POSITIVITY IN SKEW–SYMMETRIC CLUSTER ALGEBRAS OF FINITE TYPE

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Abstract. We prove that the basis of cluster monomials of a skew–symmetric cluster algebra $\mathcal{A}$ of finite type is the atomic basis of $\mathcal{A}$. This means that an element of $\mathcal{A}$ is positive if and only if it has a non–negative expansion in the basis of cluster monomials. In particular cluster monomials are positive indecomposable, i.e. they cannot be written as a sum of positive elements.
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1. Introduction

Let H be an orientation of a simply–laced Dynkin diagram, i.e. a diagram of type A, D, or E in the Cartan–Killing classification. We consider the coefficient–free cluster algebra $\mathcal{A}(H)$ associated with H. This is a $\mathbb{Z}$–subalgebra of the field $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_n)$, where n is the number of vertices of H, introduced by Fomin and Zelevinsky [13]. The algebra $\mathcal{A}(H)$ can be described as follows: for every $k \in [1, n]$ let us consider the element $x'_k \in \mathcal{F}$ defined by:

$$x'_k = \frac{\prod_{k\to j\in H_1} x_j + \prod_{i\to k\in H_1} x_i}{x_k} \tag{1}$$

(here H_1 denotes the set of arrows of H). We have $\mathcal{A}(H) = \mathbb{Z}[x_k, x'_k : k = 1, \ldots, n] \subset \mathcal{F}$ (see [2, theorem 1.18, corollary 1.19] or the survey [15, theorem 4.13]). Moreover, in [2] it is shown that standard monomials, i.e. monomials in $x_1, x'_1, \ldots, x_n, x'_n$ which do not contain the products $x_k x'_k$ ($k \in [1, n]$), form a $\mathbb{Z}$–basis of $\mathcal{A}(H)$.

Besides the basis of standard monomials, there is another basis of $\mathcal{A}(H)$ which is of particular interest to us. This is the basis $\mathbf{B}$ of cluster monomials. Let us briefly define the set $\mathbf{B}$ (see section 2 for more details). By definition, the algebra $\mathcal{A}(H)$ is the $\mathbb{Z}$–subalgebra generated by some elements of $\mathcal{F}$ called cluster variables. The cluster variables are grouped into free–generating sets for the field $\mathcal{F}$ called clusters. In particular, every cluster $\mathbf{s}$ consists of n algebraically independent rational functions $s_1, \ldots, s_n$ and $\mathcal{F} \simeq \mathbb{Q}(s_1, \ldots, s_n)$. A cluster monomial of $\mathcal{A}(H)$ is, by definition, a monomial in cluster variables belonging to the same cluster.

Cluster monomials are natural elements to consider in the additive categorification of cluster algebras via cluster categories [3]. Namely they correspond to cluster–tilting objects. This description allow Caldero and Keller in [6] to prove that cluster monomials form a $\mathbb{Z}$–basis of $\mathcal{A}(H)$. Cluster monomials are also important elements to consider in some geometric realizations of cluster algebras [2], [14], where every cluster provides a criterion for total positivity. Moreover cluster monomials belong to the dual semi–canonical basis of the coordinate ring $\mathbb{C}[N]$ of a maximal unipotent group N whose Lie algebra $\mathfrak{n}$ is the maximal unipotent subalgebra of a simple Lie algebra of type A, D or E [17, theorem 2.8]. In this paper we give further evidence of the importance of cluster monomials in the theory of cluster algebras itself, as conjectured by Fomin and Zelevinsky [15, conjecture 4.19]. As a special case of the Fomin–Zelevinsky's Laurent phenomenon [13], every element of $\mathcal{A}(H)$ is a Laurent polynomial in all its clusters. An element p of the cluster algebra

$\mathcal{A}(H)$ is called positive if its Laurent expansions in all the clusters of $\mathcal{A}(H)$ have non–negative integer coefficients. Positive elements form a semiring, i.e. sums and products of positive elements are positive. We say (see [22], [8]) that a $\mathbb{Z}$–basis $\mathbf{B}$ of $\mathcal{A}(H)$ is atomic if the semiring of positive elements consists precisely of $\mathbb{Z}_{\geq 0}$–linear combinations of elements of $\mathbf{B}$. Note that if an atomic basis exists, it is unique and it is formed by positive indecomposable elements, i.e. those elements which cannot be written as a sum of positive elements.

Theorem 1.1. *The set of cluster monomials is the atomic basis of $\mathcal{A}(H)$ when H is an orientation of a simply–laced Dynkin diagram.*

The proof of theorem 1.1 is an application of the theory of quivers with potentials developed by Derksen, Weyman and Zelevinsky in [11] and [12]. In particular, the proof does not depend on the choice of "coefficients": for every semifield $\mathbb{P}$ a cluster algebra $\mathcal{A}_{\mathbb{P}}(H)$ is defined, which is said to have coefficients in $\mathbb{P}$. Theorem 1.1 holds in $\mathcal{A}_{\mathbb{P}}(H)$ as long as the cluster monomials form a $\mathbb{Z}\mathbb{P}$–basis of it (see remark 5.1). It is only for simplifying notation that we prefer to work in the coefficient–free setting.

The existence of an atomic basis of a general cluster algebra is still an open problem. Only a few cases are known [22], [7], [8]. We trust that our proof of theorem 1.1, together with the techniques developed in [10] and [9], is an important step for the description of the positive semiring of a cluster algebra associated with any acyclic quiver.

We prove theorem 1.1 in section 5. The other sections are devoted to recalling the results used in the proof.

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2. Background on cluster algebras

In this section we recall the definition of a cluster algebra and some properties from [13].

