QUIVER HECKE SUPERALGEBRAS

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Abstract. We introduce a new family of superalgebras which should be considered as a super version of the Khovanov-Lauda-Rouquier algebras. Let I be the set of vertices of a Dynkin diagram with a decomposition $I = I_{\mathrm{even}} \sqcup I_{\mathrm{odd}}$. To this data, we associate a family of graded superalgebras R_n, the *quiver Hecke superalgebras*. When $I_{\mathrm{odd}} = \emptyset$, these algebras are nothing but the usual Khovanov-Lauda-Rouquier algebras. We then define another family of graded superalgebras RC_n, the quiver Hecke-Clifford superalgebras, and show that the superalgebras R_n and RC_n are weakly Morita superequivalent to each other. Moreover, we prove that the affine Hecke-Clifford superalgebras, as well as their degenerate version, the affine Sergeev superalgebras, are isomorphic to quiver Hecke-Clifford superalgebras RC_n after a completion.

1. Introduction

In [KL1, KL2, Rou1], Khovanov-Lauda and Rouquier independently introduced a remarkable family of graded algebras, the *Khovanov-Lauda-Rouquier algebras* or the *quiver Hecke algebras*, that categorifies the negative half of quantum groups associated with symmetrizable Kac-Moody algebras. Moreover, the cyclotomic quotients of Khovanov-Lauda-Rouquier algebras provide a categorification of integrable highest weight modules over quantum groups [KL1, KK]. An important application of the Khovanov-Lauda-Rouquier algebras is that one can derive a homogeneous presentation of the symmetric group algebras, which gives a quantization of Ariki's categorification theorem for the basic $\widehat{\mathfrak{sl}}_p$-module ([Ari, LLT, BK1, BK2, Rou1]):

$$V(\Lambda_0) \cong \bigoplus_{n\geq 0} \mathsf{K}_0(\mathbb{F}_p\mathfrak{S}_n\text{-}\mathsf{mod})_{\mathbb{C}}.$$

Considering the long history of the theory of symmetric groups, it is quite surprising that one can define such a non-trivial grading on the symmetric group algebras, which has been conjectured to exist for some time ([Rou2, Tur]), only after the discovery of Khovanov-Lauda-Rouquier algebras.

For the Dynkin diagrams of affine type A or of type A_∞, the Khovanov-Lauda-Rouquier algebras (resp. the cyclotomic Khovanov-Lauda-Rouquier algebras) are isomorphic to the affine Hecke algebras of type A (resp. the cyclotomic Hecke algebras or the Ariki-Koike

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algebras) after we take specializations and localizations. Thus the Khovanov-Lauda-Rouquier algebras can be regarded as a generalization of the affine Hecke algebras of type A. Since the Khovanov-Lauda-Rouquier algebras are graded, they fit more naturally into the categorification of quantum groups.

Indeed, in [VV1], Varagnolo and Vasserot showed that the Khovanov-Lauda-Rouquier algebras associated with symmetric generalized Cartan matrices are isomorphic to the Yoneda algebras of certain complexes of constructible sheaves, and proved that the sets of isomorphism classes of irreducible modules over the Khovanov-Lauda-Rouquier algebras correspond to Kashiwara's upper global bases (=Lusztig's dual canonical bases). In the same spirit, in [VV1], Varagnolo and Vasserot (resp. in [SVV], Shan, Varagnolo and Vasserot) introduced a version of Khovanov-Lauda-Rouquier algebras corresponding to the affine Hecke algebras of type B and C (resp. type D) and proved the Lascoux-Leclerc-Thibon-Ariki type conjecture formulated in [EK] (resp. in [KM]).

The purpose of this paper is to introduce a new family of graded superalgebras which should be considered as a super version of the Khovanov-Lauda-Rouquier algebras. Let $\mathbf{k}$ be a commutative ring with 1, and let $\big(a_{i,j}\big)_{i,j\in I}$ be a symmetrizable generalized Cartan matrix indexed by a set I with a decomposition $I = I_{\mathrm{even}} \sqcup I_{\mathrm{odd}}$ of I into the set I_{even} of even vertices and the set I_{odd} of odd vertices such that $a_{i,j} \in 2\mathbb{Z}$ for any $i \in I_{\mathrm{odd}}$. To such data we associate (see § 3.6) a family of (skew-)polynomials $Q = (Q_{i,j}(u,v))_{i,j\in I}$ satisfying the conditions given in (3.1), and then we define a family of graded $\mathbf{k}$-superalgebras R_n, the *quiver Hecke superalgebras* (Definition 3.1). When $I_{\mathrm{odd}} = \emptyset$, these algebras are nothing but the usual Khovanov-Lauda-Rouquier algebras.

Let J be an index set with an involution $c\colon J \to J$ and let $\widetilde{Q} = (\widetilde{Q}_{i,j}(u,v))_{i,j\in J}$ be a family of polynomials satisfying the conditions given in (3.5). We define another family of graded $\mathbf{k}$-superalgebras RC_n, *the quiver Hecke-Clifford superalgebras* associated with $\widetilde{Q}$. Let $\sim_c$ be the equivalence relation defined by $j \sim_c j' \Leftrightarrow j = j'$ or $j' = c(j)$, I the set of equivalence classes, and $I_{\mathrm{odd}} \subset I$ the image of $J^c := \{j \in J\,;\, c(j) = j\}$. Then we show that the corresponding superalgebras R_n and RC_n are *weakly Morita superequivalent* to each other (Theorem 3.13). Moreover, we prove that, after a completion, the affine Hecke-Clifford superalgebras are isomorphic to quiver Hecke-Clifford superalgebras RC_n with a suitable choice of $\widetilde{Q}$ (Theorem 4.4). We also show that the affine Sergeev superalgebras, a degenerate version of the affine Hecke-Clifford superalgebras, are isomorphic to quiver Hecke-Clifford superalgebras (Theorem 5.4). Consequently, the affine Hecke-Clifford superalgebras as well as the affine Sergeev superalgebras are weakly Morita superequivalent to the quiver Hecke superalgebras after a completion.

Let q be the defining parameter for the affine Hecke-Clifford superalgebras. In [BK3], Brundan-Kleshchev showed that when q^2 is a primitive $(2\,l+1)$-th root of unity, the representation theory of some blocks of the affine Hecke-Clifford superalgebras as well as that of their cyclotomic quotients is controlled by the representation theory of quantum affine algebras of type $A^{(2)}_{2l}$ at the crystal level. Later, in [Tsu], the third author showed that when q^2 is a primitive $(2l)$-th root of unity, the representation theory of some blocks is controlled by the representation theory of quantum affine algebras of type $D^{(2)}_l$. In other words, the affine Hecke-Clifford superalgebras and their cyclotomic quotients with the above defining parameters give a crystal version of the categorification theorem for the quantum affine algebras of type $A^{(2)}_{2l}$, $D^{(2)}_l$ and their highest weight modules. In this paper, by taking all the finite-dimensional representations of affine Hecke-Clifford superalgebras, we prove that the corresponding algebra are isomorphic to the quiver Hecke-Clifford superalgebras associated with (affine) Dynkin diagrams of type A_∞, B_∞, C_∞, $A_l^{(1)}$, $A_{2l}^{(2)}$, $C_l^{(1)}$ and $D_l^{(2)}$.

In the forthcoming papers, we plan to prove the conjecture that the quiver Hecke superalgebras and their cyclotomic quotients provide a categorification of the quantum Kac-Moody algebras and their highest weight modules.

It is also worthwhile to note that any irreducible supermodule over a quiver Hecke superalgebra seems to remain irreducible after forgetting its super structure, contrary to affine Hecke-Clifford superalgebras or quiver Hecke-Clifford superalgebras. Hence quiver Hecke superalgebras are more suitable for a categorification of quantum groups than quiver Hecke-Clifford superalgebras.

Finally, we would like to emphasize that the superalgebras R_n and RC_n *cannot* be reduced to the usual Khovanov-Lauda-Rouquier algebras. In the following two cases, both (cyclotomic) Khovanov-Lauda-Rouquier algebras and (cyclotomic) quiver Hecke superalgebras should categorify the same weight space of the representation of $U_v(\mathfrak{g})$, but they are neither Morita equivalent nor weakly Morita superequivalent (see § 2.4).

(1) (see Remark 5.6) Consider the case in which $\mathfrak{g} = A_2^{(2)}$ and $\operatorname{char} \mathbf{k} = 3$ with $I = \{0,1\}$, $I_{\mathrm{even}} = \{1\}$, $I_{\mathrm{odd}} = \{0\}$, where 0 is a short root. Let R_β (see (3.18)) be a direct summand of R_{11}, which categorifies $U_v^-(A_2^{(2)})_{-\beta}$ with $\beta = 8\alpha_0+3\alpha_1$. Although the Khovanov-Lauda-Rouquier algebra $\mathsf{KLR}_\beta(A_2^{(2)})$ also categorifies $U_v^-(A_2^{(2)})_{-\beta}$, $\mathsf{Irr}(\mathsf{KLR}_\beta(A_2^{(2)})\text{-}\mathsf{mod})$ and $\mathsf{Irr}(\mathrm{R}_\beta\text{-}\mathsf{mod})$ correspond to different perfect bases in the sense of [BeKa] at the specialization $v = 1$.

(2) (see Remark 5.7) When $\mathfrak{g} = A_1^{(1)}$, $\operatorname{char} \mathbf{k} = 0$ with $I = I_{\mathrm{odd}} = \{0,1\}$. there is no Morita equivalence nor weak Morita superequivalence between the cyclotomic quiver Hecke superalgebra $\mathrm{R}_4^{\Lambda_0}$ and the cyclotomic Khovanov-Lauda-Rouquier algebra $\mathsf{KLR}_4^{\Lambda_0}(A)$ whatever superalgebra structure we give to $\mathsf{KLR}_4^{\Lambda_0}(A)$ and whatever defining parameter of $\mathsf{KLR}_4(A)$ we take.

This paper is organized as follows. In Section 2, we introduce the theory of *supercategories* as a basic language, including the notion of *Clifford twist* and *weak Morita superequivalence*. In Section 3, we define the quiver Hecke superalgebras R_n and the quiver Hecke-Clifford superalgebras RC_n, and show that the superalgebras R_n and RC_n are weakly Morita superequivalent to each other. We also provide PBW-type bases for both superalgebras. In Section 4, we prove that, the affine Hecke-Clifford superalgebras are isomorphic to quiver Hecke-Clifford superalgebras after a completion. The *intertwiners* play an important role in the proof. In Section 5, by a similar argument, we show that the affine Sergeev algebras are isomorphic to quiver Hecke-Clifford superalgebras.

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2. Supercategories

2.1. **Clifford twist.** Let $\mathbf{k}$ be a commutative ring. Let us recall that a $\mathbf{k}$-*linear* category is a category $\mathscr{C}$ such that $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ is endowed with a $\mathbf{k}$-module structure for all $X, Y \in \mathscr{C}$ and the composition map $\mathrm{Hom}_{\mathscr{C}}(Y,Z)\times\mathrm{Hom}_{\mathscr{C}}(X,Y)\to\mathrm{Hom}_{\mathscr{C}}(X,Z)$ is $\mathbf{k}$-bilinear. A functor $F\colon \mathscr{C}\to\mathscr{C}'$ from a $\mathbf{k}$-linear category $\mathscr{C}$ to a $\mathbf{k}$-linear category $\mathscr{C}'$ is called $\mathbf{k}$-linear if $F\colon \mathrm{Hom}_{\mathscr{C}}(X,Y)\to\mathrm{Hom}_{\mathscr{C}'}(FX,FY)$ is $\mathbf{k}$-linear for any objects $X,Y\in\mathscr{C}$. We say that an additive category is *idempotent-complete* if $\ker(p)$ exists for any object X of $\mathscr{C}$ and $p\in\mathrm{End}_{\mathscr{C}}(X)$ such that $p^2=p$. Hence for any such p, we have $X\simeq\ker(p)\oplus\ker(1-p)$.

Definition 2.1.

(i) *A* supercategory *is a category* $\mathscr{C}$ *equipped with an endofunctor* $\Pi_{\mathscr{C}}$ *of* $\mathscr{C}$ *and a natural isomorphism* $\xi_{\mathscr{C}}\colon \Pi_{\mathscr{C}}^2 \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}}$ *such that* $\xi_{\mathscr{C}}\circ\Pi_{\mathscr{C}}=\Pi_{\mathscr{C}}\circ\xi_{\mathscr{C}}\in\mathrm{Hom}(\Pi_{\mathscr{C}}^3,\Pi_{\mathscr{C}})$.

(ii) *For a pair of supercategories* $(\mathscr{C},\Pi,\xi)$ *and* $(\mathscr{C}',\Pi',\xi')$, *a* superfunctor *from* $(\mathscr{C},\Pi,\xi)$ *to* $(\mathscr{C}',\Pi',\xi')$ *is a pair* (F,α) *of a functor* $F\colon\mathscr{C}\to\mathscr{C}'$ *and a natural isomorphism* $\alpha\colon F\circ\Pi\xrightarrow{\sim}\Pi'\circ F$ *such that the diagram*

$$\begin{tikzcd} F\circ\Pi^2 \arrow[r, "\alpha\circ\Pi"] \arrow[d, "F\circ\xi"'] & \Pi'\circ F\circ\Pi \arrow[r, "\Pi'\circ\alpha"] & \Pi'^2\circ F \arrow[d, "\xi'\circ F"] \\ F \arrow[rr, "\mathrm{id}_F"] & & F \end{tikzcd} \tag{2.1}$$

commutes. If F *is an equivalence of categories, we say that* (F,α) *is an* equivalence of supercategories.

(iii) *The notion of* $\mathbf{k}$-linear supercategories *and* $\mathbf{k}$-linear superfunctors *can be defined in a similar way.*

Throughout this paper, *by a supercategory we mean a* $\mathbf{k}$*-linear additive supercategory.*

For $\lambda\in\mathbf{k}^\times$, a supercategory $(\mathscr{C},\Pi,\xi)$ is equivalent to the supercategory $(\mathscr{C},\Pi,\lambda^2\xi)$. For a supercategory $(\mathscr{C},\Pi,\xi)$, its *reversed supercategory* $\mathscr{C}^{\mathrm{rev}}$ is the supercategory $(\mathscr{C},\Pi,-\xi)$. If $\sqrt{-1}$ exists in $\mathbf{k}$, then $\mathscr{C}^{\mathrm{rev}}$ is equivalent to $\mathscr{C}$ as a supercategory.

The *Clifford twist* of a supercategory $(\mathscr{C},\Pi,\xi)$ is the supercategory $(\mathscr{C}^{\mathrm{CT}},\Pi^{\mathrm{CT}},\xi^{\mathrm{CT}})$, where $\mathscr{C}^{\mathrm{CT}}$ is the category whose set of objects is the set of pairs (X,φ) of objects X of $\mathscr{C}$ and isomorphisms $\varphi\colon\Pi X\xrightarrow{\sim}X$ such that

$$\begin{tikzcd} \Pi^2X \arrow[rr, "\xi_X"] \arrow[dr, "\Pi\varphi"'] & & X \\ & \Pi X \arrow[ur, "\varphi"'] & \end{tikzcd} \quad \text{commutes.} \tag{2.2}$$

For a pair (X,φ) and $(X'\varphi')$ of objects of $\mathscr{C}^{\mathrm{CT}}$, $\mathrm{Hom}_{\mathscr{C}^{\mathrm{CT}}}\big((X,\varphi),(X'\varphi')\big)$ is the subset of $\mathrm{Hom}_{\mathscr{C}}(X,X')$ consisting of morphisms $f\colon X\to X'$ such that the following diagram commutes:

$$\begin{tikzcd} \Pi X \arrow[r, "\Pi f"] \arrow[d, "\varphi"'] & \Pi X' \arrow[d, "\varphi'"] \\ X \arrow[r, "f"] & X'. \end{tikzcd}$$

We define $\Pi^{\mathrm{CT}} : \mathscr{C}^{\mathrm{CT}} \to \mathscr{C}^{\mathrm{CT}}$ and $\xi^{\mathrm{CT}} \colon (\Pi^{\mathrm{CT}})^2 \to \mathrm{id}_{\mathscr{C}^{\mathrm{CT}}}$ by

$$\Pi^{\mathrm{CT}}(X, \varphi) = (X, -\varphi),$$
$$\xi^{\mathrm{CT}}_{(X,\varphi)} = \mathrm{id}_{(X,\varphi)} : (\Pi^{\mathrm{CT}})^2(X, \varphi) = (X, \varphi) \to (X, \varphi).$$

Sometimes we simply write $\mathscr{C}$ for $(\mathscr{C}, \Pi, \xi)$, $\mathscr{C}^{\mathrm{rev}}$ for its reversed supercategory and $\mathscr{C}^{\mathrm{CT}}$ for its Clifford twist.

Lemma 2.2. *Let $(\mathscr{C}, \Pi, \xi)$ be a supercategory.*

(i) *There is a canonical superfunctor $\Phi \colon \mathscr{C}^{\mathrm{CT}} \to \mathscr{C}^{\mathrm{rev}}$ defined by*

$$\Phi(X, \varphi) = X,$$
$$\Phi \circ \Pi^{\mathrm{CT}}(X, \varphi) = X \xrightarrow{\varphi^{-1}} \Pi X = \Pi^{\mathrm{rev}} \circ \Phi(X, \varphi).$$

(ii) *Assume that $\mathscr{C}$ is an additive category.*

(a) *There is a canonical superfunctor $\Psi : \mathscr{C}^{\mathrm{rev}} \to \mathscr{C}^{\mathrm{CT}}$ given by*

$$\Psi(X) = (X \oplus \Pi X, \varphi_X),$$
$$\Psi \circ \Pi^{\mathrm{rev}}(X) = \Pi X \oplus \Pi^2 X \xrightarrow{\psi_X} X \oplus \Pi X = \Pi^{\mathrm{CT}} \circ \Psi X,$$

where

$$\varphi_X = \begin{pmatrix} 0 & \xi_X \\ \mathrm{id}_{\Pi X} & 0 \end{pmatrix} : \Pi X \oplus \Pi^2 X \to X \oplus \Pi X,$$
$$\psi_X = \begin{pmatrix} 0 & -\xi_X \\ \mathrm{id}_{\Pi X} & 0 \end{pmatrix} : \Pi X \oplus \Pi^2 X \to X \oplus \Pi X.$$

(b) *There exist canonical isomorphisms*

$$\Phi \circ \Psi(X) \simeq X \oplus \Pi X \quad \text{and} \quad \Psi \circ \Phi(Z) \simeq Z \oplus \Pi^{\mathrm{CT}} Z,$$

which are functorial in $X \in \mathscr{C}$ and $Z \in \mathscr{C}^{\mathrm{CT}}$.

(iii) *There are isomorphisms $\Pi \circ \Phi \simeq \Phi \circ \Pi^{\mathrm{CT}} \simeq \Phi$ and $\Pi^{\mathrm{CT}} \circ \Psi \simeq \Psi \circ \Pi \simeq \Psi$.*

(iv) *Φ and Ψ are biadjoint to each other (i.e., Φ is a left adjoint and a right adjoint of Ψ).*

Proof. (i) Since the diagram

$$\begin{tikzcd} \Phi \circ (\Pi^{\mathrm{CT}})^2(X, \varphi) \arrow[r, "\sim"] \arrow[d, equal] & \Pi^{\mathrm{rev}} \circ \Phi \circ \Pi^{\mathrm{CT}}(X, \varphi) \arrow[r, "\sim"] \arrow[d, equal] & (\Pi^{\mathrm{rev}})^2 \circ \Phi(X, \varphi) \arrow[d, equal] \\ \Phi(X, \varphi) \arrow[r, "-\varphi^{-1}"] \arrow[d, equal] & \Pi^{\mathrm{rev}} \circ \Phi(X, -\varphi) \arrow[r, "\Pi \circ \varphi^{-1}"] \arrow[d, equal] & (\Pi^{\mathrm{rev}})^2 \circ \Phi(X, \varphi) \arrow[d, equal] \\ X \arrow[r, "-\varphi^{-1}"] & \Pi X \arrow[r, "\Pi \circ \varphi^{-1}"] & \Pi^2 X \arrow[ll, bend left, "-\xi_X"] \end{tikzcd}$$

commutes, Φ is a superfunctor from $\mathscr{C}^{\mathrm{CT}}$ to $\mathscr{C}^{\mathrm{rev}}$.

(iia) Since $\varphi_X \circ (\Pi\varphi_X) = \begin{pmatrix} 0 & \xi_X \\ \mathrm{id}_{\Pi X} & 0 \end{pmatrix} \begin{pmatrix} 0 & \Pi\xi_X \\ \mathrm{id}_{\Pi^2 X} & 0 \end{pmatrix} = \begin{pmatrix} \xi_X & 0 \\ 0 & \xi_{\Pi X} \end{pmatrix} = \xi_Z$, we see that $(Z, \varphi_X) := (X \oplus \Pi X, \varphi_X)$ is an object of $\mathscr{C}^{\mathrm{CT}}$. Since the diagram

$$\begin{tikzcd}[column sep=huge]
\Pi^2 X \oplus \Pi^3 X \arrow[r, "\Pi\psi_X = \begin{pmatrix} 0 & -\Pi\xi_X \\ \mathrm{id}_{\Pi^2 X} & 0 \end{pmatrix}"] \arrow[d, "\varphi_{\Pi X} = \begin{pmatrix} 0 & \xi_{\Pi X} \\ \mathrm{id}_{\Pi^2 X} & 0 \end{pmatrix}"'] & \Pi X \oplus \Pi^2 X \arrow[d, "-\varphi_X = \begin{pmatrix} 0 & -\xi_X \\ -\mathrm{id}_{\Pi X} & 0 \end{pmatrix}"] \\
\Pi X \oplus \Pi^2 X \arrow[r, "\psi_X = \begin{pmatrix} 0 & -\xi_X \\ \mathrm{id}_{\Pi X} & 0 \end{pmatrix}"] & X \oplus \Pi X
\end{tikzcd}$$

commutes, ψ_X is a morphism in $\mathscr{C}^{\mathrm{CT}}$.

Since the diagram

$$\begin{tikzcd}
\Psi \circ \Pi^2 X \arrow[r, "\sim"] \arrow[d, equal] & \Pi^{\mathrm{CT}} \circ \Psi \circ \Pi X \arrow[r, "\sim"] \arrow[d, equal] & (\Pi^{\mathrm{CT}})^2 \circ \Psi X \arrow[d, equal] \\
\Pi^2 X \oplus \Pi^3 X \arrow[r, "\begin{pmatrix} 0 & -\xi_{\Pi X} \\ \mathrm{id}_{\Pi^2 X} & 0 \end{pmatrix}"] \arrow[rr, bend right, "\begin{pmatrix} -\xi_X & 0 \\ 0 & -\xi_{\Pi X} \end{pmatrix}"'] & \Pi X \oplus \Pi^2 X \arrow[r, "\begin{pmatrix} 0 & -\xi_X \\ \mathrm{id}_{\Pi X} & 0 \end{pmatrix}"] & X \oplus \Pi X
\end{tikzcd}$$

commutes, Ψ is a superfunctor from $\mathscr{C}^{\mathrm{rev}}$ to $\mathscr{C}^{\mathrm{CT}}$.

(iib) is obvious.

(iii) $\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & -\mathrm{id}_{\Pi X} \end{pmatrix} \in \mathrm{End}(X \oplus \Pi X)$ gives an isomorphism $\Pi \circ \Psi(X) \xrightarrow{\sim} \Psi(X)$. Similarly, we have the other isomorphisms.

(iv) is straightforward. $\square$

In the rest of this paper, we assume that

(2.3) $$2 \text{ is invertible in } \mathbf{k}.$$

Hence for a $\mathbf{k}$-linear idempotent-complete category $\mathscr{C}$ and $p \in \mathrm{End}_{\mathscr{C}}(X)$ $(X \in \mathscr{C})$ such that $p^2 = \mathrm{id}_X$, we have the decomposition $X \simeq \ker(\mathrm{id}_X - p) \oplus \ker(\mathrm{id}_X + p)$.

Lemma 2.3. *Let $\mathscr{C}$ be a $\mathbf{k}$-linear supercategory. If $\mathscr{C}$ is idempotent-complete, then $(\mathscr{C}^{\mathrm{CT}})^{\mathrm{CT}}$ is equivalent to $\mathscr{C}$ as a supercategory.*

Proof. The underlying category of $(\mathscr{C}^{\mathrm{CT}})^{\mathrm{CT}}$ is the category $\mathscr{C}'$ whose object is a triple (X, φ, ψ), where $\varphi\colon \Pi X \xrightarrow{\sim} X$ satisfies (2.2), $\psi \in \mathrm{End}_{\mathscr{C}}(X)$ satisfies $\psi^2 = \mathrm{id}_X$ and the diagram

$$\begin{tikzcd}
\Pi X \arrow[r, "-\varphi"] \arrow[d, "\Pi\psi"] & X \arrow[d, "\psi"] \\
\Pi X \arrow[r, "\varphi"] & X
\end{tikzcd}$$

is commutative. The involution Π' of $(\mathscr{C}^{\mathrm{CT}})^{\mathrm{CT}}$ is given by $\Pi'(X, \varphi, \psi) = (X, \varphi, -\psi)$ and the isomorphism $\xi'\colon \Pi'^2 \xrightarrow{\sim} \mathrm{id}_{\mathscr{C}'}$ is given by $\Pi'^2(X, \varphi, \psi) = (X, \varphi, \psi) \xrightarrow{\mathrm{id}} (X, \varphi, \psi)$.

Since $\psi^2 = \mathrm{id}_X$, we have a direct sum decomposition $X \simeq X_+ \oplus X_-$ with $X_\pm = \ker(\mathrm{id}_X \mp \psi)$. Moreover, the commutativity of the above diagram implies that φ induces an isomorphism $\varphi_\pm\colon \Pi X_\pm \xrightarrow{\sim} X_\mp$. We define the functor $\Phi\colon \mathscr{C}' \to \mathscr{C}$ by

$$\Phi(X, \varphi, \psi) = X_+$$

and the morphism $\alpha\colon \Phi\circ\Pi' \to \Pi\circ\Phi$ by

$$\Phi\circ\Pi'\,(X,\varphi,\psi) \simeq X_- \xrightarrow[(\varphi_+)^{-1}]{\sim} \Pi X_+ \simeq \Pi\circ\Phi\,(X,\varphi,\psi).$$

Since the diagram

$$\begin{tikzcd}
\Phi\circ\Pi'^2(X,\varphi,\psi) \arrow[r, "\sim"] \arrow[d, equal] & \Pi\circ\Phi\circ\Pi'(X,\varphi,\psi) \arrow[r, "\sim"] \arrow[d, equal] & \Pi^2\circ\Phi(X,\varphi,\psi) \arrow[d, equal] \\
\Phi(X,\varphi,\psi) \arrow[r, "\sim"] \arrow[d, equal] & \Pi\circ\Phi(X,\varphi,-\psi) \arrow[r, "\sim"] \arrow[d, equal] & \Pi^2\circ\Phi(X,\varphi,\psi) \arrow[d, equal] \\
X_+ \arrow[r, "\varphi_-^{-1}"] \arrow[drr, "\mathrm{id}_{X_+}"'] & \Pi X_- \arrow[r, "\Pi\varphi_+^{-1}"] & \Pi^2 X_+ \arrow[d, "\xi_{X_+}"] \\
 & & X_+
\end{tikzcd}$$

commutes, Φ is a superfunctor from $(\mathscr{C}^{\mathrm{CT}})^{\mathrm{CT}}$ to $\mathscr{C}$.

Let $\Psi\colon \mathscr{C} \to \mathscr{C}'$ be the functor given by $\Psi(X) = (X \oplus \Pi X, \varphi_X, \psi_X)$, where $\varphi_X = \left(\begin{smallmatrix} 0 & \xi_X \\ \mathrm{id}_{\Pi X} & 0 \end{smallmatrix}\right)$ and $\psi_X = \left(\begin{smallmatrix} \mathrm{id}_X & 0 \\ 0 & -\mathrm{id}_{\Pi X} \end{smallmatrix}\right)$. Then we can easily check that Ψ is a quasi-inverse of Φ. $\square$

Let $(\mathscr{C},\Pi,\xi)$ and $(\mathscr{C}',\Pi',\xi')$ be a pair of supercategories. Let $\mathrm{Fct}(\mathscr{C},\mathscr{C}')$ be the category of functors from the category $\mathscr{C}$ to $\mathscr{C}'$. Then $\mathrm{Fct}(\mathscr{C},\mathscr{C}')$ is endowed with the structure of supercategory by $\Pi_{\mathrm{Fct}}(F) = \Pi'\circ F\circ\Pi$, and $\xi_{\mathrm{Fct}} = \xi'\circ F\circ\xi\colon \Pi^2_{\mathrm{Fct}}(F) = \Pi'^2\circ F\circ\Pi^2 \xrightarrow{\sim} F$. Then the underlying category of $\mathrm{Fct}(\mathscr{C},\mathscr{C}')^{\mathrm{CT}}$ is equivalent to the category $\mathrm{Fct}_{\mathrm{super}}(\mathscr{C},\mathscr{C}')$ of superfunctors from $\mathscr{C}$ to $\mathscr{C}'$, and its supercategory structure is given by $\Pi(F,\alpha) = (F,-\alpha)$.

The category $\mathrm{Fct}_{\mathrm{super}}(\mathscr{C},\mathscr{C}')$ is endowed with another structure of supercategory given by $\Pi(F,\alpha) = (\Pi'\circ F, \Pi'\circ\alpha)$ and $\xi(F,\alpha) = \xi'\circ F\colon \Pi^2(F,\alpha) = (\Pi'^2\circ F, \Pi'^2\circ\alpha) \xrightarrow{\sim} (F,\alpha)$.

Definition 2.4. *Let $(\mathscr{C},\Pi,\xi)$ be a supercategory such that $\mathscr{C}$ is an exact category and Π is an exact functor. The* Grothendieck group $\mathrm{K}^{\mathrm{super}}(\mathscr{C})$ *of $\mathscr{C}$ is the abelian group generated by $[X]$ (X is an object of $\mathscr{C}$) with the defining relations:*

(a) *if $0 \to X' \to X \to X'' \to 0$ is an exact sequence, then $[X] = [X'] + [X'']$,*
(b) $[\Pi X] = [X]$.

