

# Jordan Hölder theorems for derived module categories of piecewise hereditary algebras

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**Abstract.** A Jordan Hölder theorem is established for derived module categories of piecewise hereditary algebras, in particular for representations of quivers and for hereditary abelian categories of a geometric nature. The resulting composition series of derived categories are shown to be independent of the choice of bounded or unbounded derived module categories, and also of the choice of finitely generated or arbitrary modules.

## INTRODUCTION

Jordan Hölder theorems are classical and fundamental results in group theory and in module theory. Under suitable assumptions, a Jordan Hölder theorem asserts the existence of a finite ‘composition series’, the subquotients of which are ‘simple’ objects. A Jordan Hölder theorem can be formulated when the concept of ‘short exact sequence’ has been defined. Then an object may be called simple if it is not the middle term of a short exact sequence, that is, it is not an extension of another two objects in the given class of objects (groups, modules, ...). Then finite series of short exact sequences can be considered, where the given object is the middle term of the first sequence, the end terms of the first sequence are middle terms of further sequences, and so on, until simple objects are reached and the process stops. A Jordan Hölder theorem states finiteness of this process and the uniqueness of the simple constituents, up to a suitable notion of isomorphism.

About twenty years ago, the work of Cine, Parshall and Scott [9] on highest weight categories and quasi-hereditary algebras and on ‘stratifications’ of their derived module categories - i.e. on composition series in our terminology - provided a first motivation to ask for a Jordan Hölder theorem for derived categories of rings, in the following sense: A ‘short exact sequence’ of derived categories is, by definition, a recollement of triangulated categories, as defined by Beilinson, Bernstein and Deligne [7], with all three triangulated categories being derived categories of rings. A derived category is called ‘simple’ if it does not admit a non-trivial recollement. Wiedemann [43] and Happel [15] found non-trivial examples of ‘simple’ derived categories. Only recently, however, a first Jordan Hölder theorem could be established: Using methods developed in [3, 34], a Jordan Hölder theorem has been provided in [4] for unbounded derived module categories of hereditary artinian algebras. Subsequently, Liu and Yang [31] have shown that blocks of group algebras of finite groups always are derived simple.

In [4] the problem has been made more precise by showing that no positive answer can be expected when admitting arbitrary triangulated categories as factors in composition series of derived module categories. Moreover, it has been pointed out that an answer may depend on the choice of derived categories one is working with - unbounded, left bounded or bounded - and of the underlying module category - finitely generated or arbitrary modules. Examples given in [5] show that these choices really matter and in particular do have an effect on derived simpleness.

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Jordan Hölder theorems may fail for two reasons: Composition series may not be finite - an example has been given in [4] - and composition series may not be unique. The second point is much more subtle; rather sophisticated examples recently have been constructed by Chen and Xi [8]; the algebras there are not artinian.

The aim of the present article is to extend and to complement the results of [4] in at least two ways:

- We extend the range of validity of the Jordan Hölder theorem from hereditary algebras to piecewise hereditary algebras. These also include quasi-tilted algebras that are not related to hereditary algebras, but to hereditary abelian categories of a geometric nature - coherent sheaves over weighted projective lines - and the corresponding ‘canonical’ algebras.
- We show that the ‘same’ composition series are obtained when considering unbounded or bounded derived categories, finitely generated or arbitrary modules.

**Main Theorem.** *The (bounded or unbounded) derived category (using finitely generated or arbitrary modules) of a finite dimensional piecewise hereditary algebra has a finite composition series. The simple composition factors are derived categories of vector spaces over skew-fields: the endomorphism rings of the simple modules. These composition factors are unique up to ordering and Morita equivalence.*

In [4], the Jordan Hölder theorem for hereditary algebras actually has been proven in a stronger form: Any composition series can be brought into a ‘normal form’, which means that the composition series is associated with a series of homological epimorphisms, starting from the given algebra. This strong version is valid in the present more general context, too. In order to establish it, we are using the results of [4]. The proof of the Main Theorem stated above does, however, not use the special case of it shown in [4], for which we give an alternative proof here. A new ingredient compared to [4] is the concept of strong global dimension, recently investigated by Ringel [40] and by Happel and Zacharia [22]. Other key ingredients are constructions of recollements for  $D^b(\text{mod})$  and, for hereditary algebras, bijections relating recollements on different levels with each other and with further data such as exceptional objects (Theorem 5.1) and homological epimorphisms (Theorem 3.3, Theorem 5.1).

The organisation of this article is as follows: A preliminary first section recalls definitions and concepts to be used later on. The second section discusses the existence of recollements in general and the third section constructs recollements from tilting modules. In section four a collection of positive and negative examples is presented. In the final fifth section we prove the Main Theorem, which is split into several results, giving more detail than the version stated above.

## 1. PRELIMINARIES

Throughout this article, algebras are finite dimensional over a field. The reason for this restriction - when comparing to [3], where more generally artinian algebras have been investigated - is that the theory of weighted projective lines and corresponding canonical algebras is available over fields only.

Let  $A$  be a finite dimensional algebra over a field  $k$ . Then  $\text{mod-}A$  denotes the category of finite dimensional right  $A$ -modules, and  $\text{Mod-}A$  the category of all right  $A$ -modules. Let  $D^b(\text{mod-}A)$  be the bounded derived category of  $\text{mod-}A$ , and  $D(\text{Mod-}A)$  be unbounded derived category of  $\text{Mod-}A$ .

1.1. **Recollements.** Let  $\mathcal{X}, \mathcal{Y}$  and  $\mathcal{D}$  be triangulated categories.  $\mathcal{D}$  is said to be a *recollement* ([7], see also [36]) of  $\mathcal{X}$  and  $\mathcal{Y}$  if there are six triangle functors as in the following diagram

$$\mathcal{Y} \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{-i_*=i_!} \\ \xleftarrow{i^!} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{-j^!=j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{X}$$

such that

- (1)  $(i^*, i_*)$ ,  $(i_!, i^!)$ ,  $(j_!, j^!)$ ,  $(j^*, j_*)$  are adjoint pairs;
- (2)  $i_*$ ,  $j_*$ ,  $j_!$  are full embeddings;
- (3)  $i^! \circ j_* = 0$  (and thus also  $j^! \circ i_! = 0$  and  $i^* \circ j_! = 0$ );
- (4) for each  $C \in \mathcal{D}$  there are triangles

$$i_!i^!(C) \rightarrow C \rightarrow j_*j^*(C) \rightarrow$$

$$j_!j^!(C) \rightarrow C \rightarrow i_*i^*(C) \rightarrow$$

Two recollements

$$\mathcal{Y} \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{-i_*=i_!} \\ \xleftarrow{i^!} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{-j^!=j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{X} \quad \text{and} \quad \mathcal{Y}' \begin{array}{c} \xleftarrow{i'^*} \\ \xrightarrow{-i'_*=i'_!} \\ \xleftarrow{i'^!} \end{array} \mathcal{D} \begin{array}{c} \xleftarrow{j'_!} \\ \xrightarrow{-j'^!=j'^*} \\ \xleftarrow{j'_*} \end{array} \mathcal{X}'$$

are said to be *equivalent*, if the essential images of  $i_*$  and  $i'_*$ , of  $j_*$  and  $j'_*$ , and of  $j_!$  and  $j'_!$  coincide, respectively.

1.2. **Perpendicular categories, compact objects, tilting objects.** Given a triangulated category  $\mathcal{C}$  and an object  $M$  in  $\mathcal{C}$ , the smallest triangulated full subcategory of  $\mathcal{C}$  containing  $M$  and closed under taking direct summands is denoted by  $\text{tria}(M)$ . When  $\mathcal{C}$  has small coproducts, the corresponding subcategory closed under taking small coproducts is denoted by  $\text{Tria}(M)$ . The *perpendicular category* of  $M$  in  $\mathcal{C}$ , denoted by  $M^\perp$ , is by definition the full subcategory of  $\mathcal{C}$  containing of those objects  $X$  perpendicular to  $M$ , that is,  $\text{Hom}_{\mathcal{C}}(M, X[n]) = 0$  for all integers  $n$ .

Let  $A$  be a finite dimensional algebra. Write  $P_A$  for the category of finitely generated projective  $A$ -modules. It is well known that the bounded homotopy category  $K^b(P_A)$  coincides with the subcategory  $\text{tria}(A)$  of  $D^b(\text{mod-}A)$ . It is called the *compact* or *perfect subcategory* of  $D^b(\text{mod-}A)$  and  $D(\text{Mod-}A)$ . Its objects are called *compact* or *perfect*. We identify a compact object  $X$  with its '*minimal  $K^b(P_A)$ -representative*', i.e. a complex in  $K^b(P_A)$ , isomorphic to  $X$ , without direct summands of the form  $P \xrightarrow{\text{Id}} P$  or its shifts, for some  $P \in P_A$ .

Recall that an  $A$ -module  $T \in \text{mod-}A$  is a *tilting module*, if the following hold:

- (1)  $\text{pd}(T) \leq 1$ ;
- (2)  $\text{Ext}^1(T, T) = 0$ ;
- (3) There exists a short exact sequence  $0 \rightarrow A \rightarrow T_0 \rightarrow T_1 \rightarrow 0$  where  $T_i \in \text{add}(T)$ .