Let $n \geq 1$ be a positive integer. We consider an n–regular tree $\mathbf{T}_n$ and we label its edges by numbers $1, 2 \dots, n$ so that two edges adjacent to the same vertex receive different labels. We introduce the dynamic of seed mutations on $\mathbf{T}_n$. Let $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$ be the field of rational functions in n independent variables. A seed in $\mathcal{F}$ is a pair $(B, \mathbf{u})$ where B is a skew–symmetric $n \times n$ integer matrix and $\mathbf{u} = (u_1, \dots, u_n)$ is a free–generating set for $\mathcal{F}$ so that $\mathcal{F} \simeq \mathbb{Q}(u_1, \dots, u_n)$. The matrix B is called the exchange matrix of the seed Σ, while the set $\mathbf{u}$ is called its cluster. The elements of the cluster of Σ are called its cluster variables. Given a seed Σ of $\mathcal{F}$ and $k \in [1, n]$ (as is customary, we use the notation $[1, n] := (1, 2, \dots, n)$) we define a new seed $\mu_k(\Sigma) = (B', \mathbf{u}')$, called the mutation of Σ in direction k, obtained from Σ by the following rules of mutation:

(1) the matrix $B' = (b'_{ij})$ is given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k; \\ b_{ij} + sg(b_{ik})[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases} \tag{2}$$

where $sg(b)$ denotes the sign of the integer b and $[b]_+ := \max(b, 0)$.

(2) The new cluster $\mathbf{u}'$ is obtained from the cluster $\mathbf{u} = (u_1, \ldots, u_n)$ by $\mathbf{u}' = \mathbf{u} \setminus \{u_k\} \cup \{u'_k\}$ where

$$u'_k = \frac{\prod_{i=1}^n u_i^{[b_{ik}]_+} + \prod_{j=1}^n u_j^{[-b_{jk}]_+}}{u_k}. \tag{3}$$

A cluster pattern is the assignment of a seed Σ_t to every vertex of $\mathbf{T}_n$ so that whenever $t \overset{k}{\text{———}} t'$, i.e. the unique edge adjacent to t and labelled with k connects t with the vertex t', the assigned seeds Σ_t and $\Sigma_{t'}$ satisfy $\Sigma_{t'} = \mu_k(\Sigma_t)$. It is clear that a cluster pattern is uniquely determined by the choice of an "initial" seed Σ_0 and we denote it by $\mathbf{T}_n(\Sigma_0)$. By definition, the (coefficient–free skew–symmetric) cluster algebra $\mathcal{A}(\Sigma_0) = \mathcal{A}(\mathbf{T}_n(\Sigma_0))$ is the $\mathbb{Z}$–subalgebra of $\mathcal{F}$ generated by the cluster variables of the seeds of $\mathbf{T}_n(\Sigma_0)$.

We notice that the cluster pattern depends uniquely on the choice of the initial exchange matrix B of Σ_0 and we hence often write $\mathcal{A}(B)$ instead of $\mathcal{A}(\Sigma_0)$ and $\mathbf{T}_n(B)$ instead of $\mathbf{T}_n(\Sigma_0)$.

We sometimes prefer to use the language of quivers instead of the one of matrices. Let $Q = (Q_0, Q_1, t, h)$ be a finite quiver without loops and oriented 2–cycles, with vertex set Q_0 of cardinality n, with edges Q_1 and orientation given by the two maps $t, h : Q_1 \to Q_0$ which associate to an edge a its tail $t(a)$ and its head $h(a)$ and we write $t(a) \xrightarrow{a} h(a)$. We associate with Q the skew–symmetric $n \times n$ integer matrix $B(Q)$ whose ij–th entry equals the number of arrows from the vertex j to the vertex i minus the number of arrows from i to j. The map $Q \mapsto B(Q)$ is a bijection between finite quivers on n vertices with no loops and no oriented 2–cycles and $n \times n$ skew–symmetric integer matrices. We hence write $\mathcal{A}(Q)$ for $\mathcal{A}(B(Q))$ and $\mathbf{T}_n(Q)$ for $\mathbf{T}_n(B(Q))$. Notice that, in this notation, formula (1) of the introduction expresses the mutation of the cluster variable x_k of the seed $(H, (x_1, \cdots, x_n))$ of the cluster algebra $\mathcal{A}(H)$.

Every cluster $\mathbf{u} = (u_1, \ldots, u_n)$ of $\mathcal{A}(Q)$ is a free–generating set of the field $\mathcal{F}$ and hence $\mathcal{F} \simeq \mathbb{Q}(u_1, \ldots, u_n)$. In particular, every cluster variable of $\mathcal{A}(Q)$ is a rational function in every such cluster. By the famous Laurent phenomenon proved by Fomin and Zelevinsky in [13] such a rational function is actually a Laurent polynomial. We denote by $X_{k;t}^{B;t_0}$ the Laurent expansion in the seed at vertex t_0 of $\mathbf{T}_n(Q)$ whose exchange matrix is B of the k–th cluster variable of the seed at vertex t of $\mathbf{T}_n(Q)$, for $k \in [1, n]$.

A cluster algebra is called of finite type if it has only finitely many cluster variables. In [14] it is shown that $\mathcal{A}(Q)$ is of finite type if and only if the cluster pattern $\mathbf{T}_n(Q)$ contains a Dynkin quiver H (i.e. a diagram of type A, D, or E in the Cartan–Killing classification). In this case, as shown in [14], [4], [6], the connection with the representation theory of H is much deeper: there is a bijection between the indecomposable H–representations and the non–initial cluster variables of $\mathcal{A}(H)$. Such bijection is given in terms of projective varieties called quiver Grassmannians. In [5] and [12] such bijection is given also for more general quivers but with some restrictions on the involved representations. We will say more about it in the subsequent sections.

3. Background on quivers with potentials and their representations

In this section we recall some facts about the theory of quivers with potentials developed in [11].

Let $Q = (Q_0, Q_1, t, h)$ be a finite quiver. As usual, Q_0 denotes the set of vertices of Q, Q_1 is the set of edges and every edge $a \in Q_1$ is oriented $t(a) \xrightarrow{a} h(a)$. The theory of quivers with potentials produces a way to "mutate" the quiver Q. More precisely, what is going to change is the set of arrows of Q while the set of vertices is going to remain fixed. In this section we recall how this idea is formalized.

