) + d_T(\tau z) &= \dim Z - \dim Z/\text{rad } Z = \dim \text{rad } Z \\ &= \sum_{i=1}^r m_i \dim Y_i = \sum_{i=1}^r m_{y_i, z} d_T(y_i) \\ &= \sum_{i=1}^s m_{y_i, z} d_T(y_i)^+. \end{aligned}$$

Finally, we have to consider the case where z belongs to $F\mathcal{T}$. This case is dual to the previous one, now $\tau z = [X]$ for some indecomposable injective B -module X and the socle of X has dimension n_z .

The Jordan-Hölder theorem for $\text{mod } B$ shows that d_T is just the sum of the various functions $n_x h_x$ with $x \in \mathcal{T}$; namely, if y is a vertex of Γ , such that $\pi(y)$ is the isomorphism class of an indecomposable B -module N , then $h_x(y)$ is just the Jordan-Hölder multiplicity of the simple B -module S_x in N .

In the Dynkin case, we can use the cluster-algebras in order to prove our conjecture for an F -invariant cluster-additive function f provided two conditions on the position of the vertices x with $f(x) \leq 0$ are satisfied.

Proposition. *Let f be a cluster-additive function on $\Gamma = \mathbb{Z}\Delta$ with Δ a Dynkin quiver. Assume that f is F -invariant and that there is a tilting set \mathcal{T} with the following two properties:*

- (a) *If x belongs to \mathcal{T} , then $f(x) \leq 0$.*
- (b) *If $f(x) < 0$, then x belongs to the F -orbit of an element of \mathcal{T} .*

Then f is a non-negative linear combination of cluster-hammock functions.

Proof: We identify $\Gamma = \mathbb{Z}\Delta$ with the Auslander-Reiten quiver of $D^b(\text{mod } A)$ where A is a finite-dimensional hereditary algebra and where T is a tilting A -module such that \mathcal{T} is the set of isomorphism classes of indecomposable direct summands of T . Let B be the opposite endomorphism ring of T in $\mathcal{C}_A = D^b(\text{mod } A)/F$. We form the factor category $D^b(\text{mod } A)/\langle F^i T \mid i \in \mathbb{Z} \rangle$, this is the module category of a Galois cover \tilde{B} of B (with Galois group \mathbb{Z}). Thus, the Auslander-Reiten quiver $\Gamma' = \Gamma(\tilde{B})$ of $\text{mod } \tilde{B}$ is the translation subquiver obtained from Γ by deleting the F -orbits of the vertices in \mathcal{T} .

Denote by f' the restriction of f to Γ' . By assumption (b), f' takes values in \mathbb{N}_0 , is cluster-additive, thus additive on $\Gamma(\tilde{B})$ and F -invariant; thus it induces an additive function f'' on $\Gamma'' = \Gamma(B) = \Gamma(\tilde{B})/F = \Gamma'/F$ with values in \mathbb{N}_0 . According to Butler [Bu], f'' is additive on all exact sequences, thus it is a linear combination of the ‘‘hammock functions’’ h_p'' for $\text{mod } B$, where p runs through the set of indecomposable projective B -modules. If we compose these functions h_p'' with the projection $\Gamma' \rightarrow \Gamma'/F = \Gamma''$, we obtain just the restriction of h_p' to Γ' , where $p = \tau^{-1}x$ for some $x \in \mathcal{T}$. Thus, there are integers

n_p such that

$$f'' = \sum_p n_p h_p'',$$

and therefore

$$f|_{\Gamma'} = f' = \sum_p n_p h_p'|_{\Gamma'}.$$

If P' is an indecomposable projective B -module with isomorphism class p' and S' is its top (a simple B -module), then

$$n_{p'} = \sum_p n_p h_p''([S']) = f''([S']) \geq 0$$

(here we use that f' takes values in \mathbb{N}_0), thus all the coefficients n_p are non-negative.

We have seen that f and $h = \sum_p n_p h_p'$ coincide on Γ' , it remains to show that they also coincide on \mathcal{T} . Let $x \in \mathcal{T}$, thus $p = \tau^{-1}x$ is in Γ'_0 and

$$\begin{aligned} f(x) &= -f(p) + \sum_{y \in \Gamma_0} m_{y,p} f(y)^+ \\ &= -f(p) + \sum_{y \in \Gamma'_0} m_{y,p(s)} f(y)^+ \\ &= -h(p) + \sum_{y \in \Gamma'_0} m_{y,p(s)} h(y)^+ \\ &= -h(p) + \sum_{y \in \Gamma_0} m_{y,p(s)} h(y)^+ \\ &= h(x), \end{aligned}$$

where we have used that both f and h are cluster-additive, that they coincide on Γ' and have positive values only on vertices in Γ'_0 (condition (a)). This completes the proof that $f = h$.

