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BLOCKS OF GROUP ALGEBRAS ARE DERIVED SIMPLE

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ABSTRACT. A derived version of Maschke's theorem for finite groups is proved: the derived categories, bounded or unbounded, of all blocks of the group algebra of a finite group are simple, in the sense that they admit no nontrivial recollements. This result is independent of the characteristic of the base field.

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1. INTRODUCTION

Let G be any finite group and k a field. An indecomposable algebra direct summand B of the group algebra kG is called a *block*. By Maschke's theorem, all blocks of kG are simple algebras if and only if the characteristic of k does not divide the order of G. The main result of this article is a general statement about blocks that does not need any assumption on the characteristic of k:

Main Theorem. *All blocks B of kG are derived simple. More precisely, the bounded derived category $\mathcal{D}^b(\text{mod-}B)$ of finitely generated B-modules, as well as the unbounded derived category $\mathcal{D}(\text{Mod-}B)$ of all B-modules, are simple in the sense that they do not admit nontrivial recollements by derived categories of the same type.*

Simple algebras are, of course, derived simple in this sense. Thus, when the characteristic does not divide the group order the statement is an immediate corollary of Maschke's theorem.

The context of the main theorem is the following: Recollements of triangulated categories, defined by Beilinson, Bernstein and Deligne [4], can be seen as analogues of short exact sequences of these categories. We focus on (bounded or unbounded) derived categories of finite-dimensional algebras. A derived category is said to be *simple* if it is nonzero and it is not the middle term of a nontrivial recollement of derived categories. Once simpleness has been defined, one can study *stratifications*, i.e. ways of breaking up a given derived category into simple pieces using recollements. They are analogues of composition series for groups/modules. Then the question arises which objects are simple and if a Jordan–Hölder theorem holds true, that is, whether finite stratifications exist and are unique. Various positive and negative results recently have been found, see [1, 2, 6, 7]. The main theorem provides a large and quite natural supply of derived simple algebras as well as a derived Jordan–Hölder theorem for group algebras:

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Corollary. *Let G be a finite group and kG the group algebra. Then any stratification of* $\mathcal{D}$(Mod-kG) *(respectively,* $\mathcal{D}^b$(mod-kG)*) is finite. Moreover, the simple factors of any two stratifications are the same: they are precisely the derived categories of the blocks of kG.*

More generally we will prove derived simpleness for larger classes of algebras. In the case of the category $\mathcal{D}^b$(mod) we will show that all indecomposable symmetric algebras are derived simple. In the more difficult case of $\mathcal{D}$(Mod) we will show that indecomposable symmetric algebras with 'enough' cohomology (in a sense made precise below) are derived simple. Blocks of group algebras as well as indecomposable symmetric algebras of finite representation type do satisfy this condition.

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2. Recollements and derived simpleness

Let k be a field. For a finite-dimensional k-algebra A, we denote by $\mathcal{D}(\text{Mod-}A)$ the derived category of (right) A-modules, by $\mathcal{D}^b(\text{mod-}A)$ the bounded derived category of finitely generated A-modules, and by $K^b(P_A)$ the homotopy category of bounded complexes of finitely generated projective A-modules. Objects in $K^b(P_A)$ will be called *compact* complexes. We often view $K^b(P_A)$ as a triangulated full subcategory of the other two categories, and view $\mathcal{D}^b(\text{mod-}A)$ as the triangulated full subcategory of $\mathcal{D}(\text{Mod-}A)$ consisting of complexes whose total cohomology space is finite-dimensional. By abuse of notation we write $\text{Hom}_A(-,-)$ for both $\text{Hom}_{\mathcal{D}^b(\text{mod}-A)}(-,-)$ and $\text{Hom}_{\mathcal{D}(\text{Mod}-A)}(-,-)$.