We have $\mathrm{K}^{\mathrm{super}}(\mathscr{C}^{\mathrm{rev}}) \simeq \mathrm{K}^{\mathrm{super}}(\mathscr{C})$ and $\mathbb{Z}[1/2]\otimes\mathrm{K}^{\mathrm{super}}(\mathscr{C}^{\mathrm{CT}}) \simeq \mathbb{Z}[1/2]\otimes\mathrm{K}^{\mathrm{super}}(\mathscr{C})$ by Lemma 2.2.

2.2. **Superalgebras and supermodules.** Let A be a $\mathbf{k}$-*superalgebra*; i.e., a $\mathbf{k}$-algebra with a decomposition $A = A_0 \oplus A_1$ such that $A_iA_j \subset A_{i+j}$ ($i,j \in \mathbb{Z}/2\mathbb{Z}$).

Let ϕ_A be the involution of A given by $\phi_A(a) = (-1)^\varepsilon a$ for $a \in A_\varepsilon$ with $\varepsilon = 0,1$. Then the category of left A-modules $\mathrm{Mod}(A)$ is naturally endowed with a structure of supercategory. The functor Π is induced by ϕ_A. Namely, for $M \in \mathrm{Mod}(A)$, there is an additive group isomorphism $\pi\colon M \to \Pi M$ such that $a\pi(x) = \pi(\phi_A(a)x)$ for any $a \in A$, $x \in M$. The morphism $\xi_M\colon \Pi^2 M \to M$ is given by $\pi\big(\pi(x)\big) \mapsto x$ ($x \in M$).

An A-*supermodule* is an A-module M with a decomposition $M = M_0 \oplus M_1$ such that $A_iM_j \subset M_{i+j}$ ($i,j \in \mathbb{Z}/2\mathbb{Z}$). An A-linear homomorphism between A-supermodules is called *even* if it respects the $\mathbb{Z}/2\mathbb{Z}$-grading.

Let $\mathrm{Mod}_{\mathrm{super}}(A)$ be the category of A-supermodules with even A-linear homomorphisms as morphisms. Then $\mathrm{Mod}_{\mathrm{super}}(A)$ is also endowed with a structure of supercategory. The functor Π is given by the parity shift: namely, $(\Pi M)_\varepsilon := \{\pi(x)\,;\, x \in M_{1-\varepsilon}\}$ $(\varepsilon = 0, 1)$ and $a\pi(x) = \pi(\phi_A(a)x)$ where $a \in A$ and $x \in M$. The isomorphism $\xi_M \colon \Pi^2 M \to M$ is given by $\pi\pi(x) \mapsto x$ $(x \in M)$. Then there is a canonical superfunctor $\mathrm{Mod}_{\mathrm{super}}(A) \to \mathrm{Mod}(A)$. For an A-supermodule M, let $\phi_M \colon M \to M$ be the map $\phi_M|_{M_\varepsilon} = (-1)^\varepsilon \mathrm{id}_{M_\varepsilon}$. Then we have $\phi_M(ax) = \phi_A(a)\phi_M(x)$ for any $a \in A$ and $x \in M$.

Lemma 2.5. *Let A be a superalgebra in which 2 is invertible. Then we have*

$$\begin{aligned} \mathrm{Mod}(A)^{\mathrm{CT}} &\simeq \mathrm{Mod}_{\mathrm{super}}(A)^{\mathrm{rev}} \quad \textit{and} \\ (\mathrm{Mod}_{\mathrm{super}}(A)^{\mathrm{rev}})^{\mathrm{CT}} &\simeq \mathrm{Mod}(A)\,. \end{aligned}$$

Proof. Let us define the superfunctor $(\Phi, \alpha) \colon \mathrm{Mod}_{\mathrm{super}}(A)^{\mathrm{rev}} \to \mathrm{Mod}(A)^{\mathrm{CT}}$. For $M \in \mathrm{Mod}_{\mathrm{super}}(A)$, set $\Phi(M) = (M, \psi_M)$, where $\psi_M \colon \Pi M \to M$ is given by $\psi_M \pi(x) = \phi_M(x)$ $(x \in M)$. We set $\alpha_M = \psi_M \colon \Pi M \to M$. Then the diagram

$$\begin{tikzcd} \Pi^2 M \arrow[r, "\Pi\alpha_M"] \arrow[d, "\psi_{\Pi M}"'] & M \arrow[d, "-\psi_M"] \\ \Pi M \arrow[r, "\alpha_M"] & M \end{tikzcd}$$

is commutative, since for any $x \in M$ we have

$$\begin{aligned} \alpha_M \psi_{\Pi M}(\pi\pi(x)) &= \alpha_M(\phi_{\Pi M}\pi(x)) = \alpha_M(-\pi(\phi_M(x))) = -\phi_M\phi_M(x) = -x, \\ \psi_M \circ \Pi\alpha_M(\pi\pi(x)) &= \psi_M(\pi(\alpha_M(\pi(x)))) = \phi_M\phi_M(x) = x. \end{aligned}$$

Hence α_M is a morphism from $\Phi \circ \Pi M$ to $\Pi \circ \Phi M$ in $\mathrm{Mod}(A)^{\mathrm{CT}}$. Since the diagram

$$\begin{tikzcd} \Phi \circ \Pi^2 M \arrow[r, "\sim"] \arrow[d, equal] & \Pi \circ \Phi \circ \Pi M \arrow[r, "\sim"] \arrow[d, equal] & \Pi^2 \circ \Phi M \arrow[d, equal] \\ (\Pi^2 M, \psi_{\Pi^2 M}) \arrow[r, "\alpha_{\Pi M}"] \arrow[d, "-\xi_M"] & (\Pi M, -\psi_{\Pi M}) \arrow[r, "\alpha_M"] & (M, \psi_M) \arrow[d, equal] \\ (M, \psi_M) \arrow[rr, "\mathrm{id}"] & & (M, \psi_M) \end{tikzcd}$$

is commutative, we have defined a superfunctor $(\Phi, \alpha) \colon \mathrm{Mod}_{\mathrm{super}}(A)^{\mathrm{rev}} \to \mathrm{Mod}(A)^{\mathrm{CT}}$. It is straightforward to prove that Φ is an equivalence.

The second equivalence follows from the first one and Lemma 2.3. $\square$

2.3. **Tensor products.** Let A and B be superalgebras. We define the multiplication on the tensor product $A \otimes B$ by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\varepsilon_1' \varepsilon_2}(a_1 a_2) \otimes (b_1 b_2)$$

for $a_i \in A_{\varepsilon_i}$, $b_i \in B_{\varepsilon_i'}$. Then $A \otimes B$ is again a superalgebra. Note that we have $A \otimes B \cong B \otimes A$ by the supertwist map

$$A \otimes B \xrightarrow{\sim} B \otimes A, \quad a \otimes b \longmapsto (-1)^{\varepsilon_1 \varepsilon_2} b \otimes a \quad (a \in A_{\varepsilon_1},\, b \in B_{\varepsilon_2}).$$

Example 2.6.

(i) For $n \in \mathbb{Z}_{\geq 0}$, let $\mathfrak{C}_n$ be the Clifford superalgebra generated by the odd generators $C_1, \dots, C_n$ with the defining relations:

$$C_i^2 = 1 \ (i = 1, \dots n) \text{ and } C_iC_j = -C_jC_i \text{ for } i \neq j.$$

Hence $\mathfrak{C}_n \simeq (\mathfrak{C}_1)^{\otimes n}$.

(ii) Let us denote by $\mathfrak{C}_n^-$ the superalgebra with the same definition but the relations $C_i^2 = 1$ are replaced by $C_i^2 = -1$.

Lemma 2.7. *We have*

$$\mathrm{Mod}(A \otimes_{\mathbf{k}} \mathfrak{C}_1) \simeq \big(\mathrm{Mod}(A)^{\mathrm{CT}}\big)^{\mathrm{rev}}, \tag{2.4}$$

$$\mathrm{Mod}(A \otimes_{\mathbf{k}} \mathfrak{C}_1) \simeq \mathrm{Mod}_{\mathrm{super}}(A), \tag{2.5}$$

$$\mathrm{Mod}(A \otimes_{\mathbf{k}} \mathfrak{C}_1^-) \simeq \big(\mathrm{Mod}(A)^{\mathrm{rev}}\big)^{\mathrm{CT}}. \tag{2.6}$$

Proof. Let us construct an equivalence $\Phi\colon \mathrm{Mod}(A \otimes_{\mathbf{k}} \mathfrak{C}_1)^{\mathrm{rev}} \to \mathrm{Mod}(A)^{\mathrm{CT}}$. For $M \in \mathrm{Mod}(A \otimes_{\mathbf{k}} \mathfrak{C}_1)$, we set $\Phi(M) = (\Pi M \xrightarrow{\varphi_M} M)$, where $\varphi_M \colon \Pi M \to M$ is given by $\varphi_M(\pi(x)) = C_1x$. Since $C_1^2 = 1$, we have

$$\varphi_M(\Pi\varphi_M)(\pi^2(x)) = \varphi_M\big(\pi(\varphi_M(\pi(x))\big) = \varphi_M(\pi(C_1x)) = C_1^2x = x.$$

Hence $\Pi M \xrightarrow{\varphi_M} M$ is an object of $\mathrm{Mod}(A)^{\mathrm{CT}}$. Then $\varphi_{\Pi M} \colon \Pi^2 M \longrightarrow \Pi M$ is given by $\varphi_{\Pi M}(\pi^2(x)) = C_1\pi(x) = -\pi(C_1x)$. In the diagram

$$\begin{array}{ccc} \Phi \circ \Pi(M) & : & \begin{tikzcd} \Pi^2 M \arrow[r, "\varphi_{\Pi M}"] \arrow[d, "\Pi\alpha_M"'] & \Pi M \arrow[d, "\alpha_M"] \\ \Pi M \arrow[r, "-\varphi_M"] & M \end{tikzcd} \\ \big\downarrow \alpha_M & & \\ \Pi^{\mathrm{CT}} \circ \Phi(M) & : & \end{array} \tag{2.7}$$

we define α_M by $\alpha_M(\pi(x)) = C_1x$ so that we have $(\Pi\alpha_M)(\pi^2(x)) = \pi(C_1x)$. Hence the right square in (2.7) commutes. Thus this defines the morphism $\alpha\colon \Phi \circ \Pi \to \Pi^{\mathrm{CT}} \circ \Phi$. Then the diagram

$$\begin{tikzcd} \Pi^2 M \arrow[r, "\alpha_{\Pi M}"] \arrow[d, "\xi_M"'] & \Pi M \arrow[r, "\alpha_M"] & M \arrow[d, "-\mathrm{id}_M"] \\ M \arrow[rr, "\mathrm{id}_M"] & & M \end{tikzcd}$$

commutes, since

$$\alpha_M\alpha_{\Pi M}(\pi^2x) = \alpha_M(C_1\pi(x)) = -\alpha_M(\pi(C_1x)) = -C_1^2x = -x.$$

Hence (Φ, α) gives a superfunctor $\mathrm{Mod}(A \otimes_{\mathbf{k}} \mathfrak{C}_1) \to \big(\mathrm{Mod}(A)^{\mathrm{CT}}\big)^{\mathrm{rev}}$.

The second equivalence follows from the first one and Lemma 2.5. The third equivalence follows from $\mathrm{Mod}(A \otimes \mathfrak{C}_1 \otimes \mathfrak{C}_1^-) \simeq \mathrm{Mod}(A)$. ☐

Remark 2.8. (i) Let $\mathscr{C} = (\mathscr{C}, \Pi, \xi)$ be a supercategory. Let $Z(\mathscr{C})$ be the commutative ring $\mathrm{End}(\mathrm{id}_{\mathscr{C}})$. It does not depend on Π. Let $Z_{\mathrm{super}}(\mathscr{C})$ be its subring

$$\{\psi \in \mathrm{End}(\mathrm{id}_{\mathscr{C}})\,;\, \psi_{\Pi X} = \Pi(\psi_X) \in \mathrm{End}_{\mathscr{C}}(\Pi X) \text{ for any } X \in \mathscr{C}\}\,.$$

(ii) If $\mathscr{C} = \mathrm{Mod}(A)$ for a superalgebra A, then we have

$$\begin{aligned} Z(\mathscr{C}) &\simeq Z(A) := \{a \in A\,;\, ab = ba \text{ for any } b \in A\}\,, \\ Z_{\mathrm{super}}(\mathscr{C}) &\simeq Z_0(A) := A_0 \cap Z(A), \end{aligned}$$

and

$$\begin{aligned} Z(\mathscr{C}^{\mathrm{CT}}) &\simeq Z(A \otimes \mathfrak{C}_1) \\ &\simeq \{a \otimes 1 + b \otimes C_1\,;\, a, b \in A_0 \text{ and } ax = xa,\, bx = \phi_A(x)b \text{ for any } x \in A\}\,, \\ Z_{\mathrm{super}}(\mathscr{C}^{\mathrm{CT}}) &\simeq Z_0(A). \end{aligned}$$

2.4. **Weak superequivalence.** Let $\mathscr{C}$ and $\mathscr{C}'$ be supercategories. We say that $\mathscr{C}$ and $\mathscr{C}'$ are *weakly superequivalent* if there exist a pair of supercategories $\mathscr{C}_1$ and $\mathscr{C}_2$ and superequivalences $\mathscr{C} \simeq \mathscr{C}_1 \oplus \mathscr{C}_2$ and $\mathscr{C}' \simeq \mathscr{C}_1 \oplus \mathscr{C}_2^{\mathrm{CT}}$.

If $\mathscr{C}$ and $\mathscr{C}'$ are weakly superequivalent and $\mathscr{C}'$ and $\mathscr{C}''$ are weakly superequivalent, then $\mathscr{C}$ and $\mathscr{C}''$ are weakly superequivalent. Indeed, if $\mathscr{C}_1 \oplus \mathscr{C}_2 \simeq \mathscr{C}_3 \oplus \mathscr{C}_4$, then there exist supercategories $\mathscr{C}_{1,3}$, $\mathscr{C}_{1,4}$, $\mathscr{C}_{2,3}$, $\mathscr{C}_{2,4}$ and superequivalences $\mathscr{C}_i \simeq \mathscr{C}_{i,3} \oplus \mathscr{C}_{i,4}$ $(i = 1, 2)$ and $\mathscr{C}_j \simeq \mathscr{C}_{1,j} \oplus \mathscr{C}_{2,j}$ $(j = 3, 4)$ such that the composition $\mathscr{C}_1 \oplus \mathscr{C}_2 \simeq \mathscr{C}_{1,3} \oplus \mathscr{C}_{1,4} \oplus \mathscr{C}_{2,3} \oplus \mathscr{C}_{2,4} \simeq \mathscr{C}_3 \oplus \mathscr{C}_4$ is isomorphic to the given superequivalence.

Definition 2.9. *Let A and B be superalgebras.*

(i) *We say that A and B are* Morita superequivalent *if* $\mathrm{Mod}(A)$ *and* $\mathrm{Mod}(B)$ *are superequivalent.*
(ii) *We say that A and B are* weakly Morita superequivalent *if* $\mathrm{Mod}(A)$ *and* $\mathrm{Mod}(B)$ *are weakly superequivalent.*

Note that superalgebras A and B are weakly Morita superequivalent if and only if there are decompositions $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ as superalgebras and superequivalences $\mathrm{Mod}(A_1) \simeq \mathrm{Mod}(B_1)$ and $\mathrm{Mod}(A_2) \simeq \mathrm{Mod}(B_2)^{\mathrm{CT}}$. Note that A and B are Morita superequivalent if and only if there exists an (A, B)-supermodule P satisfying one of the following equivalent conditions:

(i) $N \mapsto P \otimes_B N$ is an equivalence of categories from $\mathrm{Mod}(B)$ to $\mathrm{Mod}(A)$,
(ii) $N \mapsto P \otimes_B N$ is an equivalence of categories from $\mathrm{Mod}_{\mathrm{super}}(B)$ to $\mathrm{Mod}_{\mathrm{super}}(A)$,
(iii) $M \mapsto \mathrm{Hom}_A(P, M)$ is an equivalence of categories from $\mathrm{Mod}(A)$ to $\mathrm{Mod}(B)$,
(iv) $M \mapsto \mathrm{Hom}_A(P, M)$ is an equivalence of categories from $\mathrm{Mod}_{\mathrm{super}}(A)$ to $\mathrm{Mod}_{\mathrm{super}}(B)$,
(v) $M \mapsto M \otimes_A P$ is an equivalence of supercategories from $\mathrm{Mod}(A^{\mathrm{op}})$ to $\mathrm{Mod}(B^{\mathrm{op}})$,
(vi) $N \mapsto \mathrm{Hom}_{B^{\mathrm{op}}}(P, N)$ is an equivalence of supercategories from $\mathrm{Mod}(B^{\mathrm{op}})$ to $\mathrm{Mod}(A^{\mathrm{op}})$,
(vii) P is a faithfully flat A-module of finite presentation and $B \xrightarrow{\sim} \mathrm{End}_A(P)^{\mathrm{opp}}$,
(viii) P is a faithfully flat B-module of finite presentation and $A \xrightarrow{\sim} \mathrm{End}_B(P)$.

Remark 2.10. (i) $\mathfrak{C}_n \otimes \mathfrak{C}_n^-$ is Morita superequivalent to $\mathbf{k}$.
(ii) Let A be a superalgebra and $e \in A$ a *full even idempotent*; i.e., $e \in A_0$, $e^2 = e$ and $A = AeA := \{\sum_{i=1}^n a_i e b_i\,;\, a_i, b_i \in A, n > 0\}$. Then, A and eAe are Morita superequivalent.

(iii) Assume that $\sqrt{-1}$ exists in $\mathbf{k}$. Then $\mathbf{k}$-superalgebras A and $A \otimes \mathfrak{C}_n$ are weakly Morita superequivalent. Two $\mathbf{k}$-superalgebras A and B are weakly Morita superequivalent if and only if there exist direct sum decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$ and A_1 is Morita superequivalent to B_1 and A_2 is Morita superequivalent to $B_2 \otimes \mathfrak{C}_1$ by Lemma 2.7.

2.5. **Self-associate simple objects.** Let us assume that $\mathbf{k}$ is a field of characteristic $\neq 2$, and let $\mathscr{C}$ be a $\mathbf{k}$-linear abelian supercategory.

We say that a simple object S of $\mathscr{C}$ is *self-associate* if ΠS is isomorphic to S. Let $\mathsf{Irr}(\mathscr{C})$ be the set of isomorphism classes of simple objects. Then it is divided into the set $\mathsf{Irr}^{\mathsf{SA}}(\mathscr{C})$ of self-associate simple objects and the set $\mathsf{Irr}^{\mathsf{NSA}}(\mathscr{C})$ of simple objects which are not self-associate. Note that Π gives an involution of $\mathsf{Irr}(\mathscr{C})$, and $\mathsf{Irr}^{\mathsf{SA}}(\mathscr{C})$ is the fixed point set.

Lemma 2.11. *Assume that* $\mathbf{k}$ *is algebraically closed and* $\mathscr{C}$ *satisfies the following conditions.*

(a) $\mathscr{C}$ *is an abelian category,*
(b) *any object of* $\mathscr{C}$ *has a finite length,*
(c) $\mathbf{k} \xrightarrow{\sim} \mathrm{End}_{\mathscr{C}}(S)$ *for any simple object* S *of* $\mathscr{C}$*.*

Then we have

(i) $\mathscr{C}^{\mathrm{CT}}$ *also satisfies the above conditions,*
(ii) *the superfunctors* $\Phi\colon \mathscr{C}^{\mathrm{CT}} \to \mathscr{C}^{\mathrm{rev}}$ *and* $\Psi\colon \mathscr{C}^{\mathrm{rev}} \to \mathscr{C}^{\mathrm{CT}}$ *in* Lemma 2.2 *induce two-to-one maps* $\mathsf{Irr}^{\mathsf{NSA}}(\mathscr{C}^{\mathrm{CT}}) \to \mathsf{Irr}^{\mathsf{SA}}(\mathscr{C})$ *and* $\mathsf{Irr}^{\mathsf{NSA}}(\mathscr{C}) \to \mathsf{Irr}^{\mathsf{SA}}(\mathscr{C}^{\mathrm{CT}})$*, respectively.*

Proof. (i) is obvious. Let us show (ii). Note that Φ and Ψ are exact functors, and Ψ is a left adjoint of Φ. Let $X \in \mathsf{Irr}^{\mathsf{NSA}}(\mathscr{C}^{\mathrm{CT}})$. Take a monomorphism $S \rightarrowtail M := \Phi(X)$ with a simple object S of $\mathscr{C}$. Since Ψ is a left adjoint of Ψ, we have $g\colon \Psi(S) \to X$. Since it is a non-zero morphism, g is an epimorphism. Since $\Pi \circ \Psi \simeq \Psi$ and since X is not isomorphic to $\Pi^{\mathrm{CT}} X$, g is not an isomorphism. Hence $S \oplus \Pi S \simeq \Phi \circ \Psi(S) \to M$ is not an isomorphism. Hence $S \to M$ is an isomorphism, Therefore $S \in \mathsf{Irr}^{\mathsf{SA}}(\mathscr{C})$.

Conversely, let S be a self-associate simple object of $\mathscr{C}$. Let $\varphi\colon \Pi S \xrightarrow{\sim} S$ be an isomorphism. Then $\varphi \circ \Pi(\varphi) = a\xi_S$ for some $a \in \mathbf{k}^{\times}$. Take $c \in \mathbf{k}^{\times}$ such that $c^2 = a$ and set $\varphi' = c^{-1}\varphi$. Then we have $\varphi' \circ \Pi\varphi' = \xi_S$. Then $X := (S, \varphi')$ is an object of $\mathscr{C}^{\mathrm{CT}}$. It is obviously a simple object of $\mathscr{C}^{\mathrm{CT}}$. If $(S, -\varphi') = \Pi^{\mathrm{CT}} X \to X$ is an isomorphism, then it is given by $f\colon S \xrightarrow{\sim} S$ such that $f \circ (-\varphi') = \varphi' \circ \Pi f$. Since $f = a\,\mathrm{id}_S$ for some $a \in \mathbf{k}^{\times}$ it is a contradiction. Hence X is not self-associate.

Now it is obvious that if X and X' are simple objects of $\mathscr{C}^{\mathrm{CT}}$ that are not self-associate and $\Phi(X) \simeq \Phi(X')$, then $X' \simeq X$ or $X' \simeq \Pi X$. ☐

3. Quiver Hecke superalgebras

3.1. **The quiver Hecke superalgebra** R_n**.** Recall that 2 *is assumed to be invertible in the base ring* $\mathbf{k}$. Let I be a finite set with a decomposition $I = I_{\mathrm{odd}} \sqcup I_{\mathrm{even}}$. We say that $i \in I_{\mathrm{even}}$ is an *even vertex* and $i \in I_{\mathrm{odd}}$ is an *odd vertex*. For $i \in I$, we denote the parity of i by $\mathrm{par}(i)$; i.e.,

$$\mathrm{par}(i) = \begin{cases} 0 & \text{if } i \in I_{\mathrm{even}}, \\ 1 & \text{if } i \in I_{\mathrm{odd}}. \end{cases}$$

For each $i, j \in I$, consider the $\mathbf{k}$-algebra $\mathscr{A}_{i,j} = \mathbf{k}\langle w, z\rangle / \langle zw - (-1)^{\mathrm{par}(i)\,\mathrm{par}(j)} wz \rangle$ generated by the indeterminates w, z with the defining relation

$$zw - (-1)^{\mathrm{par}(i)\,\mathrm{par}(j)} wz = 0.$$

Let $Q = (Q_{i,j}(w,z))_{i,j\in I}$ be a family of (skew-)polynomials satisfying the conditions

$$(3.1)\quad \begin{cases} \text{(i)} \ Q_{i,j}(w,z) \in \mathscr{A}_{i,j} \text{ for all } i,j \in I, \\ \text{(ii)} \ Q_{i,j}(w,z) = 0 \text{ if } i = j, \\ \text{(iii)} \ Q_{i,j}(w,z) = Q_{j,i}(z,w) \text{ for all } i,j \in I, \\ \text{(iv)} \ Q_{i,j}(w,z) = Q_{i,j}(-w,z) \text{ for all } i \in I_{\mathrm{odd}}, j \in I. \end{cases}$$

Hence $Q_{i,j}(u,v)$ belongs to the subalgebra $\widetilde{\mathscr{A}_{i,j}}$ of $\mathscr{A}_{i,j}$ generated by $w^{1+\mathrm{par}(i)}$ and $z^{1+\mathrm{par}(j)}$. Note that $\widetilde{\mathscr{A}_{i,j}}$ is commutative and isomorphic to the polynomial algebra.

Definition 3.1. *The* quiver Hecke superalgebra R_n *is the* $\mathbf{k}$*-superalgebra generated by the elements* $\{x_p\}_{1\le p\le n}$, $\{\tau_a\}_{1\le a<n}$, $\{e(\nu)\}_{\nu\in I^n}$ *with parity*

$$\mathrm{par}(e(\nu)) = 0, \quad \mathrm{par}(x_p e(\nu)) = \mathrm{par}(\nu_p), \quad \mathrm{par}(\tau_a e(\nu)) = \mathrm{par}(\nu_a)\,\mathrm{par}(\nu_{a+1})$$

and the following defining relations.

(i) $e(\mu)e(\nu) = \delta_{\mu\nu} e(\mu)$ *for all* $\mu,\nu \in I^n$, *and* $1 = \sum_{\nu\in I^n} e(\nu)$,

(ii)

$$x_p x_q e(\nu) = \begin{cases} -x_q x_p e(\nu) & \text{if } p \neq q \text{ and } \nu_p, \nu_q \in I_{\mathrm{odd}}, \\ x_q x_p e(\nu) & \text{otherwise,} \end{cases}$$

(iii) $\tau_a x_p e(\nu) = (-1)^{\mathrm{par}(\nu_p)\,\mathrm{par}(\nu_a)\,\mathrm{par}(\nu_{a+1})} x_p \tau_a e(\nu)$ *if* $p \neq a, a+1$,

(iv)

$$\begin{aligned}(\tau_a x_{a+1} - (-1)^{\mathrm{par}(\nu_a)\,\mathrm{par}(\nu_{a+1})} x_a \tau_a)e(\nu) &= (x_{a+1}\tau_a - (-1)^{\mathrm{par}(\nu_a)\,\mathrm{par}(\nu_{a+1})} \tau_a x_a)e(\nu) \\ &= \begin{cases} e(\nu) & \text{if } \nu_a = \nu_{a+1}, \\ 0 & \text{otherwise,} \end{cases}\end{aligned}$$

(v) $\tau_a^2 e(\nu) = Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1}) e(\nu)$,

(vi) $\tau_a\tau_b e(\nu) = (-1)^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})\mathrm{par}(\nu_b)\mathrm{par}(\nu_{b+1})} \tau_b\tau_a e(\nu)$ *if* $|a-b| > 1$,

(vii)

$$\begin{aligned}&(\tau_{a+1}\tau_a\tau_{a+1} - \tau_a\tau_{a+1}\tau_a)e(\nu) \\ &= \begin{cases} \dfrac{Q_{\nu_a,\nu_{a+1}}(x_{a+2},x_{a+1}) - Q_{\nu_a,\nu_{a+1}}(x_a,x_{a+1})}{x_{a+2}-x_a} e(\nu) & \text{if } \nu_a = \nu_{a+2} \in I_{\mathrm{even}}, \\ (-1)^{\mathrm{par}(\nu_a)}(x_{a+2}-x_a)\dfrac{Q_{\nu_a,\nu_{a+1}}(x_{a+2},x_{a+1}) - Q_{\nu_a,\nu_{a+1}}(x_a,x_{a+1})}{x_{a+2}^2-x_a^2} e(\nu) & \\ & \text{if } \nu_a = \nu_{a+2} \in I_{\mathrm{odd}}, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

If $I_{\mathrm{odd}} = \emptyset$, the quiver Hecke superalgebras R_n are nothing but the usual Khovanov-Lauda-Rouquier algebras. Note that, when ν_a is odd, $Q_{\nu_a,\nu_{a+1}}$ belongs to the commutative ring $\mathbf{k}[x_a^2, x_{a+1}]$, and hence we can define

$$\frac{Q_{\nu_a,\nu_{a+1}}(x_{a+2},x_{a+1}) - Q_{\nu_a,\nu_{a+1}}(x_a,x_{a+1})}{x_{a+2}^2-x_a^2}$$

as an element of $\mathbf{k}[x_a^2, x_{a+1}, x_{a+2}^2]$. For the connection with a generalized Cartan matrix, see § 3.6.

Remark 3.2. For $\left(\gamma_{i,j}\right)_{i,j\in I}$ with $\gamma_{i,j} \in \mathbf{k}^\times$, define $Q' = \left(Q'_{i,j}\right)_{i,j\in I}$ by

$$Q'_{i,j}(w,z) = \gamma_{i,j}\gamma_{j,i}Q_{i,j}(\gamma_{i,i}w, \gamma_{j,j}z).$$

Then, the assignment $x_p e(\nu) \mapsto \gamma_{\nu_p,\nu_p} x_p e(\nu)$, $\tau_a e(\nu) \mapsto \gamma^{-1}_{\nu_a,\nu_{a+1}} \tau_a e(\nu)$ induces an isomorphism $\mathrm{R}_n(Q) \xrightarrow{\sim} \mathrm{R}_n(Q')$.

The purpose of this paper is to investigate the basic properties of the quiver Hecke superalgebras R_n and show that the affine Hecke-Clifford superalgebras and affine Sergeev superalgebras are weakly Morita superequivalent to some quiver Hecke algebras after a completion.