If a cluster-additive function on Γ is a non-negative linear combination of cluster-hammock functions, then also the following properties are satisfied:

(d) Always, $f = d_T$ for some partial tilting module T .

(e) If f takes values in $\{-1\} \cup \mathbb{N}_0$, then $f = d_T$ for some multiplicity free partial tilting module T , if f takes values in $\mathbb{Z} \setminus \{0\}$, then $f = d_T$ for some tilting module T .

We end this section by giving an interpretation of the exchange property of cluster-tilting objects in a cluster category in terms of the cluster-hammock functions. Thus, suppose that we deal with a tilting set \mathcal{T} in $\mathbb{Z}\Delta$, where Δ is a Dynkin quiver. Let us look at the hammock h_x for some $x \in \mathcal{T}$. Let $\mathcal{T}' = \mathcal{T} \setminus \{x\}$. We claim that there are precisely two F -orbits of vertices of Γ which are not in the support of any function h_y with $y \in \mathcal{T}'$. Of course, one of these vertices is x itself, since $h_y(x) = 0$ for all $y \in \mathcal{T}'$. In order to find

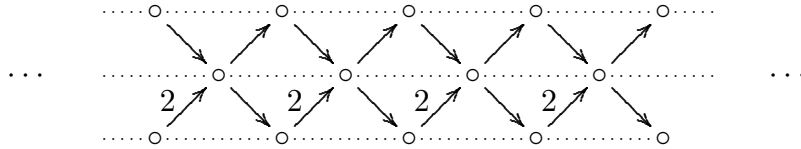
the other orbit, we only have to consider the vertices z which do not belong to $F\mathcal{T}$. As above, we know that $\pi(z)$ is the isomorphism class $[N]$ of an indecomposable B -module, say N . Now either $[N] = [S_x]$, then indeed $h_y(z) = 0$ for all $y \in \mathcal{T}'$ (since N has no composition factor of the form S_y), or else N is not isomorphic to S_x , but then N has at least one composition different from S_x , say S_y with $y \in \mathcal{T}'$, and therefore $h_y(z) \neq 0$. This shows that the second orbit consists of the vertices z such that $\pi(z) = [S_x]$. (But a warning is necessary: the position of z with $\pi(z) = [S_x]$ in the support of h_x does not only depend on h_x itself, as already the case \mathbb{A}_2 shows.)

11. Final Remarks.

1. The main results and conjectures of this note concern the translation quivers $\mathbb{Z}\Delta$ with Δ a simply laced Dynkin diagram. But there is no problem to extend the considerations to the case of an arbitrary (not necessarily simply laced) Dynkin diagram. In order to do so, we need the notion of a valued translation quiver.

A *valued translation quiver* $\Gamma = (\Gamma_0, \Gamma_1, \tau, v)$ is given by a translation quiver $(\Gamma_0, \Gamma_1, \tau)$ with the property that there is at most one arrow $x \rightarrow y$ for any pair x, y of vertices and a function $v: \Gamma_1 \rightarrow \mathbb{N}_1$ such that $v_{\tau x, \tau y} = v_{x, y}$ for all arrows $x \rightarrow y$ (we write $v_{x, y}$ or just v_{xy} instead of $v(x \rightarrow y)$). In case $v_{\tau z, y} = v_{y, z}$ for all arrows $y \rightarrow z$ with z not projective. then we may consider $(\Gamma_0, \Gamma_1, \tau, v)$ as an ordinary translation quiver by replacing any arrow $x \rightarrow y$ by $v_{x, y}$ arrows.

For example, the valued translation quiver $\mathbb{Z}\mathbb{B}_3$ has the following form (in such pictures it is sufficient to add the number v_{xy} to an arrow $x \rightarrow y$ only in case $v_{xy} \geq 2$):



The valued translation quiver $\Gamma = (\Gamma_0, \Gamma_1, \tau, v)$ is said to be *stable*, if $(\Gamma_0, \Gamma_1, \tau)$ is stable. Given a stable valued translation quiver Γ , a function $f: \Gamma_0 \rightarrow \mathbb{Z}$ should be called *cluster-additive* provided

$$f(z) + f(\tau z) = \sum_{y \in \Gamma_0} v_{yz} f(y)^+, \quad \text{for all } z \in \Gamma_0.$$

2. We should stress that cluster-additive functions are definitely also of interest when dealing with stable translation quivers which are not related to translation quivers of the form $\mathbb{Z}\Delta$ with Δ a finite directed quiver. Examples of cluster-additive functions on the translation quiver $\mathbb{Z}\mathbb{D}_\infty$ (as well as on $\mathbb{Z}\mathbb{A}_\infty$) have been exhibited in [R].

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