A complex  $X$  in  $D^b(\text{mod-}A)$  or  $D(\text{Mod-}A)$  is said to be *exceptional* if it has no nontrivial self-extension, i.e.  $\text{Hom}(X, X[n]) = 0$  for all integers  $n$ . It is said to be a *partial tilting complex*, if it is exceptional and compact, and a *tilting complex*, if in addition it generates the perfect subcategory, i.e.  $\text{tria}(X) = K^b(P_A)$ .

**1.3. Homological epimorphisms.** Recall that a ring homomorphism  $\varphi : A \rightarrow B$  is a *ring epimorphism* if and only if the induced functor  $\varphi_* : \text{mod-}B \rightarrow \text{mod-}A$  is a full embedding. Furthermore  $\varphi$  is a *homological epimorphism* if and only if the induced functor  $\varphi_* : D^b(\text{mod-}B) \rightarrow D^b(\text{mod-}A)$  is a full embedding, or equivalently  $\varphi_* : D(\text{Mod-}B) \rightarrow D(\text{Mod-}A)$  is a full embedding (cf. [13, Theorem 4.4]). In this case  $\varphi$  induces a  $D^b(\text{mod-})$  level recollement

$$D^b(\text{mod-}B) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*=i_!} \\ \xleftarrow{i^!} \end{array} D^b(\text{mod-}A) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{X}$$

and a  $D(\text{Mod-})$  level recollement

$$D(\text{Mod-}B) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*=i_!} \\ \xleftarrow{i^!} \end{array} D(\text{Mod-}A) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} \mathcal{X}'$$

for some triangulated categories  $\mathcal{X}$  and  $\mathcal{X}'$ , and the functors on the left hand side are induced by  $\varphi$ , that is,  $i^* = - \otimes_A^L B$ ,  $i^! = R\text{Hom}_A(B, -)$  and  $i_* = \varphi_*$ .

We will be interested in the case when  $\mathcal{X}$  or  $\mathcal{X}'$  is a derived module category.

**1.4. Invariants of recollements.** Suppose there is a recollement

$$D^b(\text{mod-}B) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} D^b(\text{mod-}A) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} D^b(\text{mod-}C)$$

of bounded derived categories of finitely generated modules, writing  $A$  in terms of  $B$  and  $C$ . Then the finiteness of global dimension (by [43, Lemma 2.1]) and finitistic dimension (by [17, 3.3]) are invariants. That is,  $\text{gl. dim}(A) < \infty$  if and only if  $\text{gl. dim}(B) < \infty$  and  $\text{gl. dim}(C) < \infty$ , and  $\text{fdim}(A) < \infty$  if and only if  $\text{fdim}(B) < \infty$  and  $\text{fdim}(C) < \infty$ . Denote by  $K_0(A)$  the Grothendieck group of  $\text{mod-}A$ , which is also the Grothendieck group of  $D^b(\text{mod-}A)$ . It is a free abelian group with finite rank, which equals the number of non-isomorphic simple  $A$ -modules. Given a recollement as above, there is a decomposition  $K_0(\text{mod-}A) = K_0(\text{mod-}B) \oplus K_0(\text{mod-}C)$ .

## 2. CRITERIA FOR THE EXISTENCE OF RECOLLEMENTS

Let  $A$  be a finite dimensional algebra, and  $T$  a compact exceptional complex over  $A$ . Let  $B = \text{End}_A(T)$  be the endomorphism algebra. By [24], there exists uniquely a right bounded complex  $X = \tilde{T}$  of finitely generated projective  $B$ - $A$ -bimodules, such that  $X$  as a complex of  $A$ -modules is quasi-isomorphic to  $T$ . This complex induces a pair of adjoint functors between the unbounded derived categories of  $A$  and  $B$ :

$$D(\text{Mod-}A) \begin{array}{c} \xleftarrow{F} \\ \xrightarrow{G} \end{array} D(\text{Mod-}B)$$

where  $F = - \otimes_B^L X$  and  $G = R\text{Hom}_A(X, -)$ .

**Lemma 2.1.** *With these notations:*

- (1) *The adjoint pair  $(F, G)$  restricts to  $K^b(P_A) \rightleftarrows K^b(P_B)$  if and only if  $X^{\text{tr}} := G(A)$  as a complex of (right)  $B$ -modules is compact.*
- (2) *The adjoint pair  $(F, G)$  restricts to  $D^b(\text{mod-}A) \rightleftarrows D^b(\text{mod-}B)$  if and only if  $X$  as a complex of left  $B$ -modules is compact.*

In these cases, the restrictions of  $F$  and  $G$  are again adjoint to each other.

*Proof.* (1) Since  $T$  is compact, the derived Hom-functor  $G$  is equivalent to the derived tensor functor  $-\overset{L}{\otimes}_A X^{tr}$ , where  $X^{tr} = G(A)$ . It sends  $K^b(P_A)$  to  $\text{tria}(X^{tr})$ , and it is an equivalence from  $\text{tria}(T)$  to  $K^b(P_B)$  with quasi-inverse given by the restriction of  $F$ . Hence the functor  $F$  sends  $K^b(P_B)$  to  $K^b(P_A)$ , and the functor  $G$  sends  $K^b(P_A)$  to  $K^b(P_B)$  if and only if  $X^{tr}$  as a complex of  $B$ -modules is compact.

(2) The category  $D^b(\text{mod-}A)$  is equivalent to the full subcategory of  $D(\text{Mod-}A)$  containing those complexes whose cohomology spaces are finite dimensional. When calculating the cohomology of such a complex, one may forget its module structure and view it as a chain complex of vector spaces.

We claim that the functor  $G$  sends  $D^b(\text{mod-}A)$  to  $D^b(\text{mod-}B)$ . For this we need to show that all simple  $A$ -modules have images in  $D^b(\text{mod-}B)$ . Since  $X$  as a complex of right  $A$ -modules is compact,  $X^{tr}$  as a complex of left  $A$ -modules is again compact. We take its minimal projective representative, and thus tensoring with a simple right  $A$ -module  $S$  would kill all projective modules except the projective cover of  $S$ . The compactness of  $X^{tr}$  implies the multiplicity of the corresponding projective cover in the minimal projective resolution is finite. It follows that the cohomological space of  $G(S) = S \overset{L}{\otimes}_A X^{tr}$  has finite dimension. That is,  $G(S)$  belongs to  $D^b(\text{mod-}B)$ .

Now if  $X$  as a complex of left  $B$ -modules is compact, the same arguments as above shows  $F = -\overset{L}{\otimes}_B X$  sends  $D^b(\text{mod-}B)$  to  $D^b(\text{mod-}A)$ . Conversely, suppose the functor  $F$  sends  $D^b(\text{mod-}B)$  to  $D^b(\text{mod-}A)$  and assume that  $X$  is not compact. Then there exists some indecomposable left  $B$ -projective module with infinite multiplicity in  $X$ . Tensoring its simple top (now as right  $B$ -module) with  $X$  will provide a complex with infinite dimensional cohomological space. This contradicts the assumption of  $F$ .  $\square$

*Remark.* (1) In general  $X$  as a complex of bimodules is not necessarily compact. For example, take  $A$  to be the two dimensional algebra  $k[x]/x^2$ , and  $T$  to be  $A$  itself. So  $B$  is identified with  $A$ , but  $A$  is not compact as  $A$ - $A$ -bimodule.

(2) In general  $X$  as a complex of left  $B$ -modules is not necessarily compact. For example, take  $A$  to be the quasi-hereditary algebra

$$1 \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2$$

with relation  $\beta \circ \alpha = 0$ , and  $T = P(2)$  the indecomposable projective module at 2. Then  $B = \text{End}_A(T) \cong k[x]/x^2$  and  $T$  as left  $B$ -module has infinite projective dimension.

The aim of this section is to give a ‘finitely generated modules’ version of the criterion for the existence of a recollement given in [25, Theorem 1], [35, Theorem 2]. We start with a sufficient condition.

**Proposition 2.2.** *Let  $A$  be a finite dimensional algebra. Suppose there are compact exceptional complexes  $\mathcal{C}$  and  $\mathcal{B}$  in  $K^b(P_A)$  such that*

- (1)  $\text{End}_A(\mathcal{C}) = \mathcal{C}$  and  $\text{End}_A(\mathcal{B}) = B$ ;
- (2)  $\text{Hom}_A(\mathcal{C}, \mathcal{B}[n]) = 0$  for all integers  $n$ ;
- (3)  $\mathcal{C}^\perp \cap \mathcal{B}^\perp = 0$ ;

- (4) the complex of projective  $C$ - $A$ -bimodules  $\tilde{\mathcal{C}}$  and the complex of projective  $B$ - $A$ -bimodules  $\tilde{\mathcal{B}}$ , which are quasi-isomorphic to  $\mathcal{C}$  and  $\mathcal{B}$  respectively as complexes of right  $A$ -modules, are compact as complexes of left  $C$ - and  $B$ -modules respectively.

Then  $A$  admits a recollement of the form

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

for  $B = \text{End}_A(\mathcal{B})$  and  $C = \text{End}_A(\mathcal{C})$ .

*Proof.* By 2.1 (2), the assumption implies the existence of two pairs of adjoint functors  $(i_!, i^!)$  and  $(j_!, j^!)$  as in the following partial recollement

$$D^b(\text{mod-}B) \begin{array}{c} \xrightarrow{i_!} \\ \xleftarrow{i^!} \end{array} D^b(\text{mod-}A) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^!} \end{array} D^b(\text{mod-}C)$$

where  $i_!$  and  $j_!$  are full embeddings. This ‘partial’ recollement can be completed in the same way as in the proof of [25, Theorem 1]. We omit the details.  $\square$

For a discussion of the converse direction, we need more preparations. The following lemma provides a homological characterization of compact objects. It has been stated, without proof, in [17, 3.2]. We include here a proof due to Jiaqun Wei, whom we thank for suggesting this improvement of our original proof. For an analogous statement for  $D^b(\text{Mod-}A)$  and  $K^b(\text{Proj-}A)$  see [37, Proposition 6.2].

