THE PBW FILTRATION, DEMAZURE MODULES AND TOROIDAL CURRENT ALGEBRAS

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Abstract. Let L be the basic (level one vacuum) representation of the affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$. The m-th space F_m of the PBW filtration on L is a linear span of vectors of the form $x_1 \dots x_l v_0$, where $l \leq m$, $x_i \in \widehat{\mathfrak{g}}$ and v_0 is a highest weight vector of L. In this paper we give two descriptions of the adjoint graded space L^{gr} with respect to the PBW filtration. The "top-down" description deals with a structure of L^{gr} as a representation of the abelianized algebra of generating operators. We prove that the ideal of relations is generated by the coefficients of the squared field $e_\theta(z)^2$, which corresponds to the longest root θ. The "bottom-up" description deals with the structure of L^{gr} as a representation of the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$. We prove that each quotient F_m/F_{m-1} can be filtered by graded deformations of the tensor products of m copies of $\mathfrak{g}$.

Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\widehat{\mathfrak{g}}$ be the corresponding affine Kac-Moody Lie algebra (see [K1],[Kum]). Let L be the basic representation of $\widehat{\mathfrak{g}}$, i.e. an irreducible level one module with a highest weight vector v_0 satisfying a condition $(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot v = 0$. The PBW filtration $F_\bullet$ on the space L is defined as follows:

$$F_0 = \mathbb{C}v_0, \ F_{m+1} = F_m + \mathrm{span}\{x \cdot w : \ x \in \widehat{\mathfrak{g}}, w \in F_m\}.$$

This filtration was introduced in [FFJMT] as a tool of study of vertex operators acting on the space of Virasoro minimal models (see [DMS]). In this paper we study the adjoint graded space $L^{gr} = F_0 \oplus F_1/F_0 \oplus \dots$. We describe the space L^{gr} from two different points of view.

From one hand, the space L^{gr} is a module over the abelian Lie algebra $\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$, where $\mathfrak{g}^{ab}$ is an abelian Lie algebra whose underlying vector space is $\mathfrak{g}$. The module structure is induced from the action of the algebra of generating "top-down" operators $\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$ on L. Thus L^{gr} can be identified with a polynomial ring on the space $\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ modulo certain ideal. Our first goal is to describe this ideal explicitly.

From the other hand all spaces F_m are stable with respect to the action of the subalgebra of annihilating operators $\mathfrak{g} \otimes \mathbb{C}[t]$ (the "bottom-up" operators). This gives a $\mathfrak{g} \otimes \mathbb{C}[t]$ module structure on each quotient F_m/F_{m-1}. Our second goal is to study these modules. We briefly formulate our results below.

For $x \in \mathfrak{g}$ let $x(z) = \sum_{i<0}(x \otimes t^i)z^{-i}$ be a generating function of the elements $x \otimes t^i \in \widehat{\mathfrak{g}}$, $i < 0$. These series are also called fields. They play a crucial role in the theory of vertex operator algebras (see [FK], [K2], [BF]). We will need a field which corresponds to the highest root θ of $\mathfrak{g}$. Namely let $e_\theta \in \mathfrak{g}$ be a highest weight vector in the adjoint representation. It is well known (see for instance [BF]) that the coefficients of $e_\theta(z)^2$ vanish on L. It follows immediately that the same relation holds in L^{gr}. We note also that the adjoint action of the Lie algebra $\mathfrak{g} \simeq \mathfrak{g} \otimes 1$ preserves the ideal of relations in L^{gr}. The following theorem is one of the central results of our paper.

Theorem 0.1. *L^{gr} is isomorphic to the quotient of the universal enveloping algebra $\mathrm{U}(\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}])$ by an ideal I, which is a minimal $\mathfrak{g} \otimes 1$ invariant ideal containing all coefficients of the series $e_\theta(z)^2$.*

This proves the level one case of the conjecture from [F2]. We note that for $\mathfrak{g} = \mathfrak{sl}_2$ this theorem was proved in [FFJMT]. The generalization of this theorem for higher levels and $\mathfrak{g} = \mathfrak{sl}_2$ is conjectured in [F2].

In order to prove this theorem and to make a connection with the "bottom-up" description we study the intersection of the PBW filtration with certain Demazure modules inside L. Recall (see [D]) that a Demazure module $D(\lambda) \hookrightarrow L$ is generated by extremal vector of the weight λ with an action of the universal enveloping algebra of the Borel subalgebra of $\widehat{\mathfrak{g}}$. We will only need the Demazure modules $D(N\theta)$. These modules are invariant with respect to the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ and provide a finite-dimensional approximation of the space L: $D(\theta) \hookrightarrow D(2\theta) \hookrightarrow \cdots = L$ (see [FoL1], [FoL2]; some special cases are also contained in [FF2], [Ked]). Let $F_m(N) = D(N\theta) \cap F_m$ be an intersection of the Demazure module with the m-th space of the PBW filtration. This gives a filtration on $D(N\theta)$. In order to describe the filtration $F_\bullet(N)$ we use a notion of the fusion product of $\mathfrak{g} \otimes \mathbb{C}[t]$ modules (see [FL],[FF1]) and the Fourier-Littelmann results [FoL2].

We recall that there exist two versions of the fusion procedure for modules over the current algebras. The first version constructs a graded $\mathfrak{g} \otimes \mathbb{C}[t]$ module $V_1 * \cdots * V_N$ starting from the tensor product of cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$ modules V_i. The other version also produces a graded $\mathfrak{g} \otimes \mathbb{C}[t]$ module $V_1 ** \cdots ** V_N$ but in this case all V_i are cyclic $\mathfrak{g}$ modules. The fusion modules provide a useful tool for the study of the representation theory of current and affine algebras (see for example [AKS], [Ked], [F1], [F2], [FJKLM], [FoL2]). In particular Fourier and Littelmann proved that there exists an isomorphism of $\mathfrak{g} \otimes \mathbb{C}[t]$ modules $D(N\theta) \simeq D(\theta) * \cdots * D(\theta)$ (N times). Using this theorem and the $**$-version of the fusion procedure we endow the space $D(N\theta)$ with a structure of the representation of the toroidal current algebra $\mathfrak{g} \otimes \mathbb{C}[t, u]$ (see [H], [L] and references therein for some details on the representation theory of the toroidal algebras). This allows to prove our second main theorem:

Theorem 0.2. *The $\mathfrak{g}\otimes\mathbb{C}[t]$ module $F_m(N)/F_{m-1}(N)$ can be filtered by $\binom{N}{m}$ copies of the m-th fusion power of the adjoint representation of $\mathfrak{g}$. In particular,* $\dim F_m(N)/F_{m-1}(N)=\binom{N}{m}(\dim\mathfrak{g})^m$.

The paper is organized as follows.
In Section 1 we give the definition of the PBW filtration and of the induced filtration on Demazure modules.
In Section 2 we study tensor products of cyclic $\mathfrak{g}\otimes\mathbb{C}[t]$ modules endowed with a structure of representations of toroidal algebra. This construction is applied to the Demazure modules in Section 3.
In Section 4 the functional realization of the dual space $(L^{gr})^*$ is given.
In Section 5 we combine all results of the previous sections and prove Theorems 0.1 and 0.2. We finish the paper with a list of the main notations.

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1. The PBW filtration

In this section we recall the definition and basic properties of the PBW filtration (see [FFJMT]).

Let $\mathfrak{g}$ be a simple Lie algebra, $\widehat{\mathfrak{g}}$ be the corresponding affine Kac-Moody Lie algebra:

$$\widehat{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}K\oplus\mathbb{C}d.$$

Here K is a central element, d is a degree element ($[d,x\otimes t^i]=-ix\otimes t^i$) and

$$[x\otimes t^i,y\otimes t^j]=[x,y]\otimes t^{i+j}+i\delta_{i+j,0}(x,y)K,$$

$x,y\in\mathfrak{g}$, $(\cdot,\cdot)$ is a Killing form. Let L be a basic representation of the affine Lie algebra, i.e. level one highest weight irreducible module with a highest weight vector v satisfying

$$(\mathfrak{g}\otimes\mathbb{C}[t])\cdot v_0=0,\ Kv_0=v_0,dv_0=0,\ \mathrm{U}(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])\cdot v_0=L.$$

An operator $d\in\widehat{\mathfrak{g}}$ defines a q-character of any subspace $V\hookrightarrow L$ by the formula

$$\mathrm{ch}_qV=\sum_{n\geq 0}q^n\dim\{v\in V:\ dv=nv\}. \tag{1}$$

For $x\in\mathfrak{g}$ we introduce a generating function $x(z)=\sum_{i>0}(x\otimes t^{-i})z^i$ of the elements $x\otimes t^i$, $i<0$. The representation L can be described using the function $e_\theta(z)$, where θ is a highest weight of $\mathfrak{g}$ and $e_\theta\in\mathfrak{g}$ is a highest

weight element. Namely denote by $\langle e_\theta(z)^2, \mathfrak{g}\otimes 1\rangle$ a left ideal in $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t^{-1}])$ generated by all coefficients

$$\sum_{\substack{i+j=n\\ i,j\leq -1}} (e_\theta\otimes t^i)(e_\theta\otimes t^j)$$

of the square of the series $e_\theta(z)$ and by $\mathfrak{g}\otimes 1$. Then

$$L\simeq \mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t^{-1}])/\langle e_\theta(z)^2, \mathfrak{g}\otimes 1\rangle. \tag{2}$$

Equivalently the relations $(\mathfrak{g}\otimes 1)\cdot v_0=0$ and $e_\theta(z)^2 v_0=0$ generate the ideal of relations.