3.2. **Twisted tensor product with symmetric groups.** Let A be a ring with an action of the symmetric group $\mathfrak{S}_n$ and let $Z(A)$ be the center of A. Let a_i $(i = 1, \dots, n-1)$ be elements of $Z(A)$ satisfying the following conditions:

$$wa_i = a_j \text{ for any } w \in \mathfrak{S}_n,\ 1 \le i, j < n \text{ such that } w\{i, i+1\} = \{j, j+1\}. \tag{3.2}$$

It is easy to see that this condition is equivalent to the conditions

$$\begin{cases} s_j a_i = a_i \text{ if } j \neq i-1, i+1, \\ s_{i+1} a_i = s_i a_{i+1} \text{ for } 1 \le i \le n-2. \end{cases} \tag{3.3}$$

Proposition 3.3. *Assume that a family of elements $\{a_i\}_{1\le i<n}$ of $Z(A)$ satisfies one of the equivalent conditions* (3.2) *or* (3.3)*. Let R be the ring generated by A and $\widetilde{s}_i$ $(i = 1, \dots, n-1)$ with the defining relations:*

(a) *$A \to R$ is a ring homomorphism,*
(b) *$\widetilde{s}_i \circ a = s_i(a) \circ \widetilde{s}_i$ for any $a \in A$ and $i = 1, \dots, n-1$,*
(c) *the $\widetilde{s}_i$'s satisfy the braid relations*

$$\widetilde{s}_i\widetilde{s}_j = \widetilde{s}_j\widetilde{s}_i \text{ for } |i-j| > 1 \text{ and } \widetilde{s}_{i+1}\widetilde{s}_i\widetilde{s}_{i+1} = \widetilde{s}_i\widetilde{s}_{i+1}\widetilde{s}_i. \tag{3.4}$$

(d) *$\widetilde{s}_i^2 = a_i$ for any $i = 1, \dots, n-1$.*

For a reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ of $w \in \mathfrak{S}_n$, set $\widetilde{s}_w = \widetilde{s}_{i_1} \cdots \widetilde{s}_{i_\ell}$. Then $\widetilde{s}_w$ is independent of the choice of a reduced expression and there exists a linear isomorphism

$$A \otimes \mathbb{Z}[\mathfrak{S}_n] \longrightarrow R$$

given by $a \otimes w \longmapsto a\widetilde{s}_w$.

Proof. Let

$$\varepsilon_i(w) = \begin{cases} 0 & \text{if } l(s_i w) > w, \\ 1 & \text{if } l(s_i w) < w, \end{cases}$$

where $l(w)$ denotes the length of a reduced expression of w. We define the action of $\widetilde{s}_i$ on $A \otimes \mathbb{Z}[\mathfrak{S}_n]$ by

$$\widetilde{s}_i(a \otimes w) = s_i(a) a_i^{\varepsilon_i(w)} \otimes s_i w.$$

It is enough to show that $\widetilde{s}_i$ satisfies the relations (b)–(d). Since (b) is obvious and (d) easily follows from $\varepsilon_i(w) + \varepsilon_i(s_i w) = 1$, we will show (c) only.

We shall first show $\widetilde{s}_i\widetilde{s}_j = \widetilde{s}_j\widetilde{s}_i$ for $|i-j|>1$. By the relation

$$\widetilde{s}_i\widetilde{s}_j(a\otimes w) = \widetilde{s}_i(s_j(a)a_j^{\varepsilon_j(w)}\otimes s_jw) = s_is_j(a)(s_ia_j)^{\varepsilon_j(w)}a_i^{\varepsilon_i(s_jw)}\otimes s_iw,$$

the assertion reduces to $(s_ia_j)^{\varepsilon_j(w)}a_i^{\varepsilon_i(s_jw)} = (s_ja_i)^{\varepsilon_i(w)}a_j^{\varepsilon_j(s_iw)}$, which easily follows from (3.3) and $\varepsilon_i(s_jw)=\varepsilon_i(w)$.

Let us show $\widetilde{s}_{i+1}\widetilde{s}_i\widetilde{s}_{i+1} = \widetilde{s}_i\widetilde{s}_{i+1}\widetilde{s}_i$. We have

$$\begin{aligned}
\widetilde{s}_{i+1}\widetilde{s}_i\widetilde{s}_{i+1}\big(a\otimes w\big) &= \widetilde{s}_{i+1}\widetilde{s}_i\big((s_{i+1}a)a_{i+1}^{\varepsilon_{i+1}(w)}\otimes s_{i+1}w\big)\\
&= \widetilde{s}_{i+1}\big((s_is_{i+1}a)(s_ia_{i+1})^{\varepsilon_{i+1}(w)}a_i^{\varepsilon_i(s_{i+1}w)}\otimes s_is_{i+1}w\big)\\
&= (s_{i+1}s_is_{i+1}a)(s_{i+1}s_ia_{i+1})^{\varepsilon_{i+1}(w)}(s_{i+1}a_i)^{\varepsilon_i(s_{i+1}w)}a_{i+1}^{\varepsilon_{i+1}(s_is_{i+1}w)}\otimes s_{i+1}s_is_{i+1}w\\
&= (s_{i+1}s_is_{i+1}a)a_i^{\varepsilon_{i+1}(w)}(s_{i+1}a_i)^{\varepsilon_i(s_{i+1}w)}a_{i+1}^{\varepsilon_{i+1}(s_is_{i+1}w)}\otimes s_{i+1}s_is_{i+1}w.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\widetilde{s}_i\widetilde{s}_{i+1}\widetilde{s}_i\big(a\otimes w\big)\\
&\quad= (s_is_{i+1}s_ia)(s_is_{i+1}a_i)^{\varepsilon_i(w)}(s_ia_{i+1})^{\varepsilon_{i+1}(s_iw)}a_i^{\varepsilon_i(s_{i+1}s_iw)}\otimes s_is_{i+1}s_iw\\
&\quad= (s_is_{i+1}s_ia)a_{i+1}^{\varepsilon_i(w)}(s_{i+1}a_i)^{\varepsilon_{i+1}(s_iw)}a_i^{\varepsilon_i(s_{i+1}s_iw)}\otimes s_is_{i+1}s_iw.
\end{aligned}$$

Hence the desired result follows from

$$\varepsilon_i(s_{i+1}w) = \varepsilon_{i+1}(s_iw),$$

which can be easily checked by reducing it to the $\mathfrak{S}_3$-case. □

The proof of the following lemma is straightforward.

Lemma 3.4. *Let $(c_{i,j})_{1\le i\neq j\le n}$ be a family of elements of $Z(A)$ such that $w(c_{i,j}) = c_{w(i),w(j)}$ for any $w\in\mathfrak{S}_n$. Then the $c_{i,i+1}\widetilde{s}_i$'s satisfy the braid relations* (3.4).

3.3. **The quiver Hecke-Clifford superalgebra** RC_n**.** Let J be a finite index set with an involution $c\colon J\to J$. We denote by J^c the set of fixed points $\{j\in J\,;c(j)=j\}$ and let I denote the set of equivalence classes under the equivalence relation given by $i\sim_c j \Leftrightarrow i=j$ or $i=c(j)$. We denote by pr the canonical projection $J\to I$. The symmetric group $\mathfrak{S}_n$ acts on J^n in a natural way. We define $c_p\colon J\to J$ by

$$c_p\nu = (c^{\delta_{p\ell}}\nu_\ell)_{1\le\ell\le n}\quad\text{for }\nu=(\nu_1,\dots,\nu_n)\in J^n.$$

Let $\widetilde{Q} = (\widetilde{Q}_{i,j}(u,v))_{i,j\in J}$ be a family of polynomials in $\mathbf{k}[u,v]$ satisfying the following conditions: for all $i,j\in J$, we have

$$\begin{aligned}
&\widetilde{Q}_{j,i}(u,v) = \widetilde{Q}_{i,j}(v,u),\\
&\widetilde{Q}_{ci,j}(-u,v) = \widetilde{Q}_{i,cj}(u,-v) = \widetilde{Q}_{i,j}(u,v),\\
&\widetilde{Q}_{i,j}(u,v) = 0\text{ if }\mathrm{pr}(i)=\mathrm{pr}(j).
\end{aligned}\tag{3.5}$$

In particular, $\widetilde{Q}_{i,j}(u,v) = \widetilde{Q}_{i,j}(-u,v)$ for $i\in J^c$.

Definition 3.5. *The* quiver Hecke-Clifford superalgebra RC_n *is the* $\mathbf{k}$*-superalgebra generated by the even generators* $\{y_p\}_{1\le p\le n}$, $\{\sigma_a\}_{1\le a<n}$, $\{e(\nu)\}_{\nu\in J^n}$ *and the odd generators* $\{\mathfrak{c}_p\}_{1\le p\le n}$ *with the following defining relations: for* $\mu,\nu\in J^n$, $1\le p,\ q\le n$, $1\le a\le n-1$, *we have*

(i) $e(\mu)e(\nu) = \delta_{\mu\nu}e(\mu)$, $1 = \sum_{\nu \in J^n} e(\nu)$, $y_p e(\nu) = e(\nu) y_p$, $\mathfrak{c}_p e(\nu) = e(c_p \nu)\mathfrak{c}_p$,
(ii) $y_p y_q = y_q y_p$, $\mathfrak{c}_p \mathfrak{c}_q + \mathfrak{c}_q \mathfrak{c}_p = 2\delta_{pq}$,
(iii) $\mathfrak{c}_p y_q = (-1)^{\delta_{p,q}} y_q \mathfrak{c}_p$,
(iv) $\sigma_a e(\nu) = e(s_a \nu)\sigma_a, \sigma_a \mathfrak{c}_p = \mathfrak{c}_{s_a(p)}\sigma_a$,
(v) $\sigma_a y_p e(\nu) = y_p \sigma_a e(\nu)$ *if* $p \neq a, a+1$,
(vi)

$$(\sigma_a y_{a+1} - y_a \sigma_a) e(\nu) = \begin{cases} e(\nu) & \nu_a = \nu_{a+1} \notin J^c, \\ -\mathfrak{c}_a \mathfrak{c}_{a+1} e(\nu) & \nu_a = c\nu_{a+1} \notin J^c, \\ (1 - \mathfrak{c}_a \mathfrak{c}_{a+1}) e(\nu) & \nu_a = \nu_{a+1} \in J^c, \\ 0 & \textit{otherwise}, \end{cases}$$

or equivalently,

$$\sigma_a y_{a+1} - y_a \sigma_a = \sum_{\nu_a = \nu_{a+1}} e(\nu) - \sum_{\nu_a = c\nu_{a+1}} \mathfrak{c}_a \mathfrak{c}_{a+1} e(\nu),$$

(vii)

$$(y_{a+1} \sigma_a - \sigma_a y_a) e(\nu) = \begin{cases} e(\nu) & \nu_a = \nu_{a+1} \notin J^c, \\ \mathfrak{c}_a \mathfrak{c}_{a+1} e(\nu) & \nu_a = c\nu_{a+1} \notin J^c, \\ (1 + \mathfrak{c}_a \mathfrak{c}_{a+1}) e(\nu) & \nu_a = \nu_{a+1} \in J^c, \\ 0 & \textit{otherwise}, \end{cases}$$

or equivalently,

$$y_{a+1} \sigma_a - \sigma_a y_a = \sum_{\nu_a = \nu_{a+1}} e(\nu) + \sum_{\nu_a = c\nu_{a+1}} \mathfrak{c}_a \mathfrak{c}_{a+1} e(\nu),$$

(viii) $\sigma_a^2 e(\nu) = \widetilde{Q}_{\nu_a, \nu_{a+1}}(y_a, y_{a+1}) e(\nu)$,
(ix) $\sigma_a \sigma_b = \sigma_b \sigma_a$ *if* $|a - b| > 1$,
(x)

$$\begin{aligned} &(\sigma_{a+1}\sigma_a\sigma_{a+1} - \sigma_a\sigma_{a+1}\sigma_a) e(\nu) \\ &= \begin{cases} \dfrac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})}{y_{a+2} - y_a} e(\nu) & \textit{if } \nu_a = \nu_{a+2} \notin J^c, \\[2ex] \dfrac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(-y_a, y_{a+1})}{y_{a+2} + y_a} \mathfrak{c}_a \mathfrak{c}_{a+2} e(\nu) & \textit{if } \nu_a = c\nu_{a+2} \notin J^c, \\[2ex] \dfrac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})}{y_{a+2} - y_a} e(\nu) & \\ \qquad + \dfrac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})}{y_{a+2} + y_a} \mathfrak{c}_a \mathfrak{c}_{a+2} e(\nu) & \textit{if } \nu_a = \nu_{a+2} \in J^c, \\[2ex] 0 & \textit{otherwise}, \end{cases} \end{aligned}$$

or equivalently,

$$\begin{aligned}\sigma_{a+1}\sigma_a\sigma_{a+1}-\sigma_a\sigma_{a+1}\sigma_a &= \sum_{\nu_a=\nu_{a+1}} \frac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2},y_{a+1})-\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a,y_{a+1})}{y_{a+2}-y_a}e(\nu)\\ &\quad+\sum_{\nu_a=c\nu_{a+1}} \frac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2},y_{a+1})-\widetilde{Q}_{\nu_a,\nu_{a+1}}(-y_a,y_{a+1})}{y_{a+2}+y_a}\mathfrak{c}_a\mathfrak{c}_{a+2}e(\nu).\end{aligned}$$

We first calculate the commutation relation between σ_a's and the polynomials in y_k $(k=1,\dots,n)$. Set

$$\begin{aligned}&\mathrm{e}_{a,b}=\sum_{\nu_a=\nu_b\in J}e(\nu), \quad \mathrm{e}^-_{a,b}=\sum_{\nu_a=c\nu_b\in J}e(\nu),\\ &\mathrm{e}_a=\mathrm{e}_{a,a+1}, \quad \mathrm{e}^-_a=\mathrm{e}^-_{a,a+1}.\end{aligned}$$

Hence we have

$$\begin{aligned}\sigma_a y_{a+1}-y_a\sigma_a &= \mathrm{e}_a-\mathfrak{c}_a\mathfrak{c}_{a+1}\mathrm{e}^-_a,\\ y_{a+1}\sigma_a-\sigma_a y_a &= \mathrm{e}_a+\mathfrak{c}_a\mathfrak{c}_{a+1}\mathrm{e}^-_a.\end{aligned}\tag{3.6}$$

Lemma 3.6. *For $1\le a<n$ and $f(y_1,\dots,y_n)\in\mathbf{k}[y_1,\dots,y_n]$, we have*

$$\sigma_a\circ f = s_a(f)\circ\sigma_a+\frac{f-s_a(f)}{y_{a+1}-y_a}\mathrm{e}_a-\mathfrak{c}_a\mathfrak{c}_{a+1}\frac{f-\overline{s}_af}{y_{a+1}+y_a}\mathrm{e}^-_a,\tag{3.7}$$

where

$$\begin{aligned}(s_af)(y_1,\dots,y_n)&=f(y_1,\dots,y_{a-1},y_{a+1},y_a,y_{a+2},\dots,y_n),\\ (\overline{s}_af)(y_1,\dots,y_n)&=f(y_1,\dots,y_{a-1},-y_{a+1},-y_a,y_{a+2},\dots,y_n).\end{aligned}\tag{3.8}$$

Proof. We will use induction on the degree of f. Assume that (3.7) holds for f. We will show that it holds for y_pf as well.

It is evident for $p\neq a,a+1$. If $p=a$, then

$$\begin{aligned}\sigma_a\circ(y_af)&=(y_{a+1}\sigma_a-\mathrm{e}_a-\mathfrak{c}_a\mathfrak{c}_{a+1}\mathrm{e}^-_a)\circ f\\ &=y_{a+1}\Big(s_a(f)\circ\sigma_a+\frac{f-s_a(f)}{y_{a+1}-y_a}\mathrm{e}_a-\mathfrak{c}_a\mathfrak{c}_{a+1}\frac{f-\overline{s}_af}{y_{a+1}+y_a}\mathrm{e}^-_a\Big)-f\mathrm{e}_a-\mathfrak{c}_a\mathfrak{c}_{a+1}f\mathrm{e}^-_a\\ &=s_a(y_af)\circ\sigma_a+\left(\frac{y_{a+1}(f-s_a(f))}{y_{a+1}-y_a}-f\right)\mathrm{e}_a+\mathfrak{c}_a\mathfrak{c}_{a+1}\left(\frac{y_{a+1}(f-\overline{s}_af)}{y_{a+1}+y_a}-f\right)\mathrm{e}^-_a\\ &=s_a(y_af)\circ\sigma_a+\frac{y_af-s_a(y_af)}{y_{a+1}-y_a}\mathrm{e}_a+\mathfrak{c}_a\mathfrak{c}_{a+1}\frac{-y_af+\overline{s}_a(y_af)}{y_{a+1}+y_a}\mathrm{e}^-_a,\end{aligned}$$

and if $p=a+1$, then

$$\begin{aligned}&\sigma_a\circ(y_{a+1}f)\\ &\quad=(y_a\sigma_a+\mathrm{e}_a-\mathfrak{c}_a\mathfrak{c}_{a+1}\mathrm{e}^-_a)\circ f\\ &\quad=y_a\Big(s_a(f)\circ\sigma_a+\frac{f-s_a(f)}{y_{a+1}-y_a}\mathrm{e}_a-\mathfrak{c}_a\mathfrak{c}_{a+1}\frac{f-\overline{s}_af}{y_{a+1}+y_a}\mathrm{e}^-_a\Big)+f\mathrm{e}_a-\mathfrak{c}_a\mathfrak{c}_{a+1}f\mathrm{e}^-_a\\ &\quad=s_a(y_{a+1}f)\circ\sigma_a+\left(\frac{y_a(f-s_a(f))}{y_{a+1}-y_a}+f\right)\mathrm{e}_a+\mathfrak{c}_a\mathfrak{c}_{a+1}\left(\frac{y_a(f-\overline{s}_af)}{y_{a+1}+y_a}-f\right)\mathrm{e}^-_a\\ &\quad=s_a(y_{a+1}f)\circ\sigma_a+\frac{y_{a+1}f-s_a(y_{a+1}f)}{y_{a+1}-y_a}\mathrm{e}_a+\mathfrak{c}_a\mathfrak{c}_{a+1}\frac{-y_{a+1}f+\overline{s}_a(y_{a+1}f)}{y_{a+1}+y_a}\mathrm{e}^-_a,\end{aligned}$$

which proves our assertion. $\square$

Note that the $\mathbf{k}$-algebra $(\mathbf{k}^J)^{\otimes n}$ and $\mathfrak{C}_n$ are canonically identified with $\bigoplus_{\nu\in J^n}\mathbf{k}e(\nu)$ and $\langle\{\mathfrak{c}_p\}_{1\le p\le n} \mid \mathfrak{c}_p\mathfrak{c}_q+\mathfrak{c}_q\mathfrak{c}_p=2\delta_{pq}\rangle$, respectively. We denote by $\mathcal{A}_n$ the superalgebra generated by $\{y_p, \mathfrak{c}_p, e(\nu) \mid 1\le p\le n,\ \nu\in J^n\}$ with the defining relations (i)–(iii) given above. Clearly, we have a $\mathbf{k}$-supermodule isomorphism

$$(\mathbf{k}^J)^{\otimes n}\otimes \mathbf{k}[y_1,\dots,y_n]\otimes\mathfrak{C}_n \xrightarrow{\sim} \mathcal{A}_n, \quad e(\nu)\otimes f\otimes c\longmapsto e(\nu)fc,$$

where $\nu\in J^n$, $f\in\mathbf{k}[y_1,\dots,y_n]$ and $c\in\mathfrak{C}_n$.

Remark 3.7. The algebras $\mathcal{A}_n$ and RC_n have an anti-involution that sends the generators y_p, $\mathfrak{c}_p$, σ_a, $e(\nu)$ to themselves.

3.4. **Realization of quiver Hecke-Clifford superalgebras.** Let $\mathcal{K}_n$ be the superalgebra $\mathbf{k}[y_1,\dots,y_n][(y_a-y_b)^{-1},(y_a+y_b)^{-1}; 1\le a<b\le n]\otimes_{\mathbf{k}[y_1,\dots,y_n]}\mathcal{A}_n$. The symmetric group $\mathfrak{S}_n$ acts on $\mathcal{A}_n$ and $\mathcal{K}_n$.

For $1\le a,b\le n$ with $a\neq b$, set

$$\begin{aligned}
\widetilde{Q}_{a,b} &= \sum_{\nu\in J^n}\widetilde{Q}_{\nu_a,\nu_b}(y_a,y_b)e(\nu),\\
R_{a,b} &= \widetilde{Q}_{a,b}-(y_a-y_b)^{-2}\mathrm{e}_{a,b}-(y_a+y_b)^{-2}\mathrm{e}^-_{a,b}\\
&= \sum_{\mathrm{pr}(\nu_a)\neq\mathrm{pr}(\nu_b)}\widetilde{Q}_{\nu_a,\nu_b}(y_a,y_b)e(\nu)-\sum_{\nu_a=\nu_b\notin J^c}\frac{1}{(y_a-y_b)^2}e(\nu)\\
&\qquad -\sum_{\nu_a=c\nu_b\notin J^c}\frac{1}{(y_a+y_b)^2}e(\nu)-\sum_{\nu_a=\nu_b\in J^c}2\frac{y_a^2+y_b^2}{(y_a^2-y_b^2)^2}e(\nu)\in\mathcal{K}_n.
\end{aligned}\tag{3.9}$$

Then the $R_{a,b}$'s belong to the center of $\mathcal{K}_n$ and satisfy the properties

$$R_{a,b}=R_{b,a}\quad\text{and}\quad w(R_{a,b})=R_{w(a),w(b)}\quad\text{for all } w\in\mathfrak{S}_n.$$

Let us denote by $\mathcal{KS}_n$ the $\mathbf{k}$-superalgebra generated by the $\mathbf{k}$-superalgebra $\mathcal{K}_n$ and $\widetilde{s}_k$ $(k=1,\dots,n-1)$ satisfying the defining relations:

$$\begin{cases}\text{the } \widetilde{s}_k\text{'s satisfy the braid relations (3.4)},\\ \widetilde{s}_k^2=R_{k,k+1}\text{ and},\\ \widetilde{s}_k\circ a=(s_ka)\circ\widetilde{s}_k \text{ for all } a\in\mathcal{K}_n.\end{cases}\tag{3.10}$$

Then by Proposition 3.3, we have a $\mathbf{k}$-linear isomorphism

$$\mathcal{K}_n\otimes\mathbf{k}[\mathfrak{S}_n]\xrightarrow{\sim}\mathcal{KS}_n.\tag{3.11}$$

Theorem 3.8. *Assume that $(\widetilde{Q}_{i,j})_{i,j\in J}$ satisfies the conditions* (3.5)*. Set*

$$\sigma_a e(\nu)=\begin{cases}\widetilde{s}_a e(\nu) & \text{if } \mathrm{pr}(\nu_a)\neq\mathrm{pr}(\nu_{a+1}),\\ (\widetilde{s}_a-(y_a-y_{a+1})^{-1})e(\nu) & \text{if } \nu_a=\nu_{a+1}\notin J^c,\\ (\widetilde{s}_a+(y_a+y_{a+1})^{-1}\mathfrak{c}_a\mathfrak{c}_{a+1})e(\nu) & \text{if } \nu_a=c\nu_{a+1}\notin J^c,\\ \big(\widetilde{s}_a-(y_a-y_{a+1})^{-1}+(y_a+y_{a+1})^{-1}\mathfrak{c}_a\mathfrak{c}_{a+1}\big)e(\nu) & \text{if } \nu_a=\nu_{a+1}\in J^c.\end{cases}$$

Then the σ_a's satisfy the commutation relations in Definition 3.5*. Moreover, the superalgebra homomorphism $\mathrm{RC}_n\longrightarrow\mathcal{KS}_n$ thus obtained is injective.*

Proof. For $1 \le a, b \le n$ with $a \ne b$, set

$$f_{a,b} = (y_b - y_a)^{-1}\mathrm{e}_{a,b} + (y_b + y_a)^{-1}\mathfrak{c}_a\mathfrak{c}_b\mathrm{e}^-_{a,b} \in \mathcal{K}_n, \tag{3.12}$$

and $f_a = f_{a,a+1}$. Then we have

$$\sigma_a = \widetilde{s}_a + f_a$$

and $f_{a,b}$'s satisfy the relations

$$\begin{aligned} w(f_{a,b}) &= f_{w(a),w(b)} \quad \text{for any } w \in \mathfrak{S}_n, \\ f_{a,b} &= -f_{b,a}. \end{aligned}$$

We can easily verify that $f_a\mathfrak{c}_b = \mathfrak{c}_{s_a(b)}f_a$.

Since

$$f_a^2 = (y_a - y_{a+1})^{-2}\mathrm{e}_a + (y_a + y_{a+1})^{-2}\mathrm{e}^-_a,$$

we obtain

$$R_{a,b} = \widetilde{Q}_{a,b} - f^2_{a,b}. \tag{3.13}$$

Note that the commutation relations (i)–(v) and (ix) are obvious. Moreover, (vi) follows from $f_{a,b}y_b - y_af_{a,b} = \mathrm{e}_{a,b} - \mathfrak{c}_a\mathfrak{c}_b\mathrm{e}^-_{a,b}$ and (vii) can be verified in a similar manner.

For the relation (viii), since $\widetilde{s}_a$ and f_a anti-commute with each other, we have

$$\sigma_a^2 = \widetilde{s}_a^2 + f_a^2 = R_{a,a+1} + f_a^2 = \widetilde{Q}_{a,a+1}$$

as desired, where the last equality follows from (3.13).

It remains to verify the relation (x). By the definition, we have

$$\begin{aligned} \sigma_{a+1}\sigma_a\sigma_{a+1} &= \sigma_{a+1}\sigma_a(\widetilde{s}_{a+1} + f_{a+1}) \\ &= \sigma_{a+1}(\widetilde{s}_a\widetilde{s}_{a+1} + (s_af_{a+1})\widetilde{s}_a + f_a\widetilde{s}_{a+1} + f_af_{a+1}) \\ &= \widetilde{s}_{a+1}\widetilde{s}_a\widetilde{s}_{a+1} + (s_{a+1}s_af_{a+1})\widetilde{s}_{a+1}\widetilde{s}_a + (s_{a+1}f_a)\widetilde{s}^2_{a+1} + (s_{a+1}f_a)(s_{a+1}f_{a+1})\widetilde{s}_{a+1} \\ &\quad + f_{a+1}\widetilde{s}_a\widetilde{s}_{a+1} + f_{a+1}(s_af_{a+1})\widetilde{s}_a + f_{a+1}f_a\widetilde{s}_{a+1} + f_{a+1}f_af_{a+1} \\ &= \widetilde{s}_{a+1}\widetilde{s}_a\widetilde{s}_{a+1} + f_a\widetilde{s}_{a+1}\widetilde{s}_a + f_{a+1}\widetilde{s}_a\widetilde{s}_{a+1} \\ &\quad + (f_{a+1}f_a - f_{a,a+2}f_{a+1})\widetilde{s}_{a+1} + f_{a+1}f_{a,a+2}\widetilde{s}_a + f_{a+1}f_af_{a+1} + f_{a,a+2}R_{a+1,a+2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\sigma_a\sigma_{a+1}\sigma_a \\ &\quad = \widetilde{s}_a\widetilde{s}_{a+1}\widetilde{s}_a + (s_as_{a+1}f_a)\widetilde{s}_a\widetilde{s}_{a+1} + (s_af_{a+1})\widetilde{s}^2_a + (s_af_{a+1})(s_af_a)\widetilde{s}_a \\ &\quad\quad + f_a\widetilde{s}_{a+1}\widetilde{s}_a + f_a(s_{a+1}f_a)\widetilde{s}_{a+1} + f_af_{a+1}\widetilde{s}_a + f_af_{a+1}f_a \\ &\quad = \widetilde{s}_a\widetilde{s}_{a+1}\widetilde{s}_a + f_{a+1}\widetilde{s}_a\widetilde{s}_{a+1} + f_a\widetilde{s}_{a+1}\widetilde{s}_a \\ &\quad\quad + (f_af_{a+1} - f_{a,a+2}f_a)\widetilde{s}_a + f_af_{a,a+2}\widetilde{s}_{a+1} + f_af_{a+1}f_a + f_{a,a+2}R_{a,a+1}. \end{aligned}$$

Hence our assertion would follow from the following equalities:

$$f_a f_{a+1} = f_{a,a+2} f_a + f_{a+1} f_{a,a+2}, \quad f_{a+1} f_a = f_{a,a+2} f_{a+1} + f_a f_{a,a+2},$$

$$\begin{aligned}&f_{a+1} f_a f_{a+1} - f_a f_{a+1} f_a + f_{a,a+2}(R_{a+1,a+2} - R_{a,a+1})\\ &\quad = \sum_{\nu_a=\nu_{a+1}} \frac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})}{y_{a+2} - y_a} e(\nu)\\ &\qquad + \sum_{\nu_a=c\nu_{a+1}} \frac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(-y_a, y_{a+1})}{y_{a+2} + y_a} \mathfrak{c}_a \mathfrak{c}_{a+2} e(\nu).\end{aligned} \tag{3.14}$$

In order to show the first and the second equalities, let us show

$$f_{a,b} f_{b,c} + f_{b,c} f_{c,a} + f_{c,a} f_{a,b} = 0 \quad \text{for distinct elements } a, b, c \in \{1, \dots, n\}. \tag{3.15}$$

Set $h_{a,b} = (y_b - y_a)^{-1} \mathrm{e}_{a,b}$. Then we have

$$f_{a,b} = h_{a,b} + \mathfrak{c}_a h_{a,b} \mathfrak{c}_b = h_{a,b} + \mathfrak{c}_b h_{a,b} \mathfrak{c}_a,$$

which yields

$$\begin{aligned} f_{a,b} f_{b,c} &= (h_{a,b} + \mathfrak{c}_b h_{a,b} \mathfrak{c}_a)(h_{b,c} + \mathfrak{c}_c h_{b,c} \mathfrak{c}_b) \\ &= h_{ab} h_{bc} + \mathfrak{c}_b h_{a,b} h_{b,c} \mathfrak{c}_a + \mathfrak{c}_c h_{a,b} h_{b,c} \mathfrak{c}_b + \mathfrak{c}_a h_{a,b} h_{b,c} \mathfrak{c}_c. \end{aligned}$$

Hence (3.15) is a consequence of

$$h_{a,b} h_{b,c} + h_{b,c} h_{c,a} + h_{c,a} h_{a,b} = h_{a,b} h_{b,c} h_{c,a} \big((y_a - y_c) + (y_b - y_a) + (y_c - y_b)\big) = 0.$$

Let us prove the last equality. Since $f_{a,b}^2$ belongs to the center of $\mathcal{K}_n$, by the first equality in (3.14), we have

$$\begin{aligned} &f_{a+1} f_a f_{a+1} - f_a f_{a+1} f_a + f_{a,a+2}(f_a^2 - f_{a+1}^2) \\ &\qquad = f_{a+1}(f_a f_{a+1} - f_{a+1} f_{a,a+2}) + (-f_a f_{a+1} + f_{a,a+2} f_a) f_a \\ &\qquad = f_{a+1}(f_{a,a+2} f_a) + (-f_{a+1} f_{a,a+2}) f_a = 0. \end{aligned}$$

It follows that

$$f_{a+1} f_a f_{a+1} - f_a f_{a+1} f_a + f_{a,a+2}(R_{a+1,a+2} - R_{a,a+1}) = f_{a,a+2}(\widetilde{Q}_{a+1,a+2} - \widetilde{Q}_{a,a+1}).$$

Since $\widetilde{Q}_{a,b}$ is in $Z(\mathcal{A}_n)$, we have

$$f_{a,a+2}(\widetilde{Q}_{a+1,a+2} - \widetilde{Q}_{a,a+1}) = h_{a,a+2}(\widetilde{Q}_{a+1,a+2} - \widetilde{Q}_{a,a+1}) + \mathfrak{c}_a h_{a,a+2}(\widetilde{Q}_{a+1,a+2} - \widetilde{Q}_{a,a+1}) \mathfrak{c}_{a+2},$$

as desired.