**Lemma 2.3.** *Let  $A$  be a finite dimensional algebra over a field  $k$ . Then  $X$  in  $D^b(\text{mod-}A)$  is compact if and only if for any  $Y$  in  $D^b(\text{mod-}A)$ , there exists an integer  $t_0$  with  $\text{Hom}(X, Y[t]) = 0$  for all  $t \geq t_0$ .*

*Proof.* A compact object is a bounded complex of finitely generated projective modules. Given such a object  $X$ , the condition in the statement is fulfilled. Conversely, suppose  $X$  satisfies the assumption. We identify  $D^b(\text{mod-}A)$  with  $K^{-,b}(P_A)$  and write  $X$  as a bounded-above complex of finitely generated projective modules with bounded homology  $\dots \rightarrow P_k \rightarrow P_{k-1} \rightarrow \dots \rightarrow P_m \rightarrow 0$ . Say  $H^i(X) = 0$  for all  $i \leq n$ . Set  $M = \text{Coker}(P_{n-1} \rightarrow P_n)$ , and then  $\dots \rightarrow P_{n-1} \rightarrow P_n \rightarrow 0$  is a projective resolution of  $M$ . Hence  $\text{pd}(M) < \infty$  if and only if  $X \in K^b(P_A)$ . By assumption there exists  $t_0 \geq 0$  such that

- (1)  $\text{Hom}_A(X, S[t]) = 0$  for all simple  $A$ -module  $S$  and for all  $t \geq t_0$ ;
- (2)  $\text{Hom}_A(X, S[t]) \cong \text{Hom}(M, S[t])$  for all simple  $A$ -module  $S$  and for all  $t \geq t_0$ .

Hence  $\text{Ext}_A^t(M, S) = 0$  for all simple  $A$ -module  $S$  and for all  $t \geq t_0$ . Namely  $\text{pd}(M) \leq t_0$ .  $\square$

**Corollary 2.4.** *A triangulated functor  $F : D^b(\text{mod-}A) \rightarrow D^b(\text{mod-}B)$  restricts to  $K^b(P_A) \rightarrow K^b(P_B)$  provided it has a right adjoint functor.*

Now we can show that some of the conditions in Proposition 2.2 are also necessary.

**Corollary 2.5.** *Suppose  $A$  admits a recollement of the form*

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

*with finite dimensional algebras  $B$  and  $C$ . Then  $\mathcal{C} = j_!(C)$  and  $\mathcal{B} = i_*(B)$  are compact exceptional objects satisfying*

- (1)  $\text{End}_A(\mathcal{C}) = C$  and  $\text{End}_A(\mathcal{B}) = B$ ;
- (2)  $\text{Hom}_A(\mathcal{C}, \mathcal{B}[n]) = 0$  for all integers  $n$ ;
- (3)  $\text{tria}(\mathcal{B} \oplus \mathcal{C}) = D^b(\text{mod-}A)$ .

*Proof.* Corollary 2.4 implies the compactness of  $\mathcal{C}$  and  $\mathcal{B}$ . Since  $j_!$  and  $i_*$  are full embeddings,  $\mathcal{B}$  and  $\mathcal{C}$  are exceptional and condition (1) is satisfied. Conditions (2) and (3) follow directly from the definition of recollement.  $\square$

There is still an obstruction: the adjoint pairs  $(i, i^!)$  and  $(j, j^!)$  are in general not necessarily the derived tensor- or hom-functors induced by  $i_*(B)$  and  $j_!(C)$ . This is the case, however, when the algebra  $A$  has finite global dimension. The next result asserts that, up to equivalence of recollements of  $D^b(\text{mod-}A)$ .

**Theorem 2.6.** *Let  $A$  be a finite dimensional algebra with finite global dimension. The following statements are equivalent.*

(1)  $A$  admits a recollement of the form

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

with finite dimensional algebras  $B$  and  $C$ .

(2) There exist exceptional complexes  $\mathcal{C}$  and  $\mathcal{B}$  in  $D^b(\text{mod-}A)$  such that

- (a)  $\text{End}_A(\mathcal{C}) = C$  and  $\text{End}_A(\mathcal{B}) = B$ ;
- (b)  $\text{Hom}_A(\mathcal{C}, \mathcal{B}[n]) = 0$  for all integers  $n$ ;
- (c)  $\text{tria}(\mathcal{B} \oplus \mathcal{C}) = D^b(\text{mod-}A)$ .

*Proof.* (1)  $\Rightarrow$  (2) is Corollary 2.5.

(2)  $\Rightarrow$  (1): By [25, Theorem 1], [35, Theorem 2], there is a recollement

$$D^-(\text{Mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^-(\text{Mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^-(\text{Mod-}C)$$

with  $B = \text{End}_A(\mathcal{B})$  and  $C = \text{End}_A(\mathcal{C})$ . It follows then from [25, Corollary 5] that  $B$  and  $C$  have finite global dimension since  $A$  has so. By Keller's construction [24], the bicomplexes  $\tilde{\mathcal{B}}$  and  $\tilde{\mathcal{C}}$  are right bounded and have bounded cohomologies. Hence they are compact as left  $B$ - and  $C$ -complexes respectively. That is, condition (4) in Proposition 2.2 is satisfied.  $\square$

As a corollary there is the following assertion on lifting and restricting recollements.

**Corollary 2.7.** *Let  $A, B, C$  be finite dimensional algebras.*

(1) Any recollement of bounded derived categories

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

can be lifted to a recollement of unbounded derived categories

$$D(\text{Mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}C)$$

(2) If  $A$  has finite global dimension, any recollement of unbounded derived categories

$$D(\text{Mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}C)$$

can be restricted to a recollement of bounded derived categories

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

*Proof.* (1) By Corollary 2.5, we have a pair  $(j_!(C), i_*(B))$  of compact exceptional objects, which yields the existence of a  $D(\text{Mod-})$  level recollement by [34, 5.2.9] (or [4, Theorem 2.2]).

(2) The object  $\mathcal{C} = j_!(C)$  is always compact by [34, 5.2.9] (or [4, Theorem 2.2]), and we show in [5] that  $\mathcal{B} = i_*(B)$  is compact whenever  $A$  has finite global dimension. To apply Theorem 2.6, it only remains to check  $A \in \text{tria}(\mathcal{B} \oplus \mathcal{C})$ . This is true because  $\text{Tria}(\mathcal{B} \oplus \mathcal{C}) = D(\text{Mod-}A)$  and  $\mathcal{B} \oplus \mathcal{C}$  is compact.  $\square$

### 3. CONSTRUCTING RECOLLEMENTS FROM TILTING MODULES

Let  $A$  be a finite dimensional algebra over a field  $k$ , and  $T$  a tilting module (see 1.2). Note that the  $T$ -resolution of  $A$ ,

$$(*) \quad 0 \rightarrow A \rightarrow T_0 \rightarrow T_1 \rightarrow 0$$

is not required to be minimal ( $T_i \in \text{add}(T)$ ). We fix  $T$  together with such a  $T$ -resolution of  $A$ . Our aim is to give an analogue of [3, Theorem 4.8].

For an  $A$ -module  $X$ , the (module) *perpendicular category* of  $X$ , denoted by  $\widehat{X}$ , is by definition the full subcategory of  $\text{mod-}A$ , consisting of the modules  $X$  such that  $\text{Hom}_A(X, M) = 0 = \text{Ext}_A^1(X, M)$ .

**Lemma 3.1** (cf. [10], Proposition 1.3). *The perpendicular category  $\widehat{T_1}$  of  $T_1$  is a reflective subcategory of  $\text{mod-}A$ . In other words, the full embedding  $i : \widehat{T_1} \rightarrow \text{mod-}A$  admits a left adjoint functor  $\ell$ .*

*Proof.* In [10], this statement has been proved for  $\text{Mod-}A$  by giving an explicit construction of the left adjoint functor  $\ell$ . Since  $\ell$  restricts to finite dimensional modules, the same argument works for  $\text{mod-}A$ . For the reader's convenience, we recall the construction.

It consists of two steps: given an  $A$ -module  $M$ , first the universal extension of  $T_1$  with respect to  $M$  is formed, that is, a short exact sequence  $0 \rightarrow M \rightarrow M' \rightarrow T_1^n \rightarrow 0$ , for some natural number  $n$ , such that any extension of  $\text{Ext}_A^1(T_1, M)$  is a pullback along a map  $T_1 \rightarrow T_1^n$ . Secondly, we factor out the trace of  $T_1$  in  $M'$ . The factor module provides exactly the image  $\ell(M)$  of  $M$ . In other words, the composition  $M \rightarrow M' \rightarrow \ell(M)$  is the left approximation of  $M$  in  $\widehat{T_1}$  with  $\text{Hom}(M'', N) \xrightarrow{\sim} \text{Hom}(M, N)$  for any  $N \in \widehat{T_1}$ .