In what follows we will need a certain embedding of the basic $\widehat{\mathfrak{sl}_2}$ module into L. Namely let $\mathfrak{sl}_2^\theta$ be a Lie algebra spanned by a $\mathfrak{sl}_2$-triple e_θ, f_θ and h_θ, where e_θ and f_θ are highest and lowest weight vectors in the adjoint representation of $\mathfrak{g}$. Then the restriction map defines a structure of $\widehat{\mathfrak{sl}_2^\theta}$ module on L. In particular, the space $\mathrm{U}(\widehat{\mathfrak{sl}_2^\theta})\cdot v_0$ is isomorphic to the basic representation of $\widehat{\mathfrak{sl}_2}$.

We now define a PBW filtration $F_\bullet$ on L. Namely let

$$F_0=\mathbb{C}v_0,\ F_{m+1}=F_m+\mathrm{span}\{(x\otimes t^{-i})w,\ x\in\mathfrak{g}, i>0, w\in F_m\}. \tag{3}$$

Then $F_\bullet$ is an increasing filtration converging to L. We denote the adjoint graded space by L^{gr}:

$$L^{gr}=\bigoplus_{m\geq 0} L^{gr}_m,\ L^{gr}_m=\mathrm{gr}_m F_\bullet = F_m/F_{m-1}. \tag{4}$$

In what follows we denote by $\mathfrak{g}^{ab}$ an abelian Lie algebra with the underlying vector space isomorphic to $\mathfrak{g}$.

Lemma 1.1. *The action of $\mathfrak{g}\otimes\mathbb{C}[t]$ on L induces an action of the same algebra on L^{gr}. The action of $\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]$ on L induces an action of the algebra $\mathfrak{g}^{ab}\otimes t^{-1}\mathbb{C}[t^{-1}]$ on L^{gr}.*

Proof. All spaces F_m are invariant with respect to the action of $\mathfrak{g}\otimes\mathbb{C}[t]$, since the condition $(\mathfrak{g}\otimes\mathbb{C}[t])\cdot v_0=0$ is satisfied. Hence we obtain an induced action on the quotient spaces F_m/F_{m-1}.

Operators $x\otimes t^i$, $i<0$ do not preserve F_m but map it to F_{m+1}. Therefore each element $x\otimes t^i$, $i<0$ produce an operator acting from L^{gr}_m to L^{gr}_{m+1}. An important property of these operators on L^{gr} is that they mutually commute, since the composition $(x\otimes t^i)(y\otimes t^j)$ acts from L^{gr}_m to L^{gr}_{m+2} but the commutator $[x\otimes t^i, y\otimes t^j]=[x,y]\otimes t^{i+j}$ maps L^{gr}_m to L^{gr}_{m+1}. Lemma is proved. □

The goal of our paper is to describe the structure of L^{gr} as a representation of $\mathfrak{g}\otimes\mathbb{C}[t]$ and of $\mathfrak{g}^{ab}\otimes t^{-1}\mathbb{C}[t^{-1}]$. It turns out that these two structures are closely related.

Lemma 1.2. *Let $I \hookrightarrow \mathrm{U}(\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}])$ be a minimal $\mathfrak{g}$-invariant ideal containing the coefficients of the series $e_\theta(z)^2$. Then there exists a surjective homomorphism*

$$\mathrm{U}(\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}])/I \to L^{gr} \tag{5}$$

of $\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ modules mapping 1 to v_0.

Proof. Follows from (2). □

One of our goals is to prove that the homomorphism (5) is in fact an isomorphism.

Recall (see [FoL1]) that L is filtered by the finite-dimensional Demazure modules [D]. Namely let $D(N) \hookrightarrow L$ be a Deamzure module inside L which is generated from the cyclic vector v_N of the weight $N\theta$. Thus $D(N)$ is a cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$ module with a cyclic vector v_N being an extremal vector in L of the weight $N\theta$. It is known (see [Kum], [FoL1]) that these modules are embedded into each other and the limit coincides with L:

$$L = D(1) \hookrightarrow D(2) \hookrightarrow \dots.$$

We introduce an induced PBW filtration on $D(N)$:

$$D(N) = F_0(N) \hookrightarrow F_1(N) \hookrightarrow \dots,\ F_m(N) = D(N) \cap F_m. \tag{6}$$

Obviously, each space $F_m(N)$ is $\mathfrak{g} \otimes \mathbb{C}[t]$ invariant.

Lemma 1.3. *$F_N(N) = D(N)$ but $F_{N-1}(N) \neq D(N)$.*

Proof. First equality holds since $(e_\theta \otimes t^{-1})^N v_0 = v_N$ and $D(N)$ is cyclic. In addition $v_N \notin F_{N-1}$ because all weights of F_m (as a representation of $\mathfrak{g} \simeq \mathfrak{g} \otimes 1$) are less then or equal to $m\theta$. □

Consider an adjoint graded object

$$\mathrm{gr}F(N) = \bigoplus_{m=0}^{\infty} \mathrm{gr}_m F(N),\ \mathrm{gr}_m F(N) = F_m(N)/F_{m-1}(N).$$

We note that each space $\mathrm{gr}_m F(N)$ has a natural structure of $\mathfrak{g} \otimes \mathbb{C}[t]$ module.

2. t^N-filtration

In this section we describe the filtration (6) using the generalization of the fusion product of $\mathfrak{g} \otimes \mathbb{C}[t]$ modules from [FL] and a theorem of [FoL2]. We first recall the definition of the fusion product of $\mathfrak{g} \otimes \mathbb{C}[t]$ modules.

Let V be a $\mathfrak{g} \otimes \mathbb{C}[t]$ module, c be a complex number. We denote by $V(c)$ a $\mathfrak{g} \otimes \mathbb{C}[t]$ module which coincide with V as a vector space and the action is twisted by the Lie algebra homomorphism

$$\phi(c) : \mathfrak{g} \otimes \mathbb{C}[t] \to \mathfrak{g} \otimes \mathbb{C}[t],\ x \otimes t^k \mapsto x \otimes (t+c)^k.$$

Let $V_1, \dots, V_N$ be cyclic representations of the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ with cyclic vectors $v_1, \dots, v_N$. Let $c_1, \dots, c_N$ be a set of pairwise distinct complex numbers. The fusion product $V_1(c_1) * \dots * V_N(c_N)$ is a graded deformation of

the usual tensor product $V_1(c_1)\otimes\ldots\otimes V_N(c_N)$. More precisely, let $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])_s$ be a natural grading on the universal enveloping algebra coming from the counting of the t degree. Because of the condition $c_i\neq c_j$ for $i\neq j$ the tensor product $\bigotimes_{i=1}^N V_i(c_i)$ is a cyclic $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])$ module with a cyclic vector $\otimes_{i=1}^N v_i$. Therefore the grading on $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])$ induces an increasing fusion filtration

$$\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])_{\leq s}\cdot(v_1\otimes\ldots\otimes v_N) \tag{7}$$

on the tensor product.

Definition 2.1. *The fusion product $V_1(c_1)*\cdots*V_N(c_N)$ of $\mathfrak{g}\otimes\mathbb{C}[t]$ modules V_i is an adjoint graded $\mathfrak{g}\otimes\mathbb{C}[t]$ module with respect to the filtration* (7) *on the tensor product $V_1(c_1)\otimes\ldots\otimes V_N(c_N)$. We denote the m-th graded component by $\mathrm{gr}_m(V_1(c_1)*\cdots*V_N(c_N))$.*

We note that in many special cases the $\mathfrak{g}\otimes\mathbb{C}[t]$ module structure of the fusion product does not depend on the parameters c_i (see for example [AK], [CL],[FF1],[FoL2], [FKL]). We then omit the parameters c_i and denote the corresponding fusion product simply by $V_1*\cdots*V_N$.

In what follows we will need a special but important case of the procedure described above. Namely let V_i be cyclic $\mathfrak{g}$ modules. One can extend the $\mathfrak{g}$ module structure to the $\mathfrak{g}\otimes\mathbb{C}[t]$ module structure by letting $\mathfrak{g}\otimes t\mathbb{C}[t]$ to act by zero. Then the fusion procedure produces a new graded $\mathfrak{g}\otimes\mathbb{C}[t]$ module by the following rule. One starts with a tensor product of evaluation modules $V_i(c_i)$, where $x\otimes t^k$ acts on V_i by $c_i^k x$ (we evaluate t at the point c_i). Then one constructs the fusion filtration and pass to the adjoint graded module (according to the Definition 2.1). It is convenient for us to make a special definition and introduce a special notation.