Thus we have constructed a superalgebra homomorphism $\mathrm{RC}_n \to \mathcal{KS}_n$. For each $w \in \mathfrak{S}_n$, we choose a reduced expression $s_{i_1} \cdots s_{i_\ell}$ of w, and set $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_\ell}$. Then, by the commutation relations, it is easy to see that RC_n is generated by

$$\left\{ y^a \mathfrak{c}^\eta \sigma_w e(\nu)\,;\, a \in \mathbb{Z}_{\geq 0}^n,\ \eta \in (\mathbb{Z}/2\mathbb{Z})^n,\ w \in \mathfrak{S}_n,\ \nu \in J^n \right\}$$

(see Corollary 3.9 below for the notation) as a $\mathbf{k}$-module. It is straightforward to verify that its image is linearly independent in $\mathcal{KS}_n$. Hence $\mathrm{RC}_n \to \mathcal{KS}_n$ is injective. □

Theorem 3.8 immediately implies the following corollary.

Corollary 3.9. *For each $w \in \mathfrak{S}_n$, we choose a reduced expression $s_{i_1} \cdots s_{i_\ell}$ of w, and set $\sigma_w = \sigma_{i_1} \cdots \sigma_{i_\ell}$. Then*

$$\left\{ y^a \mathfrak{c}^\eta \sigma_w e(\nu) \, ; a \in \mathbb{Z}_{\geq 0}^n, \, \eta \in (\mathbb{Z}/2\mathbb{Z})^n, \, w \in \mathfrak{S}_n, \, \nu \in J^n \right\}$$

is a basis of RC_n. *Here,* $y^a = y_1^{a_1} \cdots y_n^{a_n}$ *for* $a = (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}$ *and* $\mathfrak{c}^\eta = \mathfrak{c}_1^{\eta_1} \cdots \mathfrak{c}_n^{\eta_n}$ *for* $\eta = (\eta_1, \cdots, \eta_n) \in (\mathbb{Z}/2\mathbb{Z})^n$.

3.5. **Weak Morita superequivalence between** RC_n **and** R_n**.** Let J, $c \colon J \to J$, J^c, $\mathrm{pr} \colon J \to I$ and $\widetilde{Q} = (\widetilde{Q}_{i,j})_{i,j \in J}$ be as in the preceding subsection.

Set $I_{\mathrm{odd}} := \mathrm{pr}(J^c)$ and $I_{\mathrm{even}} := I \setminus I_{\mathrm{odd}}$. Then $\#\mathrm{pr}^{-1}(i) = 1$ or 2 according as $i \in I_{\mathrm{odd}}$ or $i \in I_{\mathrm{even}}$. Choose $J^\dagger \subset J$ such that the projection pr induces a bijection $J^\dagger \to I$ and let $\mathrm{rp} \colon I \to J^\dagger$ be its inverse.

Definition 3.10. *We define* $\mathrm{RC}_n^\dagger = e^\dagger \mathrm{RC}_n e^\dagger$, *where* $e^\dagger = \sum_{\nu \in J^{\dagger n}} e(\nu)$.

Since $\mathfrak{c}_{a_1} \cdots \mathfrak{c}_{a_k} e(\nu) \mathfrak{c}_{a_k} \cdots \mathfrak{c}_{a_1} = e(c_{a_1} \cdots c_{a_k} \nu)$, we have $\mathrm{RC}_n e^\dagger \mathrm{RC}_n = \mathrm{RC}_n$. Hence Remark 2.10 (ii) implies that RC_n and $\mathrm{RC}_n^\dagger$ are Morita superequivalent.

It is easy to see that $\mathrm{RC}_n^\dagger$ is generated by the even generators $\{y_p e^\dagger\}_{1 \leq p \leq n}$, $\{\sigma_a e^\dagger\}_{1 \leq a < n}$, $\{e(\nu)\}_{\nu \in J^{\dagger n}}$ and the odd generators $\{e^\dagger \mathfrak{c}_p e^\dagger\}_{1 \leq p \leq n}$. For simplicity, we will write y_p for $y_p e^\dagger$, σ_a for $\sigma_a e^\dagger$, $\{e(\nu)\}_{\nu \in I^n}$ for $\{e(\nu)\}_{\nu \in J^{\dagger n}}$, and $\mathfrak{c}_p$ for $\mathfrak{c}_p e^\dagger$. Then the defining relations for these generators are give as follows:

(i) $e(\mu)e(\nu) = \delta_{\mu\nu} e(\mu)$ for all $\mu, \nu \in I^n$, $1 = \sum_{\nu \in I^n} e(\nu)$,
(ii) $y_p y_q = y_q y_p$, $\mathfrak{c}_p \mathfrak{c}_q = -\mathfrak{c}_q \mathfrak{c}_p$ for all $1 \leq p < q \leq n$,
(iii) $y_p e(\nu) = e(\nu) y_p$ and $\mathfrak{c}_p e(\nu) = e(\nu) \mathfrak{c}_p$ for any $\nu \in I^n$,
(iv) $\mathfrak{c}_p e(\nu) = 0$ if $\nu_p \in I_{\mathrm{even}}$, and $\mathfrak{c}_p^2 e(\nu) = e(\nu)$ if $\nu_p \in I_{\mathrm{odd}}$,
(v) $\mathfrak{c}_p y_q = (-1)^{\delta_{pq}} y_q \mathfrak{c}_p$,
(vi) $\sigma_a e(\nu) = e(s_a \nu) \sigma_a, \sigma_a \mathfrak{c}_p = \mathfrak{c}_{s_a(p)} \sigma_a$,
(vii) $\sigma_a y_p e(\nu) = y_p \sigma_a e(\nu)$ if $p \neq a, a+1$,
(viii)

$$(\sigma_a y_{a+1} - y_a \sigma_a) e(\nu) = \begin{cases} e(\nu) & \text{if } \nu_a = \nu_{a+1} \in I_{\mathrm{even}}, \\ (1 - \mathfrak{c}_a \mathfrak{c}_{a+1}) e(\nu) & \text{if } \nu_a = \nu_{a+1} \in I_{\mathrm{odd}}, \\ 0 & \text{if } \nu_a \neq \nu_{a+1}. \end{cases}$$

(ix)

$$(y_{a+1} \sigma_a - \sigma_a y_a) e(\nu) = \begin{cases} e(\nu) & \text{if } \nu_a = \nu_{a+1} \in I_{\mathrm{even}}, \\ (1 + \mathfrak{c}_a \mathfrak{c}_{a+1}) e(\nu) & \text{if } \nu_a = \nu_{a+1} \in I_{\mathrm{odd}}, \\ 0 & \text{if } \nu_a \neq \nu_{a+1}. \end{cases}$$

(x) $\sigma_a^2 e(\nu) = \widetilde{Q}_{\nu_a, \nu_{a+1}}(y_a, y_{a+1}) e(\nu)$,
(xi) $\sigma_a \sigma_b = \sigma_b \sigma_a$ if $|a - b| > 1$,

(xii)

$$(\sigma_{a+1}\sigma_a\sigma_{a+1} - \sigma_a\sigma_{a+1}\sigma_a)e(\nu) = \begin{cases} \dfrac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2},y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a,y_{a+1})}{y_{a+2}-y_a} & \text{if } \nu_a = \nu_{a+2} \in I_{\mathrm{even}}, \\ \dfrac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2},y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a,y_{a+1})}{y_{a+2}-y_a} \\ \qquad -\mathfrak{c}_a\mathfrak{c}_{a+2}\dfrac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2},y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a,y_{a+1})}{y_{a+2}+y_a} & \text{if } \nu_a = \nu_{a+2} \in I_{\mathrm{odd}}, \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3.9 implies that the set

$$\begin{aligned} \{y^a\mathfrak{c}^\eta e(\nu)\sigma_w \mid \ & a \in \mathbb{Z}_{\ge 0}^n,\ \eta \in (\mathbb{Z}/2\mathbb{Z})^n,\ \nu \in I^n,\ w \in \mathfrak{S}_n \\ & \text{such that } \eta_p = 0 \text{ as soon as } \nu_p \in I_{\mathrm{even}}\} \end{aligned} \tag{3.16}$$

is a basis of $\mathrm{RC}_n^\dagger$.

Definition 3.11. *For $i,j \in I$, recall that $\mathscr{A}_{i,j} = \mathbf{k}\langle w,z\rangle/\langle zw - (-1)^{\mathrm{par}(i)\,\mathrm{par}(j)}wz\rangle$. We define $\widetilde{\mathscr{A}_{i,j}}$ to be the subalgebra of $\mathscr{A}_{i,j}$ generated by $w^{1+\mathrm{par}(i)}$ and $z^{1+\mathrm{par}(j)}$. Similarly, we define $\mathscr{B}_{i,j}$ to be the commutative polynomial ring $\mathbf{k}[u,v]$ and denote by $\widetilde{\mathscr{B}}_{i,j}$ its subalgebra generated by $u^{1+\mathrm{par}(i)}$ and $v^{1+\mathrm{par}(j)}$.*

Then $\widetilde{\mathscr{A}_{i,j}}$ is commutative, and $\widetilde{\mathscr{B}}_{i,j}$ and $\widetilde{\mathscr{A}_{i,j}}$ are isomorphic by the correspondence

$$u^{1+\mathrm{par}(i)} = (-1)^{\mathrm{par}(i)}w^{1+\mathrm{par}(i)}, \quad v^{1+\mathrm{par}(j)} = (-1)^{\mathrm{par}(j)}z^{1+\mathrm{par}(j)}. \tag{3.17}$$

Let $\widetilde{Q} = (\widetilde{Q}_{i,j}(u,v))_{i,j\in J}$ be as in (3.5). For $i,j \in I$, we denote by $Q_{i,j}$ the element of $\widetilde{\mathscr{A}_{i,j}} \subset \mathscr{A}_{i,j}$ corresponding to $\widetilde{Q}_{\mathrm{rp}(i),\mathrm{rp}(j)} \in \widetilde{\mathscr{B}}_{\mathrm{rp}(i),\mathrm{rp}(j)}$. Then we can easily see that $Q := (Q_{i,j})_{i,j\in I}$ satisfies the condition (3.1), and hence it defines the quiver Hecke superalgebra $\mathrm{R}_n = \mathrm{R}_n(Q)$ (see 3.1).

Remark 3.12. A different choice of $J^\dagger$ would yield a different matrix Q' which is within the re-scaling of Q given in Remark 3.2. Hence the quiver Hecke superalgebra $\mathrm{R}_n(Q)$ does not depend on the choice of $J^\dagger$ up to isomorphism.

For each $\beta = \sum_{i\in I} m_i\alpha_i \in Q_+ := \sum_{i\in I}\mathbb{Z}_{\ge 0}\alpha_i$ with $\mathrm{ht}(\beta) := \sum_{i\in I} m_i = n$, we define

$$\mathrm{R}_\beta = e_\beta \mathrm{R}_n e_\beta, \quad \mathrm{RC}_\beta = e^J(\beta)\mathrm{RC}_n e^J(\beta) \quad \text{and} \quad \mathrm{RC}_\beta^\dagger = e_\beta \mathrm{RC}_n^\dagger e_\beta, \tag{3.18}$$

where

$$\begin{aligned} I^\beta &= \{(i_1,\dots,i_n) \in I^n\,;\, \textstyle\sum_{a=1}^n \alpha_{i_a} = \beta\}, & e_\beta &= \textstyle\sum_{\nu\in I^\beta} e(\nu), \\ J^\beta &= \big\{(j_1,\dots,j_n) \in J^n\,;\, \textstyle\sum_{a=1}^n \alpha_{\mathrm{pr}(j_a)} = \beta\big\}, & e_\beta^J &= \textstyle\sum_{\nu\in J^\beta} e(\nu). \end{aligned}$$

It is easy to see that e_β is a central even idempotent and $\mathrm{R}_n = \bigoplus_{\mathrm{ht}(\beta)=n} \mathrm{R}_\beta$.

Theorem 3.13. *For each $\beta = \sum_{i\in I} m_i\alpha_i \in Q_+$, we have a $\mathbf{k}$-superalgebra isomorphism*

$$\mathrm{R}_\beta \otimes \mathfrak{C}_\ell \xrightarrow{\sim} \mathrm{RC}_\beta^\dagger,$$

where $\ell = \sum_{i \in I_{\mathrm{odd}}} m_i$. *In particular, if* $\mathbf{k}$ *contains* $\sqrt{-1}$, *then* R_β *and* RC_β *are weakly Morita superequivalent.*

For the notion of weak Morita superequivalence, see § 2.4.

Proof. To prove our theorem, we use the following strategy.

(a) We first construct the elements $\{x_p\}_{1 \le p \le n}$, $\{\tau_a\}_{1 \le a < n}$, $\{e(\nu)\}_{\nu \in I^\beta}$ in $\mathrm{RC}_\beta^\dagger$ which satisfy the defining relations for R_β.

(b) We then construct the elements $\{C_i\}_{1 \le i \le \ell}$ in $\mathrm{RC}_\beta^\dagger$ satisfying the relations $C_iC_j + C_jC_i = 2\delta_{i,j}$ $(1 \le i, j \le \ell)$ for the Clifford superalgebra.

(c) We show that the C_i's supercommute with the $x_pe(\nu)$'s and the $\sigma_ae(\nu)$'s.

(d) Finally, we prove that the resulting superalgebra homomorphism $\mathrm{R}_\beta \otimes \mathfrak{C}_\ell \xrightarrow{\sim} \mathrm{RC}_\beta^\dagger$ is an isomorphism.

(a) Let us choose $\gamma_{i,j} \in \mathbf{k}^\times$ $(i, j \in I)$ such that

$$\begin{aligned} &\text{(i) } \gamma_{i,j} = 1 \text{ if } i \in I_{\mathrm{even}} \text{ or } j \in I_{\mathrm{even}}, \\ &\text{(ii) } \gamma_{i,j}\gamma_{j,i} = -1/2 \text{ if } i, j \in I_{\mathrm{odd}} \text{ and } i \neq j, \\ &\text{(iii) } \gamma_{i,i} = 1/2 \text{ if } i \in I_{\mathrm{odd}}. \end{aligned} \tag{3.19}$$

It is obvious that such $\left(\gamma_{i,j}\right)_{i,j \in I}$ exists. We define

$$\begin{aligned} x_pe(\nu) &= \begin{cases} \mathfrak{c}_p y_p e(\nu) & \text{if } \nu_p \in I_{\mathrm{odd}}, \\ y_pe(\nu) & \text{if } \nu_p \in I_{\mathrm{even}}, \end{cases} \\ \tau_ae(\nu) &= \gamma_{\nu_a,\nu_{a+1}}(\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})}\sigma_ae(\nu). \end{aligned}$$

Then we can easily check the commutation relations

$$\begin{aligned} & x_ax_be(\nu) = (-1)^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_b)}x_bx_ae(\nu) \qquad \text{for } a \neq b, \\ & \tau_ax_pe(\nu) = (-1)^{\mathrm{par}(\nu_p)\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})}x_p\tau_ae(\nu) \quad \text{for } p \neq a, a+1, \\ & (\tau_ax_{a+1} - x_a\tau_a)e(\nu) = (x_{a+1}\tau_a - \tau_ax_a)e(\nu) = \delta_{\nu_a,\nu_{a+1}}e(\nu) \\ & \qquad\qquad\qquad\qquad \text{if } \nu_a \in I_{\mathrm{even}} \text{ or } \nu_{a+1} \in I_{\mathrm{even}}. \end{aligned}$$

Here, we have used (3.19 i). If $\nu_a \in I_{\mathrm{odd}}$ and $\nu_{a+1} \in I_{\mathrm{odd}}$, then we also have the relations

$$(\tau_ax_{a+1} + x_a\tau_a)e(\nu) = (x_{a+1}\tau_a + \tau_ax_a)e(\nu) = \delta_{\nu_a,\nu_{a+1}}e(\nu).$$

Indeed, using (3.19 ii), we have

$$\begin{aligned} (\tau_ax_{a+1} + x_a\tau_a)e(\nu) &= \gamma_{\nu_a,\nu_{a+1}}((\mathfrak{c}_a - \mathfrak{c}_{a+1})\sigma_a\mathfrak{c}_{a+1}y_{a+1} + \mathfrak{c}_ay_a(\mathfrak{c}_a - \mathfrak{c}_{a+1})\sigma_a)e(\nu) \\ &= \gamma_{\nu_a,\nu_{a+1}}(1 + \mathfrak{c}_a\mathfrak{c}_{a+1})(\sigma_ay_{a+1} - y_a\sigma_a)e(\nu) \\ &= \gamma_{\nu_a,\nu_{a+1}}\delta_{\nu_a,\nu_{a+1}}(1 + \mathfrak{c}_a\mathfrak{c}_{a+1})(1 - \mathfrak{c}_a\mathfrak{c}_{a+1})e(\nu) \\ &= \delta_{\nu_a,\nu_{a+1}}e(\nu), \end{aligned}$$

and

$$\begin{aligned} (x_{a+1}\tau_a + \tau_ax_a)e(\nu) &= \gamma_{\nu_a,\nu_{a+1}}(\mathfrak{c}_{a+1}y_{a+1}(\mathfrak{c}_a - \mathfrak{c}_{a+1})\sigma_a + (\mathfrak{c}_a - \mathfrak{c}_{a+1})\sigma_a\mathfrak{c}_ay_a)e(\nu) \\ &= \gamma_{\nu_a,\nu_{a+1}}(1 - \mathfrak{c}_a\mathfrak{c}_{a+1})(y_{a+1}\sigma_a - \sigma_ay_a)e(\nu) \\ &= \gamma_{\nu_a,\nu_{a+1}}\delta_{\nu_a,\nu_{a+1}}(1 - \mathfrak{c}_a\mathfrak{c}_{a+1})(1 + \mathfrak{c}_a\mathfrak{c}_{a+1})e(\nu) \\ &= \delta_{\nu_a,\nu_{a+1}}e(\nu) \end{aligned}$$

as expected.

Let us show the relation

$$\tau_a^2 e(\nu) = Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})e(\nu). \tag{3.20}$$

Note that

$$y_k^2 e(\nu) = -x_k^2 e(\nu) \quad \text{if } \nu_k \in I_{\mathrm{odd}}, \tag{3.21}$$

implies

$$\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})e(\nu) = Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})e(\nu).$$

Hence (3.20) is obvious unless $\nu_a, \nu_{a+1} \in I_{\mathrm{odd}}$. If $\nu_a, \nu_{a+1} \in I_{\mathrm{odd}}$, by (3.19 ii), we have

$$\tau_a^2 e(\nu) = -\gamma_{\nu_a,\nu_{a+1}}\gamma_{\nu_{a+1},\nu_a}(\mathfrak{c}_a - \mathfrak{c}_{a+1})^2\sigma_a^2 e(\nu) = \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})e(\nu).$$

Let us show the commutation relation (vii) in Definition 3.1. We first note that

$$(\mathfrak{c}_a - \mathfrak{c}_b)\mathfrak{c}_c e(\nu) = -\mathfrak{c}_{s_{ab}(c)}(\mathfrak{c}_a - \mathfrak{c}_b)e(\nu) \quad \text{if } a \neq b \text{ and } \nu_a, \nu_b \in I_{\mathrm{odd}}, \tag{3.22}$$

$$(\mathfrak{c}_a - \mathfrak{c}_b)^2 e(\nu) = (\mathrm{par}(\nu_a) + \mathrm{par}(\nu_b))e(\nu) \quad \text{if } a \neq b, \tag{3.23}$$

where s_{ab} is the transposition of a and b.

Set

$$\Delta := (\tau_{a+1}\tau_a\tau_{a+1} - \tau_a\tau_{a+1}\tau_a)e(\nu).$$

Then we have

$$\begin{aligned}
\tau_{a+1}\tau_a\tau_{a+1}e(\nu) &= \gamma_{\nu_a,\nu_{a+1}}(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})}\sigma_{a+1} \\
&\qquad \gamma_{\nu_a,\nu_{a+2}}(\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+2})}\sigma_a \\
&\qquad\quad \gamma_{\nu_{a+1},\nu_{a+2}}(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_{a+1})\mathrm{par}(\nu_{a+2})}\sigma_{a+1}e(\nu) \\
&= \gamma_{\nu_a,\nu_{a+1}}\gamma_{\nu_a,\nu_{a+2}}\gamma_{\nu_{a+1},\nu_{a+2}} \\
&\qquad (\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})}(\mathfrak{c}_a - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+2})} \\
&\qquad\quad (\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(\nu_{a+1})\mathrm{par}(\nu_{a+2})}\sigma_{a+1}\sigma_a\sigma_{a+1}e(\nu),
\end{aligned}$$

and

$$\begin{aligned}
\tau_a\tau_{a+1}\tau_a e(\nu) &= \gamma_{\nu_{a+1},\nu_{a+2}}(\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(\nu_{a+1})\mathrm{par}(\nu_{a+2})}\sigma_a \\
&\qquad \gamma_{\nu_a,\nu_{a+2}}(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+2})}\sigma_{a+1} \\
&\qquad\quad \gamma_{\nu_a,\nu_{a+1}}(\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})}\sigma_a e(\nu) \\
&= \gamma_{\nu_a,\nu_{a+1}}\gamma_{\nu_a,\nu_{a+2}}\gamma_{\nu_{a+1},\nu_{a+2}} \\
&\qquad (\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(\nu_{a+1})\mathrm{par}(\nu_{a+2})}(\mathfrak{c}_a - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+2})} \\
&\qquad\quad (\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})}\sigma_a\sigma_{a+1}\sigma_a e(\nu).
\end{aligned}$$

We can check easily

$$\begin{aligned}
&(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{s_2}(\mathfrak{c}_a - \mathfrak{c}_{a+2})^{s_0}(\mathfrak{c}_a - \mathfrak{c}_{a+1})^{s_1}e(s_a s_{a+1} s_a \nu) \\
&\qquad = (\mathfrak{c}_a - \mathfrak{c}_{a+1})^{s_1}(\mathfrak{c}_a - \mathfrak{c}_{a+2})^{s_0}(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{s_2}e(s_a s_{a+1} s_a \nu)
\end{aligned}$$

in the case $s_0 = s_1 s_2 = 0$ and the case $s_0 - 1 = s_1 - s_2 = 0$. Indeed, by (3.22), we have

$$\begin{aligned}(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^s(\mathfrak{c}_a - \mathfrak{c}_{a+2})(\mathfrak{c}_a - \mathfrak{c}_{a+1})^s e(\nu) \\ = (\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^s(-\mathfrak{c}_{a+2} + \mathfrak{c}_{a+1})^s(\mathfrak{c}_a - \mathfrak{c}_{a+2})e(\nu) \\ = (1 + \mathrm{par}(\nu_{a+1}))^s(\mathfrak{c}_a - \mathfrak{c}_{a+2})e(\nu).\end{aligned} \tag{3.24}$$

Hence we have

$$\begin{aligned}\Delta \ = \ & \gamma_{\nu_a,\nu_{a+1}}\gamma_{\nu_a,\nu_{a+2}}\gamma_{\nu_{a+1},\nu_{a+2}} \\ & (\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})}(\mathfrak{c}_a - \mathfrak{c}_{a+2})^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+2})} \\ & (\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(\nu_{a+1})\mathrm{par}(\nu_{a+2})}(\sigma_{a+1}\sigma_a\sigma_{a+1} - \sigma_a\sigma_{a+1}\sigma_a)e(\nu),\end{aligned}$$

which vanishes when $\nu_a \neq \nu_{a+2}$.

Assume $\nu_a = \nu_{a+2} = i$ and $\nu_{a+1} = j$. Then

$$\begin{aligned}\Delta \ = \ & \gamma_{i,i}\gamma_{i,j}\gamma_{j,i}(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(i)\mathrm{par}(j)}(\mathfrak{c}_a - \mathfrak{c}_{a+2})^{\mathrm{par}(i)} \\ & (\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(i)\mathrm{par}(j)}(\sigma_{a+1}\sigma_a\sigma_{a+1} - \sigma_a\sigma_{a+1}\sigma_a)e(\nu).\end{aligned}$$

If $i \in I_{\mathrm{even}}$, we have

$$\begin{aligned}\Delta &= \frac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})}{y_{a+2} - y_a}e(\nu) \\ &= \frac{Q_{\nu_a,\nu_{a+1}}(x_{a+2}, x_{a+1}) - Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})}{x_{a+2} - x_a}e(\nu).\end{aligned}$$

If $i \in I_{\mathrm{odd}}$, then

$$\begin{aligned}\Delta \ = \ & \frac{1}{2}(-\frac{1}{2})^{\mathrm{par}(j)}(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(j)}(\mathfrak{c}_a - \mathfrak{c}_{a+2})(\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(j)} \\ & (\sigma_{a+1}\sigma_a\sigma_{a+1} - \sigma_a\sigma_{a+1}\sigma_a)e(\nu).\end{aligned}$$

By (3.24), we have $(\mathfrak{c}_{a+1} - \mathfrak{c}_{a+2})^{\mathrm{par}(j)}(\mathfrak{c}_a - \mathfrak{c}_{a+2})(\mathfrak{c}_a - \mathfrak{c}_{a+1})^{\mathrm{par}(j)} = 2^{\mathrm{par}(j)}(\mathfrak{c}_a - \mathfrak{c}_{a+2})$. Hence

$$\begin{aligned}\Delta &= \frac{(-1)^{\mathrm{par}(j)}}{2}(\mathfrak{c}_a - \mathfrak{c}_{a+2})\Big(\frac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})}{y_{a+2} - y_a} \\ &\qquad - \mathfrak{c}_a\mathfrak{c}_{a+2}\frac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})}{y_{a+2} + y_a}\Big)e(\nu) \\ &= \frac{(-1)^{\mathrm{par}(j)}}{2}\big((\mathfrak{c}_a - \mathfrak{c}_{a+2})(y_{a+2} + y_a) - (\mathfrak{c}_a - \mathfrak{c}_{a+2})\mathfrak{c}_a\mathfrak{c}_{a+2}(y_{a+2} - y_a)\big) \\ &\qquad \frac{\widetilde{Q}_{\nu_a,\nu_{a+1}}(y_{a+2}, y_{a+1}) - \widetilde{Q}_{\nu_a,\nu_{a+1}}(y_a, y_{a+1})}{y_{a+2}^2 - y_a^2}e(\nu).\end{aligned}$$

Since

$$\begin{aligned}(\mathfrak{c}_a - \mathfrak{c}_{a+2})(y_{a+2} + y_a) - (\mathfrak{c}_a - \mathfrak{c}_{a+2})\mathfrak{c}_a\mathfrak{c}_{a+2}(y_{a+2} - y_a) \\ = 2\big(\mathfrak{c}_a y_a - \mathfrak{c}_{a+2}y_{a+2}\big) = 2(x_a - x_{a+2}),\end{aligned}$$

we finally obtain

$$\Delta = (-1)^{\mathrm{par}(j)}(x_a - x_{a+2})\frac{Q_{\nu_a,\nu_{a+1}}(x_{a+2}, x_{a+1}) - Q_{\nu_a,\nu_{a+1}}(x_a, x_{a+1})}{-x_{a+2}^2 + x_a^2}e(\nu).$$

The other commutation relations are obvious.

(b) For $\nu \in I^\beta$, we define $1 \le p_1(\nu) < \cdots < p_\ell(\nu) \le n$ as a unique sequence of integers such that $\{a\,;1 \le a \le n \text{ and } \nu_a \in I_{\mathrm{odd}}\} = \{p_k(\nu)\,;1 \le k \le \ell\}$. Set $C_k = \sum_{\nu \in I^\beta} \mathfrak{c}_{p_k(\nu)} e(\nu)$. It is easy to see that $C_k C_j + C_j C_k = 2\delta_{kj}$ and $C_k e(\nu) = e(\nu) C_k$.

(c) Let us show $C_k x_a e(\nu) = (-1)^{\mathrm{par}(\nu_a)} x_a e(\nu) C_k$. Since it is obvious if $\nu_a \in I_{\mathrm{even}}$, we assume that $\nu_a \in I_{\mathrm{odd}}$. By the definition, we have $C_k x_a e(\nu) = \mathfrak{c}_{p_k(\nu)} \mathfrak{c}_a y_a e(\nu)$. If $p_k(\nu) \neq a$, then $\mathfrak{c}_{p_k(\nu)}$ anticommutes with $\mathfrak{c}_a$ and commutes with $y_a e(\nu)$. If $p_k(\nu) = a$, then $\mathfrak{c}_{p_k(\nu)}$ commutes with $\mathfrak{c}_a$ and anticommutes with $y_a e(\nu)$. Hence in both cases, $\mathfrak{c}_{p_k(\nu)}$ anticommutes with $x_a e(\nu) = \mathfrak{c}_a y_a e(\nu)$.