Notice that the second step is not required when the endomorphism ring of  $T_1$  is a skew field, for in this case we can choose  $n = \dim_{\text{End}_A T_1} \text{Ext}_A^1(T_1, M)$  to obtain  $M' \in \widehat{T_1}$ , cf. [3, Appendix A.1].  $\square$

We now compute  $\ell(A)$ . The  $T$ -resolution  $(*)$  of  $A$  is a universal extension of  $T_1$  with respect to  $A$ . Indeed, applying  $\text{Hom}_A(T_1, -)$  we get a surjection  $\text{Hom}_A(T_1, T_1) \rightarrow \text{Ext}_A^1(T_1, A)$ . It follows that the left approximation of  $A$  in  $\widehat{T_1}$  is

$$\ell(A) = T_0 / \tau_{T_1}(T_0)$$

where  $\tau_{T_1}(T_0)$  is the trace of  $T_1$  in  $T_0$ . We write  $B$  for the endomorphism algebra of  $\ell(A)$ .

**Lemma 3.2** ([13], Proposition 3.8). *Notations are as above. The module  $\ell(A)$  is a projective generator of  $\widehat{T_1}$ . It determines a natural algebra homomorphism  $\varphi : A \rightarrow B$  which is a ring epimorphism such that the image of the full embedding  $\varphi_* : \text{mod-}B \rightarrow \text{mod-}A$  is equivalent to  $\widehat{T_1}$ .*



The proof is by checking directly that  $\mathrm{Hom}_A(\ell(A), M) \cong \mathrm{Hom}_A(A, M) \cong M$  for any  $M \in \widehat{T}_1$ . In particular  $B = \mathrm{Hom}_A(\ell(A), \ell(A)) \cong \ell(A)$ . This is actually an isomorphism of  $B$ - $A$ -bimodules (where  $B$  is equipped with an  $A$ -bimodule structure via  $\varphi$ ). By [2, 1.7] the ring epimorphism  $\varphi$  can be identified with the universal localisation  $A_{T_1}$  of  $A$  at  $T_1$ , see also [3, 4.1].

When does there exist a recollement of the form

$$D^b(\mathrm{mod}\text{-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\mathrm{mod}\text{-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\mathrm{mod}\text{-}C)$$

for some finite dimensional algebra  $C$ ?

**Theorem 3.3.** *Suppose the algebra homomorphism  $\varphi : A \rightarrow B$  is a homological epimorphism, and the projective dimension of  $T_1$  as a left  $C := \mathrm{End}_A(T_1)$ -module is finite. Then there is the following recollement of  $D^b(\mathrm{mod}\text{-}A)$*

$$D^b(\mathrm{mod}\text{-}B) \begin{array}{c} \longleftarrow i^* \\ \xrightarrow{-i_* = i_!} \\ \longleftarrow i^! \end{array} D^b(\mathrm{mod}\text{-}A) \begin{array}{c} \longleftarrow j_! \\ \xrightarrow{-j^! = j^*} \\ \longleftarrow j_* \end{array} D^b(\mathrm{mod}\text{-}C)$$

where  $i^* = - \otimes_A^L B$ ,  $i_* = \varphi_*$ ,  $i^! = \mathrm{RHom}_A(B, -)$ ,  $j_! = - \otimes_C^L T_1$ , and  $j^! = \mathrm{RHom}_A(T_1, -)$ .

*Proof.* We need to make a detour through the unbounded derived category  $D(\mathrm{Mod}\text{-}A)$ . Combining Example 4.5 and Theorem 4.8 in [3], we obtain a recollement

$$D(\mathrm{Mod}\text{-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\mathrm{Mod}\text{-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\mathrm{Mod}\text{-}C)$$

where the functors are as required and  $j_!(C)$  is isomorphic to  $T_1$ . For the convenience of the reader, we include more details here.

The module  $T_1$  is finite dimensional of projective dimension  $\leq 1$ , so it is compact. Hence it generates a smashing subcategory  $\mathcal{X} = \mathrm{Tria}(T_1)$  of  $D(\mathrm{Mod}\text{-}A)$  ([1, 4.5]). It follows that there is a recollement

$$\mathcal{Y} \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\mathrm{Mod}\text{-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \mathrm{Tria}(T_1)$$

for  $\mathcal{Y} = T_1^\perp$  ([34, 4.4.14]).

By [2], the universal localisation of the ring  $A$  at  $T_1$  is given by  $\varphi : A \rightarrow B$ . Because  $\varphi$  is a homological epimorphism, by [33] (more precisely, see [3, Theorem 1.8]), the recollement induced by  $\varphi$  (see 1.3) has the following form

$$D(\mathrm{Mod}\text{-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\mathrm{Mod}\text{-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} \mathrm{Tria}(T_1).$$

The two recollements are equivalent. In particular,  $\mathcal{Y} \cong D(\mathrm{Mod}\text{-}B)$ , and the functors on the left hand side are as required. On the other hand,  $T_1$  is an exceptional compact generator of  $\mathrm{Tria}(T_1)$ , and by Rickard's or by Keller's Morita Theorem,  $\mathrm{Tria}(T_1)$  is, as a triangulated category, equivalent to the derived category  $D(\mathrm{Mod}\text{-}C)$ . So the desired recollement of  $D(\mathrm{Mod}\text{-}A)$  has been obtained, with  $j_!$  and  $j^!$  as required and  $j_!(C) = T_1$ .

Now set  $\mathcal{B} = \ell(A)$  ( $\cong i_*(B)$ ) and  $\mathcal{C} = T_1$  ( $= j_!(C)$ ). Since  $\varphi : A \rightarrow B$  is a homological epimorphism, there is a full embedding  $\varphi_* : D^b(\mathrm{mod}\text{-}B) \rightarrow D^b(\mathrm{mod}\text{-}A)$  which admits a right adjoint functor (see Subsection 1.3). By Corollary 2.4,  $(\ell(A))_A \cong B_A = \varphi_*(B_B)$  is compact. It is also exceptional as  $B_B$  is so. By definition  $T_1 \in \mathrm{add}(T)$  is compact and exceptional, too. Moreover, the conditions (1)-(4) of Proposition 2.2 hold: (1) follows by construction, (2) is implied by  $\ell(A) \in \widehat{T}_1$  and  $\mathrm{pd}(T_1) \leq 1$ , (3) is a consequence of the  $D(\mathrm{Mod}\text{-})$  recollement above, and (4) follows from  $\ell(A) \cong B$  as left  $B$ -modules and from the assumption on  $C$ .  $\square$

When  $A$  has finite global dimension, there is the following simplified version.

**Proposition 3.4.** *Assume that  $A$  has finite global dimension. If the module  $\ell(A)$  is exceptional, then the algebra homomorphism  $\varphi : A \rightarrow B$  as in 3.2 is a homological epimorphism, and there is the following recollement of  $D^b(\text{mod-}A)$*

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

where the functors  $i^*$ ,  $i_*$ ,  $i^!$ ,  $j_!$  and  $j^!$  are as in the Theorem 3.3.

*Proof.* Since  $\ell(A)$  is exceptional, the map  $\varphi : A \rightarrow B$  is a homological epimorphism, see [13, 4.9]. Thus there is a  $D(\text{Mod-})$  level recollement

$$D(\text{Mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}C)$$

as in the proof of Theorem 3.3, and the statement follows from Corollary 2.7.  $\square$

Note that a Theorem of Happel [16, 3.3] is a special case when the endomorphism ring of  $\ell(A)$  is the base field  $k$ . The proof there uses the criterion of [25] which in fact has been stated for big module categories.

When the trace of  $T_1$  in  $T_0$  is trivial,  $\ell(A)$  coincides with  $T_0$  and hence it is exceptional.

**Corollary 3.5.** *Assume that  $A$  has finite global dimension. If there is no nonzero homomorphism from  $T_1$  to  $T_0$ , then there is the following recollement*

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

where  $B = \text{End}_A(T_0)$  and  $C = \text{End}_A(T_1)$ .

In particular,  $\varphi : A \rightarrow B$  is an injective homological epimorphism. (See [6] for more information on injective ring epimorphisms. This setup has been one of the motivations for the current work.) By [13, Proposition 4.13], there even exists a homological epimorphism  $\psi : A \rightarrow C := \text{End}_A(T_1)$ . Let  $\psi_* : D^b(\text{mod-}C) \rightarrow D^b(\text{mod-}A)$  be the induced full embedding. Then  $\psi_* \circ [1]$  provided the functor  $j_*$  in the recollement.

For example, take an indecomposable exceptional module  $M$  satisfying  $\text{Hom}_A(M, A) = 0$  and  $\text{End}_A(M) = k$ . The Bongartz complement of  $M$  is a universal extension  $0 \rightarrow A \rightarrow M' \rightarrow M^{\oplus n} \rightarrow 0$  where  $n = \dim_k \text{Ext}_A^1(M, A)$ . Then  $M \oplus M'$  is a tilting module and  $\text{Hom}_A(M, M') = 0$ .

#### 4. EXAMPLES

In the following some examples are given of constructing recollements from a tilting module. In particular we will see that the assumptions of Theorem 3.3 are optimal. As in the previous section,  $A$  is a finite dimensional  $k$ -algebra,  $T$  is a tilting  $A$ -module with a  $T$ -resolution of  $A$ :  $0 \rightarrow A \rightarrow T_0 \rightarrow T_1$  (where  $T_0, T_1 \in \text{add}(T)$ ),  $\ell(A) = T_0/\tau_{T_1}(T_0)$  is the left approximation of  $A$  in  $\widehat{T_1} := \{M \in \text{mod-}A : \text{Hom}_A(T_1, M) = 0 = \text{Ext}_A^1(T_1, M)\}$ ,  $B := \text{End}_A(\ell(A)) (\cong \ell(A)$  as right  $A$ -module),  $C := \text{End}_A(T_1)$ , and  $\varphi : A \rightarrow B$  is a ring epimorphism with  $\text{mod-}B \cong \widehat{T_1}$ .