Let K be a vector space. The vertex span $R = K^{Q_0}$ is defined as the space of K–functions on Q_0. There is a distinguished basis $\{e_i : i \in Q_0\}$ of idempotents of R given by $e_i(j) = \delta_{ij}$ (the Kronecker delta) for $i, j \in Q_0$. The arrow span $A = K^{Q_1}$ is the vector space of K–functions on the set of arrows. The space A has the following structure of R–bimodule: $e_i f e_j(a) = e_i(h(a))f(a)e_j(t(a))$ for every $a \in Q_1$. We identify the set of arrows Q_1 with a basis of A and for an arrow $a \in Q_1$ we denote with the same symbol a the corresponding element of A. The d–tensor power $A^d = A \otimes \cdots \otimes A$ of A has a structure of R–bimodule as well. Moreover, there is a block decomposition

$$A^d = \bigoplus_{i,j} A^d_{ij}$$

where $A^d_{ij} = e_i A^d e_j$. The R–bimodule A^d_{ij} is spanned by the elements $a_1 a_2 \cdots a_d$ such that the $a_i \in A$, $h(a_{i+1}) = t(a_i)$ for $i \in [1, d-1]$ and $t(a_d) = j$, $h(a_1) = i$. Such elements are called paths of length d from the vertex j to the vertex i. The path algebra is the tensor algebra

$$R\langle A\rangle := \bigoplus_{d=0}^{\infty} A^d$$

with the convention that $A^0 = R$. For each $i, j \in Q_0$ the R–bimodule $R\langle A\rangle_{i,j} = e_i R\langle A\rangle e_j$ is spanned by the paths from j to i and the union of all the paths form a basis of $R\langle A\rangle$ called the path basis.

For technical reasons it is convenient to consider the completed path algebra

$$R\langle\langle A\rangle\rangle = \prod_{d=0}^{\infty} A^d$$

whose elements are possibly infinite linear combinations of paths. Let

$$\mathrm{m} = \prod_{d=1}^{\infty} A^d$$

be the ideal of (linear combinations of) paths of length bigger or equal than one. The algebra $R\langle\langle A\rangle\rangle$ is a topological algebra with respect to the m–adic topology, i.e. a subset U of $R\langle\langle A\rangle\rangle$ is open if and only if for every $x \in U$ there exists $N > 0$ such that $x + \mathrm{m}^N \subset U$.

A cyclic path is a path $a_1 \cdots a_d$ such that $t(a_d) = h(a_1)$. We denote by A^d_{cyc} the span of all the cyclic paths in A^d. We define the closed subalgebra $R\langle\langle A\rangle\rangle_{cyc} \subseteq R\langle\langle A\rangle\rangle$ by

$$R\langle\langle A\rangle\rangle_{cyc} = \prod_{d=1}^{\infty} A^d_{cyc} \, .$$

A potential S is an element of $R\langle\langle A\rangle\rangle_{cyc}$. Potentials are usually considered up to cyclic equivalences: we say that two potentials $S, S' \in R\langle\langle A\rangle\rangle_{cyc}$ are cyclically equivalent [11, definition 3.2] if $S - S'$ belongs to the closure of the span of all the elements of the form $a_1 \cdots a_d - a_2 \cdots a_d a_1$ where $a_1 \cdots a_d$ is a cyclic path.

Given a potenital S in $R\langle\langle A\rangle\rangle$, the pair (A, S) is called a quiver with potential (QP for short) provided that A has no loops (i.e. $A_{i,i} = \{0\}$ for every $i \in Q_0$) and no two cyclically equivalent cyclic paths appear in the decomposition of S.

Let (A, S) and (A', S') be two QPs on the same set of vertices Q_0. A right–equivalence between (A, S) and (A', S') is an algebra isomorphism $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ such that $\varphi|_R = \mathrm{id}$ and $\varphi(S)$ is cyclically equivalent to S'. The notion of right–equivalence is very important in dealing with "mutations" of QPs that we will recall later in section 3.2. The direct sum of (A, S) and (A', S') is defined as $(A \oplus A', S + S')$. Note that this is well-defined since $R\langle\langle A\rangle\rangle \oplus R\langle\langle A'\rangle\rangle$ embeds canonically in $R\langle\langle A \oplus A'\rangle\rangle$ as a closed subalgebra.

For an element $\xi \in A^*$ we consider the cyclic derivative ∂_ξ as the operator $R\langle\langle A\rangle\rangle_{cyc} \to R\langle\langle A\rangle\rangle$ defined on a cyclic path $a_1 \cdots a_d \in A^d_{cyc}$ by

$$\partial_\xi(a_1 \cdots a_d) = \sum_{i=1}^{d} \xi(a_i) a_{i+1} \cdots a_d a_1 \cdots a_{i-1}. \tag{4}$$

Given a potential S on A, the Jacobian ideal $J(S)$ is the closure of the (two–sided) ideal in $R\langle\langle A\rangle\rangle$ generated by $\{\partial_\xi S : \xi \in A^*\}$. Notice that the closure (in the m–adic topology) of a subset $U \subset R\langle\langle A\rangle\rangle$ is given by $\overline{U} = \bigcap_{N=0}^{\infty} U + \mathrm{m}^N$. In particular, the closure of an ideal is again an ideal and hence $J(S)$ is a (two–sided) ideal of $R\langle\langle A\rangle\rangle$. The Jacobian algebra is defined as the quotient algebra $\mathcal{P}(A, S) = R\langle\langle A\rangle\rangle/J(S)$.

We notice that if two QPs (A, S) and (A', S') are right–equivalent then the corresponding Jacobian algebras $\mathcal{P}(A, S)$ and $\mathcal{P}(A', S')$ are isomorphic.

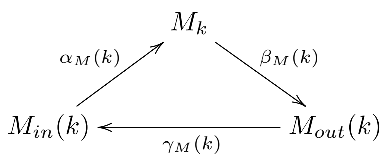
We recall the splitting theorem [11, theorem 4.6]. Let (A, S) be a QP on some set of vertices Q_0. The trivial part $S^{(2)} \in A^2$ of the potential S is, by definition, the homogeneous component of S of degree two. The QP (A, S) is called reduced if $S^{(2)} = 0$. Notice that in a reduced QP (A,S), the cyclic part A^2_{cyc} of degree two of A is allowed to be non–zero, even if $S^{(2)} = 0$. The trivial and the reduced arrow span of (A, S) are the R–bimodules given by:

$$A_{triv} = \partial S^{(2)} \quad A_{red} = A/A_{triv}$$

where $\partial S^{(2)}$ is the subspace $\{\partial_\xi S^{(2)} : \xi \in A^*\} \subseteq A$. The splitting theorem asserts that every QP (A, S) is right–equivalent to the direct sum of a trivial QP (A_{triv}, S_{triv}) and a reduced QP (A_{red}, S_{red}). Moreover, the right–equivalence class of both (A_{red}, S_{red}) and (A_{triv}, S_{triv}) is determined by the right–equivalence class of (A, S). The QP (A_{red}, S_{red}) is called the reduced part of (A, S).