Definition 2.2. *Let $V_1,\ldots,V_N$ be a set of $\mathfrak{g}$ modules. Then a $\mathfrak{g}\otimes\mathbb{C}[t]$ module $V_1(c_1)**\cdots**V_N(c_N)$ is a fusion product in the sense of the Definition* 2.1*, where V_i are equipped with a structure of $\mathfrak{g}\otimes\mathbb{C}[t]$ module via the evaluation at the origin map $x\otimes t^k\mapsto 0^k x$. We denote the m-th graded component by $\mathrm{gr}_m(V_1(c_1)**\cdots**V_N(c_N))$.*

Remark 2.1. *Let us stress the main difference between Definitions* 2.1 *and* 2.2*. The Definition* 2.1 *gets as an input a set of cyclic representations of the current algebra $\mathfrak{g}\otimes\mathbb{C}[t]$ and as a result produces a t-graded $\mathfrak{g}\otimes\mathbb{C}[t]$ module. The input of the Definitions* 2.2 *is a set of cyclic $\mathfrak{g}$ modules and an output is again a t-graded $\mathfrak{g}\otimes\mathbb{C}[t]$ module.*

We now recall a theorem from [FoL2] which uses the fusion procedure to construct the Demazure module $D(N)$ starting from $D(1)$.

Theorem 2.1. [FoL2] *The N-th fusion power $D(1)^{*N}$ is independent on the evaluation parameters c_i. The $\mathfrak{g}\otimes\mathbb{C}[t]$ modules $D(1)^{*N}$ and $D(N)$ are isomorphic.*

We recall that as a $\mathfrak{g}$ module $D(1)$ is isomorphic to the direct sum of trivial and adjoint representations, $D(1)\simeq\mathbb{C}\oplus\mathfrak{g}$. The trivial representation is

annihilated by $\mathfrak{g}\otimes\mathbb{C}[t]$, the adjoint representation is annihilated by $\mathfrak{g}\otimes t^2\mathbb{C}[t]$ and $\mathfrak{g}\otimes\mathbb{C}[t]$ maps $\mathfrak{g}$ to $\mathbb{C}$.

Our idea is to combine the theorem above and Definition 2.2 with $\mathfrak{g}$ being the current algebra $\mathfrak{g}\otimes\mathbb{C}[u]$. In fact the Definition 2.2 works for arbitrary $\mathfrak{g}$ and produces a representation of an algebra with an additional current variable. In particular starting from the $\mathfrak{g}\otimes\mathbb{C}[u]$ modules $V_i = D(1)$ and an N tuple of pairwise distinct complex numbers $c_1,\dots,c_N$ one gets a new bi-graded $\mathfrak{g}\otimes\mathbb{C}[t,u]$ module. The resulting module can be obtained from the Demazure module $D(N)$ by a rather simple procedure which we are going to describe now.

Let $V_1,\dots,V_N$ be cyclic representations of the algebra $\mathfrak{g}\otimes\mathbb{C}[t]/\langle t^2\rangle$. Hence $V_1(c_1)*\dots*V_N(c_N)$ is a cyclic $\mathfrak{g}\otimes\mathbb{C}[t]/\langle t^{2N}\rangle$ module. We consider a decreasing filtration $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])^j$ on the universal enveloping algebra defined by

$$\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])^0=\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t]),\ \mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])^{j+1}=(\mathfrak{g}\otimes t^N\mathbb{C}[t])\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])^j.$$

This filtration induces a decreasing filtration G^j on the fusion product $V_1(c_1)*\dots*V_N(c_N)$ (since it is cyclic $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])$ module). $G^\bullet$ will be also reffered to as a t^N-filtration. Consider the adjoint graded space

$$\mathrm{gr}G^\bullet=\bigoplus_{j=0}^{N}\mathrm{gr}^jG^\bullet,\ \mathrm{gr}^jG^\bullet=G^j/G^{j+1}. \tag{8}$$

Since each space G^j is $\mathfrak{g}\otimes\mathbb{C}[t]$ invariant one gets a structure of $\mathfrak{g}\otimes\mathbb{C}[t]/\langle t^N\rangle$ module on each space G^j/G^{j+1}. In addition an element from $\mathfrak{g}\otimes t^N\mathbb{C}[t]/\langle t^{2N}\rangle$ produces a degree 1 operator on (8) mapping $\mathrm{gr}^jG^\bullet$ to $\mathrm{gr}^{j+1}G^\bullet$. We thus obtain a structure of $\mathfrak{g}\otimes\mathbb{C}[t,u]/\langle t^N,u^2\rangle$ module on (8), where $\mathfrak{g}\otimes u\mathbb{C}[t]$ denotes an algebra of degree one operators on $\mathrm{gr}G^\bullet$ coming from the action of $\mathfrak{g}\otimes t^N\mathbb{C}[t]/\langle t^{2N}\rangle$.

On the other hand let us consider the modules V_i as representations of the Lie algebra $\mathfrak{g}\otimes\mathbb{C}[u]/\langle u^2\rangle$ (simply replacing t by u). We denote these modules as V_i^u. Then the bi-graded tensor product $V_1^u(c_1)**\dots**V_N^u(c_N)$ is a representation of the Lie algebra $\mathfrak{g}\otimes\mathbb{C}[t,u]/\langle t^N,u^2\rangle$.

Proposition 2.1. *We have an isomorphism of* $\mathfrak{g}\otimes\mathbb{C}[t,u]/\langle t^N,u^2\rangle$ *modules*

$$\mathrm{gr}G^\bullet\simeq V_1^u(c_1)**\dots**V_N^u(c_N). \tag{9}$$

Proof. The idea of the proof is as follows. We start with a tensor product $V_1\otimes\dots\otimes V_N$ and apply the fusion filtration. Afterwards we apply the t^N-filtration $G^\bullet$. Combining these operations with certain changes of the basis of current algebra we arrive at the definition of the bi-graded module $V_1^u**\dots**V_N^u$. We give the details below.

For an element $x\otimes t^i\in\mathfrak{g}\otimes\mathbb{C}[t]$ let $(x\otimes t^i)^{(j)}$ be an operator on the tensor product $V_1\otimes\dots\otimes V_N$ defined by

$$(x\otimes t^i)^{(j)}=\mathrm{Id}^{\otimes j-1}\otimes(x\otimes t^i)\otimes\mathrm{Id}^{\otimes N-j},$$

i.e. $(x\otimes t^i)^{(j)}$ acts as $x\otimes t^i$ on V_j and as an identity operator on the other factors. In order to construct the fusion product one starts with the operators

$$A(x\otimes t^i)=\sum_{j=1}^{N}(x\otimes (t+c_j)^i)^{(j)}, \tag{10}$$

where c_j are pairwise distinct numbers. These operators define an action of the algebra $\mathfrak{g}\otimes\mathbb{C}[t]$ on the tensor product $V_1(c_1)\otimes\ldots\otimes V_N(c_N)$. Since $x\otimes t^i$ with $i>1$ vanish on V_j we obtain

$$A(x\otimes t^i)=\sum_{j=1}^{N}c_j^i(x\otimes 1)^{(j)}+\sum_{j=1}^{N}ic_j^{i-1}(x\otimes t)^{(j)}. \tag{11}$$

As the next step we pass to the adjoint graded module with respect to the fusion filtration. By definition operators of the form

$$A(x\otimes t^k)+\text{ linear combination of } A(x\otimes t^l), l<k \tag{12}$$

act in the adjoint graded module in the same way as $A(x\otimes t^k)$ (the lower degree term vanish after passing to the adjoint graded space). We are going to fix special linear changes in (12) for $N\le k<2N$ which make the expressions for $A(x\otimes t^k)$ simpler.

Let $\alpha_0,\alpha_1,\ldots,\alpha_{N-1}$ be numbers such that for all $1\le j\le N$

$$c_j^N+\sum_{i=0}^{N-1}\alpha_i c_j^i=0. \tag{13}$$

We state that

$$A(x\otimes t^{N+s})-\sum_{i=0}^{N-1}\alpha_i A(x\otimes t^{s+i})=\sum_{j=1}^{N}c_j^s(x\otimes t)^{(j)}\prod_{k\neq j}(c_j-c_k) \tag{14}$$

for all $0\le s\le N-1$. We first check this for $s=0$. Let

$$f(x)=x^N+\alpha_{N-1}x^{N-1}+\cdots+\alpha_0. \tag{15}$$

Then $f(x)=\prod_{j=1}^{N}(x-c_j)$. Therefore, for the derivative $f'(x)$ one gets

$$f'(c_i)=Nc_i^{N-1}+\sum_{k=0}^{N-1}k\alpha_k c_i^{k-1}=\prod_{j\neq i}(c_i-c_j). \tag{16}$$

This proves (14) for $s=0$. The $s>0$ case follows easily.

Using formula (14) we replace operators $A(x\otimes t^i)$, $0\le i<2N$ by operators $B(x\otimes t^i)$ as follows

$$B(x\otimes t^i)=\sum_{j=1}^{N}c_j^i(x\otimes 1)^{(j)}+\sum_{j=1}^{N}ic_j^{i-1}(x\otimes t)^{(j)},\ 0\le i<N, \tag{17}$$

$$B(x\otimes t^{N+i})=\sum_{i=1}^{N}c_i^s(x\otimes t)^{(i)}\prod_{k\neq j}(c_j-c_k),\ 0\le i<N, \tag{18}$$

thus performing the linear change (12). So we redefine half of the operators $A(x\otimes t^i)$ and leave the other half unchanged.