**Example 4.1.** *In general, the ring epimorphism  $\varphi : A \rightarrow B$  need not be a homological epimorphism.*

Let  $A$  be the path algebra of the quiver

$$1 \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2 \begin{array}{c} \xleftarrow{\gamma} \\ \xrightarrow{\delta} \end{array} 3$$

with relations  $\alpha \circ \gamma = 0$ ,  $\delta \circ \beta = 0$ ,  $\beta \circ \alpha = \gamma \circ \delta$  and  $\delta \circ \gamma = 0$ . This is the same example as in [16, 1.5]. The indecomposable projective  $A$ -modules are

$$P_1 = \begin{pmatrix} 1 & & \\ 2 & & \\ 1 & & \end{pmatrix}, \quad P_2 = \begin{pmatrix} & 2 & \\ & & 3 \\ & & 2 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Take  $T = P_1 \oplus P_2 \oplus T_1$  where  $T_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$  is the quotient of  $P_2$  factoring out  $P_3$ . It is clear that  $T$  is a tilting module and

$$0 \rightarrow A = P_1 \oplus P_2 \oplus P_3 \rightarrow P_1 \oplus P_2 \oplus P_2 \rightarrow T_1 \rightarrow 0$$

is a  $T$ -resolution of  $A$ . The trace of  $T_1$  in  $P_1$  and  $P_2$  is isomorphic to  $T_1$  and 2 respectively. So the left approximation of  $A$  in  $\widehat{T_1}$  is  $\ell(A) = T_0/\tau_{T_1}(T_0) \cong 1 \oplus \begin{pmatrix} & 2 \\ 1 & 3 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ . Note that  $\text{Ext}_A^2(1, \begin{pmatrix} 2 \\ 3 \end{pmatrix}) \neq 0$ . So,  $\ell(A)$  as an  $A$ -module is not exceptional, and hence  $\varphi$  cannot be a homological epimorphism.

**Example 4.2.** *In general,  $T_1$  as a left  $C$ -module may have infinite projective dimension.*

This is an example from [25]. Let  $A$  be the path algebra of the quiver

$$1 \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2$$

with relation  $\alpha \circ \beta \circ \alpha = 0$ . So the indecomposable projective  $A$ -modules are

$$P_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Take  $T = A = P_1 \oplus P_2$  the regular module and

$$0 \rightarrow A = P_1 \oplus P_2 \rightarrow T_0 = P_1 \oplus P_2 \oplus P_2 \rightarrow T_1 = P_2 \rightarrow 0$$

as  $T$ -resolution of  $A$  (not minimal). Then  $\ell(A) = T_0/\tau_{T_1}(T_0) = S_1$ . It has no self-extension. In fact,  $B = \text{End}_A(\ell(A)) \cong k$  and  $\varphi : A \rightarrow B$  is a homological epimorphism. But  $C = \text{End}_A(T_1) \cong k[x]/(x^2)$  and  $T_1$  as a left  $C$ -module is isomorphic to  $C \oplus k$ , where  $k$  is the simple  $C$ -module with infinite projective dimension.

**Example 4.3.** *Here the conditions of Theorem 3.3 are satisfied.*

Let  $A$  be the path algebra of the quiver

$$1 \begin{array}{c} \xleftarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2$$

with relation  $\beta \circ \alpha = 0$ . So the indecomposable projective  $A$ -modules are

$$P_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

The global dimension of  $A$  is 2. Take  $T = P_2 \oplus S_2$ . It is a tilting module with the following resolution of  $A$ :

$$0 \rightarrow A = P_1 \oplus P_2 \rightarrow T_0 = P_2 \oplus P_2 \rightarrow S_2 \rightarrow 0.$$

Then  $\ell(A) = T_0/\tau_{T_1}(T_0) = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ . It has no self-extension. By Proposition 3.4, the ring epimorphism  $\varphi : A \rightarrow B \cong M_2(k)$  is homological. Indeed it sends  $e_i$  to  $E_{ii}$  (for  $i = 1, 2$ ),  $\alpha$  to  $E_{21}$  and  $\beta$  to 0. On the other hand,  $C = \text{End}_A(T_1) \cong k$  and  $T_1 \cong k$  is projective as  $C$ -module. So there is a recollement

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

with  $D^b(\text{mod-}B) \cong D^b(\text{mod-}k) \cong D^b(\text{mod-}C)$ .

**Example 4.4.** *The standard stratification of quasi-hereditary algebras.*

Recall [9] that a two-sided ideal  $J$  of a finite dimensional algebra  $A$  is a *heredity ideal*, if  $J = AeA$  is generated by some idempotent  $e$  and  $J$  is projective as  $A$ -module and  $eAe$  is a semi-simple algebra. The algebra  $A$  is called *quasi-hereditary*, if there exists a chain  $0 = J_0 \subset J_1 \subset \dots \subset J_s = A$  of two-sided ideals of  $A$ , such that  $J_i/J_{i-1}$  is a heredity ideal of  $A/J_{i-1}$  for all  $i \geq 1$ . Such a chain is called a *heredity chain* of  $A$  (not necessarily unique). By Parshall and Scott [36, Theorem 2.7(b)], an ideal  $J = AeA$  appearing in a heredity chain induces a recollement of the form

$$D^b(\text{mod-}A/AeA) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}eAe).$$

This fits in our setup.

Let  $A$  be a quasi-hereditary algebra. In particular it has finite global dimension. Let  $e = e^2$  be an idempotent in  $A$  such that  $J = AeA$  is an ideal in a heredity chain of  $A$ . Take  $T = A$  with

$$0 \rightarrow A \rightarrow T_0 = A \oplus eA \rightarrow T_1 = eA \rightarrow 0$$

the  $T$ -resolution of  $A$  (not minimal). So  $\ell(A) = T_0/\tau_{T_1}(T_0) = A/AeA$ . Note that  $eAe$  and  $A/AeA$  are again quasi-hereditary algebras. Hence  $A/AeA$  is exceptional and  $\varphi : A \rightarrow A/AeA$  is a homological epimorphism. Proposition 3.4 reasserts the existence of the standard recollement. In this case all the functors can be written down explicitly:  $i^* = -\overset{L}{\otimes}_A A/AeA$ ,  $i^! = R\text{Hom}_A(A/AeA, -)$ ,  $j_! = \overset{L}{\otimes}_{eAe} eA$  and  $j_* = R\text{Hom}_{eAe}(Ae, -)$ .

*Remark.* The algebra  $A$  in Example 4.3 is quasi-hereditary with a heredity chain  $0 \subset Ae_1A \subset A$ . However, the induced standard recollement is not equivalent to the recollement in Example 4.3. This shows the recollements obtained from a tilting module depends on the choice of the resolution of  $A$ .

**Example 4.5.** *An injective homological epimorphism.*

Let  $A$  be the path algebra of  $1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ . So  $A$  is hereditary with indecomposable projective modules

$$P_1 = \begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}, \quad P_2 = \begin{smallmatrix} 0 \\ 2 \\ 1 \end{smallmatrix}, \quad P_3 = \begin{smallmatrix} 0 \\ 0 \\ 3 \end{smallmatrix}.$$

Take  $T = P_1 \oplus P_3 \oplus S_3$ . It is a tilting module with a resolution of  $A$ :

$$0 \rightarrow A = P_1 \oplus P_2 \oplus P_3 \rightarrow T_0 = P_1 \oplus P_3 \oplus P_3 \rightarrow T_1 = S_3 \rightarrow 0.$$

Clearly,  $T_1$  does not map non-trivially to  $T_0$ . The endomorphism ring of  $\ell(A) = T_0$  is the path algebra of

$$1 \xrightarrow{\alpha} 2 \begin{array}{c} \xrightarrow{\beta} \\ \xleftarrow{\gamma} \end{array} 3$$

with relations  $\beta \circ \gamma = e_3$  and  $\gamma \circ \beta = e_2$ . Then  $A$  embeds into  $B$ , which is Morita equivalent to the path algebra of  $1 \rightarrow 2$ .

## 5. HEREDITARY AND PIECEWISE HEREDITARY ALGEBRAS

In the first subsection, the focus will be on hereditary algebras, i.e. algebras of global dimension one. In the second subsection, the remaining case of weighted projective lines will be considered. Combining the results will yield a Jordan Hölder theorem both in the small world of bounded derived categories of finitely generated modules and in the large world of unbounded derived categories of (possibly infinitely generated) modules.

**5.1. Hereditary algebras.** The first result states that any recollement of a finite dimensional hereditary algebra, bounded or unbounded, is uniquely determined by the same datum, namely a compact and exceptional object. It is inspired by [20, Proposition 3] and [4, Theorem 2.5 and Corollary 3.3]. Bijections between homological epimorphisms and recollements as well as various other bijections have already been established by Krause and Stovicek [27, Theorem 8.1], in a different way. Throughout this section  $k$  is an arbitrary field.