A *recollement* [4] of triangulated categories is a diagram of triangulated categories and triangle functors

$$\mathcal{C}' \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*=i_!\;} \\ \xleftarrow{\;i^!\;} \end{array} \mathcal{C} \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^!=j^*\;} \\ \xleftarrow{\;j_*\;} \end{array} \mathcal{C}''$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$, (j^*, j_*) are adjoint pairs;
(2) i_*, j_*, $j_!$ are full embeddings;
(3) $i^! \circ j_* = 0$ (and thus also $j^! \circ i_! = 0$ and $i^* \circ j_! = 0$);
(4) for each $C \in \mathcal{C}$ there are triangles

$$i_!i^!(C) \to C \to j_*j^*(C) \to i_!i^!(C)[1],$$
$$j_!j^!(C) \to C \to i_*i^*(C) \to j_!j^!(C)[1].$$

We are particularly interested in recollements of the following two forms

$$\mathcal{D}^b(\text{mod-}B) \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*=i_!\;} \\ \xleftarrow{\;i^!\;} \end{array} \mathcal{D}^b(\text{mod-}A) \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^!=j^*\;} \\ \xleftarrow{\;j_*\;} \end{array} \mathcal{D}^b(\text{mod-}C) \qquad \text{(R1)}$$

and

$$\mathcal{D}(\text{Mod-}B) \begin{array}{c} \xleftarrow{\;i^*\;} \\ \xrightarrow{\;i_*=i_!\;} \\ \xleftarrow{\;i^!\;} \end{array} \mathcal{D}(\text{Mod-}A) \begin{array}{c} \xleftarrow{\;j_!\;} \\ \xrightarrow{\;j^!=j^*\;} \\ \xleftarrow{\;j_*\;} \end{array} \mathcal{D}(\text{Mod-}C) \qquad \text{(R2)}$$

with A, B and C being finite-dimensional algebras over k. By [2, Corollary 2.3] and [11, 5.2.9], a recollement of the form (R1) always implies the existence of a recollement of the form (R2).

Lemma 2.1. *Let A be a finite-dimensional k-algebra admitting a recollement of the form (R2). Then $j_!j^!(A)$, $i_*i^*(A)$ and $i_*(B)$ all belong to $\mathcal{D}^b(\text{mod-}A)$.*

Proof. There is a canonical triangle

$$j_!j^!(A) \to A \to i_*i^*(A) \to j_!j^!(A)[1],$$

which yields an isomorphism $\text{Hom}_A(A, i_*i^*(A)[n]) \cong \text{Hom}_A(i_*i^*(A), i_*i^*(A)[n])$ by applying $\text{Hom}_A(-, i_*i^*(A)[n])$. Note that there is an isomorphism between the n-th cohomology space $H^n(i_*i^*(A))$ of $i_*i^*(A)$ and the Hom-space $\text{Hom}_A(A, i_*i^*(A)[n])$ and that i_* is a full embedding. Hence

$$H^n(i_*i^*(A)) \cong \text{Hom}_A(i_*i^*(A), i_*i^*(A)[n]) \cong \text{Hom}_B(i^*(A), i^*(A)[n]).$$

By [11, 4.3.6, 4.4.8], $i^*(A)$ belongs to and generates $K^b(P_B)$ (i.e. $K^b(P_B)$ is the smallest triangulated subcategory of $\mathcal{D}(\text{Mod-}B)$ containing $i^*(B)$ and closed under taking direct summands). Since B is a finite-dimensional algebra, it follows by dévissage that the space of self extensions $\text{Hom}_B(i^*(A), i^*(A)[n])$ of $i^*(A)$ is finite-dimensional and vanishes for all but finitely many integers n. Combining this observation with the above isomorphism, we obtain that $i_*i^*(A)$ has finite-dimensional total cohomology space. Therefore it belongs to $\mathcal{D}^b(\text{mod-}A)$. So do $j_!j^!(A)$, thanks to the canonical triangle, and as well as $i_*(B)$, because $i_*(B)$ and $i_*i^*(A)$ generate each other in finitely many steps. □

Derived simpleness of a finite-dimensional k-algebra A was introduced by Wiedemann [14] (see also [1]). By definition A is said to be *derived simple* with respect to $\mathcal{D}^b(\text{mod})$ respectively $\mathcal{D}(\text{Mod})$, if A is nontrivial and there are no nontrivial recollements of the form (R1) respectively (R2), namely, none of the full embedding i_*, $j_!$ and $j_!$ is a triangle equivalence. We also say that A is $\mathcal{D}^b(\text{mod})$*-simple* respectively $\mathcal{D}(\text{Mod})$*-simple* for short.