Let us show

$$C_k \tau_a e(\nu) = (-1)^{\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})} \tau_a e(\nu) C_k. \tag{3.25}$$

It is obvious if $\nu_a \in I_{\mathrm{even}}$ or $\nu_{a+1} \in I_{\mathrm{even}}$. Assume that $\nu_a, \nu_{a+1} \in I_{\mathrm{odd}}$. Then we have

$$C_k \tau_a e(\nu) = C_k e(s_a \nu) \tau_a e(\nu) = \mathfrak{c}_{p_k(s_a\nu)} (\mathfrak{c}_a - \mathfrak{c}_{a+1}) \gamma_{\nu_a,\nu_{a+1}} \sigma_a e(\nu).$$

Since $p_k(s_a\nu) = p_k(\nu)$, by (3.22), we have

$$\mathfrak{c}_{p_k(s_a\nu)} (\mathfrak{c}_a - \mathfrak{c}_{a+1}) \sigma_a = -(\mathfrak{c}_a - \mathfrak{c}_{a+1}) \mathfrak{c}_{s_a p_k(\nu)} \sigma_a = -(\mathfrak{c}_a - \mathfrak{c}_{a+1}) \sigma_a \mathfrak{c}_{p_k(\nu)},$$

which implies (3.25).

Now assume that one of ν_a or ν_{a+1} belongs to I_{odd} and the other belongs to I_{even}. In this case $p_k(s_a\nu) = s_a(p_k(\nu))$ holds, and hence we have

$$C_k \tau_a e(\nu) = \mathfrak{c}_{p_k(s_a\nu)} \sigma_a e(\nu) = \mathfrak{c}_{s_a(p_k(\nu))} \sigma_a e(\nu) = \sigma_a \mathfrak{c}_{p_k(\nu)} e(\nu) = \sigma_a e(\nu) C_k.$$

Thus we have constructed a superalgebra homomorphism $\mathrm{R}_n \otimes \mathfrak{C}_\ell \to \mathrm{RC}_n^\dagger$.

(d) It is obvious that $\mathrm{R}_n \otimes \mathfrak{C}_\ell$ is generated by

$$\left\{ x^a \tau_w e(\nu) \otimes C^\eta\,; a \in \mathbb{Z}_{\ge 0}^n,\ w \in \mathfrak{S}_n,\ \nu \in I^n,\ \eta \in (\mathbb{Z}/2\mathbb{Z})^\ell \right\}$$

as a $\mathbf{k}$-module. Its image by the homomorphism $\mathrm{R}_n \otimes \mathfrak{C}_\ell \to \mathrm{RC}_n^\dagger$ forms a basis of $\mathrm{RC}_n^\dagger$ as a $\mathbf{k}$-module by (3.16). Hence $\mathrm{R}_n \otimes \mathfrak{C}_\ell \to \mathrm{RC}_n^\dagger$ is bijective. ☐

Remark 3.14. We have constructed I and $Q = \big(Q_{i,j}(w,z)\big)_{i,j \in I}$ starting from J and $\big(\widetilde{Q}_{i,j}(u,v)\big)_{i,j \in J}$. Conversely, we can construct J and $\big(\widetilde{Q}_{i,j}(u,v)\big)_{i,j \in J}$ starting from I and $Q = \big(Q_{i,j}(w,z)\big)_{i,j \in I}$. Assume that $I = I_{\mathrm{even}} \sqcup I_{\mathrm{odd}}$ and $Q = \big(Q_{i,j}(w,z)\big)_{i,j \in I}$ are given so that (3.1) is satisfied. Then set $J = (I_{\mathrm{even}} \times \{0,1\}) \sqcup (I_{\mathrm{odd}} \times \{0\})$. Let $\mathrm{pr}\colon J \to I$ be the canonical map, and let c be the involution given by $c(i,\varepsilon) = (i, 1-\varepsilon)$ for $i \in I_{\mathrm{even}}$ and $c(i,0) = (i,0)$ for $i \in I_{\mathrm{odd}}$. For $i,j \in I$, let $\widetilde{\mathscr{A}_{i,j}} \subset \mathscr{A}_{i,j}$ and $\widetilde{\mathscr{B}}_{i,j} \subset \mathscr{B}_{i,j}$ be the algebras as in Definition 3.11, and let $\widetilde{\mathscr{A}_{i,j}} \simeq \widetilde{\mathscr{B}}_{i,j}$ be the isomorphism given in (3.17). Let $Q'_{i,j} \in \widetilde{\mathscr{B}}_{i,j}$ be the element corresponding to $Q_{i,j} \in \widetilde{\mathscr{A}_{i,j}}$. Define

$$\widetilde{Q}_{(i,\varepsilon),(i',\varepsilon')}(u,v) = Q'_{i,i'}((-1)^\varepsilon u, (-1)^{\varepsilon'} v) \ \text{ for } (i,\varepsilon),(i',\varepsilon') \in J.$$

Then $(\widetilde{Q}_{j,j'})_{j,j' \in J}$ satisfies the condition (3.5).

Combining Theorem 3.13 with Remark 3.14 and Corollary 3.9, we immediately obtain the following corollary.

Corollary 3.15. *Let $I = I_{\mathrm{even}} \sqcup I_{\mathrm{odd}}$ and let $Q = \big(Q_{i,j}(w,z)\big)_{i,j\in I}$ be a family of skew polynomials satisfying the conditions* (3.1)*. Let $\mathrm{R}_n(Q)$ be the associated quiver Hecke $\mathbf{k}$-superalgebra. For each $w \in \mathfrak{S}_n$, we choose a reduced expression $s_{i_1}\cdots s_{i_\ell}$ of w and write $\tau_w = \tau_{i_1}\cdots\tau_{i_\ell}$. Then*

$$\big\{x^a\tau_w e(\nu)\,;\, a\in\mathbb{Z}_{\ge 0}^n,\ w\in\mathfrak{S}_n,\ \nu\in I^n\big\}$$

is a basis of $\mathrm{R}_n(Q)$.

3.6. **Quiver Hecke superalgebra associated with generalized Cartan matrices.** Let I be a finite index set with a decomposition $I = I_{\mathrm{odd}} \sqcup I_{\mathrm{even}}$ and let $A = (a_{i,j})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix indexed by I. We assume that

$$a_{i,j}\in 2\mathbb{Z} \text{ if } i\in I_{\mathrm{odd}}. \tag{3.26}$$

For $i,j\in I$, let $S_{i,j}$ be the set of (r,s) where r and s are integers satisfying the following conditions:

$$\begin{cases} \text{(i) } 0\le r\le -a_{i,j},\ 0\le s\le -a_{j,i},\\ \text{(ii) } a_{j,i}r + a_{i,j}s = -a_{i,j}a_{j,i},\\ \text{(iii) } r\in 2\mathbb{Z} \text{ if } i\in I_{\mathrm{odd}},\\ \text{(iv) } s\in 2\mathbb{Z} \text{ if } j\in I_{\mathrm{odd}}.\end{cases} \tag{3.27}$$

Let $\{t_{i,j;r,s}\}_{i\neq j,\,(r,s)\in S_{i,j}}$ be a family of indeterminates such that $t_{i,j;r,s} = t_{j,i;s,r}$ and $t_{i,j;-a_{i,j},0}$ is invertible, and let $\mathbf{k}_A$ be the algebra $\mathbb{Z}[1/2][\{t_{i,j;r,s}\}][\{(t_{i,j,-a_{i,j},0})^{-1}\}]$. We take $Q = (Q_{i,j})_{i,j\in I}$, where

$$Q_{i,j}(w,z) = \sum_{(r,s)\in S_{i,j}} t_{i,j;r,s}w^rz^s \in \mathbf{k}_A\langle w,z\rangle/\langle zw-(-1)^{\mathrm{par}(i)\,\mathrm{par}(j)}wz\rangle$$

for $i\neq j$ and $Q_{i,j}=0$ for $i=j$. We write $\mathrm{R}_n = \mathrm{R}_n(A) = \mathrm{R}_n(Q)$ for the associated quiver Hecke $\mathbf{k}_A$-superalgebra.

Let $J = I\times\{0\}\sqcup I_{\mathrm{even}}\times\{1\}$ and $\widetilde{Q} = (\widetilde{Q}_{i,j})_{i,j\in J}$ associated with I and Q as in Remark 3.14. Then we can define the quiver Hecke-Clifford $\mathbf{k}_A$-superalgebra $\mathrm{RC}_n = \mathrm{RC}_n(A) := \mathrm{RC}_n(\widetilde{Q})$. Note that R_n and RC_n are $(\mathbb{Z}\times(\mathbb{Z}/2\mathbb{Z}))$-graded via the following assignment.

$$\begin{aligned}&\deg e(\nu) = (0;0),\quad \deg x_pe(\nu) = ((\alpha_{\nu_p},\alpha_{\nu_p});\mathrm{par}(\nu_a)),\\ &\deg \tau_ae(\nu) = (-(\alpha_{\nu_a},\alpha_{\nu_{a+1}});\mathrm{par}(\nu_a)\mathrm{par}(\nu_{a+1})),\end{aligned}$$

and

$$\begin{aligned}&\deg y_pe(\nu) = ((\alpha_{\nu_p},\alpha_{\nu_p});0),\quad \deg \mathfrak{c}_pe(\nu) = (0;1),\\ &\deg \sigma_ae(\nu) = (-(\alpha_{\nu_a},\alpha_{\nu_{a+1}});0).\end{aligned}$$

If $\mathbf{k}_A\to\mathbf{k}$ is a ring homomorphism, we can consider the superalgebra $\mathbf{k}\otimes_{\mathbf{k}_A}\mathrm{R}_n$, which will be called the quiver Hecke superalgebra associated with a generalized Cartan matrix A.

3.7. **Cyclotomic quiver Hecke superalgebras.** Let $Q^\vee := \oplus_{i\in I}\mathbb{Z}h_i$ be the coroot lattice with the bilinear form $Q^\vee \times Q \to \mathbb{Z}$ by $\langle h_i, \alpha_j\rangle = a_{i,j}$. Let $P := \mathrm{Hom}(Q^\vee, \mathbb{Z})$ be the weight lattice, and $\big(\Lambda_i\big)_{i\in I}$ the dual basis of $\big(h_i\big)_{i\in I}$. Set

$$P_+ := \{\lambda \in P\,;\, \lambda(h_i) \geq 0 \text{ for any } i \in I\}\,.$$

For $\Lambda \in P_+$, set $x_1^\Lambda = \sum_{\nu\in I^n} x_1^{\Lambda(h_{\nu_1})} e(\nu) \in \mathrm{R}_n$. We define

$$\mathrm{R}_n^\Lambda = \mathrm{R}_n/\mathrm{R}_n x_1^\Lambda \mathrm{R}_n = \mathrm{R}_n/\big(\sum_{\nu\in I^n} x_1^{\Lambda(h_{\nu_1})} e(\nu)\mathrm{R}_n\big),$$

$$R_\beta^\Lambda = \mathrm{R}_\beta/\mathrm{R}_\beta x_1^\Lambda \mathrm{R}_\beta \quad \text{for } \beta \in Q_+,$$

and call them the *cyclotomic quiver Hecke superalgebras*. Similarly, we can define the *cyclotomic quiver Hecke-Clifford superalgebras* RC_n^Λ and $\mathrm{RC}_\beta^\Lambda$. By Theorem 3.13 the superalgebras R_β^Λ and $\mathrm{RC}_\beta^\Lambda$ are weakly Morita superequivalent. Note that RC_n^Λ and R_n^Λ are finitely generated $\mathbf{k}$-modules.

3.8. **Completion of** RC_n**.** For the data J, c, $\big(\widetilde{Q}_{i,j}\big)_{i,j\in J}$ as in § 3.3, let RC_n be the associated quiver Hecke-Clifford superalgebra. Let $\mathfrak{a}$ be the ideal of $\mathbf{k}[y_1, \ldots, y_n]$ generated by $y_1, \ldots, y_n$. Then we have

$$\text{for any } s \in \mathrm{RC}_n, \text{ there exists } m \text{ such that } s\mathfrak{a}^{k+m} \subset \mathfrak{a}^k\mathrm{RC}_n.$$

Indeed, it is enough to verify this for the generators, and it is obvious for y_p and $\mathfrak{c}_p$. The case for $s = \sigma_a$ follows from Lemma 3.6.

Hence we can easily see that the superalgebra structure of RC_n induces a superalgebra structure on $\varprojlim_k \mathrm{RC}_n/\mathfrak{a}^k\mathrm{RC}_n$.

Definition 3.16. *We define the completion* $\widehat{\mathrm{RC}}_n$ *of* RC_n *to be*

$$\widehat{\mathrm{RC}}_n = \varprojlim_k \mathrm{RC}_n/\mathfrak{a}^k\mathrm{RC}_n \simeq \mathbf{k}[[y_1, \ldots, y_n]] \ \otimes_{\mathbf{k}[y_1,\ldots,y_n]} \mathrm{RC}_n.$$

Then the formula (3.7) holds in $\widehat{\mathrm{RC}}_n$ for all $f \in \mathbf{k}[[y_1, \ldots, y_n]]$. The algebra $\widehat{\mathrm{RC}}_n$ contains $\bigoplus_{\nu\in J} \mathbf{k}[[y_1, \ldots, y_n]]e(\nu)$ as a subalgebra.

4. Relation to the affine Hecke-Clifford superalgebras

In the rest of this article, $\mathbf{k}$ is an algebraically closed field of characteristic $\neq 2$. We fix a non-zero element $q \in \mathbf{k}^\times$ and set $\xi = q - q^{-1}$. We assume

$$(q^2)^2 \neq 1. \tag{4.1}$$

The purpose of this section is to prove that the affine Hecke-Clifford superalgebras are isomorphic to quiver Hecke-Clifford superalgebras after a completion.

4.1. **Affine Hecke-Clifford superalgebra.** For a more comprehensive treatment of the affine Hecke-Clifford superalgebra and its cyclotomic quotients, see [BK3] and the references therein.

Definition 4.1. *Let $n \geq 0$ be an integer. The* affine Hecke-Clifford superalgebra $\mathcal{AHC}_n$ *of degree n is the* $\mathbf{k}$*-superalgebra generated by the even generators $X_1^{\pm 1}, \dots, X_n^{\pm 1}, T_1, \dots, T_{n-1}$ and the odd generators $C_1, \dots, C_n$ with the following relations:*
(i) $X_i X_i^{-1} = X_i^{-1} X_i = 1$, $X_i X_j = X_j X_i$ *for all* $1 \leq i, j \leq n$,
(ii) $C_i C_j + C_j C_i = 0$ $(i \neq j)$ *and* $C_i^2 = 1$,
(iii) $T_i^2 = \xi T_i + 1$, $T_i T_j = T_j T_i$ $(|i-j| > 1)$, $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$,
(iv) $T_i C_j = C_j T_i$ $(j \neq i, i+1)$, $T_i C_i = C_{i+1} T_i$ *for all* $1 \leq i \leq n-1$,
(v) $C_i X_j = X_j C_i$ *for all* $i \neq j$ *and* $C_i X_i^{\pm 1} = X_i^{\mp 1} C_i$,
(vi) $T_i X_j = X_j T_i$ $(j \neq i, i+1)$, $(T_i + \xi C_i C_{i+1}) X_i T_i = X_{i+1}$.

It is known that $\mathcal{AHC}_n$ has a PBW-type basis.

Proposition 4.2. *For each $w \in \mathfrak{S}_n$, let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression and set $T_w = T_{i_1} \cdots T_{i_\ell}$ (which is independent of the choice of a reduced expression). Then $\mathcal{AHC}_n$ is a free left $\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$-module with a basis*

$$\{C^\eta T_w e(\nu) \,;\, \eta \in (\mathbb{Z}/2\mathbb{Z})^n,\ w \in \mathfrak{S}_n,\ \nu \in J^n\},$$

where $C^\eta = C_1^{\eta_1} \cdots C_n^{\eta_n}$ for $\eta = (\eta_1, \dots, \eta_n) \in (\mathbb{Z}/2\mathbb{Z})^n$.

4.2. **Intertwiners.** We recall the definition of intertwiners of $\mathcal{AHC}_n$ ([Naz],[Kle, §14.8] ??):

$$\Phi_a = T_a + \xi \frac{X_{a+1}}{X_a - X_{a+1}} + \xi C_a C_{a+1} \frac{X_a X_{a+1}}{X_a X_{a+1} - 1} \in \mathbf{k}(X_1, \dots, X_n) \underset{\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]}{\otimes} \mathcal{AHC}_n. \tag{4.2}$$

They satisfy the relations:

$$\Phi_a X_i = X_{s_a(i)} \Phi_a \quad \text{and} \quad \Phi_a C_i = C_{s_a(i)} \Phi_a. \tag{4.3}$$

Setting

$$K(u, v) = u^2 - (q^2 + q^{-2}) uv + v^2 + 4(q - q^{-1})^2,$$

$$F(X_a, X_{a+1}) = \frac{X_a^2 X_{a+1}^2\, K(X_a + X_a^{-1},\, X_{a+1} + X_{a+1}^{-1})}{(X_a - X_{a+1})^2 (X_a X_{a+1} - 1)^2},$$

we have

$$\Phi_a^2 = F(X_a, X_{a+1}). \tag{4.4}$$

Note that by setting $w_k = X_k + X_k^{-1} = 2 \dfrac{\lambda_k + \lambda_k^{-1}}{q + q^{-1}}$ $(k = 1, 2)$, we have

$$\begin{aligned} K(w_1, w_2) &= \frac{4}{(q + q^{-1})^2 \lambda_1^2 \lambda_2^2} (\lambda_1 - q^2 \lambda_2)(\lambda_1 - q^{-2} \lambda_2)(\lambda_1 \lambda_2 - q^2)(\lambda_1 \lambda_2 - q^{-2}), \\ F(X_1, X_2) &= \frac{(\lambda_2 - q^2 \lambda_1)(\lambda_2 - q^{-2} \lambda_1)(\lambda_2 - q^2 \lambda_1^{-1})(\lambda_2 - q^{-2} \lambda_1^{-1})}{(\lambda_2 - \lambda_1)^2 (\lambda_2 - \lambda_1^{-1})^2}. \end{aligned} \tag{4.5}$$

Moreover, the Φ_a's satisfy the braid relations:

$$\Phi_a \Phi_c = \Phi_c \Phi_a \quad (|a - c| > 1), \qquad \Phi_a \Phi_{a+1} \Phi_a = \Phi_{a+1} \Phi_a \Phi_{a+1}.$$

4.3. **Localization of $\mathcal{AHC}_n$.** For a finite-dimensional representation of $\mathcal{AHC}_n$, we consider the simultaneous eigenspaces of the X_i's. Let us denote by $\mathbb{A}^1$ the one-dimensional affine space and by $\mathbb{T}$ the algebraic torus $\mathbb{A}^1 \setminus \{0\}$. We denote by $\mathbb{T}_X$ the variety $\mathbb{T}$ with X as a coordinate. It plays a role of the set of eigenvalues of X_k. Let $c\colon \mathbb{T}_X \to \mathbb{T}_X$ be the involution of $\mathbb{T}_X$ given by $c(X) = X^{-1}$ (corresponding to Definition 4.1 (v)).

For an algebraic variety X over $\mathbf{k}$ and its $\mathbf{k}$-valued point x, let us denote by $\mathscr{O}_{X,x}$ the germ of the structure sheaf $\mathscr{O}_X$ at x, by $\widehat{\mathscr{O}}_{X,x}$ its completion, and by $\mathsf{Frac}(\widehat{\mathscr{O}}_{X,x})$ its fraction field. For p such that $1 \le p \le n$, let us denote by $c_p\colon \mathbb{T}^n \to \mathbb{T}^n$ the involution

$$(X_1, \dots, X_n) \mapsto (X_1, \dots, X_{p-1}, X_p^{-1}, X_{p+1}, \dots, X_n),$$

and by the same letter c_p the induced isomorphism $\mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n,q}) \xrightarrow{\sim} \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n,c_p(q)})$ ($q \in \mathbb{T}^n$). For $1 \le p < n$, we denote by $s_p\colon \mathbb{T}^n \to \mathbb{T}^n$ the involution

$$(X_1, \dots, X_n) \mapsto (X_1, \dots, X_{p-1}, X_{p+1}, X_p, X_{p+2}, \dots, X_n),$$

and by the same letter s_p the induced isomorphism $\mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n,q}) \xrightarrow{\sim} \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n,s_p(q)})$. Similarly, we denote by $\overline{s}_p\colon \mathbb{T}^n \to \mathbb{T}^n$ the involution

$$(X_1, \dots, X_n) \mapsto (X_1, \dots, X_{p-1}, X_{p+1}^{-1}, X_p^{-1}, X_{p+2}, \dots, X_n),$$

and the induced isomorphism $\mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n,q}) \xrightarrow{\sim} \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n,\overline{s}_p(q)})$. Clearly, we have $\overline{s}_p = c_p c_{p+1} s_p$.

Let us denote by $\mathcal{HC}_n$ the $\mathbf{k}$-superalgebra generated by C_p $(1 \le p \le n)$, T_a $(1 \le a < n)$ with (ii)–(iv) in Definition 4.1 as the defining relations. The superalgebra $\mathcal{HC}_n$ is called the *Hecke-Clifford superalgebra* and can be regarded as a subsuperalgebra of $\mathcal{AHC}_n$.

Definition 4.3. *Let J be a finite subset of $\mathbb{T}_X$ invariant under c, and let $X\colon J \to \mathbb{T}_X$ be the inclusion map. We define the commutative $\mathbf{k}$-algebras*

$$\mathscr{O}_n = \bigoplus_{\nu \in J^n} \widehat{\mathscr{O}}_{\mathbb{T}^n,X(\nu)} e(\nu), \quad \mathscr{K}_n = \bigoplus_{\nu \in J^n} \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n,X(\nu)}) e(\nu),$$

where $X(\nu) = (X(\nu_1), \dots, X(\nu_n)) \in \mathbb{T}^n$ for $\nu = (\nu_1, \dots, \nu_n)$. We define the algebra structure on

$$\mathcal{KHC}_n = \mathscr{K}_n \otimes_{\mathbf{k}} \mathcal{HC}_n$$

by

$$\begin{aligned} &C_p e(\nu) f = e(c_p\nu) c_p(f) C_p \quad (1 \le p \le n), \\ &T_a e(\nu) f = e(s_a\nu) s_a(f) T_a - \xi \frac{e(\nu)f - e(s_a\nu) s_a f}{X_a X_{a+1}^{-1} - 1} - \xi C_a C_{a+1} \frac{e(\nu) f - e(\overline{s}_a \nu)\overline{s}_a(f)}{X_a^{-1} X_{a+1}^{-1} - 1} \end{aligned} \tag{4.6}$$

for $\nu \in J^n$, $1 \le a < n$, $f \in \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n,X(\nu)})$.

We define $\mathcal{OHC}_n$ to be the subsuperalgebra $\mathscr{O}_n \otimes \mathcal{HC}_n$ of $\mathcal{KHC}_n$.

Let us denote by $\mathfrak{a}_J$ the ideal of $\mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ consisting of functions which vanish on $\{X(\nu)\,;\, \nu \in J^n\}$. Then we have

$$\begin{aligned} \mathscr{O}_n &\simeq \varprojlim_k \mathbf{k}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]/\mathfrak{a}_J^k, \\ \mathcal{OHC}_n &\simeq \varprojlim_k \mathcal{AHC}_n/\mathfrak{a}_J^k \mathcal{AHC}_n, \\ \mathcal{KHC}_n &\simeq \mathscr{K}_n \otimes_{\mathscr{O}_n} \mathcal{OHC}_n. \end{aligned}$$

Thus there exists a $\mathbf{k}$-superalgebra homomorphism $\mathcal{AHC}_n \to \mathcal{OHC}_n$, which is injective if J is not empty.

Let $\mathcal{AHC}_n$-mod be the category of $\mathcal{AHC}_n$-modules finite-dimensional over $\mathbf{k}$. Let $\mathcal{C}(J)$ be the full subcategory of $\mathcal{AHC}_n$-mod consisting of $M \in \mathcal{AHC}_n$-mod whose X_k-eigenvalues lie in $\{X(i)\,;\, i \in J\}$ for all $1 \le k \le n$. For such an M and $\nu \in J^n$, let $e(\nu)$ be the projection operator of M onto the simultaneous generalized eigenspace of $\big(X_k\big)_{1 \le k \le n}$ with $\big(X(\nu_k)\big)_{1 \le k \le n}$ as eigenvalues. Then M has a natural structure of $\mathcal{OHC}_n$-module. Thus $\mathcal{C}(J)$ is equivalent to the category $\mathcal{OHC}_n$-mod of $\mathcal{OHC}_n$-modules finite-dimensional over $\mathbf{k}$. We have

$$\Phi_a e(\nu) = e(s_a\nu)\Phi_a \text{ for any } 1 \le a < n. \tag{4.7}$$

Indeed, this formula must hold since Φ_a is an intertwiner, and the definition (4.6) is equivalent to $\Phi_a e(\nu) f = e(s_a\nu) s_a(f) \Phi_a$ for $f \in \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n, X(\nu)})$.

4.4. **Dynkin diagram.** Let us denote by $\mathbb{A}^1_w$ the one-dimensional affine space $\mathbb{A}^1$ with the coordinate w. Let $\mathrm{pr}\colon \mathbb{T}_X \to \mathbb{A}^1_w$ be the map $\mathrm{pr}(X) = X + X^{-1}$. Hence the fiber of pr is of the form $\{j, c(j)\}$ for some $j \in \mathbb{T}_X$. Let $\mathbb{T}_\lambda$ be the variety $\mathbb{T}$ with the coordinate λ. Let $g\colon \mathbb{T}_\lambda \to \mathbb{A}^1_w$ be the map $g(\lambda) = 2\dfrac{\lambda + \lambda^{-1}}{q + q^{-1}}$. Thus these three varieties are related by

$$\mathbb{T}_X \xrightarrow{\mathrm{pr}} \mathbb{A}^1_w \xleftarrow{g} \mathbb{T}_\lambda.$$

We will regard $\mathbb{A}^1_w$ as the set of vertices of a Dynkin diagram. We consider a $\mathbb{Q}$-vector space with $\{\alpha_w\}_{w \in \mathbb{A}^1_w}$ as a basis, and define an inner product on it as follows:

$$\begin{cases} (\alpha_w, \alpha_w) = \begin{cases} 1 & \text{if } \#\mathrm{pr}^{-1}(w) = 1, \\ 4 & \text{if } \#g^{-1}(w) = 1, \\ 2 & \text{otherwise,} \end{cases} \\ \text{if } w_1 \neq w_2 \text{ and } K(w_1, w_2) = 0, \text{ then} \\ \qquad (\alpha_{w_1}, \alpha_{w_2}) = \begin{cases} -2 & \text{if } (\alpha_{w_1}, \alpha_{w_1}) = 4 \text{ or } (\alpha_{w_2}, \alpha_{w_2}) = 4, \\ -1 & \text{if } (\alpha_{w_1}, \alpha_{w_1}),\ (\alpha_{w_2}, \alpha_{w_2}) \neq 4, \end{cases} \\ \text{if } w_1 \neq w_2 \text{ and } K(w_1, w_2) \neq 0, \text{ then} \\ \qquad (\alpha_{w_1}, \alpha_{w_2}) = 0. \end{cases} \tag{4.8}$$

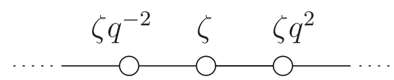
Note that by (4.5), $K(w_1, w_2) = 0$ if and only if $\lambda_1 = q^{\pm 2}\lambda_2$ or $\lambda_1 = q^{\pm 2}\lambda_2^{-1}$ when writing $w_k = g(\lambda_k) = 2\dfrac{\lambda_k + \lambda_k^{-1}}{q + q^{-1}}$ $(k = 1, 2)$. The generalized Cartan matrix $\big(a_{w,w'}\big)_{w,w' \in \mathbb{A}^1_w}$ is given by $a_{w,w'} = 2(\alpha_w, \alpha_{w'})/(\alpha_w, \alpha_w)$.

Set $(\mathbb{A}^1_w)_{\mathrm{odd}} = \{\mathrm{pr}(\pm 1)\} = \{g(\pm q)\}$ and $(\mathbb{A}^1_w)_{\mathrm{even}} = \mathbb{A}^1_w \setminus \mathbb{A}^1_{\mathrm{odd}}$. Then $\mathbb{A}^1_w$ decomposes into the even part and the odd part.

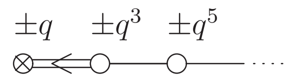
The connected components of the Dynkin diagram $\mathbb{A}^1_w$ are classified as follows. (The odd vertices are marked by $\times$.)

(i) when q^2 is not a root of unity, there are three types of connected components.

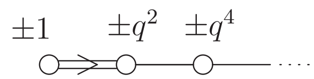
(a) $\big\{g(\zeta q^{2k})\,;\, k \in \mathbb{Z}\big\}$ for some $\zeta \notin \pm q^{\mathbb{Z}}$, where $\pm q^{\mathbb{Z}} = \big\{\pm q^k\,;\, k \in \mathbb{Z}\big\}$. The Dynkin diagram is of type A_∞.



(b) $\big\{g(\varepsilon q^{2k+1})\,;\, k \in \mathbb{Z}_{\geq 0}\big\}$ $(\varepsilon = \pm 1)$. The Dynkin diagram is of type B_∞.

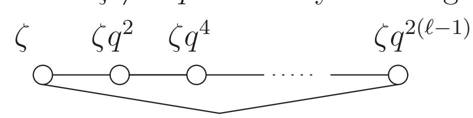


(c) $\big\{g(\varepsilon q^{2k})\,;\, k \in \mathbb{Z}_{\geq 0}\big\}$ $(\varepsilon = \pm 1)$. The Dynkin diagram is of type C_∞.

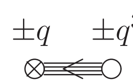


(ii) When q^2 is a primitive ℓ-th root of unity, there are three types of connected components.

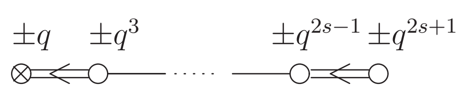
(a) $\big\{g(\zeta q^{2k})\,;\, k \in \mathbb{Z}/\ell\mathbb{Z}\big\}$ for some $\zeta \notin \pm q^{\mathbb{Z}}$. The Dynkin diagram is of type $A^{(1)}_{\ell-1}$.



(b) $\{g(\pm q), \dots, g(\pm q^{2s+1})\}$ when ℓ is odd ($\ell = 2s + 1$ with $s \geq 1$). In this case $q^{2s+1} = \pm 1$. The Dynkin diagram is of type $A^{(2)}_{2s}$.