3.1. **QP–representations.** A QP–representation is, by definition, a quadruple (A, S, M, V) where (A, S) is a QP, M is a finite dimensional $\mathcal{P}(A, S)$–module and $V = (V_i)_{i \in Q_0}$ is a collection of finite dimensional vector spaces. Sometimes we say that the pair $\mathcal{M} = (M, V)$ is a decorated representation of the QP (A, S). Thus V is a finite dimensional R–bimodule while $M = (M_i)_{i\in Q_0}$ is a finite dimensional representation of the quiver Q whose arrow span is A, which is annihilated by all the cyclic derivatives of the potential S. For every arrow $a \in A$ we denote by a_M the action of a on M. For every vertex $k \in Q_0$ and arrows a and b such that $h(a) = t(b) = k$ there is an element $\partial_{ba} S \in e_{t(a)} R\langle\langle A\rangle\rangle e_{h(b)}$ from the vertex $h(b)$ to the vertex $t(a)$ defined similarly to (4). Such an element acts on M as a linear map $\gamma_{ba} = (\partial_{ba}(S))_M : M_{h(b)} \to M_{t(a)}$. This gives rise to a triangle of linear maps

(5)



where

$$M_{in}(k) = \bigoplus_{a \in Q_1 : h(a)=k} M_{t(a)}, \quad M_{out}(k) = \bigoplus_{b \in Q_1 : t(b)=k} M_{h(b)}$$

and

$$\alpha_M(k) = \sum_{a \in Q_1 : h(a)=k} a_M, \quad \beta_M(k) = \sum_{b \in Q_1 : t(b)=k} b_M, \quad \gamma_M(k) = \sum_{a,b : h(a)=t(b)=k} \gamma_{ba}\,.$$

Moreover, the linear maps satisfy the following relations [11, lemma 10.6]:

$$\alpha_M(k) \circ \gamma_M(k) = 0 = \gamma_M(k) \circ \beta_M(k). \tag{6}$$

Given two decorated QP–representations $\mathcal{M} = (M, V)$ and $\mathcal{M}' = (M', V')$ of the same QP (A, S), their direct sum is the decorated representation of (A, S) given by $\mathcal{M} \oplus \mathcal{M}' = (M \oplus M', V \oplus V')$.

A QP–representation $\mathcal{M} = (A, S, M, V)$ is called positive if $V = \{0\}$, and negative if $M = 0$. The negative simple representation at vertex k is the negative QP–representation $\mathcal{S}_k^- = \mathcal{S}_k^-(A, S) = (A, S, \{0\}, V)$, whose decoration V consists of a one dimensional vector space at vertex k and zero elsewhere.

A right–equivalence between two QP–representations $\mathcal{M} = (A, S, M, V)$ and $\mathcal{M}' = (A', S', M', V')$ is a triple (φ, ψ, η) of maps such that: $\varphi : R\langle\langle A\rangle\rangle \to R\langle\langle A'\rangle\rangle$ is a right–equivalence between (A, S) and (A', S'); $\psi : M \to M'$ is a vector space isomorphism such that $\psi \circ u_M = \varphi(u)_{M'} \circ \psi$; $\eta : V \to V'$ is an isomorphism of R–bimodules.

Let (A, S) be a QP and let (A_{red}, S_{red}) be its reduced part. For every trivial QP (C, T) the natural embedding $R\langle\langle A_{red}\rangle\rangle \to R\langle\langle A_{red} \oplus C\rangle\rangle$ induces an isomorphism of Jacobian algebras $\mathcal{P}(A_{red}, S_{red}) \to \mathcal{P}(A_{red} \oplus C, S_{red} + T)$ [11, proposition 4.5]. Let $\varphi : R\langle\langle A_{red} \oplus C\rangle\rangle \to R\langle\langle A\rangle\rangle$ be a right equivalence between $(A_{red} \oplus C, S_{red} \oplus T)$ and (A, S). Given a QP–representation $\mathcal{M} = (A, S, M, V)$, its reduced part is defined as the QP–representation $\mathcal{M}_{red} = (A_{red}, S_{red}, M', V)$ where $M' = M$ as K–vector space and for $u \in R\langle\langle A_{red}\rangle\rangle$ the action is given by $u_M = \varphi(u)_M$. The right–equivalence class of $\mathcal{M}_{red}$ is determined by that of $\mathcal{M}$ [11, proposition 10.5].

The **g**–vector of a QP–representation $\mathcal{M}$ is, by definition, the vector $\mathbf{g}_{\mathcal{M}} = (g_1, \ldots, g_n) \in \mathbb{Z}^n$ $(n = |Q_0|)$ whose k–th component is given by

$$g_k = \dim \ker\gamma_M(k) - \dim M_k + \dim V_k \tag{7}$$

for every $k = 1, 2, \ldots, n$ (in the notations of (5)). In particular it follows that for every two QP–representations $\mathcal{M}$ and $\mathcal{M}'$ we have

$$\mathbf{g}_{\mathcal{M}\oplus\mathcal{M}'} = \mathbf{g}_{\mathcal{M}} + \mathbf{g}_{\mathcal{M}'}. \tag{8}$$

We notice that if A is acyclic, i.e. $R\langle\langle A\rangle\rangle_{cyc} = \{0\}$, then $\gamma_M(k) = 0$ and hence the **g**–vector of a positive QP–representation M equals $\mathbf{g}_M = -E_A\mathbf{dim}(M)$ where $E_A = (e_{ij})$ is the Euler matrix of A (see e.g. [1]) defined by $e_{ii} = 1$ and $e_{kj} = -\dim A_{jk}$ $(j \neq k)$ and $\mathbf{dim}M = (\dim M_i)_{i\in Q_0}$.