A $\mathcal{D}(\text{Mod})$-simple algebra is always $\mathcal{D}^b(\text{mod})$-simple since, as mentioned above, a recollement of the form (R1) always induces a recollement of the form (R2). The converse is in general not true, an example can be found in [3].

An algebra is said to be *indecomposable* if it is not isomorphic to a direct product of two nonzero algebras. Clearly, if an algebra is decomposable, then a nontrivial decomposition of the algebra yields a nontrivial recollement. Hence a decomposable algebra is never derived simple in any sense.

3. The bounded case

Let A be a finite-dimensional k-algebra. Recall that A is said to be a *symmetric algebra*, if DA is isomorphic to A as A-A-bimodules, where $D = \text{Hom}_k(-, k)$ is the k-dual. In particular, an A-module is projective if and only if it is injective. For equivalent definitions of symmetric algebras see Curtis and Reiner [8]. Group algebras of finite groups form an important class of symmetric algebras.

Lemma 3.1 ([12] Corollary 3.2)**.** *Let A be a symmetric finite-dimensional k-algebra. Then there is a bifunctorial isomorphism*

$$D\operatorname{Hom}_A(P,M)\cong \operatorname{Hom}_A(M,P)$$

for $P\in K^b(P_A)$, and $M\in\mathcal{D}(\text{mod-}A)$.

The main result of this section is the following.

Theorem 3.2. *A finite-dimensional indecomposable symmetric k-algebra is $\mathcal{D}^b(mod)$-simple.*

Proof. Let A be a finite-dimensional symmetric k-algebra and assume that there exists a nontrivial recollement of the form (R1). By [2, Corollary 2.3], $j_!j^!(A)$ and $i_*i^*(A)$ belong to $K^b(P_A)$. Now apply Lemma 3.1 to $P=j_!j^!(A)$ and $M=i_*i^*(A)[n]$ $(n\in\mathbb{Z})$. We obtain

$$D\operatorname{Hom}_A(j_!j^!(A),i_*i^*(A)[n])\cong \operatorname{Hom}_A(i_*i^*(A)[n],j_!j^!(A))$$

for all integers n. The left hand side always vanishes as $j^*i_*=0$, and hence so does the right hand side. This means $i_*i^*(A)$ and $j_!j^!(A)$ are orthogonal to each other and the canonical triangle

$$j_!j^!(A)\to A\to i_*i^*(A)\to j_!j^!(A)[1]$$

splits, i.e. $A\cong j_!j^!(A)\oplus i_*i^*(A)$. In particular, $A=\operatorname{End}_A(A)$ is decomposed as the product of $\operatorname{End}_A(j_!j^!(A))$ and $\operatorname{End}_A(i_*i^*(A))$. □

Remark. Lemma 3.1 says that the triangulated category $K^b(P_A)$ is 0-Calabi–Yau. The above proof shows that this 0-Calabi–Yau triangulated category is simple, in the sense that it admits no nontrivial recollements of triangulated categories. Notice that in the proof the Calabi–Yau dimension is not important. Thus with the same proof one shows that an indecomposable d-Calabi–Yau triangulated category is simple $(d\in\mathbb{Z})$[1]. In fact, it admits no nontrivial stable t-structures. Cluster categories of connected quivers are examples of indecomposable 2-Calabi-Yau categories, and hence are simple.