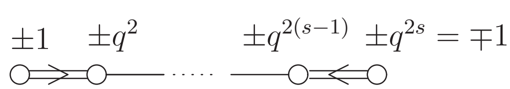


$((q^2)^3 = 1)$

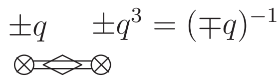


$(s > 1)$

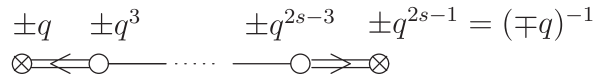
(c) $\{g(\pm 1), g(\pm q^2), \dots, g(\pm q^{2s)})\}$ when ℓ is even ($\ell = 2s$ with $s \geq 2$). In this case $q^{2s} = -1$. The Dynkin diagram is of type $C^{(1)}_s$.



(d) $\{g(\pm q), g(\pm q^3), \ldots, g(\pm q^{2s-3}), g(\pm q^{2s-1})\}$ where ℓ is even ($\ell = 2s$ with $s \geq 2$). In this case, $q^{2s} = -1$. The Dynkin diagram is of type $D_s^{(2)}$.



$(s = 2,\ (q^2)^2 = -1)$



$(s > 2)$

Observe that any vertex of $\mathbb{A}^1_w$ has one or two edges, and $\{g(q), g(-q), g(1), g(-1)\}$ is the set of points with one edge. Note that the odd vertices have the minimal length.

Let J be a finite subset of $\mathbb{T}_X$ invariant under c, and let $X\colon J \to \mathbb{T}_X$ be the inclusion map. Let I be the image of the composition $J \xrightarrow{X} \mathbb{T}_X \xrightarrow{\mathrm{pr}} \mathbb{A}^1_w$, and we also denote by $\mathrm{pr}\colon J \to I$ the surjection. Set $J^c := \{j \in J \,;\, c(j) = j\}$ and $I_{\mathrm{odd}} = \mathrm{pr}(J^c) = I \cap \{g(\pm q)\}$, $I_{\mathrm{even}} = I \setminus I_{\mathrm{odd}}$.

We choose a function $h\colon I \to \mathbb{T}_\lambda$ such that

$$(4.9)\qquad \begin{cases} \bullet\ g(h(i)) = i, \\ \bullet\ \text{if } (\alpha_i, \alpha_{i'}) < 0, \text{ then } h(i) = q^{\pm 2} h(i') \text{ for any } i, i' \in I, \end{cases}$$

Such a function h exists. Indeed, it is enough to show it for a connected component of $\mathbb{A}_w$. Since any vertices has at most two edges, the assertion is obvious when the connected component has no loop, and the remaining case (iia) can be checked directly.

We define the map $\lambda\colon J \to \mathbb{T}_\lambda$ by $\lambda(j) = h(\mathrm{pr}(j))$. Hence we have $X(j) + X(j)^{-1} = 2\dfrac{\lambda(j) + \lambda(j)^{-1}}{q + q^{-1}}$. Since $h\colon I \to \mathbb{T}_\lambda$ is injective, we have

$$(4.10)\qquad \text{for } j, j' \in J, \text{ we have } \lambda(j) = \lambda(j') \text{ if and only if } j = j' \text{ or } j = c(j').$$

We choose a function $\varepsilon\colon J \to \{0, 1\}$ such that $\varepsilon^{-1}(0) \to I$ is bijective. Hence we have

$$\varepsilon(cj) = 1 - \varepsilon(j) \text{ for } j \in J \setminus J^c, \text{ and } \varepsilon(j) = 0 \text{ for } j \in J^c.$$

For $i, j \in J$, we write (i, j) for $(\alpha_{\mathrm{pr}(i)}, \alpha_{\mathrm{pr}(j)})$ and $a_{i,j} := 2(i, j)/(i, i)$. For $i, j \in J$ such that $i \neq j$ and $\varepsilon(i) = \varepsilon(j) = 0$, we choose

$$(4.11)\qquad \widetilde{Q}_{i,j}(u, v) = \begin{cases} \pm(u^{-a_{i,j}} - v^{-a_{j,i}}) & \text{if } (i, j) < 0, \\ 1 & \text{if } (i, j) = 0, \end{cases}$$

where we choose $\pm$ such that $\widetilde{Q}_{i,j}(u, v) = \widetilde{Q}_{j,i}(v, u)$.

We can extend uniquely the definition of $\widetilde{Q}_{i,j}$ for all $i, j \in J$ such that the conditions in (3.5) hold; namely,

$$(4.12)\qquad \widetilde{Q}_{i,j}(u, v) = \widetilde{Q}_{c^{\varepsilon(i)}(i),\, c^{\varepsilon(j)}(j)}((-1)^{\varepsilon(i)} u,\ (-1)^{\varepsilon(j)} v).$$

Then we can associate to the data $\{\widetilde{Q}_{i,j}\}_{i,j \in J}$ the quiver Hecke-Clifford superalgebra RC_n and its completion $\widehat{\mathrm{RC}}_n$.

The following theorem is a main result of this section.

Theorem 4.4. *We have* $\widehat{\mathrm{RC}}_n \simeq \mathcal{OHC}_n$.

The rest of this subsection is devoted to the proof of this theorem.

4.5. **Proof of Theorem 4.4.**

4.5.1. *Strategy of the proof.* We will construct the elements $y_k \in \mathscr{O}_n \subset \mathcal{OHC}_n$ ($k = 1, \ldots, n$) and $\widetilde{s}_a \in \mathcal{KHC}_n$ ($a = 1, \ldots, n-1$) such that

$$(4.13)\qquad \begin{cases} \text{(i)}\ y_p e(\nu) \in (\widehat{\mathscr{O}}_{\mathbb{T}^n, X(\nu)})^\times \big(X_p - X(\nu_p)\big) e(\nu), \\ \text{(ii)}\ C_a y_p e(\nu) = (-1)^{\delta_{a,p}} y_p e(c_a\nu) C_a, \\ \text{(iii)}\ \widetilde{s}_a^2 = R_{a,a+1} \text{ (see (3.9))}, \\ \text{(iv)}\ \{\widetilde{s}_a\}_{1 \le a < n} \text{ satisfies the braid relations (see (3.4))}, \\ \text{(v)}\ \widetilde{s}_a C_k = C_{s_a(k)} \widetilde{s}_a,\ \widetilde{s}_a y_k = y_{s_a(k)} \widetilde{s}_a, \\ \text{(vi)}\ \text{setting } \sigma_a := \widetilde{s}_a + f_{a,a+1} \text{ (see (3.12)), the element } \sigma_a e(\nu) \text{ belongs} \\ \qquad \text{to } T_a e(\nu) (\widehat{\mathscr{O}}_{\mathbb{T}^n, X(\nu)})^\times + \mathscr{O}_n \langle C_1, \cdots, C_n \rangle. \end{cases}$$

Setting $\mathfrak{c}_k = C_k$, the elements y_p, $\mathfrak{c}_p$ ($p = 1, \ldots, n$), σ_a ($a = 1, \ldots, n-1$) satisfy the defining relations of RC_n by Theorem 3.8. Hence we obtain a homomorphism $F\colon \widehat{\mathrm{RC}}_n \to \mathcal{OHC}_n$. By Corollary 3.9, $\{\mathfrak{c}^\eta \sigma_w\}_{\eta \in (\mathbb{Z}/2\mathbb{Z})^n,\, w \in \mathfrak{S}_n}$ is a basis of $\widehat{\mathrm{RC}}_n$ as a left $\mathscr{O}_n$-module. On the other hand, it is easy to see that its image by F forms a basis of $\mathcal{OHC}_n$ as a left $\mathscr{O}_n$-module by Proposition 4.2. Hence we conclude that $F\colon \widehat{\mathrm{RC}}_n \to \mathcal{OHC}_n$ is an isomorphism.

Now we will construct $y_k \in \mathscr{O}_n$ ($k = 1, \ldots, n$) and $\widetilde{s}_a \in \mathcal{KHC}_n$ ($a = 1, \ldots, n-1$) which satisfy the condition (4.13).

4.5.2. *Definition of* $y_p\, e(\nu)$.

We will construct $y_p\, e(\nu)$ such that

$$\begin{aligned} &y_p e(\nu) \in (\widehat{\mathscr{O}}_{\mathbb{T}^n, X(\nu)})^\times (X_p - X(\nu_p)) e(\nu), \\ &C_a y_p e(\nu) = (-1)^{\delta_{ap}} y_p e(c_a \nu) C_a. \end{aligned}$$

Let C be the subscheme of $\mathbb{T}_X \times \mathbb{T}_\lambda$ defined by the equation

$$X + X^{-1} = 2\frac{\lambda + \lambda^{-1}}{q + q^{-1}}.$$

Then C is a smooth curve and we have a Cartesian product

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{T}_\lambda \\ \big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathbb{T}_X & \xrightarrow{\mathrm{pr}} & \mathbb{A}_w. \end{array}$$

Then for every $i \in J$, it induces the injective homomorphisms

$$(4.14)\qquad \widehat{\mathscr{O}}_{\mathbb{T}_X, X(i)} \rightarrowtail \widehat{\mathscr{O}}_{C, (X(i), \lambda(i))} \leftarrowtail \widehat{\mathscr{O}}_{\mathbb{T}_\lambda, \lambda(i)}.$$

Let us set $\mathscr{O}_i := \widehat{\mathscr{O}}_{\mathbb{T}_X, X(i)} = \mathbf{k}[[X - X(i)]]$ and $\widetilde{\mathscr{O}}_i := \widehat{\mathscr{O}}_{C,\, (X(i),\, \lambda(i))}$, and we regard $\mathscr{O}_i$ as a subalgebra of $\widetilde{\mathscr{O}}_i$. The algebra $\mathscr{O}_i$ is a discrete valuation ring with the maximal ideal $\mathfrak{m}_i := \mathscr{O}_i(X - X(i))$. The indeterminates X and λ are considered as elements of $\widetilde{\mathscr{O}}_i$ and they satisfy $X + X^{-1} = 2\dfrac{\lambda+\lambda^{-1}}{q+q^{-1}}$. Let $c_i\colon \mathscr{O}_i \to \mathscr{O}_{c(i)}$ be the homomorphism induced by the map $X \mapsto X^{-1}$. For $1 \le a \le n$, let us denote by $q_{a,\nu}\colon \mathscr{O}_{\nu_a} \longrightarrow \widehat{\mathscr{O}}_{\mathbb{T}^n,X(\nu)}e(\nu)$ the homomorphism induced by the a-th projection $(X_1,\dots,X_n) \mapsto X_a$.

Recall that $\varepsilon\colon J \to \{0,1\}$ is a map such that $\varepsilon^{-1}(0) \to I$ is bijective. For all $i \in J$ with $\varepsilon(i) = 0$, we will construct $y_i \in \mathscr{O}_i$ such that

$$\text{(4.15)} \qquad \begin{cases} \text{(i) } y_i \in \mathscr{O}_i^{\times}(X - X(i)), \\ \text{(ii) if } c(i) = i, \text{ then } c_i(y_i) = -y_i, \end{cases}$$

and define

$$y_a e(\nu) = \begin{cases} q_{a,\nu}(y_{\nu_a}) & \text{if } \varepsilon(\nu_a) = 0, \\ -c_a q_{a,c_a\nu}(y_{c(\nu_a)}) & \text{if } \varepsilon(\nu_a) = 1. \end{cases}$$

Then we have $y_p e(\nu) \in (\widehat{\mathscr{O}}_{\mathbb{T}^n,X(\nu)})^{\times}\,(X_p - X(\nu_p))e(\nu)$ and $C_a y_p e(\nu) = (-1)^{\delta_{ap}} y_p e(c_a\nu) C_a$.

Now, we will construct y_i for $i \in J$ such that $\varepsilon(i) = 0$. We choose $\psi(\lambda) \in \widehat{\mathscr{O}}_{\mathbb{T}_\lambda,1}$ such that

$$\text{(4.16)} \qquad \begin{aligned} &\psi \in (\widehat{\mathscr{O}}_{\mathbb{T},1})^{\times}(\lambda - 1), \\ &\psi(\lambda^{-1}) = -\psi(\lambda). \end{aligned}$$

For example, we can take $(\lambda-1)/(\lambda+1)$ or $\lambda - \lambda^{-1}$ as $\psi(\lambda)$.

• Case $\lambda(i) \ne \pm 1, \pm q, \pm q^{-1}$.

The projections $C \to \mathbb{T}_X$ and $C \to \mathbb{T}_\lambda$ are étale at $(X(i), \lambda(i))$, and hence we have isomorphisms

$$\mathbf{k}[[X - X(i)]] = \mathscr{O}_i \xrightarrow{\sim} \widetilde{\mathscr{O}}_i = \widehat{\mathscr{O}}_{C,\,(X(i),\lambda(i))} \xleftarrow{\sim} \widehat{\mathscr{O}}_{\mathbb{T}_\lambda,\lambda(i)} = \mathbf{k}[[\lambda - \lambda(i)]].$$

We define $y_i \in \mathscr{O}_i$ by $y_i = \psi(\lambda(i)^{-1}\lambda)$.

• Case $\lambda(i) = \pm q, \pm q^{-1}$.

In this case, we have $X(i) = \pm 1$ and

$$\mathbf{k}[[X - X(i)]] = \mathscr{O}_i = \widehat{\mathscr{O}}_{\mathbb{T},X(i)} \xrightarrow{\sim} \widetilde{\mathscr{O}}_i = \widehat{\mathscr{O}}_{C,\,(X(i),\lambda(i))} \hookleftarrow \widehat{\mathscr{O}}_{\mathbb{T}_\lambda,\lambda(i)} = \mathbf{k}[[\lambda - \lambda(i)]].$$

Then $\widehat{\mathscr{O}}_{\mathbb{T}_\lambda,\lambda(i)}$ is identified with $\{f(X) \in \mathscr{O}_i\,;\, c(f) = f\}$. (Recall that $c(f)(X) = f(X^{-1})$.) Since $\mathscr{O}_i\psi(\lambda(i)^{-1}\lambda) = \mathfrak{m}_i^2$, there exists $y_i \in \mathscr{O}_i$ such that $y_i^2 = \psi(\lambda(i)^{-1}\lambda)$. Then y_i generates the maximal ideal $\mathfrak{m}_i$. Since $c(y_i)^2 = y_i^2$ and $c(y_i)$ cannot be equal to y_i, we obtain $c(y_i) = -y_i$.

• Case $\lambda(i) = \pm 1$.

In this case, we have

$$\widehat{\mathscr{O}}_{\mathbb{T}_X,X(i)} \rightarrowtail \widehat{\mathscr{O}}_{C,\,(X(i),\pm 1)} \xleftarrow{\sim} \widehat{\mathscr{O}}_{\mathbb{T}_\lambda,\,\pm 1}.$$

Then $\mathscr{O}_i$ is identified with $\Big\{f(\lambda) \in \widehat{\mathscr{O}}_{\mathbb{T}_\lambda,\,\pm 1}\,;\, f(\lambda^{-1}) = f(\lambda)\Big\}$. Hence $\psi(\lambda(i)^{-1}\lambda)^2$ belongs to $\mathscr{O}_i$. We define $y_i \in \mathscr{O}_i$ by $y_i = \psi(\lambda(i)^{-1}\lambda)^2$. Then we have $\mathfrak{m}_i = \mathscr{O}_i y_i$.

4.5.3. *Definition of* $\widetilde{s}_a e(\nu)$. We will define $\widetilde{s}_a e(\nu)$ for $1 \le a < n$ and $\nu \in J^n$.

In the preceding subsection, we have constructed $y_i \in \mathscr{O}_i$ such that $y_a e(\nu) = q_{a,\nu}(y_{\nu_a})$ for any $\nu \in J^n$. For $i, j \in J$, let us denote

$$\mathscr{O}_{i,j} = \widehat{\mathscr{O}}_{\mathbb{T}^2,(X(i),X(j))} \text{ and } \widetilde{\mathscr{O}}_{i,j} = \widehat{\mathscr{O}}_{C\times C,\, ((X(i),\lambda(i)),(X(j),\lambda(j)))}.$$

We regard $\mathscr{O}_{i,j}$ as a subalgebra of $\widetilde{\mathscr{O}}_{i,j}$. Let $r^1_{i,j}\colon \widetilde{\mathscr{O}}_i \to \widetilde{\mathscr{O}}_{i,j}$ and $r^2_{i,j}\colon \widetilde{\mathscr{O}}_j \to \widetilde{\mathscr{O}}_{i,j}$ be the algebra homomorphisms induced by the first and second projections from $C \times C$ to C, respectively. We write y_1 for $r^1_{i,j}(y_i) \in \mathscr{O}_{i,j}$ and y_2 for $r^2_{i,j}(y_j) \in \mathscr{O}_{i,j}$. Similarly, we define $\lambda_1 \in \widetilde{\mathscr{O}}_{i,j}$ and $\lambda_2 \in \widetilde{\mathscr{O}}_{i,j}$ as $r^1_{i,j}(\lambda)$ and $r^2_{i,j}(\lambda)$. Note that $X_1, X_2 \in \mathscr{O}_{i,j}$ may be regarded as $r^1_{i,j}(X)$ and $r^2_{i,j}(X)$.

Let $c_1\colon \mathscr{O}_{i,j} \to \mathscr{O}_{c(i),j}$ and $c_2\colon \mathscr{O}_{i,j} \to \mathscr{O}_{i,c(j)}$ be the isomorphisms induced by $(X_1, X_2) \mapsto (X_1^{-1}, X_2)$ and $(X_1, X_2) \mapsto (X_1, X_2^{-1})$, respectively. We denote by $s_{12}\colon \mathscr{O}_{i,j} \to \mathscr{O}_{j,i}$ the isomorphism induced by $\mathbb{T}^2 \ni (X_1, X_2) \mapsto (X_2, X_1) \in \mathbb{T}^2$.

Let $\psi_{a,\nu}\colon \mathscr{O}_{\nu_a,\nu_{a+1}} \longrightarrow \widehat{\mathscr{O}}_{\mathbb{T}^n,X(\nu)} e(\nu)$ be the algebra homomorphism induced by the projection $(X_1, \dots, X_n) \longmapsto (X_a, X_{a+1})$. Let us define $R_{i,j} \in \mathscr{O}_{i,j}$ by

$$R_{i,j} = \begin{cases} \widetilde{Q}_{i,j}(y_1, y_2) & \text{if } j \neq i, c(i), \\ -(y_1 - y_2)^{-2} & \text{if } j = i \notin J^c, \\ -(y_1 + y_2)^{-2} & \text{if } j = c(i) \notin J^c, \\ -(y_1 - y_2)^{-2} - (y_1 + y_2)^{-2} = -2\dfrac{y_1^2 + y_2^2}{(y_1^2 - y_2^2)^2} & \text{if } j = i \in J^c. \end{cases} \tag{4.17}$$

Then we have $R_{a,a+1} e(\nu) = \psi_{a,\nu}(R_{\nu_a,\nu_{a+1}})$.

Let us recall (see Remark 4.2)

$$\Phi_a^2 = F(X_a, X_{a+1}).$$

Lemma 4.5. *For $i, j \in J$ such that $\varepsilon(i) = \varepsilon(j) = 0$, the following statements hold.*

(i) *$F(X_1, X_2)^{-1} R_{i,j}$ belongs to ${\mathscr{O}_{i,j}}^{\times}$.*

(ii) *$F(X_1, X_2)^{-1} R_{i,i} - \left(\dfrac{X_1 X_2^{-1} - 1}{\xi(y_1 - y_2)}\right)^2$ belongs to $\mathscr{O}_{i,i}(X_1 - X_2)$.*

We will prove this lemma later: (i) by case-by-case verification and (ii) as a consequence of (i). Admitting this lemma for a while, we shall construct $\widetilde{s}_a e(\nu)$ and prove the theorem.

Lemma 4.6. *For $i, j \in J$ such that $\varepsilon(i) = \varepsilon(j) = 0$, there exists a family of elements $G_{i,j} \in \mathscr{O}_{i,j}$ satisfying the following conditions:*

$$\begin{cases} \text{(a) } G_{i,j} \cdot s_{12} G_{j,i} = F(X_1, X_2)^{-1} R_{i,j}, \\ \text{(b) } G_{i,i} - \dfrac{X_1 X_2^{-1} - 1}{\xi(y_1 - y_2)} \in \mathscr{O}_{i,i}(X_1 - X_2), \\ \text{(c) if } c(i) = i, \text{ then } c_1\big(G_{i,j}\big) = G_{i,j}, \text{ and if } c(j) = j, \text{ then } c_2\big(G_{i,j}\big) = \\ \quad G_{i,j}. \end{cases} \tag{4.18}$$

Proof. Assume first that $i \neq j$. In order to see (a), it is enough to take $G_{j,i} = 1$ and $G_{i,j} = F(X_1, X_2)^{-1} R_{i,j}$. Indeed, we have $G_{j,i} \cdot s_{12} G_{i,j} = F(X_1, X_2)^{-1} R_{j,i}$ because $F(X_1, X_2)$ is invariant under s_{12} and $s_{12} R_{i,j} = R_{j,i}$.

If $i = j$, then by Lemma 4.5, there exists a unique $G_{i,i}$ satisfying $G_{i,i}^2 = F(X_1, X_2)^{-1} R_{i,i}$ and the condition (b). Since $F(X_1, X_2)^{-1} R_{i,i}$ is invariant under s_{12}, we derive that

$(s_{12}G_{i,i})^2 = G_{i,i}^2$. Since $G_{i,i}$ and $s_{12}G_{i,i}$ take the same non-zero value at $(X(i), X(j))$, we have $s_{12}G_{i,i} = G_{i,i}$. Hence (a) is satisfied.

Let us show (c). Assume $c(i) = i$. Since $F(X_1, X_2)^{-1}R_{i,j}$ is invariant under c_1, we have $c_1(G_{i,j}) = \pm G_{i,j}$. Since $G_{i,j}$ and $c_1G_{i,j}$ take the same non-zero value at $(X(i), X(j))$, $c_1G_{i,j} = G_{i,j}$ holds. Similarly, we have $c_2G_{i,j} = G_{i,j}$ if $c(j) = j$. □

Thus we have proved that there exists $\big(G_{i,j}\big)_{i,j}$ ($i, j \in J$ with $\varepsilon(i) = \varepsilon(j) = 0$) which satisfies the conditions (4.18). We extend the definition of $G_{i,j}$ for all $i, j \in J$ by

$$G_{i,j} = c_1^{\varepsilon(i)} c_2^{\varepsilon(j)} G_{c^{\varepsilon(i)}(i),\, c^{\varepsilon(j)}(j)}.$$

Then $(G_{i,j})_{i,j\in J}$ satisfies (4.18) (a), (4.18) (b) and

$$c_1(G_{i,j}) = G_{c(i),j} \text{ and } c_2(G_{i,j}) = G_{i,c(j)} \tag{4.19}$$

as well.

Now we define $\widetilde{s}_a \in \mathcal{KHC}_n$ by

$$\widetilde{s}_a e(\nu) = \Phi_a e(\nu)\psi_{a,\nu}(G_{\nu_a,\nu_{a+1}}). \tag{4.20}$$

Let us verify the conditions in (4.13). By Lemma 3.4, the $\widetilde{s}_a$'s satisfy the braid relations. By the construction, we have

$$\widetilde{s}_a \cdot e(\nu) f = s_a(f) e(s_a\nu) \cdot \widetilde{s}_a \text{ for any } f \in \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{T}^n, X(\nu)}).$$

The condition (4.19) implies that

$$\widetilde{s}_a C_b = C_{s_a(b)} \widetilde{s}_a.$$

Hence we have proved the conditions (iv) and (v) in (4.18).

The hypothesis (4.18) (a) implies

$$\widetilde{s}_a^2 e(\nu) = R_{a,a+1} e(\nu).$$

Indeed, $s_a\big(\psi_{a,s_a\nu}(G_{\nu_{a+1},\nu_a})\big) = \psi_{a,\nu}(s_{12}G_{\nu_{a+1},\nu_a})$ and we have

$$\begin{aligned}
\widetilde{s}_a^2 e(\nu) &= \widetilde{s}_a e(s_a\nu) \cdot \widetilde{s}_a e(\nu) = \Phi_a e(s_a\nu)\psi_{a,s_a\nu}(G_{\nu_{a+1},\nu_a})\Phi_a e(\nu)\psi_{a,\nu}(G_{\nu_a,\nu_{a+1}}) \\
&= \Phi_a^2 e(\nu) s_a\big(\psi_{a,s_a\nu}(G_{\nu_{a+1},\nu_a})\big)\psi_{a,\nu}(G_{\nu_a,\nu_{a+1}}) \\
&= \Phi_a^2 e(\nu)\psi_{a,s_a\nu}\big(s_{12}G_{\nu_{a+1},\nu_a} \cdot G_{\nu_a,\nu_{a+1}}\big) \\
&= F(X_a, X_{a+1})e(\nu)\big(F(X_a, X_{a+1})^{-1}R_{a,a+1}e(\nu)\big) = R_{a,a+1}e(\nu).
\end{aligned}$$

Using the condition (4.18) (b), let us show the condition (4.13) (vi):

$$\sigma_a e(\nu) := \widetilde{s}_a e(\nu) + f_{a,a+1} e(\nu) \in T_a e(\nu)(\widehat{\mathscr{O}}_{\mathbb{T}^n, X(\nu)})^{\times} + \mathscr{O}_n\langle C_1, \cdots, C_n\rangle.$$

Setting $i = \nu_a$ and $j = \nu_{a+1}$ and writing $G_{i,j}$ for $\psi_{a,\nu}(G_{\nu_a,\nu_{a+1}})$, we have

$$\begin{aligned}
\sigma_a e(\nu) &= \Big(\widetilde{s}_a - (y_a - y_{a+1})\delta_{i,j} + (y_a + y_{a+1})^{-1} C_a C_{a+1}\delta_{c(i),j}\Big) e(\nu) \\
&= \Big(T_a + \xi(X_a X_{a+1}^{-1} - 1)^{-1} - C_a C_{a+1}\xi(X_a^{-1} X_{a+1}^{-1} - 1)^{-1}\Big) e(\nu) G_{i,j} \\
&\qquad -(y_a - y_{a+1})^{-1}\delta_{i,j} e(\nu) - C_a C_{a+1}(y_a + y_{a+1})^{-1}\delta_{c(i),j} e(\nu) \\
&= T_a e(\nu) G_{i,j} + \Big(\xi(X_a X_{a+1}^{-1} - 1)^{-1} G_{i,j} - (y_a - y_{a+1})^{-1}\delta_{i,j}\Big) e(\nu) \\
&\qquad + C_a\Big(-\xi(X_a^{-1} X_{a+1} - 1)^{-1}(c_{a+1} G_{i,j}) + (-y_a + y_{a+1})^{-1}\delta_{c(i),j}\Big) C_{a+1} e(\nu) \\
&= T_a e(\nu) G_{i,j} + \Big(\xi(X_a X_{a+1}^{-1} - 1)^{-1} G_{i,j} - (y_a - y_{a+1})^{-1}\delta_{i,j}\Big) e(\nu) \\
&\qquad + C_a\Big(\xi\big((X_a X_{a+1}^{-1} - 1)^{-1} + 1\big) G_{i,c(j)} - (y_a - y_{a+1})^{-1}\delta_{i,c(j)}\Big) e(c_{a+1}\nu) C_{a+1}.
\end{aligned}$$

Here, the last equality follows from (4.19). Hence it is enough to show that

$$\xi(X_a X_{a+1}^{-1} - 1)^{-1} G_{i,j} - (y_a - y_{a+1})^{-1}\delta_{i,j} \in \mathscr{O}_n. \tag{4.21}$$

If $i \neq j$, then $(X_a X_{a+1}^{-1} - 1)^{-1} e(\nu) \in \mathscr{O}_n$ and hence (4.21) holds.

If $i = j$, then

$$\xi(X_a X_{a+1}^{-1} - 1)^{-1} G_{i,j} - (y_a - y_{a+1})^{-1} = \xi(X_a X_{a+1}^{-1} - 1)^{-1}\Big(G_{i,j} - (X_a X_{a+1}^{-1} - 1)\xi^{-1}(y_a - y_{a+1})^{-1}\Big)$$

belongs to $\mathscr{O}_n$ by (4.18) (b). Thus we have verified all the conditions in (4.13), and hence $\widehat{\mathrm{RC}}_n$ and $\mathcal{OHC}_n$ are isomorphic.

4.5.4. *Proof of Lemma 4.5.* Now, we shall prove Lemma 4.5 for $i, j \in J$ such that $\varepsilon(i) = \varepsilon(j) = 0$.

Let us first derive (ii) admitting (i). Let η be a generic point of the irreducible subscheme $\{X_1 = X_2\}$ of $\mathrm{Spec}(\mathscr{O}_{i,i})$, and let $(\mathscr{O}_{i,i})_\eta$ be the localization of $\mathscr{O}_{i,i}$ at η. Then (i) implies that $S := F(X_1, X_2)^{-1} R_{i,i} - \Big(\dfrac{X_1 X_2^{-1} - 1}{\xi(y_1 - y_2)}\Big)^2$ belongs to $\mathscr{O}_{i,i}$. Hence in order to see (ii), it is enough to show that S belongs to $(\mathscr{O}_{i,i})_\eta (X_1 - X_2)$.

We have, modulo $(\mathscr{O}_{i,i})_\eta (X_1 - X_2)$

$$\begin{aligned}
& F(X_1, X_2)^{-1}(X_1 X_2^{-1} - 1)^{-2} \\
&= \frac{(X_2 - X_1^{-1})^2}{(X_1 + X_1^{-1})^2 - (q^2 + q^{-2})(X_1 + X_1^{-1})(X_2 + X_2^{-1}) + (X_2 + X_2^{-1})^2 + 4(q - q^{-1})^2} \\
&\equiv \frac{(X_1 - X_1^{-1})^2}{(2 - q^2 - q^{-2})(X_1 + X_1^{-1})^2 + 4(q - q^{-1})^2} \\
&= \frac{(X_1 - X_1^{-1})^2}{-(q - q^{-1})^2 (X_1 - X_1^{-1})^2} = -\xi^{-2}
\end{aligned}$$

Since $R_{i,j} \equiv -(y_1 - y_2)^{-2} \mod (\mathscr{O}_{i,j})_\eta$, we obtain

$$F(X_1, X_2)^{-1} R_{i,j} \equiv \frac{(X_1 X_2^{-1} - 1)^2}{\xi^2 (y_1 - y_2)^2} \mod (\mathscr{O}_{i,j})_\eta (X_1 - X_2).$$

It only remains to prove (i). Let us show it by case-by-case verification. Let us recall that

$$F(X_1,X_2)^{-1} = \frac{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_1^{-1})^2}{(\lambda_2-q^2\lambda_1)(\lambda_2-q^{-2}\lambda_1)(\lambda_2-q^2\lambda_1^{-1})(\lambda_2-q^{-2}\lambda_1^{-1})}. \tag{4.22}$$

Note that (see (4.9)) for $i,j\in J$ such that $\varepsilon(i)=\varepsilon(j)=0$, we have

(a) $\lambda(j)=\lambda(i)^{-1}$ implies $i=j$,
(b) $(\alpha_{\mathrm{pr}(i)},\alpha_{\mathrm{pr}(j)})<0$ implies $\lambda(j)=q^{\pm2}\lambda(i)$,

- Case $\lambda(j)\neq\lambda(i),q^{\pm2}\lambda(i)$.