In order to construct the left hand side of (9) one first apply the fusion filtration to the algebra of operators $B(x\otimes t^i)$ and afterwards proceed with the t^N-filtration. The last step means that the subtraction of a linear combination of the operators $B(x\otimes t^{N+i})$, $0\le i<N$ from $B(x\otimes t^i)$, $0\le i<N$ does not change the structure of the resulting module. Redefining the operators (17) we arrive at the following operators:

$$C(x\otimes t^i)=\sum_{j=1}^{N}c_j^i(x\otimes 1)^{(j)},\ 0\le i<N, \tag{19}$$

$$C(x\otimes t^{N+i})=\sum_{i=j}^{N}c_j^i(x\otimes t)^{(j)},\ 0\le i<N. \tag{20}$$

Note that we can remove the constants $\prod_{k\neq j}(c_j-c_k)$ from $B(x\otimes t^{N+i})$ since this procedure corresponds simply to rescaling the variable t in each V_i.

Summarizing all the formulas above we arrive at the following two step construction of the left hand side of (9):

- apply the fusion procedure to the operators $C(x\otimes t^i)$, $0\le i<2N$,
- attach a u-degree 1 to each of the operators $C(x\otimes t^{N+i})$, $0\le i<0$.

In order to construct the right hand side of (9) one uses the same procedure with $C(x\otimes t^{N+i})$ being operators which correspond to $x\otimes ut^i$. Thus we have shown that the adjoint graded to the fusion product with respect to the t^N-filtration is isomorphic to to the module $V_1**\cdots**V_N$. Proposition is proved. □

Corollary 2.1. *The fusion product $D(1)**\cdots**D(N)$ does not depend on the evaluation parameters.*

Corollary 2.2. *The fusion product of the adjoint representations*

$$\mathfrak{g}(c_1)**\cdots**\mathfrak{g}(c_N)$$

is independent on the parameters $c_1,\dots,c_N$.

Proof. By definition the zeroth graded component with respect to the t grading of the module

$$D(1)(c_1)**\cdots**D(1)(c_N)$$

is isomorphic to the fusion product $\mathfrak{g}(c_1)**\cdots**\mathfrak{g}(c_N)$. From the Proposition above we obtain an isomorphism

$$\mathfrak{g}(c_1)**\cdots**\mathfrak{g}(c_n) \simeq D(N)/(\mathfrak{g}\otimes t^N\mathbb{C}[t])D(N) \tag{21}$$

for any $c_1,\dots,c_n$. Thus the left hand side is independent on c_i. □

We finish this section introducing an "energy" operator $\bar{d}$ on the fusion product. The operator $\bar{d}$ acts by a constant m on the graded component

$$\mathrm{gr}_m(V_1(c_1)*\cdots*V_N(c_N)).$$

The operator $\bar{d}$ defines a q character of $V_1(c_1)*\cdots*V_n(c_n)$ by the standard formula formula

$$\mathrm{ch}_q V_1(c_1)*\cdots*V_n(c_n) = \sum_{n\geq 0} q^n \dim\{v:\ \bar{d}v = nv\}.$$

An analogous formula defines a character of $V_1(c_1)**\cdots**V_n(c_n)$. We note that this grading has nothing to do with a Cartan grading coming from the action of the Cartan subalgebra. We do not consider the latter grading in this paper.

Remark 2.2. *Operators $d\in\widehat{\mathfrak{g}}$ and $\bar{d}$ are different. In particular, one has*

$$[d, x\otimes t^i] = -ix\otimes t^i,\ [\bar{d}, x\otimes t^i] = ix\otimes t^i.$$

Nevertheless these two operators are closely related in certain cases. In particular let $V_i\simeq D(1)$ for all i. Then the fusion module is independent on the evaluation parameters and $D(1)^{*N}$ is isomorphic to the Demazure module $D(N)$ (see [FoL2]*). By the very definition we have an embedding $D(N)\hookrightarrow L$. Thus both operators d and $\bar{d}$ are acting on $D(N)$. It is easy to see that they are related by a simple identity $\bar{d} = N^2 - d$.*

3. Demazure modules

In this section we study the fusion filtration on the tensor product $D(1)^{\otimes N}$ and the induced PBW filtration on the Demazure modules $D(N)$. We also derive some connections between these filtrations.

Let $D^u(1)$ be a $\mathfrak{g}\otimes\mathbb{C}[u]/\langle u^2\rangle$ module obtained from $D(1)$ by substituting u instead of t. In particular $(D^u(1))^{**N}$ is a (t,u) bi-graded representation of the algebra $\mathfrak{g}\otimes\mathbb{C}[t,u]/\langle t^N, u^2\rangle$. Here t-grading is exactly the fusion grading $\mathrm{gr}_j D^u(1)*\cdots*D^u(1)$ and the u-grading comes from the grading on $D^u(1)$ which assignes degree zero to $\mathfrak{g}$ and degree one to $\mathbb{C}v_0$. We consider the decomposition

$$(D^u(1))^{**N} = \bigoplus_{m=0}^{N} W(m)$$

into the graded components with respect to the u-grading. (The u-grading is bounded from above by N since the u-grading in each of $D(1)$ could be either 0 or 1). Note that each space $W(m)$ is a representation of $\mathfrak{g}\otimes\mathbb{C}[t]$. We want to show that $W(m)$ can be filtered by certain number of copies of

the fusion product $\mathfrak{g}^{**N-m}$. The precise statement is given in the following proposition.

Proposition 3.1. *Let $D^u(1)^{**N}$ be a bi-graded tensor product of N copies of $\mathfrak{g}\otimes\mathbb{C}[u]$-module $D(1)$. Then*

- *For any $0\le m\le N$ the $\mathfrak{g}\otimes\mathbb{C}[t]$ module $W(m)$ can be filtered by $\binom{N}{m}$ copies of the $\mathfrak{g}\otimes\mathbb{C}[t]$-module $\mathfrak{g}^{*N-m}$.*
- *The cyclic vectors of the modules $\mathfrak{g}^{**N-m}$ above are the images of the vectors*

$$f^\theta\otimes ut^{i_1}\dots f^\theta\otimes ut^{i_m}v_N,\ 0\le i_1\le\dots\le i_m\le N-m. \tag{22}$$

Remark 3.1. *We first give a non rigorous but conceptual explanation of the statement of the proposition above. Recall that $D^u(1)$ is isomorphic to $\mathfrak{g}\oplus\mathbb{C}$ as a $\mathfrak{g}$ module. Let $v_1,v_0\in D(1)$ be highest weight vectors of $\mathfrak{g}$ and $\mathbb{C}$ respectively. Then $(f_\theta\otimes u)v_1=v_0$ and $(f_\theta\otimes u)^2v_0=0$. Therefore after making a fusion $D^u(1)^{**N}$ the tensor product of N copies of 2-dimensional vector space* $\mathrm{span}\{v_0,v_1\}$ *will be deformed into the N-fold fusion product of two dimensional representations of the algebra $\mathbb{C}[f_\theta]$. The set of vectors* (22) *represents a basis of this fusion product (see* [CP], [FF1]*). Hence $D^u(1)^{*N}$ is equal to the $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])$ span of the vectors of the form* (22)*. We now want to describe the space $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])\cdot w_m$, where w_m is of the form* (22) *with exactly m factors. Note that w_m is a linear combination of the vectors of the form*

$$v_{i_1}\otimes\dots\otimes v_{i_N},$$

where i_α equal 0 or 1 and the number of α such that $i_\alpha=0$ is equal to m. This means that the space $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])\cdot w_m$ is embedded into the direct sum of $\binom{N}{m}$ copies of the tensor product $\mathfrak{g}^{\otimes m}$. Hence it is natural to expect that after passing to the adjoint graded object with respect to the fusion filtration one arrives at the fusion product $\mathfrak{g}^{**m}$. This is not exactly true. In order to make the statement precise one additional filtration is needed (that is the reason why $W(m)$ is not the direct sum of the fusion products but rather can be filtered by these modules).

We now give the proof of the Proposition 3.1.

Proof. As a starting point we note the isomorphism of $\mathfrak{g}\otimes\mathbb{C}[t]$-modules

$$W(0)\simeq\mathfrak{g}^{**N}.$$

In fact $D(1)\simeq\mathfrak{g}\oplus\mathbb{C}v_0$ with cyclic vector being the highest weight vector of $\mathfrak{g}$. An algebra $\mathfrak{g}\otimes u$ maps $\mathbb{C}$ to zero and $\mathfrak{g}$ to $\mathbb{C}$. In particular

$$(f_\theta\otimes u)\cdot v_1=v_0. \tag{23}$$

Hence if we do not apply operators with positive powers of u (i.e. we consider the space $W(0)$) we arrive at the usual fusion product of N copies of $\mathfrak{g}$.