As a corollary, we establish a derived Jordan–Hölder theorem for symmetric algebras, which is on the existence and uniqueness of finite stratifications. Roughly speaking, a *stratification* is a way of breaking up a given derived category into simple pieces using recollements. More rigorously, a stratification is a full rooted binary tree whose root is the given derived category, whose nodes are derived categories and whose leaves are simple (they are called the *simple factors* of the stratification) such that a node is a recollement of its two child nodes unless it is a leaf. For a finite-dimensional algebra, a *block* is an indecomposable algebra direct summand. The number of blocks is a derived invariant.

Corollary 3.3. *Let A be a finite-dimensional symmetric algebra. Then any stratification of $\mathcal{D}^b(\text{mod-}A)$ is finite. Moreover, the simple factors of any two stratifications are the same: they are precisely the bounded derived categories of the blocks of A.*

[1]We thank Bernhard Keller for pointing out this to us.

Proof. Suppose the algebra A has s blocks A_i with $A = \bigoplus_{i=1}^s A_i$. Suppose a recollement of $\mathcal{D}^b(\text{mod-}A)$ of the form (R1) is given. The block decomposition of A yields a decomposition of its derived category: $\mathcal{D}^b(\text{mod-}A) = \bigoplus_i \mathcal{D}^b(\text{mod-}A_i)$. In particular, $j_!(C)$ is a direct sum $\bigoplus_i X_i$ of $X_i \in \mathcal{D}^b(\text{mod-}A_i)$, and hence C, being isomorphic to $\text{End}_A(j_!(C))$, admits a block decomposition $C = \bigoplus_i C_i$ such that $j_!(C_i) = X_i$. Similarly, the algebra B admits a block decomposition $B = \bigoplus_i B_i$ such that $i_*(B_i) \in \mathcal{D}^b(\text{mod-}A_i)$. Fix an $i = 1, \ldots, s$. For an indecomposable object $M \in \mathcal{D}^b(\text{mod-}C_i)$, considered as an object in $\mathcal{D}^b(\text{mod-}C)$, there exists an $n \in \mathbb{Z}$ such that $\text{Hom}_C(C_i, M[n]) \neq 0$. Since $j_!$ is fully faithful, it follows that $\text{Hom}_A(X_i, j_!(M)[n]) \neq 0$, implying that $j_!(M) \in \mathcal{D}^b(\text{mod-}A_i)$. Therefore, $j_!$ restricts to a triangle functor $j_! : \mathcal{D}^b(\text{mod-}C_i) \to \mathcal{D}^b(\text{mod-}A_i)$. Similarly, i_* restricts to a triangle functor $i_* : \mathcal{D}^b(\text{mod-}B_i) \to \mathcal{D}^b(\text{mod-}A_i)$. For an object $N \in \mathcal{D}^b(\text{mod-}A_i)$, we have $\text{Hom}_C(C_j, j^*(N)) = \text{Hom}_A(j_!(C_j), N) = \text{Hom}_A(X_j, N) = 0$ for $j \neq i$, implying that $j^*(N) \in \mathcal{D}^b(\text{mod-}C_i)$. Therefore, j^* restricts to a triangle functor $j^* : \mathcal{D}^b(\text{mod-}A_i) \to \mathcal{D}^b(\text{mod-}C_i)$. Similarly, j_*, i^* and $i^!$ respectively restricts to triangle functors $j_* : \mathcal{D}^b(\text{mod-}C_i) \to \mathcal{D}^b(\text{mod-}A_i)$, $i^* : \mathcal{D}^b(\text{mod-}A_i) \to \mathcal{D}^b(\text{mod-}B_i)$ and $i^! : \mathcal{D}^b(\text{mod-}A_i) \to \mathcal{D}^b(\text{mod-}B_i)$. It follows that the diagram below is a recollement

$$\mathcal{D}^b(\text{mod-}B_i) \begin{array}{c} \xleftarrow{i^*} \\ \xrightarrow{i_*=i_!} \\ \xleftarrow{i^!} \end{array} \mathcal{D}^b(\text{mod-}A_i) \begin{array}{c} \xleftarrow{j_!} \\ \xrightarrow{j^!=j^*} \\ \xleftarrow{j_*} \end{array} \mathcal{D}^b(\text{mod-}C_i).$$