  In this case, $F(X_1,X_2)\in\mathscr{O}_{i,j}^{\times}$ and $R_{i,j}=\widetilde{Q}_{i,j}=1$.

- Case $\lambda(j)=q^{\pm2}\lambda(i)$ and $\lambda(i),\lambda(j)\neq\pm1,\pm q,\pm q^{-1}$.

  We may assume that $\lambda(j)=q^2\lambda(i)$. In this case, $y_1=\psi(\lambda(i)^{-1}\lambda_1)$, $y_2=\psi(\lambda(j)^{-1}\lambda_2)$, and

$$R_{i,j}=\widetilde{Q}_{i,j}(y_1,y_2)=\pm(y_1-y_2)\in{\mathscr{O}_{i,j}}^{\times}(\lambda(i)^{-1}\lambda_1-\lambda(j)^{-1}\lambda_2)={\mathscr{O}_{i,j}}^{\times}(\lambda_2-q^2\lambda_1).$$

Therefore, since

$$F(X_1,X_2)^{-1}R_{i,j}\in{\mathscr{O}_{i,j}}^{\times}\frac{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_1^{-1})^2}{(\lambda_2-q^{-2}\lambda_1)(\lambda_2-q^2\lambda_1^{-1})(\lambda_2-q^{-2}\lambda_1^{-1})},$$

$F(X_1,X_2)^{-1}R_{i,j}$ belongs to $\mathscr{O}_{i,j}^{\times}$.

- Case $(\lambda(i),\,\lambda(j))=\pm(1,q^{2c})$ or $\pm(q^{2c},1)$ for some $c=1,-1$

  We may assume that $(\lambda(i),\,\lambda(j))=\pm(1,q^{2c})$. Thus we have $y_1=\psi(\lambda(i)^{-1}\lambda_1)^2$. Moreover, we have $\mathrm{pr}(j)=g(\pm q)$ if and only if $(q^2)^3=1$.

  Let us first assume $(q^2)^3=1$. In this case, $y_2^2=\psi(\lambda(j)^{-1}\lambda_2)$, and

$$\begin{aligned}
R_{i,j} &= \widetilde{Q}_{i,j}(y_1,y_2)=\pm(y_1-y_2^4)=\pm\big(\psi(\lambda(i)^{-1}\lambda_1)^2-\psi(\lambda(j)^{-1}\lambda_2)^2\big)\\
&= \pm\big(\psi(\lambda(i)^{-1}\lambda_1)-\psi(\lambda(j)^{-1}\lambda_2)\big)\big(\psi(\lambda(i)^{-1}\lambda_1)+\psi(\lambda(j)^{-1}\lambda_2)\big)\\
&= \pm\big(\psi(\lambda(i)^{-1}\lambda_1)-\psi(\lambda(j)^{-1}\lambda_2)\big)\big(-\psi(\lambda(i)\lambda_1^{-1})+\psi(\lambda(j)^{-1}(\lambda_2))\big)\\
&\in {\mathscr{O}_{i,j}}^{\times}(\lambda(j)^{-1}\lambda_2-\lambda(i)^{-1}\lambda_1)(\lambda(j)^{-1}\lambda_2-\lambda(i)\lambda_1^{-1})\\
&= {\mathscr{O}_{i,j}}^{\times}(\lambda_2-\lambda(j)\lambda(i)^{-1}\lambda_1)(\lambda_2-\lambda(i)\lambda(j)\lambda_1^{-1})\\
&= {\mathscr{O}_{i,j}}^{\times}(\lambda_2-q^{2c}\lambda_1)(\lambda_2-q^{2c}\lambda_1^{-1}).
\end{aligned}$$

Hence we have

$$F(X_1,X_2)^{-1}R_{i,j}\ \in\ {\mathscr{O}_{i,j}}^{\times}\frac{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_1^{-1})^2}{(\lambda_2-q^{-2c}\lambda_1)(\lambda_2-q^{-2c}\lambda_1^{-1})}=\mathscr{O}_{i,j}^{\times}.$$

Now assume that $(q^2)^3\neq1$. In this case, $y_1=\psi(\lambda(i)^{-1}\lambda_1)^2$, $y_2=\psi(\lambda(j)^{-1}\lambda_2)$, $R_{i,j}=\widetilde{Q}(y_1,y_2)=y_2^2-y_1=\psi(\lambda(i)^{-1}\lambda_1)^2-\psi(\lambda(j)^{-1}\lambda_2)^2$. Hence, similarly to the case of $(q^2)^3=1$, we have $R_{i,j}\in{\mathscr{O}_{i,j}}^{\times}(\lambda_2-q^{2c}\lambda_1)(\lambda_2-q^{2c}\lambda_1^{-1})$ and hence we have

$$F(X_1,X_2)^{-1}R_{i,j}\ \in\ {\mathscr{O}_{i,j}}^{\times}\frac{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_1^{-1})^2}{(\lambda_2-q^{-2c}\lambda_1)(\lambda_2-q^{-2c}\lambda_1^{-1})}=\mathscr{O}_{i,j}^{\times}.$$

• Case $(\lambda(i),\lambda(j)) = \pm(q^c, q^{3c})$ or $\pm(q^{3c}, q^c)$ for some $c = 1, -1$.

Assume that $(\lambda(i),\lambda(j)) = \pm(q^c, q^{3c})$. First note that $\lambda(j) = \pm 1$ if and only if $(q^2)^3 = 1$, and $\mathrm{pr}(j) = g(\pm q)$ if and only if $(q^2)^4 = 1$.
The case $(q^2)^3 = 1$ was already treated. Assume that $(q^2)^4 = 1$. Then $q^4 = -1$ and hence $\lambda(j) = \mp q^{-c}$. In this case, $y_1^2 = \psi(\lambda(i)^{-1}\lambda_1)$ and $y_2^2 = \psi(\lambda(j)^{-1}\lambda_2)$, and $R_{i,j} = \pm(y_1^2 - y_2^2)$. Hence $R_{i,j} \in \mathscr{O}_{i,j}{}^{\times}(\lambda(i)^{-1}\lambda_1 - \lambda(j)^{-1}\lambda_2) = \mathscr{O}_{i,j}{}^{\times}(\lambda_2 - q^{2c}\lambda_1)$. Therefore, we have

$$F(X_1,X_2)^{-1}R_{i,j} \quad \in \quad \mathscr{O}_{i,j}{}^{\times}\frac{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_1^{-1})^2}{(\lambda_2-q^{-2c}\lambda_1)(\lambda_2-q^2\lambda_1^{-1})(\lambda_2-q^{-2}\lambda_1^{-1})} = \mathscr{O}_{i,j}^{\times}.$$

Assume now $(q^2)^k \neq 1$ for $k = 3, 4$. In this case, $y_1^2 = \psi(\lambda(i)^{-1}\lambda_1)$, $y_2 = \psi(\lambda(j)^{-1}\lambda_2)$, and $R_{i,j} = \widetilde{Q}_{i,j}(y_1, y_2) = \pm(y_1^2 - y_2) \in \mathscr{O}_{i,j}{}^{\times}(\lambda_2 - q^{2c}\lambda_1)$. Therefore, since

$$F(X_1,X_2)^{-1}R_{i,j} \quad \in \quad \mathscr{O}_{i,j}{}^{\times}\frac{(\lambda_2-\lambda_1)^2(\lambda_2-\lambda_1^{-1})^2}{(\lambda_2-q^{-2c}\lambda_1)(\lambda_2-q^2\lambda_1^{-1})(\lambda_2-q^{-2}\lambda_1^{-1})},$$

$F(X_1,X_2)^{-1}R_{i,j}$ belongs to $\mathscr{O}_{i,j}^{\times}$.

• Case $i = j$ and $\lambda(i) \neq \pm 1, \pm q^{\pm 1}$.

In this case, $y_k = \psi(\lambda(i)^{-1}\lambda_k)$ $(k = 1, 2)$ and $R_{i,j} = -(y_1 - y_2)^{-2} \in \mathscr{O}_{i,j}{}^{\times}(\lambda_1 - \lambda_2)^{-2}$. Then by (4.22)

$$\begin{aligned}F(X_1,X_2)^{-1}R_{i,i} \quad &\in \quad \mathscr{O}_{i,j}{}^{\times}\frac{(\lambda_2-\lambda_1^{-1})^2}{(\lambda_2-q^2\lambda_1)(\lambda_2-q^{-2}\lambda_1)(\lambda_2-q^2\lambda_1^{-1})(\lambda_2-q^{-2}\lambda_1^{-1})}\\ &= \quad \mathscr{O}_{i,j}{}^{\times}.\end{aligned}$$

• Case $i = j$ and $\lambda(i) = \pm q^c$ for some $c = 1, -1$.

In this case, $y_k^2 = \psi(\lambda(i)^{-1}\lambda_k)$ $(k = 1, 2)$, and

$$\begin{aligned}R_{i,j} \quad &= \quad -2(y_1^2+y_2^2)(y_1^2-y_2^2)^{-2}\\ &= \quad -2\big(\psi(\lambda(i)^{-1}\lambda_2)-\psi(\lambda(i)\lambda_1^{-1})\big)\big(\psi(\lambda(i)^{-1}\lambda_1)-\psi(\lambda(i)^{-1}\lambda_2)\big)^{-2}\\ &\in \quad \mathscr{O}_{i,j}{}^{\times}(\lambda_2-q^{2c}\lambda_1^{-1})(\lambda_2-\lambda_1)^{-2}.\end{aligned}$$

Hence by (4.22), we have

$$F(X_1,X_2)^{-1}R_{i,j} \quad \in \quad \mathscr{O}_{i,j}{}^{\times}\frac{(\lambda_2-\lambda_1^{-1})^2}{(\lambda_2-q^2\lambda_1)(\lambda_2-q^{-2}\lambda_1)(\lambda_2-q^{-2c}\lambda_1^{-1})}$$

and it belongs to $\mathscr{O}_{i,j}{}^{\times}$.

• Case $i = j$ and $\lambda(i) = \pm 1$.

In this case, $y_k = \psi(\lambda(i)^{-1}\lambda_k)^2$ $(k = 1, 2)$, and

$$\begin{aligned}R_{i,j} = -(y_1-y_2)^{-2} \quad &\in \quad \mathscr{O}_{i,j}{}^{\times}\big(\psi(\lambda(i)^{-1}\lambda_2)-\psi(\lambda(i)^{-1}\lambda_1)\big)^{-2}\big(\psi(\lambda(i)^{-1}\lambda_2)-\psi(\lambda(i)\lambda_1^{-1})\big)^{-2}\\ &= \quad \mathscr{O}_{i,j}{}^{\times}(\lambda_2-\lambda_1)^{-2}(\lambda_2-\lambda_1^{-1})^{-2}.\end{aligned}$$

Hence

$$F(X_1,X_2)^{-1}R_{i,j} \quad \in \quad \mathscr{O}_{i,j}{}^{\times}\frac{1}{(\lambda_2-q^2\lambda_1)(\lambda_2-q^{-2}\lambda_1)(\lambda_2-q^2\lambda_1^{-1})(\lambda_2-q^{-2}\lambda_1^{-1})}$$

and it belongs to $\mathscr{O}_{i,j}^{\times}$.

Thus we have proved Lemma 4.5, which completes the proof of Theorem 4.4. □

4.6. **The case $q^2 = -1$.** Let us briefly explain what happens in the case $q^2 = -1$. In this case also, we can prove that $\mathcal{OHC}_n$ is isomorphic to $\widehat{\mathrm{RC}}_n$ by a similar argument. We have a factorization $K(u, v) = (u+v-4)(u+v+4)$. Therefore $\mathrm{pr}\colon \mathbb{T}_X \to \mathbb{A}^1_w$ is the same but g is the identity. The Dynkin diagram structure of $\mathbb{A}^1_w$ is given by the same formula (4.8) as in the $(q^2)^2 \neq 1$ case. The odd vertices are $w = \pm 2$. We will explain which types of Dynkin diagrams appear.

Let $s_1(w) = 4-w$ and $s_2(w) = -4-w$ be the automorphisms of $\mathbb{A}^1_w$. Let G be the group generated by s_1 and s_2. Then it is an affine reflection group. The connected components of $\mathbb{A}^1_w$ are nothing but the G-orbits. They are described as follows

(i) When $\operatorname{char} \mathbf{k} = 0$, there are two types of connected components.
  (a) $G\zeta$ ($\zeta \notin 2\mathbb{Z}$). The Dynkin diagram is of type A_∞.
  (b) $G\gamma$ with $\gamma = \pm 2$. The Dynkin diagram is of type B_∞.
(ii) When $\operatorname{char} \mathbf{k} = p > 2$, there are two types of connected components. Note that $(s_1 s_2)^p = \mathrm{id}$.
  (a) $G\zeta$ ($\zeta \notin \mathbb{F}_p$). The Dynkin diagram is of type $A^{(1)}_{2p-1}$.
  (b) $\mathbb{F}_p$. The Dynkin diagram is of type $D^{(2)}_p$.

4.7. **Cyclotomic Hecke-Clifford superalgebras.** For a subset I_1 of $\mathbb{A}^1_w$, set $J_1 = \mathrm{pr}^{-1}(I_1)$. For $\Lambda = \sum_{i \in I_1} m_i \Lambda_i$ with $m_i \in \mathbb{Z}_{\geq 0}$ (see § 3.7), we set

$$f_\Lambda(X_1) = \prod_{i \in J_1} (X_1 - X(i))^{m_{\mathrm{pr}(i)}},$$

and

$$\mathcal{AHC}^\Lambda_n = \mathcal{AHC}_n / \big(\mathcal{AHC}_n f_\Lambda(X_1) \mathcal{AHC}_n\big).$$

We call $\mathcal{AHC}^\Lambda_n$ the *cyclotomic Hecke-Clifford superalgebra*. Then we have the following lemma.

Lemma 4.7. *Let $n \geq 1$ and Λ as above.*

(i) *$\mathcal{AHC}^\Lambda_n$ is a finite-dimensional $\mathbf{k}$-module.*
(ii) *Set $I_n = \{g(q^{2k}\lambda)\,;\, g(\lambda) \in I_1$ and $-n < k < n\}$ and $J_n = \mathrm{pr}^{-1}(I_n)$. Then for any finite-dimensional $\mathcal{AHC}^\Lambda_n$-module M, the eigenvalues of $X_k|_M$ belongs to $\{X(i)\,;\, i \in J_n\}$.*

Hence by Theorem 4.4,we have

Corollary 4.8. *If J is a subset of $\mathbb{T}_X$ invariant under c and containing J_n, then the cyclotomic Hecke-Clifford superalgebra $\mathcal{AHC}^\Lambda_n$ is isomorphic to the cyclotomic quiver Hecke-Clifford superalgebra RC^Λ_n.*

Note that the Hecke-Clifford algebra $\mathcal{HC}_n$ is isomorphic to $\mathcal{AHC}^{\Lambda_{i_0}}_n$, where $i_0 \in I$ is an odd vertex ($i_0 = \mathrm{pr}(\pm 1) = g(\pm q)$).

5. Relations to affine Sergeev superalgebras

In this section, we will prove that the affine Sergeev superalgebra, a degenerate version of the affine Hecke-Clifford superalgebra, is also isomorphic to a quiver Hecke-Clifford superalgebra. Since the proof is parallel to the affine Hecke-Clifford superalgebra case, the discussion will be brief. We still assume that $\mathbf{k}$ is an algebraically closed field of characteristic $\neq 2$.

5.1. Affine Sergeev superalgebras.

Definition 5.1 ([JN]). *For an integer $n \geq 0$, the* affine Sergeev superalgebra $\overline{\mathcal{AHC}}_n$ *is the $\mathbf{k}$-superalgebra generated by the even generators $x_1, \dots, x_n, t_1, \dots, t_{n-1}$ and the odd generators $C_1, \dots, C_n$ with the following defining relations.*

(i) $x_i x_j = x_j x_i$ *for all* $1 \leq i, j \leq n$,
(ii) $C_i^2 = 1$, $C_i C_j + C_j C_i = 0$ *for all* $1 \leq i \neq j \leq n$,
(iii) $t_i^2 = 1$, $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$, $t_i t_j = t_j t_i$ $(|i-j| \geq 2)$,
(iv) $t_i C_j = C_{s_i(j)} t_i$,
(v) $C_i x_j = x_j C_i$ *for all* $1 \leq i \neq j \leq n$,
(vi) $C_i x_i = -x_i C_i$ *for all* $1 \leq i \leq n$,
(vii) $t_i x_i = x_{i+1} t_i - 1 - C_i C_{i+1}$, $t_i x_{i+1} = x_i t_i + 1 - C_i C_{i+1}$ *for all* $1 \leq i \leq n-1$,
(viii) $t_i x_j = x_j t_i$ *if* $j \neq i, i+1$.

Remark 5.2. ([Naz],[Kle, §14.8]) The intertwiners of $\overline{\mathcal{AHC}}_n$ are defined by

$$\varphi_a = t_a + (x_a - x_{a+1})^{-1} - (x_a + x_{a+1})^{-1} C_a C_{a+1} \in \mathbf{k}(x_1, \dots, x_n) \otimes_{\mathbf{k}[x_1,\dots,x_n]} \overline{\mathcal{AHC}}_n \tag{5.1}$$

and they satisfy the relations

$$\varphi_a x_i = x_{s_a(i)} \varphi_a \quad \text{and} \quad \varphi_a C_i = C_{s_a(i)} \varphi_a. \tag{5.2}$$

Setting

$$K(u,v) = (u-v)^2 - 2(u+v) \quad \text{and} \quad F(x_a, x_{a+1}) = K(x_a^2, x_{a+1}^2)/(x_a^2 - x_{a+1}^2)^2,$$

we have

$$\Phi_a^2 = F(x_a, x_{a+1}). \tag{5.3}$$

By setting $u = \lambda^2 - 1/4$ and $v = \mu^2 - 1/4$, we have

$$K(u,v) = (\lambda - \mu - 1)(\lambda - \mu + 1)(\lambda + \mu - 1)(\lambda + \mu + 1). \tag{5.4}$$

Let $\mathbb{A}_x^1$ be the one-dimensional affine space $\mathbb{A}^1$ with the coordinate x and let $c\colon \mathbb{A}_x^1 \to \mathbb{A}_x^1$ be the involution of $\mathbb{A}_x^1$ given by $c(x) = -x$ (corresponding to Definition 5.1 (vi)).

For $1 \leq p \leq n$, let us denote by $c_p\colon \mathbb{A}^n \to \mathbb{A}^n$ the involution

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{p-1}, -x_p, x_{p+1}, \dots, x_n),$$

and by the same letter the induced isomorphism $\mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n,q}) \xrightarrow{c_p} \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n,c_p(q)})$. For $1 \leq p < n$, we denote by $s_p\colon \mathbb{A}^n \to \mathbb{A}^n$ the involution

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{p-1}, x_{p+1}, x_p, x_{p+2}, \dots, x_n),$$

and by the same letter s_p the induced isomorphism $\mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n,q}) \to \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n,s_p(q)})$. Similarly, we denote by $\overline{s}_p\colon \mathbb{A}^n \to \mathbb{A}^n$ the involution

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{p-1}, -x_{p+1}, -x_p, x_{p+2}, \dots, x_n),$$

and by the same letter $\overline{s}_p$ the induced isomorphism $\mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n,q}) \to \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n,\overline{s}_p(q)})$.

Let us denote by $\overline{\mathcal{HC}}_n$ the $\mathbf{k}$-superalgebra generated by C_p $(1 \le p \le n)$, t_a $(1 \le a < n)$ with the defining relations (ii)–(iv) in Definition 5.1. The superalgebra $\overline{\mathcal{HC}}_n$ is called the *Sergeev superalgebra* and can be regarded as a subsuperalgebra of $\overline{\mathcal{AHC}}_n$.

Definition 5.3. *Let J be a finite subset of $\mathbb{A}^1_x$ invariant by c, and let $x \colon J \to \mathbb{T}_X$ be the inclusion map. We define*

$$\mathscr{O}_n = \bigoplus_{\nu \in J^n} \widehat{\mathscr{O}}_{\mathbb{A}^n, x(\nu)} e(\nu), \quad \mathscr{K}_n = \bigoplus_{\nu \in J^n} \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n, x(\nu)}) e(\nu),$$

where $x(\nu) = (x(\nu_1), \cdots, x(\nu_n)) \in \mathbb{A}^n$. We define the algebra structure on

$$\overline{\mathcal{KHC}}_n = \mathscr{K}_n \otimes_{\mathbf{k}} \overline{\mathcal{HC}}_n$$

by

$$(5.5) \qquad \begin{aligned} C_p e(\nu) f &= e(c_p \nu) c_p(f) C_p \quad (1 \le p \le n), \\ t_a e(\nu) f &= e(s_a \nu) s_a(f) t_a + \frac{e(\nu) f - e(s_a \nu) s_a f}{x_{a+1} - x_a} + C_a C_{a+1} \frac{e(\nu) f - e(\overline{s}_a \nu) \overline{s}_a(f)}{x_{a+1} + x_a} \end{aligned}$$

for $\nu \in J^n$, $1 \le a < n$ and $f \in \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n, x(\nu)})$.

We define $\overline{\mathcal{OHC}}_n$ to be the subsuperalgebra of $\overline{\mathcal{KHC}}_n$ generated by $\mathscr{O}_n$ and $\overline{\mathcal{HC}}_n$.

Thus there exists a $\mathbf{k}$-superalgebra homomorphism $\overline{\mathcal{AHC}}_n \to \overline{\mathcal{OHC}}_n$. We have

$$(5.6) \qquad \varphi_a e(\nu) = e(s_a \nu) \varphi_a \text{ for any } 1 \le a < n.$$

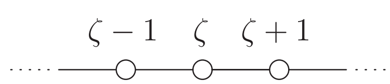
Let us denote by $\mathbb{A}^1_w$ the space $\mathbb{A}^1$ with the coordinate w. Let $\mathrm{pr} \colon \mathbb{A}^1_x \to \mathbb{A}^1_w$ be the map $\mathrm{pr}(x) = x^2$. Let $\mathbb{A}^1_\lambda$ be the affine space $\mathbb{A}^1$ with the coordinate λ. Let $g \colon \mathbb{A}^1_\lambda \to \mathbb{A}^1_w$ be the map $g(\lambda) = \lambda^2 - 1/4$.

5.2. **Dynkin diagram.** We regard $\mathbb{A}^1_w$ as a Dynkin diagram by the same formula (4.8). Note that $g(\lambda_1)$ and $g(\lambda_2)$ $(\lambda_1 \ne \pm\lambda_2)$ are connected if and only if $\lambda_1 = \lambda_2 \pm 1$ or $\lambda_1 = -\lambda_2 \pm 1$.

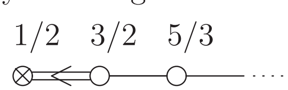
The connected components of the Dynkin diagram $\mathbb{A}^1_w$ are classified as follows. Here the odd vertices are marked by $\times$.

(i) when $\operatorname{char} \mathbf{k} = 0$, there are three types of connected components.

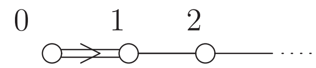
(a) $\{g(\zeta + k)\,;\, k \in \mathbb{Z}\}$ for some $\zeta \notin \mathbb{Z}/2$. The Dynkin diagram is of type A_∞.



(b) $\left\{g(\tfrac{1}{2} + k)\,;\, k \in \mathbb{Z}_{\ge 0}\right\}$. The Dynkin diagram is of type B_∞.

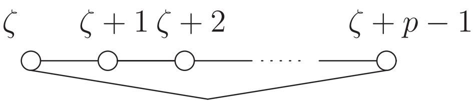


(c) $\{g(k)\,;\, k \in \mathbb{Z}_{\ge 0}\}$. The Dynkin diagram is of type C_∞.

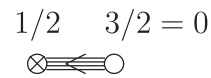


(ii) When $\operatorname{char} \mathbf{k} = p > 2$, there are three types of connected components.

(a) $\{g(\zeta + k)\,;\, k \in \mathbb{F}_p\}$ for some $\zeta \notin \mathbb{F}_p$. The Dynkin diagram is of type $A^{(1)}_{p-1}$.



(b) $\{g(\frac{1}{2}), \ldots, g(\frac{p}{2})\}$ of type $A^{(2)}_{p-1}$.



$(p = 3)$

$(p > 3)$

Let us take a finite subset $J \subset \mathbb{A}^1_x$ invariant under the involution c and set $I = \operatorname{pr}(J) \subset \mathbb{A}^1_w$. Set $J^c := \{j \in J\,;\, c(j) = j\}$, $I_{\text{odd}} = \operatorname{pr}(J^c)$ and $I_{\text{even}} = I \setminus \{0\}$. We choose a function $h\colon I \to \mathbb{A}^1_\lambda$ such that

$$\begin{cases} \bullet\ g(h(i)) := h(i)^2 - 1/4 = i, \\ \bullet\ \text{if } (\alpha_i, \alpha_{i'}) < 0, \text{ then } h(i') = h(i) \pm 1, \\ \bullet\ h(g(1/2)) = 1/2. \end{cases} \tag{5.7}$$

We define $\lambda\colon J \to \mathbb{A}^1_\lambda$ by $\lambda(j) = h(\operatorname{pr}(j))$, and $x\colon J \to \mathbb{A}^1_x$ by $x(j) = j$. We choose $\varepsilon\colon J \to \{0, 1\}$ such that $\varepsilon^{-1}(0) \to I$ is bijective. For $i, j \in I$, we write (i, j) for $(\alpha_{\operatorname{pr}(i)}, \alpha_{\operatorname{pr}(j)})$ and define $\widetilde{Q}_{i,j}$ by the same formula (4.11) and (4.12).

Then we associate to the data $\{\widetilde{Q}_{i,j}\}_{i,j\in J}$ the quiver Hecke-Clifford superalgebra RC_n and its completion $\widehat{\mathrm{RC}}_n$.

Theorem 5.4. *We have* $\widehat{\mathrm{RC}}_n \simeq \overline{\mathcal{OHC}}_n$.

5.3. **Proof of Theorem 5.4.**

5.3.1. *Strategy of the proof.* By Theorem 3.8, it is sufficient to construct the elements $y_k \in \mathscr{O}_n$ $(k = 1, \ldots, n)$ and $\widetilde{s}_a \in \overline{\mathcal{KHC}}_n$ $(a = 1, \ldots, n-1)$ such that

$$\begin{cases} \text{(i)}\ y_p e(\nu) \in (\widehat{\mathscr{O}}_{\mathbb{A}^n, x(\nu)})^\times\, (x_p - x(\nu_p)) e(\nu), \\ \text{(ii)}\ C_a y_p e(\nu) = (-1)^{\delta_{a,p}} y_p e(c_a \nu) C_a, \\ \text{(iii)}\ \widetilde{s}_a^2 = R_{a,a+1} \ \text{(see (3.9))}, \\ \text{(iv)}\ \{\widetilde{s}_a\}_{1 \le a < n} \text{ satisfies the braid relations (see (3.4))}, \\ \text{(v)}\ \widetilde{s}_a C_k = C_{s_a(k)} \widetilde{s}_a,\ \widetilde{s}_a y_k = y_{s_a(k)} \widetilde{s}_a, \\ \text{(vi)}\ \text{setting } \sigma_a := \widetilde{s}_a + f_{a,a+1} \text{ (see (3.12)), the element } \sigma_a e(\nu) \text{ belongs to} \\ \qquad t_a e(\nu) (\widehat{\mathscr{O}}_{\mathbb{A}^n, x(\nu)})^\times + \mathscr{O}_n \langle C_1, \ldots, C_n \rangle. \end{cases} \tag{5.8}$$

5.3.2. *Definition of* $y_p\, e(\nu)$.

We will construct $y_p e(\nu) \in (\widehat{\mathscr{O}}_{\mathbb{A}^n, x(\nu)})^{\times}\, (x_p - x(\nu_p))e(\nu)$. Let C be the curve defined by the equation $x^2 = \lambda^2 - 1/4$. We have two projections:

$$\mathbb{A}^1_x \longleftarrow C \longrightarrow \mathbb{A}^1_\lambda.$$

For every $i \in J$, they induce the injective homomorphisms

$$\widehat{\mathscr{O}}_{\mathbb{A}^1, x(i)} \rightarrowtail \widehat{\mathscr{O}}_{C, (x(i), \lambda(i))} \leftarrowtail \widehat{\mathscr{O}}_{\mathbb{A}^1, \lambda(i)}. \tag{5.9}$$

Let us set $\mathscr{O}_i = \widehat{\mathscr{O}}_{\mathbb{A}^1, x(i)}$ and $\widetilde{\mathscr{O}}_i = \widehat{\mathscr{O}}_{C, (x(i), \lambda(i))}$, and we regard $\mathscr{O}_i$ as a subalgebra of $\widetilde{\mathscr{O}}_i$. Let $c_i \colon \mathscr{O}_i \to \mathscr{O}_{c(i)}$ be the homomorphism induced by the map $x \mapsto -x$. For $1 \le a \le n$, let us denote by $q_{a,\nu} \colon \mathscr{O}_{\nu_a} \longrightarrow \widehat{\mathscr{O}}_{\mathbb{A}^n, x(\nu)} e(\nu)$ the homomorphism induced by the a-th projection $(x_1, \ldots, x_n) \mapsto x_a$.