3.2. **Mutations of QPs.** We recall the mutation of a quiver with potential (A, S) on some set of vertices Q_0. Let $k \in Q_0$ be a vertex such that no oriented 2–cycles in A start (and end) at k, i.e. either $A_{ik} = 0$ or $A_{ki} = 0$ for all $i \in Q_0$. Let us also assume that there are no components of the potential S that start (and end) at the vertex k (if this is the case, it is sufficient to replace S with a cyclically equivalent potential). We define the "premutation" of (A, S) as the QP $(\tilde{A}, \tilde{S})$ on the same set of vertices Q_0 as (A, S) defined as follows: the new arrow span $\tilde{A}$ is given in three steps:

(1) take all the arrows of A which do not start or end at k;
(2) for every path ba such that $h(a) = t(b) = k$ add a new arrow $[ba] \in e_{h(b)}\tilde{A}e_{t(a)}$;
(3) replace every arrow a in e_kA (i.e. ending at k) or in Ae_k (i.e. starting at k) by an opposite arrow a^*.

The potential $\tilde{S}$ on $\tilde{A}$ is given by

$$\tilde{S} = [S] + \Delta_k$$

where $[S]$ is obtained by replacing in S every path ba such that $h(a) = t(b) = k$, with the arrow $[ba]$ (recall that there are no components of S starting at k); the element Δ_k is defined by

$$\Delta_k = \sum a^*[ba]b^*,$$

where the sum is taken over all the paths ba such that $h(a) = t(b) = k$. Now the mutation $\mu_k(A,S)$ of the QP (A,S) at vertex k is defined as the reduced part $(\tilde{A}_{red}, \tilde{S}_{red})$ of $(\tilde{A}, \tilde{S})$. In view of the Splitting Theorem, the operation $(A,S) \mapsto \mu_k(A,S)$ is well–defined on the set of right equivalence classes of QPs.

We remark that by [11, theorem 5.7] $\mu_k^2(A,S)$ is right–equivalent to (A,S) and hence μ_k is an involution on the set of right–equivalence classes of QPs.

In order to perform mutations of a QP (A,S) in all the vertices, the arrow span A is assumed to be 2–acyclic i.e. for every vertex k either $A_{ik} = \{0\}$ or $A_{ki} = \{0\}$ for every vertex i. Even in this case the mutation can create 2–cycles. A QP is called non–degenerate if this does not happen and any sequence of mutations does not create 2–cycles. If the field K is uncountable then for every arrow span A there exists a potential S such that (A,S) is non–degenerate [11, corollary 7.4].

We notice that a 2–acyclic arrow span A with no loops, can be encoded by a $n \times n$ skew–symmetric integer matrix $B = B(A)$ whose ij–th component b_{ij} is given by

$$b_{ij} = \mathrm{dim}A_{ij} - \mathrm{dim}A_{ji}.$$

It can be shown [11, proposition 7.1] that in this case the mutation $\mu_k(A,S) = (A',S')$ translates into the matrix mutation $B(A') = \mu_k(B(A))$ given by (2).

A QP (A,S) is called rigid if every potential S' on A is cyclically equivalent to an element of J(S). Rigid potentials have several nice properties: if (A,S) is rigid then also $\mu_k(A,S)$ is rigid [11, corollary 6.11]; moreover, rigid potentials are 2–acyclic [11, proposition 8.1]. It follows that rigid potentials are non–degenerate.

If Q is acyclic, with arrow span A, the only possibility is that the potential is zero and the QP (A,0) is rigid. Moreover for every arrow span A' such that $B(A')$ is mutation–equivalent via (2) to $B(A)$, there exists a potential S' such that (A',S') is reduced and rigid and (A',S') is unique up to right–equivalences. In the special case of a Dynkin quiver H such choice for the potential S' is explicitly given in [12, section 9]. Let us recall it.

Let H be a Dynkin quiver and let A be an arrow span on the set of vertices of H such that the corresponding matrix $B = B(A)$ is mutation equivalent to $B(H)$. In [14] it is shown that the ij–th component b_{ij} of B satisfies $b_{ij} \leq 1$, i.e. the space A_{ij} is either zero or it is one dimensional. For $d \geq 3$, a d–chordless cycle in A is a d–cycle whose vertices can be labeled by $\mathbb{Z}/d\mathbb{Z}$ so that the edges are precisely labeled by pairs $\{i, i+1\}$, $i \in \mathbb{Z}/d\mathbb{Z}$. In [14, proposition 9.7], it is shown that all the d–chordless cycles of A are cyclically oriented. A potential S of A is called primitive if it is a linear combination of all the chordless cycles of A. In [11, proposition 9.1] it is shown that the QP (A,S) is rigid for every primitive potential S of A. In type A and D this is a special case of a general construction due to Labardini–Fragoso [19].

3.3. **Mutations of QP–representations.** Let (A,S) be a non–degenerate QP on a set of vertices Q_0 and let $k \in Q_0$. Let $\mathcal{M} = (M,V)$ be a decorated representation of (A,S). We are going to define a QP representation $\mu_k(\mathcal{M})$ which is a decorated representation of the mutated QP $\mu_k(A,S)$. We define $\overline{\mathcal{M}} = (\overline{M} = (\overline{M}_i : i \in Q_0), \overline{V} = (\overline{V}_i : i \in Q_0))$ by

$$\overline{M}_i = M_i, \quad \overline{V}_i = V_i \quad (i \neq k)$$

and
(9)
$$\overline{M}_k = \mathrm{im}\gamma \oplus \frac{\ker\, \alpha_M(k)}{\mathrm{im}\ \gamma_M(k)} \oplus \frac{\ker\, \gamma_M(k)}{\mathrm{im}\ \beta_M(k)} \oplus V_k, \quad \overline{V}_k = \frac{\ker\, \beta_M(k)}{\ker\, \beta_M(k) \cap \mathrm{im}\ \alpha_M(k)}.$$