By Theorem 3.2, the blocks A_i are derived simple. Therefore for each i, either $B_i = A_i$ and $C_i = 0$ or $B_i = 0$ and $C_i = A_i$, up to derived equivalence. In particular, up to derived equivalence, B and C are algebra direct summands of A. Therefore, the desired result follows by induction on the number s of blocks of A. □

4. The unbounded case

In this section we partially generalise Theorem 3.2 to the unbounded derived category. Let A be a finite-dimensional k-algebra. Consider the following condition

(#) for any finitely generated non-projective A-module M, there are infinitely many integers n with $\text{Ext}_A^n(M, M) \neq 0$.

We will prove that an indecomposable finite-dimensional symmetric algebra satisfying (#) is $\mathcal{D}(\text{Mod})$-simple.

Here is an example of a symmetric algebra which satisfies (#). Let A be the quotient of the path algebra of the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

by the ideal generated by $\alpha\beta\alpha$ and $\beta\alpha\beta$. Up to isomorphism A has four indecomposable non-projective modules given by the following representations

$$k \rightleftarrows 0\ , \quad 0 \rightleftarrows k\ , \quad k \underset{1}{\overset{0}{\rightleftarrows}} k\ , \quad k \underset{0}{\overset{1}{\rightleftarrows}} k\ .$$

It is easy to check that

$$\operatorname{Ext}^n_A(M,M) \;=\; \begin{cases} k & \text{if } n\geq 0 \text{ and } n\equiv 0,3 \pmod 4,\\ 0 & \text{otherwise,}\end{cases}$$

for each of the above modules M.

More generally, the condition (#) is satisfied by the following two classes of algebras, by [13, Proposition 3.2, Example 3.1]

(1) group algebras of finite groups over k;
(2) self-injective k-algebras of finite representation type.

Recall that a finite-dimensional algebra is said to be *self-injective* if all projective modules are injective.

Proposition 4.1. *Let A be a finite-dimensional self-injective k-algebra satisfying the condition (#). Assume that A admits a nontrivial recollement of the form (R2). Then $i_*i^*(A)$ belongs to $K^b(P_A)$.*

Proof. Let A be a finite-dimensional k-algebra satisfying the condition (#). We claim that for any $X\in\mathcal{D}^b(\text{mod-}A)$ we have either $X\in K^b(P_A)$ or there are infinitely many integers n such that $\operatorname{Hom}_A(X,X[n])\neq 0$. By Lemma 2.1, the object $i_*i^*(A)$ belongs to $\mathcal{D}^b(\text{mod-}A)$, and hence $\operatorname{Hom}_A(i_*i^*(A),i_*i^*(A)[n])\cong H^n(i_*i^*(A))$ vanishes for all but finitely many $n\in\mathbb{Z}$. Therefore $i_*i^*(A)$ must be in $K^b(P_A)$.

Next we prove the claim. Let $X\in\mathcal{D}^b(\text{mod-}A)$. Without loss of generality, we assume that X is a minimal complex of finitely generated projective A-modules which is bounded from the right. Here minimality means that for any $p\in\mathbb{Z}$ the image of each differential $d^p: X^p\to X^{p+1}$ lies in the radical of X^{p+1}. Since X has bounded cohomology, there is an integer N such $H^n(X)=0$ for $n\leq N$. Up to shift, we may assume that $N=0$. Let X'' be the subcomplex of X with $(X'')^n=X^n$ for $n>0$ and $(X'')^n=0$ for $n\leq 0$ and let X' be the corresponding quotient complex. Then there is a triangle

$$X''\to X\to X'\to X''[1].$$

Note that $X''\in K^b(P_A)$, and X' has cohomology concentrated in degree 0, and hence it is the minimal projective resolution of a finitely generated A-module, say M.