For all $i \in J$ such that $\varepsilon(i) = 0$, we will construct $y_i \in \mathscr{O}_i$ such that

$$\begin{cases} \text{(i)} \ y_i \in \mathscr{O}_i^{\times}(x - x(i)), \\ \text{(ii)} \ \text{if } c(i) = i, \text{ then } c_i(y_i) = -y_i, \end{cases} \tag{5.10}$$

and we define $y_a e(\nu) = q_{a,\nu}((-c_{\nu_a})^{-\varepsilon(\nu_a)} y_{c^{\varepsilon(\nu_a)}\nu_a})$.

Then we have

$$\begin{aligned} &y_p e(\nu) \in (\widehat{\mathscr{O}}_{\mathbb{A}^n, x(\nu)})^{\times}\, (x_p - x(\nu_p))e(\nu), \\ &C_a y_p e(\nu) = (-1)^{\delta_{ap}} y_p e(c_a \nu) C_a. \end{aligned}$$

- Case $\lambda(i) \neq 0, 1/2$.

In this case, we have isomorphisms

$$\mathbf{k}[[x - x(i)]] = \mathscr{O}_i \xrightarrow{\sim} \widetilde{\mathscr{O}}_i = \widehat{\mathscr{O}}_{C, (\lambda(i), x(i))} \xleftarrow{\sim} \widehat{\mathscr{O}}_{\mathbb{A}^1_\lambda, \lambda(i)} = \mathbf{k}[[\lambda - \lambda(i)]].$$

We define $y_i \in \mathscr{O}_i$ by $y_i = \lambda - \lambda(i)$.

- Case $\lambda(i) = 1/2$.

In this case, we have $x(i) = 0$,

$$\mathbf{k}[[x]] = \mathscr{O}_i = \widehat{\mathscr{O}}_{\mathbb{A}^1_x, 0} \xrightarrow{\sim} \widetilde{\mathscr{O}}_i = \widehat{\mathscr{O}}_{C, (0, 1/2)} \leftarrowtail \widehat{\mathscr{O}}_{\mathbb{A}^1_\lambda, 1/2} = \mathbf{k}[[\lambda - 1/2]],$$

and $\lambda - \frac{1}{2} = \frac{1}{2}(\sqrt{1 + 4x^2} - 1) = x^2 + \cdots \in \mathbf{k}[[x^2]]$. Take $y_i = x + \cdots \in \mathbf{k}[[x]] = \mathscr{O}_i$ such that $y_i^2 = \lambda$. Since $y_i \in x\mathbf{k}[[x^2]]$, we have $c_i(y_i) = -y_i$.

- Case $\lambda(i) = 0$.

In this case, we have $x(i)^2 = -1/4$ and

$$\mathbf{k}[[x - x(i)]] = \widehat{\mathscr{O}}_{\mathbb{A}^1_x, x(i)} \rightarrowtail \widehat{\mathscr{O}}_{C, (x(i), 0)} \xleftarrow{\sim} \widehat{\mathscr{O}}_{\mathbb{A}^1_\lambda, 0} = \mathbf{k}[[\lambda]].$$

Since $x^2 - x(i)^2 = \lambda^2$, we have $\mathscr{O}_i = \mathbf{k}[[x - x(i)]] = \mathbf{k}[[\lambda^2]]$. We define $y_i \in \mathbf{k}[[x - x(i)]]$ by $y_i = \lambda^2$.

5.3.3. *Definition of* $\widetilde{s}_a e(\nu)$. We will define $\widetilde{s}_a e(\nu)$ for $1 \le a < n$ and $\nu \in J^n$.

For $i, j \in J$, let us denote $\mathscr{O}_{i,j} = \widehat{\mathscr{O}}_{\mathbb{A}^2,(x(i),x(j))}$ and $\widetilde{\mathscr{O}}_{i,j} = \widehat{\mathscr{O}}_{C\times C,((x(i),\lambda(i)),(x(j),\lambda(j)))}$. We regard $\mathscr{O}_{i,j}$ as a subalgebra of $\widetilde{\mathscr{O}}_{i,j}$. Let $r^1_{i,j}\colon \widetilde{\mathscr{O}}_i \to \widetilde{\mathscr{O}}_{i,j}$ and $r^2_{i,j}\colon \widetilde{\mathscr{O}}_j \to \widetilde{\mathscr{O}}_{i,j}$ be the algebra homomorphism induced by the first and second projections from $C \times C$ to C, respectively. We write y_1 for $r^1_{i,j}(y_i) \in \mathscr{O}_{i,j}$ and y_2 for $r^2_{i,j}(y_j) \in \mathscr{O}_{i,j}$. Similarly, we define $\lambda_1 \in \widetilde{\mathscr{O}}_{i,j}$ and $\lambda_2 \in \widetilde{\mathscr{O}}_{i,j}$ as $r^1_{i,j}(\lambda)$ and $r^2_{i,j}(\lambda)$.

Let $c_1\colon \mathscr{O}_{i,j} \to \mathscr{O}_{c(i),j}$ and $c_2\colon \mathscr{O}_{i,j} \to \mathscr{O}_{i,c(j)}$ be the isomorphisms induced by $(x_1, x_2) \mapsto (-x_1, x_2)$ and $(x_1, x_2) \mapsto (x_1, -x_2)$, respectively. Let $s_{12}\colon \mathscr{O}_{i,j} \to \mathscr{O}_{j,i}$ be the homomorphism induced by $\mathbb{A}^2 \ni (x_1, x_2) \mapsto (x_2, x_1) \in \mathbb{A}^2$.

Let $\psi_{a,\nu}\colon \mathscr{O}_{\nu_a,\nu_{a+1}} \longrightarrow \widehat{\mathscr{O}}_{\mathbb{A}^n,x(\nu)}e(\nu)$ be the algebra homomorphism induced by the projection $(x_1, \dots, x_n) \longmapsto (x_a, x_{a+1})$. We define $R_{i,j} \in \mathscr{O}_{i,j}$ by the same formula (4.17). Let us recall (see Remark 5.2)

$$\varphi_a^2 = F(x_a, x_{a+1}).$$

Lemma 5.5. *For $i, j \in J$ such that $\varepsilon(i) = \varepsilon(j) = 0$, we have*

(i) $F(x_1, x_2)^{-1}R_{i,j}$ *belongs to* $\mathscr{O}_{i,j}{}^{\times}$,

(ii) *if $i = j$, then* $F(x_1, x_2)^{-1}R_{i,j} - \left(\dfrac{x_1 - x_2}{y_1 - y_2}\right)^2$ *belongs to* $\mathscr{O}_{i,j}(x_1 - x_2)$.

Admitting this lemma for a while, we will construct $\widetilde{s}_a e(\nu)$ and prove the theorem. By this lemma, we can choose $G_{i,j} \in \mathscr{O}_{i,j}$ for $i, j \in J$, satisfying the following conditions:

$$\tag{5.11} \begin{cases} \text{(a) } G_{i,j} \cdot s_{12}G_{j,i} = F(x_1, x_2)^{-1}R_{i,j}, \\ \text{(b) if } i = j, \text{ then } G_{i,j} - \dfrac{x_1 - x_2}{y_1 - y_2} \in (x_1 - x_2)\mathscr{O}_{i,j}, \\ \text{(c) } c_1\big(G_{i,j}\big) = G_{c(i),j} \text{ and } c_2\big(G_{i,j}\big) = G_{i,c(j)}. \end{cases}$$

Now we define

$$\tag{5.12} \widetilde{s}_a e(\nu) = \varphi_a e(\nu)\psi_{a,\nu}(G_{\nu_a,\nu_{a+1}}).$$

By the construction, we have

$$\begin{aligned} &\widetilde{s}_a \cdot e(\nu) f = s_a(f) e(s_a\nu) \cdot \widetilde{s}_a \text{ for any } f \in \mathsf{Frac}(\widehat{\mathscr{O}}_{\mathbb{A}^n,x(\nu)}), \\ &\widetilde{s}_a C_b = C_{s_a(b)}\widetilde{s}_a, \\ &\widetilde{s}_a^2 e(\nu) = R_{a,a+1}e(\nu). \end{aligned}$$

Let us verify the condition (5.8) (vi):

$$\sigma_a e(\nu) := \widetilde{s}_a e(\nu) + f_{a,a+1}e(\nu) \in t_a e(\nu)(\widehat{\mathscr{O}}_{\mathbb{A}^n,x(\nu)})^{\times} + \mathscr{O}_n\langle C_1, \cdots, C_n\rangle.$$

Setting $i = \nu_a$ and $j = \nu_{a+1}$ and writing $G_{i,j}$ for $\psi_{a,\nu}(G_{\nu_a,\nu_{a+1}})$, we have

$$\begin{aligned}
\tau_a e(\nu) &= \Big(\widetilde{s}_a - (y_a - y_{a+1})\delta_{i,j} + (y_a + y_{a+1})^{-1} C_a C_{a+1} \delta_{c(i),j}\Big) e(\nu) \\
&= \Big(t_a + (x_a - x_{a+1})^{-1} - (x_a + x_{a+1})^{-1} C_a C_{a+1}\Big) e(\nu) G_{i,j} \\
&\qquad - (y_a - y_{a+1})\delta_{i,j} e(\nu) + (y_a + y_{a+1})^{-1} C_a C_{a+1} \delta_{c(i),j} e(\nu) \\
&= t_a e(\nu) G_{i,j} + \Big(G_{i,j}(x_a - x_{a+1})^{-1} - (y_a - y_{a+1})\delta_{i,j}\Big) e(\nu) \\
&\qquad + C_a \Big(-(c_{a+1} G_{i,j})(-x_a + x_{a+1})^{-1} + (-y_a + y_{a+1})^{-1} \delta_{c(i),j}\Big) C_{a+1} e(\nu) \\
&= t_a e(\nu) G_{i,j} + \Big(G_{i,j}(x_a - x_{a+1})^{-1} - (y_a - y_{a+1})\delta_{i,j}\Big) e(\nu) \\
&\qquad + C_a \Big(G_{i,c(j)}(x_a - x_{a+1})^{-1} - (y_a - y_{a+1})^{-1} \delta_{i,c(j)}\Big) e(c_{a+1}\nu) C_{a+1}
\end{aligned}$$

belongs to $t_a e(\nu)(\widehat{\mathscr{O}}_{\mathbb{A}^n,x(\nu)})^\times + \mathscr{O}_n\langle C_1, \cdots, C_n\rangle$ by (5.11) (b). Thus we have verified all the conditions in (5.8), and hence $\widehat{\mathrm{RC}}_n$ and $\overline{\mathcal{OHC}}_n$ are isomorphic.

5.3.4. *Proof of Lemma 5.5.* Now, we will prove Lemma 5.5 for $i, j \in J$ such that $\varepsilon(i) = \varepsilon(j) = 0$.

Let us first derive (ii) admitting (i). Let ξ be the generic point of $\{x_1 = x_2\} \subset \mathrm{Spec}(\mathscr{O}_{i,j})$, and let $(\mathscr{O}_{i,j})_\xi$ be the localization of $\mathscr{O}_{i,j}$ at ξ. By (i), $F(x_1,x_2)^{-1} R_{i,j} - \Big((x_1 - x_2)/(y_1 - y_2)\Big)^2$ belongs to $\mathscr{O}_{i,j}$, and hence it is enough to show that $F(x_1,x_2)^{-1} R_{i,j} - \Big((x_1 - x_2)/(y_1 - y_2)\Big)^2$ belongs to $(\mathscr{O}_{i,j})_\xi (x_1 - x_2)$.

We have

$$F(x_1,x_2)^{-1}(x_1 - x_2)^{-2} = \frac{(x_1 + x_2)^2}{(x_1^2 - x_2^2)^2 - 2(x_1^2 + x_2^2)^2} \equiv -1 \mod (\mathscr{O}_{i,j})_\xi (x_1 - x_2).$$

Since $R_{i,j} \equiv -(y_1 - y_2)^{-2} \mod (\mathscr{O}_{i,j})_\xi$, we obtain

$$F(x_1,x_2)^{-1} R_{i,j} \equiv \frac{(x_1 - x_2)^2}{(y_1 - y_2)^2} \mod (\mathscr{O}_{i,j})_\xi (x_1 - x_2).$$

It only remains to prove (i). We will use case-by-case check-up. Recall that

$$F(x_1,x_2)^{-1} = \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2)^2}{(\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 + 1)}. \tag{5.13}$$

Note that (see (5.7)) for $i, j \in J$ such that $\varepsilon(i) = \varepsilon(j) = 0$, we have

(a) $\lambda(j) = \pm\lambda(i)$ implies $i = j$,
(b) $(\alpha_{\mathrm{pr}(i)}, \alpha_{\mathrm{pr}(j)}) < 0$ implies $\lambda(j) = \lambda(i) \pm 1$,
(c) $\lambda(g(1/2)) = 1/2$.

- Case $\lambda(j) \neq \lambda(i), \lambda(i) \pm 1$.
  In this case, $F(x_1,x_2) \in \mathscr{O}_{i,j}^\times$ and $R_{i,j} = 1$.
- Case $\lambda(j) = \lambda(i) \pm 1$ and $\lambda(i), \lambda(j) \neq 0, 1/2$.

Set $\lambda(j) = \lambda(i) + c$ with $c = \pm 1$. In this case, $y_1 = \lambda_1 - \lambda(i)$, $y_2 = \lambda_2 - \lambda(j)$, and $R_{i,j} = \widetilde{Q}_{i,j}(y_1, y_2) = \pm(y_1 - y_2) = \pm(\lambda_1 - \lambda_2 + c)$. Therefore,

$$F(x_1, x_2)^{-1} R_{i,j} = \pm \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2)^2}{(\lambda_1 - \lambda_2 - c)(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 + 1)}$$

belongs to $\mathscr{O}_{i,j}^{\times}$.

- Case $(\lambda(i), \lambda(j)) = (1/2, 3/2)$ or $(3/2, 1/2)$.

Assume that $(\lambda(i), \lambda(j)) = (1/2, 3/2)$. First note that $\lambda(j) = 0$ if and only if $\operatorname{char} \mathbf{k} = 3$.

Let us first assume $\operatorname{char} \mathbf{k} = 3$. In this case, $\lambda_1 - 1/2 = y_1^2$, $y_2 = (\lambda_2 - \lambda(j))^2$, and

$$R_{i,j} = \widetilde{Q}_{i,j}(y_1, y_2) = \pm(y_1^4 - y_2) = \pm\big((\lambda_1 - 1/2)^2 - (\lambda_2 - 3/2)^2\big) = \pm(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 + 1).$$

Hence we have

$$F(x_1, x_2)^{-1} R_{i,j} = \pm \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2)^2}{(\lambda_1 - \lambda_2 - 1)(\lambda_1 + \lambda_2 - 1)}$$

belongs to $\mathscr{O}_{i,j}^{\times}$.

Assume that $\operatorname{char} \mathbf{k} \neq 3$. In this case, $\lambda_1 = (y_1 - 1/2)^2$, $y_2 = \lambda_2 - 3/2$, and $R_{i,j} = \widetilde{Q}_{i,j}(y_1, y_2) = \pm(y_1^2 - y_2) = \pm(\lambda_1 - \lambda_2 + 1)$. Therefore

$$F(x_1, x_2)^{-1} R_{i,j} = \pm \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2)^2}{(\lambda_1 - \lambda_2 - 1)(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 - 1)}$$

belongs to $\mathscr{O}_{i,j}^{\times}$.

- Case $(\lambda(i), \lambda(j)) = (0, \pm 1)$ or $(\pm 1, 0)$.

We may assume that $(\lambda(i), \lambda(j)) = (0, c)$ with $c = \pm 1$. In this case, $y_1 = \lambda_1^2$, $y_2 = \lambda_2 - c$,

$$R_{i,j} = \widetilde{Q}_{i,j}(y_1, y_2) = \pm(y_2^2 - y_1) = \pm\big((\lambda_2 - c)^2 - \lambda_1^2\big) = \pm(\lambda_1 - \lambda_2 + c)(\lambda_1 + \lambda_2 - c).$$

Hence

$$F(x_1, x_2)^{-1} R_{i,j} = \pm \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 + \lambda_2)^2}{(\lambda_1 - \lambda_2 - c)(\lambda_1 + \lambda_2 + c)}$$

belongs to $\mathscr{O}_{i,j}^{\times}$.

- Case $i = j$ and $\lambda(i) \neq 0, 1/2$.

In this case, $y_k = \lambda_k - \lambda(i)$ $(k = 1, 2)$ and $R_{i,j} = -(y_1 - y_2)^{-2} = -(\lambda_1 - \lambda_2)^{-2}$. Then by (5.13)

$$F(x_1, x_2)^{-1} R_{i,i} = \frac{-(\lambda_1 + \lambda_2)^2}{(\lambda_1 - \lambda_2 - 1)(\lambda_1 - \lambda_2 + 1)(\lambda_1 + \lambda_2 - 1)(\lambda_1 + \lambda_2 + 1)}$$

belongs to $\mathscr{O}_{i,j}^{\times}$.

- Case $i = j$ and $\lambda(i) = 1/2$.

In this case, $y_k^2 = \lambda_k - 1/2$ $(k = 1, 2)$, and

$$R_{i,j} = -2(y_i^2 + y_2^2)(y_1^2 - y_2^2)^{-2} = -2(\lambda_1 + \lambda_2 - 1)(\lambda_1 - \lambda_2)^{-2}.$$

Hence by (5.13), we have

$$F(x_1,x_2)^{-1}R_{i,j} = \frac{(\lambda_1+\lambda_2)^2}{(\lambda_1-\lambda_2-1)(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2+1)}$$

and it belongs to $\mathscr{O}_{i,j}^{\times}$.

- Case $i=j$ and $\lambda(i)=0$.
  In this case, $y_k=\lambda_k^2$ $(k=1,2)$, and $R_{i,j}=-(y_1-y_2)^{-2}=-(\lambda_1^2-\lambda_2^2)^{-2}$. Hence

$$F(x_1,x_2)^{-1}R_{i,j} = \frac{-1}{(\lambda_1-\lambda_2-1)(\lambda_1-\lambda_2+1)(\lambda_1+\lambda_2-1)(\lambda_1+\lambda_2+1)}$$

belongs to $\mathscr{O}_{i,j}^{\times}$.

This completes the proof of Theorem 5.4. □

Similarly to the case of cyclotomic Hecke-Clifford superalgebras, we can define the notion of cyclotomic Sergeev superalgebras.

For $\Lambda=\sum_{i\in I_1} m_i\Lambda_i$ with $m_i\in\mathbb{Z}_{\ge 0}$ (see § 3.7), we set

$$f'_\Lambda(X_1)=\prod_{i\in J}(x_1-x(i))^{m_{\mathrm{pr}(i)}},$$

and define

$$\overline{\mathcal{AHC}}_n^{\Lambda}=\overline{\mathcal{AHC}}_n/\big(\overline{\mathcal{AHC}}_n f'_\Lambda(X_1)\overline{\mathcal{AHC}}_n\big).$$

We call $\overline{\mathcal{AHC}}_n^{\Lambda}$ the *cyclotomic Sergeev superalgebra*.

Then, taking I large enough comparing to $\mathrm{supp}(\Lambda):=\{i\in I\,;m_i\neq 0\}$, the cyclotomic Sergeev superalgebra $\overline{\mathcal{AHC}}_n^{\Lambda}$ is isomorphic to the cyclotomic quiver Hecke-Clifford superalgebra RC_n^{Λ}.

Note that the Sergeev algebra $\overline{\mathcal{HC}}_n$ is isomorphic to $\overline{\mathcal{AHC}}_n^{\Lambda_{i_0}}$, where $i_0\in I$ is a unique odd vertex $0\in\mathbb{A}_w^1$.

For a generalized Cartan matrix A, we denote by $\mathsf{KLR}_n(A)$ the Khovanov-Lauda-Rouquier algebra associated with A. It is nothing but $\mathrm{R}_n(A)$ with $I_{\mathrm{odd}}=\emptyset$. The algebra $\mathsf{KLR}_n(A)$ depends on the parameter $Q=(Q_{i,j})_{i,j\in I}$ as in §3.6. By the rescaling in Remark 3.2, we can easily see that the $\mathbf{k}$-algebra $\mathsf{KLR}_n(A)$ is unique up to isomorphism when A is of finite type or affine type except the case when the Dynkin diagram has a cycle, i.e., when A is of $A_{n-1}^{(1)}$ type $(n\ge 2)$.

Remark 5.6. Consider the case $\mathrm{char}\,\mathbf{k}=3$ and $A=A_2^{(2)}$ (see the picture (ii) (b) in §5.2). Take a block subsuperalgebra B of the affine Sergeev superalgebra $\overline{\mathcal{AHC}}_{11}$ which categorifies $U^-(\mathfrak{g}(A))_{-\beta}$ with $\beta=8\alpha_0+3\alpha_1$, where α_0 is the short root. Although $\mathsf{KLR}_\beta(A)$ also categorifies $U^-(\mathfrak{g}(A))_{-\beta}$, the set of irreducible objects $\mathsf{Irr}(\mathsf{KLR}_\beta(A)\text{-}\mathsf{mod})$ in the category $\mathsf{KLR}_\beta(A)\text{-}\mathsf{mod}$ of finite-dimensional $\mathsf{KLR}_\beta(A)$-modules and the set of irreducible objects $\mathsf{Irr}(B\text{-}\mathsf{smod})$ in the category $B\text{-}\mathsf{smod}$ of finite-dimensional B-supermodules correspond to different perfect bases.

Let us explain in more detail. By [BK3] (see also [Kle, part II]), the cyclotomic Sergeev superalgebras $\overline{\mathcal{AHC}}_n^{\Lambda_0}\simeq\mathrm{RC}_n^{\Lambda_0}(A)$ categorify the irreducible highest module

$V(\Lambda_0)$. Namely, denoting by $\mathrm{RC}_n^{\Lambda_0}(A)\text{-}\mathsf{smod}$ the supercategory of finite-dimensional $\mathrm{RC}_n^{\Lambda_0}(A)$-supermodules, we have

$$\begin{aligned}&\bigoplus_{n\geq 0} K^{\mathrm{super}}(\mathrm{RC}_n^{\Lambda_0}(A)\text{-}\mathsf{smod})_{\mathbb{C}} \simeq V(\Lambda_0),\\ &\textstyle\bigsqcup_{n\geq 0} \mathsf{Irr}^{\mathrm{super}}(\mathrm{RC}_n^{\Lambda_0}(A)\text{-}\mathsf{smod}) \simeq B(\Lambda_0) \simeq \mathsf{RP}_3\end{aligned} \tag{5.14}$$

where $\mathsf{Irr}^{\mathrm{super}}(\mathrm{RC}_n^{\Lambda_0}(A)\text{-}\mathsf{smod})$ is the set of equivalence classes of irreducible $\mathrm{RC}_n^{\Lambda_0}(A)$-supermodules by the equivalence relation $\sim$ given by $S \sim S' \Leftrightarrow S \simeq S'$ or $S \simeq \Pi(S')$. The first isomorphism in (5.14) is a $U(\mathfrak{g}(A))$-module isomorphism and the second isomorphism is a $U_v(\mathfrak{g}(A))$-crystal isomorphism. Recall that RP_3 is the set of all *3-restricted 3-strict partitions*, which is equivalent to the set of *reduced Young walls of type* $A_2^{(2)}$ in [Kang 03]. A partition $\lambda = (\lambda_1, \dots, \lambda_r)$ is 3-restricted 3-strict if the following conditions are satisfied.

- $\lambda_k = \lambda_{k+1}$ implies $\lambda_k \in 3\mathbb{Z}$,
- $\lambda_k - \lambda_{k+1} < 3$ if $\lambda_k \in 3\mathbb{Z}$,
- $\lambda_k - \lambda_{k+1} \leq 3$ if $\lambda_k \notin 3\mathbb{Z}$.

Let $\mathbf{k}\mathfrak{S}_n^-$ denote the spin symmetric group superalgebra of order n. Then the cyclotomic Sergeev superalgebra $\overline{\mathcal{AHC}}_n^{\Lambda_0}$ is isomorphic to the tensor product $\mathbf{k}\mathfrak{S}_n^- \otimes \mathfrak{C}_n$ of the spin symmetric group superalgebra $\mathbf{k}\mathfrak{S}_n^-$ and the Clifford superalgebra $\mathfrak{C}_n$ by Sergeev and Yamaguchi [Ser2, Yam]. Hence (5.14) holds also for $\mathbf{k}\mathfrak{S}_n^-$.

For each $\lambda \in \mathsf{RP}_3 \cong B(\Lambda_0)$, we denote by $V_\lambda^{\mathsf{spin}}$ the corresponding isomorphism class of irreducibles of $\mathbf{k}\mathfrak{S}_{|\lambda|}^-$.

On the other hand, by [KK, LV] we have

$$\bigoplus_{n\geq 0} \mathsf{K}_0(\mathsf{KLR}_n^{\Lambda_0}(A)\text{-}\mathsf{mod})_{\mathbb{C}} \cong V(\Lambda_0), \qquad \bigsqcup_{n\geq 0} \mathsf{Irr}(\mathsf{KLR}_n^{\Lambda_0}(A)\text{-}\mathsf{mod}) \cong B(\Lambda_0), \tag{5.15}$$

where the left isomorphism is as $U(\mathfrak{g}(A))$-modules and the right isomorphism is as $U_v(\mathfrak{g}(A))$-crystals. For each $\lambda \in \mathsf{RP}_3 \cong B(\Lambda_0)$, we denote by V_λ^{KLR} the corresponding isomorphism class of irreducibles of $\mathsf{KLR}_n^{\Lambda_0}(A)$.

If both $\mathsf{Irr}(\mathsf{KLR}_\beta(A)\text{-}\mathsf{mod})$ and $\mathsf{Irr}(B\text{-}\mathsf{smod})$ correspond to the same perfect basis on $U(\mathfrak{g}(A))$-module $L(\Lambda_0)$, then we must have

$$\dim V_\lambda^{\mathsf{spin}} / \dim V_\lambda^{\mathsf{KLR}} = 2^{[(1+\gamma_1(\lambda))/2]}$$

for any $\lambda \in \mathsf{RP}_3$ (see [Kle, Lemma 22.3.8]). Here, $\gamma_1(\lambda) = \sum_{i\in I_{\mathrm{even}}} m_i = n - \ell$ if the weight of λ is $\Lambda_0 - \beta$ and $\beta = \sum_{i\in I} m_i\alpha_i$ ($n = \mathrm{ht}(\beta) = \sum_{i\in I} m_i$ and $\ell = \sum_{i\in I_{\mathrm{odd}}} m_i$).

A computer calculation shows that for $\lambda = (6,4,1)$ ($\gamma_1(\lambda) = 3$), we have $\dim V_\lambda^{\mathsf{KLR}} = 648$ while it is known[1] that $\dim V_\lambda^{\mathsf{spin}} = 2880 = 4 \times 720$.

Remark 5.7. Consider the case (ii) (d) in §4.4, $A = D_2^{(2)} = A_1^{(1)}$ indexed by $I = I_{\mathrm{odd}} = \{0, 1\}$ and assume $q = \exp(2\pi\sqrt{-1}/8) \in \mathbf{k}$ with $\mathrm{char}\,\mathbf{k} = 0$. By the isomorphism described in [Rou1, §3.2.1] or Remark 3.2, it is enough to consider $\mathsf{KLR}_4^{\Lambda_0}(A)$ for $Q_{0,1}(u,v) = u^2 - auv + v^2$ for some $a \in \mathbf{k}$.

[1] Historically, it was first miscalculated as $\dim V_\lambda^{\mathsf{spin}} = 2592 = 4\times 648$ in [MY]. If it were correct, observing such a direct discrepancy between the Khovanov-Lauda-Rouquier algebras and the spin symmetric groups must become more difficult.

By [BK1, BK2, KK, Rou1, Tsu], the family of superalgebras $\{\mathrm{RC}_n^{\Lambda_0}(A)\}_{n\geq 0} \simeq \{\mathcal{AHC}_n^{\Lambda_0}(q)\}_{n\geq 0}$ (resp. $\{\mathsf{KLR}_n^{\Lambda_0}(A)\}_{n\geq 0}$) categorifies $U(\mathfrak{g}(A))$-module (resp. $U_v(\mathfrak{g}(A))$-module) $V(\Lambda_0)$. However, there is no Morita equivalence between $\mathrm{RC}_4^{\Lambda_0}(A)$ and $\mathsf{KLR}_4^{\Lambda_0}(A)$ nor weak Morita superequivalence between $\mathrm{RC}_4^{\Lambda_0}(A)$ and $\mathsf{KLR}_4^{\Lambda_0}(A)$ whatever superalgebra structure we give to $\mathsf{KLR}_4^{\Lambda_0}(A)$ and for any choice of parameters $a \in \mathbf{k}$.

The algebras $\mathrm{RC}_4^{\Lambda_0}(A)$ and $\mathsf{KLR}_4^{\Lambda_0}(A)$ are not Morita equivalent because we have (for any $a \in \mathbf{k}$)

$$\dim Z(\mathrm{RC}_4^{\Lambda_0}(A)) = 4 \neq 5 = \dim Z(\mathsf{KLR}_4^{\Lambda_0}(A))$$

by a computer calculation. (See Remark 2.8 for Z.) Since it can be easily seen that $\mathsf{Irr}(\mathsf{KLR}_4^{\Lambda_0}(A)\text{-}\mathsf{mod})$ consists of 2 irreducible modules of dimension $1, 4$ for any $a \in \mathbf{k}$, these two irreducibles are self-associate (see § 2.5) for any superalgebra structure on $\mathsf{KLR}_4^{\Lambda_0}(A)$. Moreover, $\mathsf{Irr}(\mathrm{RC}_4^{\Lambda_0}(A)\text{-}\mathsf{mod})$ also consists of two self-associate irreducible modules. Hence two supercategories $\mathrm{RC}_4^{\Lambda_0}(A)\text{-}\mathsf{mod}$ and $\left(\mathsf{KLR}_4^{\Lambda_0}(A)\text{-}\mathsf{mod}\right)^{\mathrm{CT}}$ cannot be superequivalent for any superalgebra structure on $\mathsf{KLR}_4^{\Lambda_0}(A)$ by Lemma 2.11. Since $\mathsf{KLR}_4^{\Lambda_0}(A)$ has no non-trivial block decomposition, $\mathrm{RC}_4^{\Lambda_0}(A)$ and $\mathsf{KLR}_4^{\Lambda_0}(A)$ cannot be weakly Morita superequivalent.

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