The action of an arrow $c \in \tilde{A}$ (see section 3.2) on $\overline{M}$ is defined as follows. If c is not incident to k then $c_{\overline{M}} = c_M$ and $[ba]_{\overline{M}} = b_M \circ a_M$ for every arrows a and b of A such that $h(a) = t(b) = k$. It remains to define the linear maps

$$(10) \qquad \overline{M}_{out} = M_{in}(k) \xleftarrow{\overline{\beta}=(\beta_1,\beta_2,\beta_3,\beta_4)} \overline{M}_k \xleftarrow{\overline{\alpha}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix}} \overline{M}_{in} = M_{out}(k)$$

in the corresponding triangle (5) for $\overline{\mathcal{M}}$ at vertex k (in (10) we express both $\overline{\alpha}$ and $\overline{\beta}$ in the matrix form corresponding to the decomposition (9) of $\overline{M}_k$). These maps are defined in the following natural way: we choose splitting data $\rho : M_{out}(k) \to \ker\, \gamma_M(k)$ such that $\rho \circ \iota = \mathrm{id}_{\ker\, \gamma_M(k)}$ (where ι denotes the inclusion map) and $\sigma : {\ker\, \alpha_M(k)}/{\mathrm{im}\ \gamma_M(k)} \to \ker\, \alpha$ such that $\pi \circ \sigma = \mathrm{id}_{\ker\, \alpha_M(k)/\mathrm{im}\ \gamma_M(k)}$ (where π denotes the natural projection). The components α_i and β_i are defined by:

$$\begin{array}{ll} \beta_1 = \iota, & \alpha_1 = -\gamma \\ \beta_2 = \iota \circ \sigma, & \alpha_2 = 0 \\ \beta_3 = 0, & \alpha_3 = -\pi \circ \rho \\ \beta_1 = 0, & \alpha_1 = 0 \end{array}$$

This choice makes $\overline{\mathcal{M}}$ a decorated representation of the premutation $(\tilde{A}, \tilde{S})$ of (A, S) [11, proposition 10.7]. Moreover, a different choice of the splitting data ρ and ι would produce an isomorphic representation [11, proposition 10.9]. The mutation of $\mathcal{M}$ is $\mu_k(\mathcal{M}) := \overline{\mathcal{M}}_{red}$, i.e. it is the reduced part of $\overline{\mathcal{M}}$ and hence a decorated representation of $\mu_k(A, S)$. The right–equivalence class of $\mu_k(\mathcal{M})$ is determined by the right–equivalence class of $\mathcal{M}$ [11, proposition 10.10]. Moreover, $\mu_k^2(\mathcal{M})$ is right–equivalent to $\mathcal{M}$ [11, theorem 10.13].

3.4. **Some mutation–invariants.** Let $\mathcal{M} = (M, V)$ and $\mathcal{N} = (N, W)$ be decorated representations of the same nondegenerate QP (A, S). We consider the following number [12, section 7]

$$(11) \qquad E^{inj}(\mathcal{M}, \mathcal{N}) = \mathrm{dimHom}_{\mathcal{P}(A,S)}(M, N) + \mathbf{dim}(M) \cdot \mathbf{g}_{\mathcal{N}}$$

where $\mathbf{dim}(M) = (\dim\, M_i)_{i \in Q_0}$ is the dimension vector of the positive part M of $\mathcal{M}$, $\mathbf{g}_{\mathcal{N}}$ is the $\mathbf{g}$–vector of $\mathcal{N}$ whose k–th entry is given by (7) and $\cdot$ denotes the usual scalar product of vectors.

The E–invariant of a QP–representation $\mathcal{M}$ is the number

$$E(\mathcal{M}) := E^{inj}(\mathcal{M}, \mathcal{M}).$$

By [12, theorem 7.1], this number is invariant under mutations, i.e. $E(\mu_k(\mathcal{M})) = E(\mathcal{M})$. We notice that $E(\mathcal{S}_k^-, \mathcal{S}_k^-) = 0$ and hence $E(\mathcal{M}) = 0$ for every QP–representation mutation equivalent to $\mathcal{S}_k^-$.

We now recall the homological interpretation of the E–invariant given in [12, section 10]. Let us assume that the QP (A, S) has the following property:

(12) the potential S belongs to the path algebra $R\langle A\rangle$, and the two–sided ideal J_0 of $R\langle A\rangle$ generated by all the cyclic derivatives $\partial_a S$ contains some power m^N of the ideal m.

Under these assumption the Jacobian algebra coincides with $R\langle A\rangle/J_0$ and it is finite dimensional. Condition (12) is satisfied by a rigid QP (A, S) mutation equivalent to a Dynkin quiver in view of the explicit description of all such QPs recalled above.

Let $\mathcal{M} = (M, 0)$ and $\mathcal{N} = (N, 0)$ be two positive representations of a QP (A, S) which satisfy (12). The following formula holds [12, corollary 10.9]

$$E^{inj}(M, N) = \dim\mathrm{Hom}(\tau^{-1}N, M). \tag{13}$$

where τ is the Auslander–Reiten translate.

4. Quiver with potentials and cluster algebras

Let K be the field of complex numbers. Let $n \geq 1$ and let $\mathbf{T}_n(B)$ be a cluster pattern as in section 2 associated with a skew–symmetric $n \times n$ integer matrix B. Let A be a set of arrows such that $B(A) = B$ and let S be a potential in $R\langle\langle A\rangle\rangle_{cyc}$ such that (A, S) is a non–degenerate QP. In this section we recall the following two results from [12]:

(1) for every decorated representation $\mathcal{M}$ of (A, S) there is a corresponding a Laurent polynomial $X_{\mathcal{M}} \in \mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$.
(2) There exists a family $\{\mathcal{M}_{k,t}^{B,t_0}\}$ of decorated representations of (A, S) such that
   - for every $k \in [1, n]$, $\mathcal{M}_{k,t_0}^{B,t_0} = \mathcal{S}_k^- = \mathcal{S}_k^-(A, S)$ (defined in section 3.1);
   - for every vertex $t \in \mathbf{T}_n(B)$ and every index $k \in [1, n]$ we have
   $$X_{\mathcal{M}_{k,t}^{B,t_0}} = X_{k,t}^{B,t_0}$$
   (defined in section 2). In particular, $X_{\mathcal{S}_k^-} = x_k$.