We now introduce a decreasing filtration $W^j(m)$ such that the adjoint graded object is isomorphic to the direct sum of $\binom{N}{m}$ copies of $\mathfrak{g}^{**m}$. Let

$$w_{i_1,\dots,i_m} = f^\theta \otimes ut^{i_1} \dots f^\theta \otimes ut^{i_m} v_N,\ 0 \le i_1 \le \dots \le i_m \le N-m.$$

We set

$$W^n(m) = \mathrm{U}(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot \mathrm{span}\{w_{i_1,\dots,i_m} :\ i_1 + \dots + i_m \ge n\}.$$

In particular, $W^0(m) = W(m)$ and each space $W^n(m)$ is $\mathfrak{g} \otimes \mathbb{C}[t]$ invariant. We state that the adjoint graded space

$$W^0(m)/W^1(m) \oplus W^1(m)/W^2(m) \oplus \dots \tag{24}$$

is isomorphic to the direct sum of $\binom{N}{m}$ copies of the modules $\mathfrak{g}^{*N-m}$. Moreover the highest weight vectors of these modules are exactly the images of $w_{i_1,\dots,i_m}$. We prove this statement for $m = 1$. The proof for other m is very similar.

For $m = 1$ we have

$$w_i = f_\theta \otimes ut^i = \sum_{j=1}^N c_j^i v_1^{\otimes j-1} \otimes v_0 \otimes v_1^{\otimes N-j},\ i = 0, \dots, N-1. \tag{25}$$

We want to show that

$$W^i(1)/W^{i+1}(1) \simeq \mathfrak{g}^{*N-1},\ W_i(1) = \mathrm{U}(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot \mathrm{span}\{w_i, \dots, w_{N-1}\}. \tag{26}$$

Let $\alpha_{i,j}$, $1 \le i, j \le N$ be some numbers. Denote

$$\tilde{W}_i(1) = \mathrm{U}(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot \mathrm{span}\{\sum_{j=1}^N \alpha_{i+1,j} w_i, \dots, \sum_{j=1}^N \alpha_{N,j} w_{N-1}\}.$$

$\tilde{W}_i(1)$ differs from $W_i(1)$ be replacing vectors w_i by some linear combination of vectors w_j. We state that

$$W^i(1)/W^{i+1}(1) \simeq \tilde{W}^i(1)/\tilde{W}^{i+1}(1) \tag{27}$$

as $\mathfrak{g} \otimes \mathbb{C}[t]$ modules. In fact adding to w_i a linear combination of w_j with $j < i$ does not change the $\mathfrak{g} \otimes \mathbb{C}[t]$ module structure because of the fusion filtration. Because of the filtration (24) this is still true if one adds a linear combination of w_j with $j > i$. Thus we conclude that we can replace each vector w_i by an arbitrary linear combination. In particular there exist numbers $\alpha_{i,j}$ such that

$$\sum_{j=1}^N \alpha_{i-1,j} w_i = v_0^{\otimes i} \otimes v_1 \otimes v_0^{\otimes N-1-i}.$$

Since v_1 is a highest weight vector of the adjoint representation of $\mathfrak{g}$ and v_0 spans the trivial representation we arrive at the fact that

$$\tilde{W}^i(1)/\tilde{W}^{i+1}(1) \simeq \mathfrak{g}^{**N-1}.$$

Because of the isomorphism (27) the $m = 1$ case of the proposition is proved. □

We are now going to connect the filtrations $G^\bullet(N)$ and the induced PBW filtration $F_\bullet(N)$ on the Demazure modules $D(N)$. We use Proposition 2.1 for $V_i = D(1)$.

Lemma 3.1. $G^m \hookrightarrow F_{N-m}(N)$

Proof. We first note that for the cyclic vector $v_N \in D(N)$ we have

$$v_N = (e_\theta \otimes t^{-N})^N v_0. \tag{28}$$

Therefore $G^0 = F_N(N) = D(N)$ and our lemma is true for $m = 0$.

In general we need to prove that

$$x_1 \otimes t^{N+i_1} \dots x_m \otimes t^{N+i_m} v \in F_{N-m} \tag{29}$$

for any $v \in D(N)$ and $x_1, \dots, x_m \in \mathfrak{g}$, $i_1, \dots, i_m \geq 0$. Since $v \in D(N)$ there exists an element $r \in \mathrm{U}(\mathfrak{g} \otimes \mathbb{C}[t]$ such that $v = rv_N$. Because of (28) the inclusion (29) follows from the following statement:

$$(x_1 \otimes t^{N+i_1}) \dots (x_m \otimes t^{N+i_m})(y_1 \otimes t^{j_1}) \dots (y_m \otimes t^{j_m})(e_\theta \otimes t^{-N})^N v_0 \in F_{N-m} \tag{30}$$

for arbitrary $x_\alpha, y_\beta \in \mathfrak{g}$ and $i_\alpha, j_\beta, n \geq 0$. Since $(\mathfrak{g} \otimes \mathbb{C}[t]) \cdot v_0 = 0$ the expression

$$y_1 \otimes t^{j_1} \dots y_n \otimes t^{j_n} (e_\theta \otimes t^{-N})^N v_0$$

is equal to a linear combination of the vectors of the form

$$z_1 \otimes t^{-N+l_1} \dots z_N \otimes t^{-N+l_N} v_0 \tag{31}$$

where $z_i \in \mathfrak{g}$ and $l_i \geq 0$. One can easily see that because of the condition $(\mathfrak{g} \otimes \mathbb{C}[t])v_0 = 0$ the expression of the form

$$x \otimes t^{N+i} z_1 \otimes t^{-N+l_1} \dots z_N \otimes t^{-N+l_N} v_0$$

can be rewritten as a linear combination of the monomials of the form

$$z_1 \otimes t^{-N+l_1} \dots z_{N-1} \otimes t^{-N+1+l_{N-1}} v_0. \tag{32}$$

Iterating this procedure we arrive at (30). This proves (29) and hence our lemma is proved. □

Recall a notation for q-binomial coefficients

$$\binom{n}{m}_q = \frac{(q)_n}{(q)_m (q)_{n-m}}, \ (q)_a = (1-q)\dots(1-q^a).$$

Corollary 3.1. *The following character inequality holds:*

$$\mathrm{ch}_q \mathrm{gr}_m F_\bullet(N)) \geq q^{m^2} \binom{N}{m}_q \mathrm{ch}_{q^{-1}} \mathfrak{g}^{**m}. \tag{33}$$

Proof. Because of the Lemma above it suffices to show that

$$\mathrm{ch}_q G^m / G^{m+1} = q^{(N-m)^2} \binom{N}{m}_q \mathrm{ch}_{q^{-1}} \mathfrak{g}^{*N-m}.$$

Note that $dv_N = N^2 v_N$. Therefore the q character of the space of cyclic vectors $w_{i_1,\dots,i_m}$ from the Proposition 3.1 (where the q-degree of u is fixed to be equal to N according to Proposition 2.1) is given by $q^{m^2}\binom{N}{m}_q$. Multiplying by the "reversed" character of $\mathfrak{g}^{*N-m}$ (see Remark 2.2) we arrive at our Corollary. □

Corollary 3.2. *The following character inequality holds:*

$$\mathrm{ch}_q \mathrm{gr}_m F_\bullet \geq q^{m^2} \frac{1}{(q)_m} \mathrm{ch}_{q^{-1}} \mathfrak{g}^{**m}. \tag{34}$$

Proof. Follows from limit formulas $\lim_{N\to\infty} D(N) = L$ and

$$\lim_{N\to\infty} \binom{N}{m}_q = \frac{1}{(q)_m}.$$

□

Remark 3.2. *In the next section we prove that* (34) *is in fact an equality. In Section* 5 *we prove that* (33) *is also an equality.*

4. Dual functional realization

We now consider the restricted dual space to the "expected" PBW filtered space L^{gr}. Let

$$\mathrm{U}^{ab} = \mathbb{C}[\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]]$$

be a space of polynomial functions on the space $\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ (recall that $\mathfrak{g}^{ab}$ is an abelian Lie algebra with the underlying vector space isomorphic to $\mathfrak{g}$). The algebra U^{ab} is an abelinization of the universal enveloping algebra $\mathrm{U}(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$ of the Lie algebra of generating operators. We note that $\mathfrak{g}$ acts on the space $\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ via the adjoint representation on $\mathfrak{g}^{ab}$.