**Theorem 5.1.** *Let  $A$  be a finite dimensional hereditary algebra over a field  $k$ .*

*There are one to one correspondences between the equivalence classes of the following:*

- (1) *Exceptional objects in  $D^b(\text{mod-}A)$ .*
- (2) *Homological epimorphisms  $A \rightarrow B$ , where  $B$  is a finite dimensional algebra.*
- (3) *Recollements of the form*

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

- (4) *Recollements of the form*

$$D^b(\text{Mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{Mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{Mod-}C)$$

- (5) *Recollements of the form*

$$D^-(\text{Mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^-(\text{Mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^-(\text{Mod-}C)$$

- (6) *Recollements of the form*

$$D(\text{Mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}C)$$

where  $B$  and  $C$  in (3) to (6) are finite dimensional algebras.

Here two exceptional objects  $X$  and  $Y$  are said to be *equivalent*, if they generate the same triangulated category, i.e.  $\text{tria}(X) = \text{tria}(Y)$ . Two homological epimorphisms  $\varphi : A \rightarrow B$  and  $\varphi' : A \rightarrow B'$  are *equivalent*, when the essential images of the full embeddings  $\varphi_* : D^b(\text{mod-}B) \rightarrow D^b(\text{mod-}A)$  and  $\varphi'_* : D^b(\text{mod-}B') \rightarrow D^b(\text{mod-}A)$  coincide. The equivalence of two recollements has been defined in Section 1.1.

*Proof.* (1)  $\Rightarrow$  (2), (3): Let  $X$  be an exceptional complex in  $D^b(\text{mod-}A)$ . The proof of [4, Corollary 3.3] carries over to produce recollements on  $D^b(\text{mod-})$  level. In particular, the recollement in (3) is induced by the homological epimorphism in (2) with  $C = \text{End}_A(X)$ , and the essential image of  $j_!$  is  $\text{tria}(X)$ .

(2)  $\Rightarrow$  (1): Starting from a homological epimorphism  $\varphi : A \rightarrow B$ , we would like to get an exceptional object  $X \in D^b(\text{mod-}A)$  such that the essential image of  $\varphi_* : D^b(\text{mod-}B) \rightarrow D^b(\text{mod-}A)$  equals the right perpendicular category  $X^\perp$  of  $X$ , or equivalently  $\text{tria} X$  equals the left perpendicular category of  $\varphi_*(B)$ . Serre duality is well-known to hold in  $D^b(\text{mod-}A)$ , that is

$\text{Hom}_A(M, N) \cong D\text{Hom}_A(N, SM)$  for  $M, N \in D^b(\text{mod-}A)$ , where  $D = \text{Hom}_k(-, k)$ ,  $S = \nu$  the Nakayama functor, and also  $S = \tau \circ [1]$  where  $\tau$  is the Auslander-Reiten translation. Therefore the left perpendicular category of  $\varphi_*(B)$  coincides with the right perpendicular category of  $S^{-1} \circ \varphi_*(B)$ . It is clear that  $\varphi_*(B)$  is a partial tilting  $A$ -module. Since  $S$  is an autoequivalence of  $D^b(\text{mod-}A)$ ,  $S^{-1} \circ \varphi_*(B)$  is exceptional in  $D^b(\text{mod-}A)$ . Now apply (1)  $\Rightarrow$  (3) to  $S^{-1} \circ \varphi_*(B)$ . Note that  $\text{End}_A(S^{-1}(\varphi_*(B))) \cong \text{End}_A(\varphi_*(B)) \cong \text{End}_B(B) \cong B$ . We obtain a homological ring epimorphism  $A \rightarrow B'$  as well as the induced recollement

$$D^b(\text{mod-}B') \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}B)$$

with  $\text{tria}(i_*(B')) = \text{Im}(i_*) = (S^{-1} \circ \varphi_*(B))^\perp$ . So  $i_*(B')$ , i.e.  $B'$  viewed as an  $A$ -module via the homological epimorphism  $A \rightarrow B'$ , is the exceptional object we are looking for.

(4)  $\Leftrightarrow$  (5) follows from [25, Proposition 4 and Corollary 6], see also [4, Lemma 4.1].

(3)  $\Rightarrow$  (5): Given a recollement as in (3), the objects  $j_!(C)$  and  $i_*(B)$  guarantee the existence of a recollement of the form (5), by the characterisations in [25, Theorem 1] and [35, Theorem 2].

(5)  $\Rightarrow$  (6) follows from [4, Lemma 4.3].

(6)  $\Rightarrow$  (1): Given a  $D(\text{Mod-})$  level recollement, we get back a compact and exceptional object  $j_!(C)$ , following [34, 5.2.9], [4, Theorem 2.2].  $\square$

*Remark.* (1) In the proof of (2)  $\Rightarrow$  (1), we have obtained a recollement

$$D^b(\text{mod-}B') \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}B).$$

Comparing with the recollement in (3), it is not difficult to see that  $B'$  is derived equivalent to  $C$ . It is a general phenomenon for algebras of finite global dimension, where Serre duality holds, that the two sides of a recollement can be switched.

(2) The following fact is also implicit in the proof: given an exceptional object  $X$  in  $D^b(\text{mod-}A)$ , there exists a homological epimorphism  $\varphi : A \rightarrow B$  such that the essential image of the full embedding  $\varphi_* : D^b(\text{mod-}B) \rightarrow D^b(\text{mod-}A)$  is  $\text{tria} X$ . In other words  $B$  is derived equivalent to  $\text{End}_A(X)$ . So in a recollement of  $D^b(\text{mod-}A)$ , there are three homological epimorphisms hidden, corresponding to  $j_!(C)$ ,  $i_*(B)$  and  $j_*(C)$  respectively.

**Corollary 5.2.** *Let  $A$  be a finite dimensional hereditary algebra with  $n$  nonisomorphic simple modules. Let  $T \in D^b(\text{mod-}A)$  be multiplicity-free and exceptional. The following assertions are equivalent:*

- (1)  $T$  is a tilting complex;
- (2) The number of indecomposable direct summands of  $T$  equals  $n$ ;
- (3) The perpendicular category  $\text{tria}(T)^\perp$  in  $D^b(\text{mod-}A)$  vanishes.

*Proof.* (1)  $\Rightarrow$  (2): If  $T$  is tilting, then  $A$  is derived equivalent to the endomorphism algebra  $B$  of  $T$ . So they have the same number of non-isomorphic simple modules. This number of  $B$  equals the number of indecomposable direct summands of  $T$ .

(2)  $\Rightarrow$  (3): By Theorem 5.1,  $T$  generates a recollement

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

where  $C = \text{End}_A(T)$ . It follows then from Proposition 1.4 that  $K_0(B) = 0$ . Hence  $\text{tria}(T)^\perp \cong D^b(\text{mod-}B)$  must be trivial.

(3)  $\Rightarrow$  (1) is straightforward from Theorem 5.1. □

Combining Proposition 5.1 and [4, Theorem 6.1], we obtain the derived Jordan-Hölder theorem for hereditary algebras on  $D^b(\text{mod-})$ ,  $D^b(\text{Mod-})$  and  $D^-(\text{Mod-})$  levels, as well as on  $D(\text{Mod-})$  level.

**Corollary 5.3.** *Let  $A$  be a finite dimensional hereditary algebra and let  $S_1, \dots, S_n$  be the representatives of isomorphism classes of simple  $A$ -modules. Denote by  $D_i$  the endomorphism rings of  $S_i$  ( $1 \leq i \leq n$ ). Then  $D^b(\text{mod-}A)$  ( $D^b(\text{Mod-}A)$ ,  $D^-(\text{Mod-}A)$ ) has a stratification with  $D^b(\text{mod-}D_i)$  ( $D^b(\text{Mod-}D_i)$ ,  $D^-(\text{Mod-}D_i)$  respectively) ( $1 \leq i \leq n$ ) being the factors. Moreover, any stratification of  $D^b(\text{mod-}A)$  ( $D^b(\text{Mod-}A)$ ,  $D^-(\text{Mod-}A)$ ) has precisely these factors, up to ordering and derived equivalence.*

**5.2. Weighted projective lines and canonical algebras.** Recall that a finite dimensional algebra  $A$  over a field  $k$  is called *piecewise hereditary*, if there exists a hereditary and abelian category  $\mathcal{H}$  such that the bounded derived categories  $D^b(\text{mod-}A)$  and  $D^b(\mathcal{H})$  are equivalent as triangulated categories. In other words, there exists a tilting complex  $T$  in  $D^b(\mathcal{H})$  with endomorphism ring being  $A$ .

In order to proceed inductively, we need the following result, which will follow immediately from Lemma 5.6 below. Another proof can be based on [40, Corollary 3], where it is shown that a finite dimensional algebra  $A$  over a field is piecewise hereditary if and only if for each indecomposable object  $X$  in  $D^b(\text{mod-}A)$ , there is no ‘path’ from  $X[1]$  to  $X$ .

**Proposition 5.4.** *Suppose there is a recollement of finite dimensional algebras*

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C) .$$

*If  $A$  is piecewise hereditary, then  $B$  and  $C$  are also piecewise hereditary.*

A direct consequence is the following analogue of [14, Corollary III.6.5], where it has been shown that the endomorphism algebra of a partial tilting module over a finite dimensional hereditary algebra is a tilted algebra.