Let us start by recalling 1). Let M be a finite–dimensional (complex) representation of a finite quiver Q. Given a dimension vector $\mathbf{e} \in \mathbb{Z}^n_{\geq 0}$, the quiver Grassmannian $Gr_{\mathbf{e}}(M)$ is the collection of all subrepresentations of M of dimension vector $\mathbf{e}$. It is closed inside the product of the usual Grassmannians $\prod_{i\in Q_0} Gr_{e_i}(M_i)$ and it is hence a projective variety. We denote by $\chi(Gr_{\mathbf{e}}(M))$ its Euler–Poincaré characteristic. The F–polynomial $F_{\mathcal{M}}$ of a QP–representation $\mathcal{M} = (A, S, M, V)$ is defined as the generating function of $\chi(Gr_{\mathbf{e}}(M))$:

$$F_M(y_1, \dots, y_n) = \sum_{\mathbf{e}} \chi(Gr_{\mathbf{e}}(M)) y_1^{e_1} \cdots y_n^{e_n}. \tag{14}$$

In particular, the F–polynomial of a negative QP–representation is 1. In [12, proposition 3.2], it is shown that

$$F_{\mathcal{M}\oplus\mathcal{M}'} = F_{\mathcal{M}} F_{\mathcal{M}'}. \tag{15}$$

Let $\mathbf{b}_1, \dots, \mathbf{b}_n$ be the columns of the matrix B. The desired Laurent polynomial $X_{\mathcal{M}}$ is defined by:

$$X_{\mathcal{M}} = F_{\mathcal{M}}(\mathbf{x}^{\mathbf{b}_1}, \dots, \mathbf{x}^{\mathbf{b}_n})\mathbf{x}^{\mathbf{g}_{\mathcal{M}}}, \tag{16}$$

where $\mathbf{g}_{\mathcal{M}}$ is the $\mathbf{g}$–vector of $\mathcal{M}$ defined in (7) and we use the notation $\mathbf{x}^{\mathbf{c}} = x_1^{c_1} \cdots x_n^{c_n}$ for $\mathbf{c} = (c_1, \dots, c_n)$. In particular, we can rewrite (16) as follows:

$$X_{\mathcal{M}} = \sum_{\mathbf{e}} \chi(Gr_{\mathbf{e}}(M))\mathbf{x}^{\mathbf{g}_M + B\mathbf{e}}. \tag{17}$$

From (15) and (8) it follows that the map $\mathcal{M} \mapsto X_{\mathcal{M}}$ has the following property:

$$X_{\mathcal{M}\oplus\mathcal{M}'} = X_{\mathcal{M}} X_{\mathcal{M}'} \tag{18}$$

for every decorated representation $\mathcal{M}$ and $\mathcal{M}'$ of (A, S). Moreover, it can be shown that the map $\mathcal{M} \mapsto X_{\mathcal{M}}$ is constant on right–equivalence classes of QP–representations.

The expression (17) is a sum of Laurent monomials and hence it is a Laurent polynomial in the variables $x_1, \dots, x_n$. Its reduced form is hence a rational function whose denominator is a monomial $x_1^{d_1} \cdots x_n^{d_n}$. The integer vector

$\mathbf{d}(X_{\mathcal{M}}) = (d_1, \dots, d_n)$ is called the denominator vector of $X_{\mathcal{M}}$. For example, $\mathbf{d}(X_{\mathcal{S}_k^-}) = \mathbf{d}(1/x_k^{-1}) = (0, \dots, -1, \dots, 0)$, -1 at the k–th position. In [12, corollary 5.5], it is shown that

$$d_i \leq dim(M_i) \quad \text{for every } i \in [1, n]. \tag{19}$$

Let us recall 2). Let t be a vertex of $\mathbf{T}_n(B)$ and let $\Sigma_t = (B', \mathbf{u})$ be the corresponding seed (see section 2). Since $\mathbf{T}_n(\Sigma)$ is a tree, there is a unique path

$$t_0 \overset{k_1}{\longrightarrow} t_1 \overset{k_2}{\longrightarrow} \cdots \overset{k_{m-1}}{\longrightarrow} t_{m-1} \overset{k_m}{\longrightarrow} t_m = t$$

which connects t_0 with t. Let

$$(A_t, S_t) = \mu_{k_m} \circ \cdots \circ \mu_{k_2} \circ \mu_{k_1}(A, S). \tag{20}$$

In particular (A_t, S_t) is a non–degenerate QP and $B(A_t) = B'$. Recall that $X_{k,t}^{B,t_0}$ denotes the Laurent expansion of the k–th cluster variable u_k of Σ_t in the seed $\Sigma_{t_0} = (B, \mathbf{x})$. We have

$$u_k = X_{k,t}^{B,t_0} = \mu_{k_m} \circ \cdots \circ \mu_{k_2} \circ \mu_{k_1} x_k$$

Let $\mathcal{S}_k^- = \mathcal{S}_k^-(A_t, S_t)$ be the negative simple representation of (A_t, S_t). We define

$$\mathcal{M}_{k,t}^{B,t_0} := \mu_{k_1} \circ \cdots \circ \mu_{k_{m-1}} \circ \mu_{k_m} \mathcal{S}_k^- \tag{21}$$

By [12, theorem 5.1], $X_{\mathcal{M}_{k,t}^{B,t_0}} = X_{k,t}^{B,t_0}$. The family $\{\mathcal{M}_{k,t}^{B,t_0}\}$ is the desired family.

We notice that by (18) the same description holds for cluster monomials. For instance the Laurent expansion of a cluster monomial $b = u_1^{a_1} \cdots u_n^{a_n}$ is given by

$$b = X_{\mathcal{M}_{1,t}^{B,t_0 \oplus a_1} \oplus \mathcal{M}_{2,t}^{B,t_0 \oplus a_2} \oplus \cdots \oplus \mathcal{M}_{n,t}^{B,t_0 \oplus a_n}} \tag{22}$$

and we define

$$\mathcal{M}_b^{B,t_0} = \mathcal{M}_{1,t}^{B,t_0 \oplus a_1} \oplus \mathcal{M}_{2,t}^{B,t_0 \oplus a_2} \oplus \cdots \oplus \mathcal{M}_{n,t}^{B,t_0 \oplus a_n}. \tag{23}$$

The representations $\mathcal{M}_{k,t}^{B,t_0}$ have the following remarkable property: in view of (21), $E(\mathcal{M}_{k,t}^{B,t_0}) = 0$ for every k and t. For a QP–representation $\mathcal{M}$ such that $E(\mathcal{M}) = 0$ we have [12, corollary 5.5]:

$$\text{either } M_i = \{0\} \text{ or } V_i = \{0\} \text{ for every } i \in [1, n]. \tag{24}$$

In particular, this holds for the family $\{\mathcal{M}_{k,t}^{B,t_0}\}$.