Let $I \in \mathrm{U}^{ab}$ be a minimal $\mathfrak{g}$ invariant ideal, which contains all coefficients of $e^\theta(z)^2$, i.e. all elements of the form

$$\sum_{1\leq i\leq n} (e_\theta \otimes t^{-i})(e_\theta \otimes t^{-n-1+i}),\ n = 1, 2, \dots.$$

Denote

$$L' = \mathrm{U}^{ab}/I.$$

This space can be decomposed according to the number of variables in a monomial:

$$L' = \bigoplus_{m=0}^{\infty} L'_m = \bigoplus_{m=0}^{\infty} \mathrm{span}\{x_1 \otimes t^{i_1} \dots x_m \otimes t^{i_m},\ x_i \in \mathfrak{g}^{ab}\}. \tag{35}$$

The operator $d \in \widehat{\mathfrak{g}}$ induces a degree operator on L'. We denote this operator by the same symbol. There exists a surjective homomorphism of $\mathfrak{g}^{ab} \otimes t^{-1}\mathbb{C}[t^{-1}]$ modules $L' \to L^{gr}$ (see Lemma 1.2). Our goal is to show that

$L' \simeq L^{gr}$. Let $(L'_m)^*$ be a restricted dual space (with respect to the d grading). We construct the functional realization of L'_m using currents

$$x(z) = \sum_{i>0} (x \otimes t^{-i}) z^i$$

for $x \in \mathfrak{g}$. Following [FFJMT] we consider a map

$$\alpha_m : {L'_m}^* \to \mathbb{C}[z_1, \dots, z_m] \otimes \mathfrak{g}^{\otimes m}, \ \phi \mapsto r_\phi$$

from the dual space to L'_m to the space of polynomials in m variables with values in the m-th tensor power of the space $\mathfrak{g}$. This map is given by the formula

$$\langle r_\phi, x_1 \otimes \ldots \otimes x_m \rangle = \phi(x_1(z_1) \ldots x_m(z_m)), \tag{36}$$

where brackets in the left hand side denotes the product of non degenerate Killing forms on n factors $\mathfrak{g}$. Our goal is to describe the image of α_m. We first formulate the conditions on r_ϕ which follow from the definition of L' and then prove that these conditions determine the image of α_m. We prepare some notations first.

Consider the decomposition of the tensor square $\mathfrak{g} \otimes \mathfrak{g}$ into the direct sum of $\mathfrak{g}$ modules:

$$\mathfrak{g} \otimes \mathfrak{g} = V_{2\theta} \oplus \bigwedge^2 \mathfrak{g} \oplus S^2\mathfrak{g}/V_{2\theta}, \tag{37}$$

where $V_{2\theta}$ is a highest weight $\mathfrak{g}$-module with a highest weight 2θ. We have an embedding

$$V_{2\theta} \simeq \mathrm{U}(\mathfrak{g}) \cdot (e_\theta \otimes e_\theta) \hookrightarrow S^2\mathfrak{g}.$$

Lemma 4.1. *For the module $\mathfrak{g} * *\mathfrak{g}$ we have*

$$\mathrm{gr}_0(\mathfrak{g} * *\mathfrak{g}) = V_{2\theta}, \ \mathrm{gr}_1(\mathfrak{g} * *\mathfrak{g}) = \bigwedge^2 \mathfrak{g}, \ \mathrm{gr}_2(\mathfrak{g} * *\mathfrak{g}) = S^2\mathfrak{g}/V_{2\theta}$$

and all other graded components vanish.

Proof. We first show that $\mathrm{gr}_n(\mathfrak{g} * *\mathfrak{g}) = 0$ for $n > 2$. Let c_1, c_2 be evaluation constants which appear in the Definition 2.2 of the fusion product. Recall that $\mathfrak{g} * \mathfrak{g}$ is independent on the evaluation parameters. So we set $c_1 = 1$, $c_2 = 0$. Then the second space of the fusion filtration is given by

$$\mathrm{U}(\mathfrak{g}) \cdot (\mathrm{span}\{[x_1, [x_2, e_\theta]], \ x_1, x_2 \in \mathfrak{g}\} \otimes e_\theta), \tag{38}$$

where $\mathrm{U}(\mathfrak{g})$ acts on the tensor product $\mathfrak{g} \otimes \mathfrak{g}$ diagonally. But $[f_\theta, [f_\theta, e_\theta]] = -2f_\theta$ and hence

$$\mathrm{span}\{[x_1, [x_2, e_\theta]], \ x_1, x_2 \in \mathfrak{g}\} = \mathfrak{g}$$

(since the left hand side is $\mathfrak{n}_-$-invariant and contains the lowest weight vector f_θ of the adjoint representation). Therefore (38) coincides with $\mathfrak{g} \otimes \mathfrak{g}$.

We now compute three nontrivial graded components $\mathrm{gr}_0(\mathfrak{g}**\mathfrak{g})$, $\mathrm{gr}_1(\mathfrak{g}**\mathfrak{g})$ and $\mathrm{gr}_2(\mathfrak{g} * *\mathfrak{g})$. From the definition of the fusion filtration we have

$$\mathrm{gr}_0(\mathfrak{g} * *\mathfrak{g}) = \mathrm{U}(\mathfrak{g}) \cdot (e_\theta \otimes e_\theta) \simeq V(2\theta).$$

We now redefine the evaluation parameters by setting $c_1 = 1$, $c_2 = -1$. Then the formula for the operators $x \otimes t$ acting on $\mathfrak{g} \otimes \mathfrak{g}$ is given by $x \otimes \mathrm{Id} - \mathrm{Id} \otimes x$. These operators map $S^2(\mathfrak{g})$ to $\bigwedge^2 \mathfrak{g}$ and vice versa.We conclude that

$$\mathrm{gr}_1(\mathfrak{g} * *\mathfrak{g}) \hookrightarrow \bigwedge^2 \mathfrak{g},\ \mathrm{gr}_2(\mathfrak{g} * *\mathfrak{g}) \hookrightarrow S^2(\mathfrak{g}).$$

Now our Lemma follows from the equality $\mathrm{gr}_n(\mathfrak{g} * *\mathfrak{g}) = 0$ for $n > 2$. □

Lemma 4.2. *For any $\phi \in (L'_m)^*$ the image r_ϕ is divisible by $z_1 \dots z_m$ and satisfies the vanishing condition*

$$\langle r_\phi, V_{2\theta} \otimes \mathfrak{g}^{\otimes m-2} \rangle_{z_1 = z_2} = 0 \tag{39}$$

and the symmetry condition

$$\sigma r = r,\ \sigma \in S_m \tag{40}$$

where the permutation group S_m acts simultaneously on the set of variables $z_1, \dots, z_m$ and on the tensor product $\mathfrak{g}^{\otimes m}$.

Proof. The product $z_1 \dots z_m$ comes from the condition that the highest weight vector is annihilated by $\mathfrak{g} \otimes \mathbb{C}[t]$, so for any $x \in \mathfrak{g}$ the series $x(z)$ starts with z. The condition (39) follows from the relation $e_\theta(z)^2 = 0$ and the $\mathfrak{g}$-invariance of the ideal I. The symmetry condition follows from the commutativity of the algebra U^{ab}. □

We denote by

$$V_m \hookrightarrow z_1 \dots z_m \mathbb{C}[z_1, \dots, z_m] \otimes \mathfrak{g}^{\otimes m}$$

the space of functions satisfying conditions (39) and (40). In the following lemma we endow V_m with structures of representation of the ring of symmetric polynomials

$$P_m^{sym} = \mathbb{C}[z_1, \dots, z_m]^{S_m}$$

and of the current algebra $\mathfrak{g} \otimes \mathbb{C}[t]$.

Lemma 4.3. *There exists natural structures of representations of P_m^{sym} and of $\mathfrak{g} \otimes \mathbb{C}[t]$ on V_m defined by the following rule:*

- *P_m^{sym} acts on V_m by multiplications on $\mathbb{C}[z_1, \dots, z_m]^{sym}$.*
- *The Lie algebra $\mathfrak{g} \otimes \mathbb{C}[t]$ acts on V_m by the formula*

$$x \otimes t^k \text{ acts as } \sum_{i=1}^n z_i^k \otimes x^{(i)}. \tag{41}$$

The actions of P_m^{sym} and of $\mathfrak{g} \otimes \mathbb{C}[t]$ commute.

Lemma 4.3 gives a structure of the $\mathfrak{g} \otimes \mathbb{C}[t]$ module on the quotient space $V_m / P_{m+}^{sym} V_m$, where the subscript $+$ denotes the space of polynomials of positive degree. We will show that the dual to this module is isomorphic to $\mathfrak{g}^{**m}$. We first consider the $m = 2$ case.

Lemma 4.4. *We have an isomorphism of representations of* $\mathfrak{g}\otimes\mathbb{C}[t]$

$$\left(V_2/P_+^{sym}V_2\right)^*\simeq\mathfrak{g}*{*}\mathfrak{g}.$$

Proof. Let r be an element of V_2. Using the decomposition (37) we write $r=r_0+r_1+r_2$, where

$$r_0\in z_1z_2\mathbb{C}[z_1,z_2]\otimes V_{2\theta},$$
$$r_1\in z_1z_2\mathbb{C}[z_1,z_2]\otimes\Lambda^2\mathfrak{g},$$
$$r_2\in z_1z_2\mathbb{C}[z_1,z_2]\otimes S^2\mathfrak{g}/V_{2\theta}.$$

Then the conditions (39) and (40) are equivalent to

$$r_0\in z_1z_2(z_1-z_2)^2P^{sym}\otimes V_{2\theta},$$
$$r_1\in z_1z_2(z_1-z_2)P^{sym}\otimes\Lambda^2\mathfrak{g},$$
$$r_2\in z_1z_2P^{sym}\otimes S^2\mathfrak{g}/V_{2\theta}.$$

Therefore the dual quotient space $(V_2/P_+^{sym}V_2)^*$ is isomorphic to $\mathfrak{g}*{*}\mathfrak{g}$ as a vector space. It is straightforward to check that the $\mathfrak{g}\otimes\mathbb{C}[t]$ modules structures are also isomorphic. □