**Corollary 5.5.** *The endomorphism algebra of a partial tilting complex over a finite dimensional hereditary algebra is piecewise hereditary.*

*Proof.* Let  $T$  be a partial tilting complex over  $A$ , a finite dimensional hereditary algebra. By Proposition 3.4 (1)  $\Rightarrow$  (3), it induces a recollement

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

where  $B$  and  $C$  are finite dimensional algebras and  $C = \text{End}_A(T)$ . The statement follows then from Proposition 5.4. □

Recall the definition of strong global dimension ([42], [22]). Let  $A$  be a finite dimensional algebra. We define the *length* of a compact complex  $X \in K^b(P_A)$ , denoted by  $\text{length}(X)$ , to be the length of its minimal  $K^b(P_A)$ -representative. More precisely, suppose

$$0 \rightarrow P_{-s} \rightarrow P_{-s+1} \rightarrow \dots \rightarrow P_{r-1} \rightarrow P_r \rightarrow 0$$

is the minimal  $K^b(P_A)$ -representative of  $X$  (where  $P_i$  are finitely generated projective modules and  $-s \leq r$  are integers). Then  $\text{length}(X) := s + r$ . The *strong global dimension* of  $A$ , denoted by  $\text{s.gl.dim}(A)$ , is defined to be the supremum of the lengths of all indecomposable compact

complexes over  $A$ . If  $A$  has finite strong global dimension, then it has finite global dimension. Happel and Zacharia [22, Theorem 3.2] have shown that  $A$  is piecewise hereditary if and only if it has finite strong global dimension.

The following is a partial analogue of [43, Lemma 2.1] (for global dimension) and [17, 3.3] (for finitistic dimension).

**Lemma 5.6.** *Suppose there is a recollement of finite dimensional algebras*

$$D^b(\text{mod-}B) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} D^b(\text{mod-}A) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} D^b(\text{mod-}C),$$

where the algebra  $A$  has finite strong global dimension. Then the algebras  $B$  and  $C$  also have finite strong global dimensions.

*Proof.* Assume  $A$  has finite strong global dimension say  $d$ . We will show  $\text{s. gl. dim}(B) < \infty$  (and the proof for  $C$  is similar). By Corollary 2.4, the full embedding  $i_* : D^b(\text{mod-}B) \rightarrow D^b(\text{mod-}A)$  restricted to the perfect subcategories  $i_* : K^b(P_B) \rightarrow K^b(P_A)$ . Take an arbitrary indecomposable complex  $X$  in  $K^b(P_B)$  with a minimal projective resolution

$$0 \rightarrow P_{-s} \rightarrow P_{-s+1} \rightarrow \dots \rightarrow P_{r-1} \rightarrow P_r \rightarrow 0$$

where  $P_i$  are finitely generated projective  $B$ -modules and  $-s \leq r \in \mathbb{Z}$ . We claim that

$$r = \max\{n : \text{Hom}_{D^b(\text{mod-}B)}(B, X[n]) \neq 0\},$$

$$s = \max\{n : \text{Hom}_{D^b(\text{mod-}B)}(X, B[n]) \neq 0\}.$$

Indeed, the first equality is implied by  $\text{Hom}_{D^b(\text{mod-}B)}(B, X[n]) \cong H^n(X)$ . Moreover it is clear that  $\text{Hom}_{D^b(\text{mod-}B)}(X, B[n]) \cong \text{Hom}_{K^b(P_B)}(X, B[n])$ , which is trivial whenever  $n > s$ . To see that  $\text{Hom}_{K^b(P_B)}(X, B[s])$  does not vanish, one takes a map  $f : P_{-s} \rightarrow B$  which is identity restricted to a common indecomposable direct summand of  $P_{-s}$  and  $B$ , and is zero elsewhere.

Since  $i_*$  is a full embedding,  $i_*(X)$  is again indecomposable and hence  $\text{length}(i_*(X)) \leq \text{s. gl. dim}(A) = d$ . Since  $i_*(B)$  is compact, it has finite length say  $t$ . Up to shift (which does not change the length of a complex), we assume the nonzero components of  $i_*(X)$  are concentrated in positions between 0 and  $d$ , and those of  $i_*(B)$  are between  $k$  and  $k+t$  for some integer  $k$ . Therefore

$$\max\{n : \text{Hom}_A(i_*(X), i_*(B)[n]) \neq 0\} \leq k+t,$$

$$\max\{n : \text{Hom}_A(i_*(B), i_*(X)[n]) \neq 0\} \leq d-k.$$

But  $\text{Hom}_B(X, B[n]) \cong \text{Hom}_A(i_*(X), i_*(B)[n])$  and  $\text{Hom}_B(B, X[n]) \cong \text{Hom}_A(i_*(B), i_*(X)[n])$ . Hence  $s \leq k+t$  and  $r \leq d-k$ . By definition

$$\text{length}(X) = s+r \leq d+t = \text{s. gl. dim}(A) + \text{length}(i_*(B)).$$

Then  $X$  being arbitrary implies that  $\text{s. gl. dim}(B) \leq \text{s. gl. dim}(A) + \text{length}(i_*(B))$ , in particular it is finite.  $\square$

In contrast to the situation for global and finitistic dimension, the converse of the statement is unfortunately wrong. For an example we choose the quasi-hereditary algebra  $A$  in 4.3 given by

$$1 \cdot \begin{array}{c} \xrightarrow{\alpha} \\ \xleftarrow{\beta} \end{array} \cdot 2 \quad [\beta \circ \alpha = 0].$$

It has infinite strong global dimension, for there exist compact complexes of arbitrary length

$$\dots \rightarrow P(2) \rightarrow P(2) \rightarrow \dots \rightarrow P(2) \rightarrow P(1).$$

But the quasi-hereditary structure gives a standard recollement, where  $D^b(\text{mod-}k)$  is on both sides.



Now we are ready to prove the general Jordan Hölder theorem for bounded derived categories of finitely generated modules over piecewise hereditary algebras over arbitrary base fields.

**Theorem 5.7.** *Let  $A$  be a finite dimensional piecewise hereditary algebra over a field  $k$ . Let  $S_1, \dots, S_n$  be the representatives of isomorphism classes of simple  $A$ -modules. Denote by  $D_i$  the endomorphism rings of  $S_i$  ( $1 \leq i \leq n$ ). Then  $D^b(\text{mod-}A)$  has a stratification with  $D^b(\text{mod-}D_i)$  ( $1 \leq i \leq n$ ) being the factors. Moreover, any stratification of  $D^b(\text{mod-}A)$  has precisely these factors, up to derived equivalence.*

*Proof.* Without loss of generality we may assume the algebra  $A$  and the hereditary category  $\mathcal{H}$  to be connected. Moreover, replacing  $A$  by a derived equivalent algebra, if necessary, we may assume - by [18, 29] - that  $\mathcal{H}$  either is  $\text{mod-}H$ , the module category of a finite dimensional hereditary  $k$ -algebra  $H$ , or it is  $\text{coh}(X)$ , the category of coherent sheaves on an exceptional curve  $X$  (which is a weighted projective line in the sense of [13] when  $k$  is algebraically closed). In the second case, there is a ‘standard’ tilting object  $T$  in  $\mathcal{H} = \text{coh}(X)$  with endomorphism ring being a canonical algebra in the sense of [38] (see for example [28, 2.4]). To summarise: the algebra  $A$  is derived equivalent to an indecomposable hereditary algebra or to an indecomposable canonical algebra.

Recall that an object in  $D^b(\mathcal{H})$  is called exceptional if it has no self-extension. A sequence of indecomposable and exceptional objects  $(E_1, E_2, \dots, E_m)$  is called exceptional, if  $\text{Hom}(E_i, E_j) = 0 = \text{Ext}^1(E_i, E_j)$  for all  $i > j$ . An exceptional sequence is called complete if the length  $m$  equals to the rank  $n$  of  $A$  (i.e. the number of non-isomorphic simple modules). As  $\mathcal{H}$  is hereditary, using the method of [21, 4.1.4.2], the indecomposable direct summands of a partial tilting complex can be rearranged into an exceptional sequence (c.f. [4, 2.5]). Moreover this exceptional sequence is complete if and only if the partial tilting complex is a full tilting complex.

On the set of complete exceptional sequences in  $D^b(\mathcal{H})$  there is an action of  $\mathbb{Z}^n \rtimes B_n$ , where  $B_n$  is the braid group with  $n - 1$  generators acting by mutations. This action is moreover transitive. In the case of hereditary algebras this has been shown by [39] (extending the result for the algebraically closed case in [11]), and in the case of exceptional curves by [28] (extending the result for the algebraically closed case in [32]). It follows that the list of endomorphism rings of the indecomposable objects of a complete exceptional sequence in  $D^b(\mathcal{H})$  is an invariant. Therefore it is just the list  $(D_1, \dots, D_n)$  of the endomorphism rings of non-isomorphic simple  $A$ -modules.

The existence of a stratification of  $D^b(\text{mod-}A) \cong D^b(\mathcal{H})$  as claimed follows from the directedness of finite dimensional hereditary algebras and of canonical algebras (or indeed of all piecewise hereditary algebras, since Happel’s argument in [14, Lemma IV.1.10] works in general). Here,  $A$  directed means that the quiver of  $A$  has no oriented cycles, or equivalently that  $A$  has a simple projective module  $eA$ , for some idempotent  $e = e^2 \in A$  and the quotient algebra  $A/AeA$  is again directed. The two-sided ideal  $AeA$  is semisimple and projective as a right module. Therefore, the quotient map  $A \rightarrow A/AeA$  is a homological epimorphism inducing a recollement, which is a special case of the canonical recollement for a quasi-hereditary algebra discussed in Example 4.4. By induction we get the stratification as claimed.