It is remarkable that, while the definition of the family $\{\mathcal{M}_{k,t}^{B,t_0}\}$ depends on the choice of a non–degenerate QP (A,S), the cluster algebra $\mathcal{A}(B)$ only depends on the initial exchange matrix $B = B(A)$. In general, two potentials S and S' on A such that (A, S) and (A, S') are non–degenerate can be very different and give rise to non–isomorphic Jacobian algebras.

5. Proof of theorem 1.1

In view of [6, corollary 3], the set $\mathbf{B}$ of cluster monomials form a $\mathbb{Z}$–basis of $\mathcal{A}(H)$. In view of [20, 18, 21] the cluster monomials are positive. We hence prove that $\mathbf{B}$ is the atomic basis of $\mathcal{A}(H)$ i.e. positive elements of $\mathcal{A}(H)$ are non–negative linear combinations of cluster monomials. We say that a Laurent monomial $x_1^{a_1} \cdots x_n^{a_n}$ in some variables $x_1, \dots, x_n$ is *proper* if there is at least an index i such that $a_i < 0$. A shown in [22] the following lemma implies theorem 1.1.

Lemma 5.1. *For every cluster $\mathcal{C}$ of $\mathcal{A}(H)$ and every cluster monomial b which is not a monomial in the elements of $\mathcal{C}$, the expansion of b in $\mathcal{C}$ is a sum of proper Laurent monomials.*

We hence prove lemma 5.1. Let $\mathbf{T}_n(H)$ be the cluster pattern associated with H (see section 2). In particular, n denotes the number of vertices of H. Let $\Sigma_0 = (H, \mathbf{x})$ be the initial seed at some vertex s of $\mathbf{T}_n(H)$. Let t_0 be a vertex of $\mathbf{T}_n(H)$ and let $\Sigma_{t_0} = (B, \mathbf{u})$ be the corresponding seed in $\mathbf{T}_n(H)$. To such a vertex, there is also associated a QP $(A, S) = (A_{t_0}, S_{t_0})$, which is mutation equivalent to the QP $(H, 0)$ by (20). In section 3.2 we have recalled the explicit description of (A, S) and we have noticed that it is rigid and hence non–degenerate.

Let b be a cluster monomial of $\mathcal{A}(H)$ and let $\mathcal{M}_b^{B,t_0}$ be the corresponding decorated representation of (A, S) given by (23). We show that the Laurent polynomial $X_{\mathcal{M}_b}^{B,t_0} = b$ given by (22) is a sum of proper Laurent monomials.

Let us first consider the case when $\mathcal{M}_b^{B,t_0} = (M, V)$ is not a positive representation, i.e. there is an $i \in Q_0$ such that $V_i \neq \{0\}$. Therefore the monomial b has the form $x_i b'$ for another cluster monomial b'. For such an index i, since $E(\mathcal{M}_b^{B,t_0}) = 0$ and in view of (24), we have that $M_i = \{0\}$. In view of (19), the i–th entry d_i of the denominator vector of $X_{\mathcal{M}_b^{B,t_0}}$ is zero as well. It follows that if we prove the lemma for b' then the lemma holds also for b.

We hence assume that $\mathcal{M}_b^{B,t_0}$ is a positive representation of (A, S), i.e. $\mathcal{M}_b^{B,t_0} = (M, 0)$ and M is a finite–dimensional $\mathcal{P}(A, S)$–module. In view of (17), we have

$$X_{\mathcal{M}_b^{B,t_0}} = \sum_{\mathbf{e}} \chi(Gr_{\mathbf{e}}(M))\mathbf{x}^{\mathbf{g}_M + B\mathbf{e}}.$$

We show that if there exists a non–zero subrepresentation N of M of dimension vector $\mathbf{e}$ then the vector $\mathbf{g}_M + B\mathbf{e}$ has at least one negative entry. Since B is skew–symmetric, the scalar product $\mathbf{e} \cdot (\mathbf{g}_M + B\mathbf{e}) = \mathbf{e} \cdot \mathbf{g}_M$. We hence show that the number $\mathbf{e} \cdot \mathbf{g}_M$ is negative. In view of (13) we have

$$E(M) := \mathbf{dim}(M) \cdot \mathbf{g}_M + \mathrm{dimHom}(M, M) = 0 = \mathrm{dimHom}(\tau^{-1}M, M).$$

Moreover, again by (13), we have

$$E^{inj}(N, M) := \mathbf{e} \cdot \mathbf{g}_M + \mathrm{dimHom}(N, M) = \mathrm{dimHom}(\tau^{-1}M, N)$$

Since N is a subrepresentation of M, there is an injection

$$\mathrm{Hom}(\tau^{-1}M, N) \to \mathrm{Hom}(\tau^{-1}M, M)$$

and so $E^{inj}(N, M) \leq E(M) = 0$. It follows that $E^{inj}(N, M) = 0$ and hence $\mathbf{e} \cdot \mathbf{g}_M = -\mathrm{dimHom}(N, M) < 0$ as desired. It remains to discuss the case $\mathbf{e} = 0$. We hence prove that the vector $\mathbf{g}_{\mathcal{M}}$ has at least one negative entry. We take the scalar product $\mathbf{dim}(M) \cdot \mathbf{g}_{\mathcal{M}} = -\mathrm{dimHom}(M, M) < 0$, as desired.

Remark 5.1. *In view of the separation formula* [16, corollary 6.3] *and of the explicit expression for the F–polynomials and the* $\mathbf{g}$*–vectors of a cluster monomial in every cluster (recalled in section 4) we can prove lemma 5.1 in a cluster algebra* $\mathcal{A}_{\mathbb{P}}(H)$ *with arbitrary coefficients* $\mathbb{P}$*. We know that cluster monomials of* $\mathcal{A}_{\mathbb{P}}(H)$ *are positive (i.e. their Laurent expansions in every cluster have coefficients in* $\mathbb{Z}_{\geq 0}\mathbb{P}$*). If we assume that the they form a* $\mathbb{ZP}$*–basis of* $\mathcal{A}_{\mathbb{P}}(H)$ *then they are an atomic basis for the same reasons as in the coefficient–free setting (see* [22]*).*

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