We will also need a nonsymmetric version of the space $V_m/P_{m+}^{sym}V_m$. Namely let W_m be a subspace of $z_1\dots z_m\mathbb{C}[z_1,\dots,z_m]\otimes\mathfrak{g}^{\otimes m}$ satisfying

$$F_{z_i=z_j}\hookrightarrow\mathbb{C}[z_1,\dots,z_m]|_{z_1=z_2}\otimes\mu_{i,j}(S^2\mathfrak{g}/V_{2\theta}\otimes\mathfrak{g}^{\otimes m-2}),\tag{42}$$

$$\partial_{z_i}F_{z_i=z_j}\hookrightarrow\mathbb{C}[z_1,\dots,z_m]|_{z_1=z_2}\otimes\mu_{i,j}(S^2(\mathfrak{g})/V_{2\theta}\otimes\mathfrak{g}^{\otimes m-2})\tag{43}$$

for all $1\le i<j\le m$, where $\mu_{i,j}$ is an operator on the tensor power $\mathfrak{g}^{\otimes m}$ which inserts the first two factors on the i-th and j-th places respectively:

$$\mu_{i,j}(v_1\otimes v_2\otimes v_2\otimes\dots\otimes v_n)=$$
$$v_3\otimes\dots\otimes v_{i-1}\otimes v_1\otimes v_i\otimes\dots\otimes v_{j-2}\otimes v_2\otimes v_{j-1}\dots\otimes v_n.$$

The natural action of the polynomial ring $P_m=\mathbb{C}[z_1,\dots,z_m]$ commutes with the action of the current algebra, defined as in Lemma 4.3. Therefore we obtain a structure of $\mathfrak{g}\otimes\mathbb{C}[t]$ module on the quotient space $W_m/P_{m+}W_m$, where P_{m+} is the ring of positive degree polynomials. As proved in [FFJMT] the symmetric and nonsymmetric constructions produce the same module (see Proposition 4.1 below for the precise statement). We illustrate this in the case $n=2$.

Lemma 4.5. $(W_2/P_{2+}W_2)^*\simeq\mathfrak{g}*{*}\mathfrak{g}$.

Proof. We use the decomposition $r=r_0+r_1+r_2$ as in Lemma 4.4 for $r\in W_m$. The conditions (42) and (43) are equivalent to the following conditions on r_i:

$$r_0\in(z_1-z_2)^2z_1z_2\mathbb{C}[z_1,z_2]\otimes V_{2\theta},$$
$$r_1\in(z_1-z_2)z_1z_2\mathbb{C}[z_1,z_2]\otimes\Lambda^2\mathfrak{g},$$
$$r_2\in z_1z_2\mathbb{C}[z_1,z_2]\otimes S^2\mathfrak{g}/V_{2\theta}.$$

After passing to the quotient with respect to the action of the algebra P_{2+} we arrive at the isomorphism of vector spaces $(W_2/P_{2+}W_2)^* \simeq \mathfrak{g} * \mathfrak{g}$. It is straightforward to check that this is in fact an isomorphism of $\mathfrak{g} \otimes \mathbb{C}[t]$ modules. □

The following Proposition is proved in [FFJMT], Lemma 5.8 for $\mathfrak{g} = \mathfrak{sl}_2$.

Proposition 4.1. *There exists an isomorphism of $\mathfrak{g} \otimes \mathbb{C}[t]$-modules*

$$V_m/P^{sym}_{m+}V_m \simeq W_m/P_{m+}W_m.$$

Proof. The proof of the general case differs from the one from [FFJMT] by the replacement of the decomposition of the tensor square of the adjoint representation of $\mathfrak{sl}_2$ by the general decomposition (37). □

Remark 4.1. *We note that the definition of the space W_m is a bit more involved than the definition of V_m. The reason is that the polynomials used to construct W_m are not symmetric. In particular this forces to add the condition* (43) *in order to get the isomorphism $V_m/P^{sym}_{m+}V_m \simeq W_m/P_{m+}W_m$..*

Proposition 4.2. *The module W_m/P_+W_m is cocyclic with a cocylcic vector being a class*

$$r_m = z_1 \dots z_m \prod_{1\le i<j\le m} (z_i - z_j)^2 \otimes (e_\theta)^{\otimes m}.$$

Proof. We first show that if $r \in W_m$ is a nonzero element satisfying $r \in V_{m\theta} \otimes \mathbb{C}[z_1, \dots, z_m]$ then either $r \in P_{m+}W_m$ or there exists an element of the universal enveloping algebra of $\mathfrak{g} \otimes \mathbb{C}[t]$ which sends r to r_m (here we embed $V_{m\theta}$ into $\mathfrak{g}^{\otimes m}$ as an irreducible component containing $(e_\theta)^{\otimes m}$). In fact from the conditions (42), (43) and the assumption $r \in V_{m\theta}\otimes\mathbb{C}[z_1, \dots, z_m]$ we obtain that r is divisible by the product $\prod_{1\le i<j\le m}(z_i - z_j)^2$. If $r \neq P_{m+}W_m$ then we obtain

$$r = x \otimes \prod_{1\le i<j\le m} (z_i - z_j)^2$$

with some $x \in V_{m\theta}$. Since $V_{m\theta}$ is irreducible the $\mathrm{U}(\mathfrak{g})$ orbit of x contains $(e_\theta)^{\otimes m}$.

Thus it suffices to show that any element $r \in W_m/P_+W_m$ is contained in the $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])$ orbit of the image of $V_{m\theta}\otimes\mathbb{C}[z_1, \dots, z_m]$ in the quotient space. We prove that if $(a \otimes t)r = 0$ for all $a \in \mathfrak{g}$ then $r \in V_{m\theta} \otimes P_m$. This would imply the previous statement since $W_m/P_{m+}W_m$ is finite dimensional.

So let $r \in W_m$ be some element and $\bar{r} \in W_m/P_{m+}W_m$ be its class. Assume that $(a \otimes t)r = 0$ for all $a \in \mathfrak{g}$. We first show that $r \in V_{2\theta} \otimes \mathfrak{g}^{\otimes n-2} \otimes P_m$. In fact there exists a polynomial $l \in P_m$ such that the following holds in W_m:

$$(a \otimes t)r = l(z_1, \dots, z_m)r_1$$

for some $r_1 \in W_m$. Consider the space of functions $W^{1,2}_m$ which satisfy conditions (42) and (43) only for $i = 1$, $j = 2$. (In particular $W_m \hookrightarrow W^{1,2}_m$).

We have an isomorphism of $\mathfrak{g}\otimes\mathbb{C}[t]$ modules

$$W_m^{1,2}\simeq W_2\otimes\mathfrak{g}^{\otimes m-2}\otimes\mathbb{C}[z_3,\dots,z_m].$$

The equality $(a\otimes t)r=lr_1$ gives

$$((a^{(1)}\otimes z_1+a^{(2)}\otimes z_2)r=lr_1-(\sum_{i=3}^{m}a^{(i)}\otimes z_i)r.$$

Using Lemma 4.5 we obtain $r\in V_{2\theta}\otimes\mathfrak{g}^{\otimes n-2}\otimes P_m$.

The same procedure can be done for all pairs $1\le i<j\le n$. Now our proposition follows from the following equality in $\mathfrak{g}^{\otimes n}$:

$$\bigcap_{1\le i<j\le n}\mu_{i,j}V_{2\theta}\otimes\mathfrak{g}^{\otimes n-2}=V_{n\theta}.$$

□

The dual module W_m/P_mW_m is cyclic. We denote by r'_m a cyclcic vector which corresponds to the cocyclic vector r_m.

Proposition 4.3. *There exists a surjective homomorphism of* $\mathfrak{g}\otimes\mathbb{C}[t]$ *modules*

$$\mathfrak{g}^{**m}\to(W_m/P_{m+}W_m)^* \tag{44}$$

sending a cyclic vector $e_\theta^{\otimes m}$ *to* r'_m.

Proof. A relation in the fusion product means that some expression of the form

$$\sum\alpha_{i_1\dots i_s}x_1\otimes t^{i_1}\dots x_s\otimes t^{i_s} \tag{45}$$

with fixed t degree can be expressed in $\mathfrak{g}^{\otimes m}$ via a linear combination of monomials of lower t-degree. The coefficients of the expression of (45) in terms of the lower degree monomials are polynomials in evaluation parameters $c_1,\dots,c_n$. Therefore by the very definition of the action of P_m on W_m the operator of the form (45), which is a relation in the fusion product, vanishes on $(W_m/P_{m+}W_m)^*$. □

Note that W_m and V_m are naturally graded by the degree grading on P_m (P_m^{sym}). This grading defines a q character of the space V_m/P_{m+}^{sym}.

Corollary 4.1. $\mathrm{ch}_qV_m/P_{m+}V_m\le q^{m^2}\mathrm{ch}_{q^{-1}}\mathfrak{g}^{**m}$.

Proof. Follows from the Proposition 4.3 and Proposition 4.1. Note that the factor m^2 is a degree of the cyclic vector r'_m. □

5. Proofs of the main statements

Proposition 5.1. $\mathrm{ch}_q(L_m)^* \leq \mathrm{ch}_q V_m \leq q^{m^2}\mathrm{ch}_{q^{-1}}\mathfrak{g}^{**m}/(q)_m$.