Uniqueness of the stratification will be shown by induction on the number  $n$  (the rank of  $A$ ) of isomorphism classes of simple  $A$ -modules. When  $n = 1$ , there is nothing to show. Now assume  $n \geq 2$ . Given a recollement of  $A$

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

by finite dimensional algebras  $B$  and  $C$ , it follows from Corollary 5.4 and Subsection 1.4 that  $B$  and  $C$  are also piecewise hereditary, and hence directed, with rank strictly smaller than

$n$ . It follows from the structure of the recollement that the indecomposable direct summands of  $j_!(C) \oplus i_*(B)$  form a complete exceptional sequence in  $D^b(\text{mod-}A)$ . Note that for directed algebras  $B$  and  $C$ , the endomorphism ring of a simple module is the same as the endomorphism ring of its projective cover. Therefore the list of endomorphism rings of non-isomorphic simple  $C$ -modules and non-isomorphic simple  $B$ -modules coincides with that of non-isomorphic simple  $A$ -modules, i.e.  $\{D_1, \dots, D_n\}$ . The assertion follows by induction.  $\square$

The proof underlines the close link between recollements of bounded derived categories and exceptional sequences. Exceptional sequences have been used heavily by Bondal, Orlov and others when studying derived categories. Recently, Ingalls and Thomas have classified exceptional sequences by combinatorial objects (non-crossing partitions) in certain situations related to tame quivers. This classification carries over to all hereditary algebras, see [26, Section 6], where exceptional sequences are related also to thick subcategories that can occur in recollements of derived categories of hereditary algebras. In the case of hereditary algebras, this provides an alternative point of view on stratifications.

Note that the above proof is independent of [4, Theorem 6.1]. But the proof there yields the stronger fact that any stratification of  $D(\text{Mod-}A)$  can be rearranged into a chain of increasing derived module categories, via a sequence of homological epimorphisms. We will obtain this stronger version also for piecewise hereditary algebras. For that we have to prove in the setting of [4, Proposition 3.1], that  $G$  can be chosen to be a piecewise hereditary algebra provided that  $A$  is piecewise hereditary. The case when  $A$  is derived equivalent to a hereditary algebra follows from [4, Corollary 3.3]. Now we consider the case when  $A$  is derived equivalent to a canonical algebra.

**Proposition 5.8.** *Let  $A$  be a canonical algebra, and  $E$  an indecomposable exceptional object in  $D^b(\text{mod-}A)$ . Then there exists a piecewise hereditary algebra  $B$  that fits into a recollement of the form*

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

where  $C = \text{End}_A(E)$ . The recollement is induced by a homological epimorphism.

*Proof.* Let  $D^b(\text{mod-}A) = D^b(\mathcal{H})$  where  $\mathcal{H} = \text{coh}(X)$  for some exceptional curve  $X$ . Without loss of generality we assume  $E$  lies in  $\mathcal{H}$ . By [21, 4.1] any endomorphism of  $E$  is either a monomorphism or an epimorphism. If  $E$  is a torsion sheaf, i.e. it has finite length, then any endomorphism of  $E$  must be an isomorphism. If  $E$  is a bundle, considering the rank (degree, respectively) shows that any monomorphic (respectively, epimorphic) endomorphism of  $E$  must be an isomorphism. Therefore, the endomorphism ring of  $E$  is a skew-field.

Let  $T$  be the standard tilting object in  $\mathcal{H}$  with  $A$  being the endomorphism ring. Adjusting by using tubular mutation (see for example [29, 32, 41, 30]), we can assume that  $\text{Hom}_{\mathcal{H}}(E, T) = 0 = \text{Ext}^1_{\mathcal{H}}(T, E)$ . Applying [13, Proposition 6.5], we obtain that the perpendicular category  $\widehat{E} := \{Y : \text{Hom}_{\mathcal{H}}(E, Y) = 0 = \text{Ext}^1_{\mathcal{H}}(E, Y)\}$  of  $E$  in  $\mathcal{H}$  admits a tilting object say  $T'$ . Indeed,  $T' = \ell(T)$  is constructed by the universal extension of  $T$  and  $E$ : since  $\text{Hom}_{\mathcal{H}}(E, T) = 0$  and  $T$  is a tilting object,  $\text{Ext}^1_{\mathcal{H}}(E, T)$  must be nonzero, say of dimension  $m$  over the skew-field  $\text{End}_{\mathcal{H}}(E)$ . Then the universal extension

$$0 \rightarrow T \rightarrow T' \rightarrow E^{\oplus m} \rightarrow 0$$

provides  $T' = \ell(T)$  as the approximation of  $T$  in the perpendicular category  $\widehat{E}$  (cf. the proof of 3.1 or [3, Appendix A.1]). It is straightforward to check that  $\widehat{E}$  is a hereditary and abelian subcategory of  $\mathcal{H}$ .

Write  $E^\perp := \{Y \in D^b(\mathcal{H}) : \text{Hom}_{D^b(\mathcal{H})}(E, Y[k]) = 0, \forall k \in \mathbb{Z}\}$  for the perpendicular category of  $E$  in the bounded derived category. Since  $\mathcal{H}$  is hereditary, it is clear that  $E^\perp = \{Y[k] : Y \in \widehat{E}, k \in \mathbb{Z}\} \cong D^b(\widehat{E})$ , which is equivalent to  $D^b(\text{mod-}B)$  for  $B = \text{End}(T')$  as triangulated categories, since  $T'$  is a tilting object. As a compact exceptional object in the unbounded derived module category  $D(\text{Mod-}A)$ ,  $E$  generates a recollement on the unbounded derived category level

$$D(\text{Mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D(\text{Mod-}C)$$

where  $C = \text{End}(E)$  (see the proof of Theorem 3.3). Now the global dimension of the canonical algebra  $A$  is finite, and the image of  $B$  in  $D(\text{Mod-}A)$  is  $T'$  which is compact. By Corollary 2.7 such a recollement can be restricted to  $D^b(\text{mod-})$  level.

For the last statement, it suffices to prove that  $i^*(A)$  is an exceptional object, see [3, 1.7]. By construction, the universal extension  $0 \rightarrow T \rightarrow T' \rightarrow E^{\oplus m} \rightarrow 0$  gives rise to the canonical triangle

$$j_!j^!(A) \rightarrow A \rightarrow i_*i^*(A) \rightarrow j_!j^!(A)[1]$$

in which  $i_*i^*(A) \cong T'$ . Since  $i_*$  is fully faithful and  $T'$  is a tilting object in  $\mathcal{H}$ , we obtain  $\text{Hom}_B(i^*(A), i^*(A)[n]) \cong \text{Hom}_{D^b(\mathcal{H})}(T', T'[n]) = 0$  for all  $n \neq 0$ .  $\square$

*Remark.* We have shown the perpendicular category of an indecomposable exceptional sheaf  $E$  in  $\mathcal{H} = \text{coh}(X)$  is derived equivalent to a quasitilted algebra. When  $E$  is a bundle, Hübner [23, Theorem 5.4] shows the perpendicular category  $\widehat{E}$  is equivalent to the module category of some hereditary algebra. When  $E$  is a simple torsion sheaf, Geigle and Lenzing [13] (see also [3, Example 3.2]) showed  $\widehat{X}$  is equivalent to the category of coherent sheaves on another exceptional curve with reduced weights.

More generally, take an object  $E$  in  $D^b(\text{coh}(X))$  without self-extensions (not necessarily indecomposable). Without loss of generality we assume it is multiplicity-free. Then its indecomposable direct summands can be ordered into an exceptional sequence. It follows by induction that there exists some piecewise hereditary algebra  $B$  fitting into a recollement

$$D^b(\text{mod-}B) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}A) \begin{array}{c} \longleftarrow \\ \rightleftarrows \\ \longrightarrow \end{array} D^b(\text{mod-}C)$$

where  $C = \text{End}_A(E)$  (c.f. proof of [4, Corollary 3.3]). Now in the setting of [4, Proposition 3.1], if we start with a piecewise hereditary algebra  $A$ , then  $G$  can be chosen to be again piecewise hereditary (in particular, it is an ordinary algebra). Therefore any stratification of  $A$  can be rearranged into a chain of increasing derived module categories corresponding to homological epimorphisms (c.f. proof of [4, Theorem 6.1]).

At this point we have proved (1)  $\Rightarrow$  (3) in Proposition 5.1 for piecewise hereditary algebras. As piecewise hereditary algebras have finite global dimension, the equivalences (1)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4)  $\Leftrightarrow$  (5)  $\Leftrightarrow$  (6) in Proposition 5.1 hold true. Combining this information with Theorem 5.7, we obtain the Jordan Hölder theorem for piecewise hereditary algebras on different levels.

**Corollary 5.9.** *Let  $A$  be a piecewise hereditary algebra over a field  $k$ . Then Theorem 5.7 holds true also for recollements on  $D^-(\text{Mod-})$ ,  $D^b(\text{Mod-})$  and  $D(\text{Mod-})$  levels.*

Now Corollaries 5.2 and 5.5 extend to piecewise hereditary algebras. We obtain:

**Corollary 5.10.** *The endomorphism algebra of a partial tilting complex of a piecewise hereditary algebra is again piecewise hereditary.*

*Proof.* Let  $A$  be a piecewise hereditary algebra and  $X$  a partial tilting complex. So  $X$  is exceptional in  $D^b(\text{mod-}A)$ . We have just shown that  $X$  determines a recollement of  $A$  with  $C := \text{End}(X)$  on the right hand side. It follows from Corollary 5.4 that  $C$  is piecewise hereditary.  $\square$

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