Case 1: M is projective. This implies that X' is a stalk complex, and hence $X\in K^b(P_A)$.

Case 2: M is not projective. The above triangle gives us two long exact sequences

$$\ldots\to\operatorname{Hom}_A(X,X''[n])\to\operatorname{Hom}_A(X,X[n])\to\operatorname{Hom}_A(X,X'[n])\to\ldots$$

$$\ldots\to\operatorname{Hom}_A(X',X'[n])\to\operatorname{Hom}_A(X,X'[n])\to\operatorname{Hom}_A(X'',X'[n])\to\ldots$$

Recall that $X''\in K^b(P_A)$ is also a bounded complex of finitely generated injective modules. Therefore there exist only finitely many integers n such that $\operatorname{Hom}_A(X,X''[n])\neq 0$ (respectively, $\operatorname{Hom}_A(X'',X'[n])\neq 0$). So from the above two long exact sequences we see that

$$\operatorname{Hom}_A(X,X[n])\cong\operatorname{Hom}_A(X,X'[n])\cong\operatorname{Hom}_A(X',X'[n])$$

for all but finitely many integers n. Now the claim follows from the condition (#) since $\mathrm{Hom}_A(X', X'[n]) = \mathrm{Ext}^n_A(M, M)$. □

Now we are ready to prove the main results of this section.

Theorem 4.2. *Let A be a finite-dimensional symmetric k-algebra satisfying the condition (#). If A is indecomposable, then it is $\mathcal{D}(\mathrm{Mod})$-simple.*

Proof. Let A be finite-dimensional symmetric satisfying the condition (#). It follows from Proposition 4.1 that $i_*i^*(A)$ belongs to $K^b(P_A)$. Now we proceed as in the proof of Theorem 3.2 to show that A is decomposable. □

Corollary 4.3. *The following two classes of finite-dimensional symmetric algebras are $\mathcal{D}(\mathrm{Mod})$-simple:*

(1) *blocks of group algebras of finite groups over k;*
(2) *indecomposable symmetric k-algebras of finite representation type.*

Proof. This is an immediate consequence of Theorem 4.2. For completeness, we give a proof for the assertion that group algebras of finite groups satisfy the condition (#). Let G be a finite group and let $A = kG$ be the group algebra. For a finitely generated A-module M, define

$$\mathrm{Ext}^{\cdot}(M, M) = \begin{cases} \bigoplus_{n\geq 0} \mathrm{Ext}^n_A(M, M) & \text{if } \mathrm{char}(k) = 2, \\ \bigoplus_{n\geq 0} \mathrm{Ext}^{2n}_A(M, M) & \text{otherwise.} \end{cases}$$

Let k also denote the trivial module. Then the graded k-algebra $H^{\cdot}(G, k) = \mathrm{Ext}^{\cdot}(k, k)$ is commutative Noetherian. Let M be a finitely generated A-module. The tensor product $-\otimes_k M$ induces an algebra homomorphism

$$\varphi : H^{\cdot}(G, k) \to \mathrm{Ext}^{\cdot}(M, M).$$

It is known that if M is non-projective, then the Krull dimension of $H^{\cdot}(G, k)/\ker(\varphi)$ is greater than or equal to 1 (see for example [5, Section 2.24, 2.25]), which implies that it is infinite-dimensional. As a consequence, $\mathrm{Ext}^{\cdot}(M, M)$ is infinite-dimensional, implying the condition (#). □

We have a $\mathcal{D}(\mathrm{Mod})$-counterpart of Corollary 3.3: an unbounded derived Jordan–Hölder theorem for symmetric algebras satisfying the condition (#).

Corollary 4.4. *Let A be a finite-dimensional symmetric algebra satisfying the condition (#). Then any stratification of $\mathcal{D}(\mathrm{Mod}\text{-}A)$ is finite. Moreover, the simple factors of any two stratifications are the same: they are precisely the derived categories of the blocks of A.*

Proof. Similar to the proof for Corollary 3.3. □

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