Proof. We know that

$$\mathrm{ch}_q L_m^* \leq \mathrm{ch}(L'_m)^* = \mathrm{ch}_q V_m.$$

Because of the surjection (44) the character of V_m^* is smaller than or equal to $q^{m^2}\mathrm{ch}_{q^{-1}}\mathfrak{g}^{**m}/(q)_m$ (since $1/(q)_m$ is the character of the space of symmetric polynomials in m variables). This proves the Proposition. □

Theorem 5.1. $L_m \simeq L'_m$.

Proof. Corollary 3.2 provides an inequality

$$\mathrm{ch}_q L_m \geq \frac{q^{m^2}}{(q)_m}\mathrm{ch}_{q^{-1}}\mathfrak{g}^{**m}.$$

Now from the Proposition 5.1 we obtain

$$\frac{q^{m^2}}{(q)_m}\mathrm{ch}_{q^{-1}}\mathfrak{g}^{**m} \leq \mathrm{ch}_q L_m \leq \mathrm{ch}_q L'_m \leq \frac{q^{m^2}}{(q)_m}\mathrm{ch}_{q^{-1}}\mathfrak{g}^{**m}.$$

Theorem is proved. □

Corollary 5.1. *The dual module* $(V_m/P_{m+}^{sym}V_m)^*$ *and* $\mathfrak{g}^{**m}$ *are isomorphic as* $\mathfrak{g}\otimes\mathbb{C}[t]$ *modules.*

Proof. Follows from Propositions 4.3, 4.1 and Theorem 5.1. □

Corollary 5.2. *The adjoint graded space* L^{gr} *to the PBW filtration is isomorphic to the quotient*

$$\mathrm{U}(\mathfrak{g}^{ab}\otimes\mathbb{C}[t^{-1}])/I,$$

where I *is a minimal* $\mathfrak{g}$ *invariant ideal containing the coefficients of the current* $e_\theta(z)^2 = 0$.

Remark 5.1. *Corollary* 5.2 *is a generalization of the* $\mathfrak{sl}_2$ *case from* [FFJMT]. *It also proves a level* 1 *conjecture from* [F2].

Corollary 5.3. *The action of the polynomial ring* P_m^{sym} *on* V_m *is free.*

Proof. Follows from the isomorphism $(L_m/P_{m+}^{sym})^* \simeq \mathfrak{g}^{**m}$ and the character equality

$$\mathrm{ch}_q L_m = \frac{q^{m^2}}{(q)_m}\mathrm{ch}_{q^{-1}}\mathfrak{g}^{**m} = q^{m^2}\mathrm{ch}_q P_m^{sym}\cdot\mathrm{ch}_{q^{-1}}\mathfrak{g}^{**m}.$$

□

Recall vectors

$$w_{i_1,\dots,i_m} = e_\theta\otimes t^{i_1}\dots e_\theta\otimes t^{i_m}v_0 \in L.$$

Let $\bar{w}_{i_1,\dots,i_m}\in L_m$ be the images of these vectors.

Corollary 5.4. *L_m is generated from the set of vectors*

$$\bar{w}_{i_1,\dots,i_m},\ -m \geq i_1 \geq \dots \geq i_m \tag{46}$$

with an action of the algebra $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])$.

Proof. Recall an element

$$r_m = z_1 \dots z_m \prod_{1\leq i<j\leq m} (z_i - z_j)^2 \otimes e_\theta^{\otimes N} \in V_m.$$

Since r_m is a cocyclic vector of V_m/P_{m+}^{sym} and the polynomial algebra acts freely on V_m we obtain that the space $P_{m+}^{sym} r_m$ is cocyclic space of V_m, which means that the for any vector $v \in V_m$ the space $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])\cdot v$ has a nontrivial intersection with $P_{m+}^{sym} r_m$. We note that $P_{m+}^{sym} r_m$ coincides with the subspace of V_m of the $\mathfrak{g}$ weight $m\theta$. Dualizing this construction we obtain that the subspace of L_m of the $\mathfrak{g}$ weight $-m\theta$ is $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])$ cyclic. This space is linearly spanned by the set (46). Corollary is proved. □

Corollary 5.5. *The space* $\mathrm{gr}_m F_\bullet(N)$ *is generated by the set of vectors*

$$\bar{w}_{i_1,\dots,i_m},\ -m \geq i_1 \geq \dots \geq i_m \geq -N \tag{47}$$

with an action of the algebra $\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])$.

Proof. Introduce an increasing filtration $J_\bullet$ on L_m:

$$J_n = \mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])\cdot \mathrm{span}\{\bar{w}_{i_1,\dots,i_m} :\ -i_1 - \dots - i_m \leq m^2 + n\}.$$

Corollary 5.1 provides an isomorphism of $\mathfrak{g}\otimes\mathbb{C}[t]$ modules

$$\mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])\cdot \bar{w}_{i_1,\dots,i_m} / \left(J_{-i_1-\dots-i_m-1} \cap \mathrm{U}(\mathfrak{g}\otimes\mathbb{C}[t])\cdot \bar{w}_{i_1,\dots,i_m}\right) \simeq \mathfrak{g}^{**m}.$$

Thus, since all vectors of the from (47) belong to $F_m(N)$, we obtain

$$\dim \mathrm{gr}_m F_\bullet(N) \geq \binom{N}{m} (\dim \mathfrak{g})^m.$$

Because of the equality

$$\dim D(N) = (1 + \dim \mathfrak{g})^N = \sum_{m=0}^{N} \binom{N}{m} (\dim \mathfrak{g})^m$$

we conclude that $\dim \mathrm{gr}_m F_\bullet(N) = \binom{N}{m}(\dim \mathfrak{g})^m$ and hence the whole space $\mathrm{gr}_m F_\bullet(N)$ is generated from the vectors (47). □

Proposition 5.2. *The induced PBW filtration $F_\bullet(N) \hookrightarrow D(N)$ coincides with the t^N-filtration $G^\bullet$, i.e. $F_m(N) = G^{N-m}$.*

Proof. Recall (see Lemma 3.1) that $G^{N-m} \hookrightarrow F_m(N)$. Since $v_0 = (f_\theta \otimes t^N)^N v_N$ we obtain that $\bar{w}_{i_1,\dots,i_m} \in G^{N-m}$ for $-m \geq i_1 \geq \dots \geq i_m \geq -N$. Therefore Corollary 5.5 gives $F_m(N) \hookrightarrow G^{N-m}$. Proposition is proved. □

Corollary 5.6. *The graded component* $\mathrm{gr}_m F_\bullet(N)$ *is filtered by* $\binom{N}{m}$ *copies of* $\mathfrak{g}^{**m}$. *The character of the space of cyclic vectors of those fusions is equal to* $q^{m^2}\binom{N}{m}_q$.

We summarize all above in the following theorem:

Theorem 5.2. *Let* $F_\bullet$ *be the PBW filtration on the level one vacuum* $\widehat{\mathfrak{g}}$ *module* L. *Then*

a). $\mathrm{gr}_m F_\bullet$ *is filtered by the fusion modules* $\mathfrak{g}^{**m}$.
b). *The character of the space of cyclic vectors of these* $\mathfrak{g}^{**m}$ *is equal to* $\frac{q^{m^2}}{(q)_m}$.
c). *The induced PBW filtration on Demazure modules* $D(N)$ *coincides with the double fusion filtration coming from the Theorem* 2.1.
d). *The defining relation in* L^{gr} *is* $e_\theta(z)^2 = 0$.

A list of the main notations

$\mathfrak{g}$ – simple finite dimensional algebra;
$\widehat{\mathfrak{g}}$ – corresponding affine Kac-Moody algebra;
θ – a highest weight of the adjoint representation of $\mathfrak{g}$;
$e_\theta, f_\theta \in \mathfrak{g}$ – highest and lowest weight vectors of the adjoint representation;
L – a basic (vacuum level one) representation of $\widehat{\mathfrak{g}}$;
$v_0 \in L$ – a highest weight vector of L;
$v_N \in L$ – an extremal vector of the weight $N\theta$;
$D(N) \hookrightarrow L$ – a Demazure module with a cyclic vector v_N;
$\mathrm{gr}_m A_\bullet$ – m-th graded component of the adjoint graded space with respect to the filtration $A_\bullet$;
$F_\bullet$ – (increasing) PBW filtration on L;
$F_\bullet(N)$ – induced PBW filtration on $D(N)$;
$G^\bullet$ – decreasing t^N-filtration on $D(N)$;
$V_1 * \cdots * V_N$ – adjoint graded $\mathfrak{g} \otimes \mathbb{C}[t]$ module with respect to the fusion filtration on the tensor product of cyclic $\mathfrak{g} \otimes \mathbb{C}[t]$ modules;
$V_1 ** \cdots ** V_N$ – adjoint graded $\mathfrak{g} \otimes \mathbb{C}[t]$ module with respect to the fusion filtration on the tensor product of cyclic $\mathfrak{g}$ modules;
$\mathrm{gr}_m(V_1 * \cdots * V_N)$ – m-th graded component with respect to the fusion filtration;
ch_q – a character defined by an operator d.

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