

# IRREDUCIBILITY CRITERIA OF SCHUR-TYPE AND PÓLYA-TYPE

K. GYÓRY, L. HAJDU AND R. TIJDEMAN

*To the memory of Professor E. Hlawka*

ABSTRACT. Let  $f(x) = (x - a_1) \cdots (x - a_m)$ , where  $a_1, \dots, a_m$  are distinct rational integers. In 1908 Schur raised the question whether  $f(x) \pm 1$  is irreducible over the rationals. One year later he asked whether  $(f(x))^{2^k} + 1$  is irreducible for every  $k \geq 1$ . In 1919 Pólya proved that if  $P(x) \in \mathbb{Z}[x]$  is of degree  $m$  and there are  $m$  rational integer values  $a$  for which  $0 < |P(a)| < 2^{-N}N!$  where  $N = \lceil m/2 \rceil$ , then  $P(x)$  is irreducible. A great number of authors have published results of Schur-type or Pólya-type afterwards. Our paper contains various extensions, generalizations and improvements of results from the literature. It consists of three parts and we mention here a typical result from each part. In Theorem 2.1 we describe the form of the factors of polynomials of the shape  $f(x)g(x) + c$ , where  $g(x)$  is a polynomial and  $c$  is a constant such that  $|c|$  is small with respect to the degree of  $f(x)g(x)$ . In Theorem 5.1 a Pólya-type result is established when the ground ring is the ring of integers of an arbitrary imaginary quadratic number field. In Part 3 we obtain irreducibility results for polynomials of the form  $g(f(x))$  where  $g(x)$  is a monic irreducible polynomial which is of degree  $\leq 3$  or is of CM-type. Besides elementary arguments we apply methods and results from algebraic number theory, interpolation theory and diophantine approximation.

## 1. INTRODUCTION

In 1908 Schur [33] raised the question of the irreducibility of polynomials of the form

$$P_{\pm}(x) := (x - a_1)(x - a_2) \cdots (x - a_m) \pm 1$$

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where  $a_1, a_2, \dots, a_m$  are distinct rational integers. One year later Westlund [47] and Flügel [16] found that  $P_-(x)$  is always irreducible over  $\mathbb{Q}$ , and that  $P_+(x)$  can be reducible only if, for some  $c \in \mathbb{Z}$ ,

$$P_+(x - c) = x(x - 2) + 1 = (x - 1)^2$$

or

$$P_+(x - c) = x(x - 1)(x - 2)(x - 3) + 1 = (x(x - 3) + 1)^2.$$

We call polynomials  $P(x)$  and  $Q(x)$  with integral coefficients *equivalent* if  $P(x) = Q(x - c)$  for some integer  $c$ . Clearly, equivalent polynomials are at the same time reducible or irreducible in  $\mathbb{Z}[x]$ .

In 1919 Pólya [30] found the following irreducibility criterion. If  $P(x) \in \mathbb{Z}[x]$  is of degree  $m$  and there are  $m$  values  $a \in \mathbb{Z}$  for which

$$0 < |P(a)| < 2^{-N} N!$$

where  $N := \lceil m/2 \rceil$ , then  $P(x)$  is irreducible over  $\mathbb{Q}$ . This result implies that  $P_{\pm}(x)$  is irreducible over  $\mathbb{Q}$  for  $m > 6$ . By a different method Pólya proved that a polynomial  $P(x) \in \mathbb{Z}[x]$  of odd degree  $m$  is irreducible if  $m \geq 17$  and  $|P(x)| = p$  for  $m$  different integral arguments, where  $p$  is a rational prime.

Write

$$(1) \quad f(x) = (x - a_1)(x - a_2) \cdots (x - a_m)$$

where the integers  $a_1, a_2, \dots, a_m$  are distinct. Schur [34] (see also [6]) also asked whether  $(f(x))^{2^k} + 1$  is irreducible for  $k \geq 1$ . In 1926 A. Brauer, R. Brauer and Hopf [7] answered the question in the affirmative for  $k = 1$  and 2, and treated many other polynomials of the type  $g(f(x))$  where  $g(x)$  is an irreducible polynomial of low degree. For example, they treated the irreducibility of  $g(f(x))$  for  $g(x) = ax^2 + 1, ax^4 + 1, ax^6 + 1$  where  $a \in \mathbb{Z}_{>0}$  and  $g(x) = x^8 + 1$ . See also Ille [26]. Similarly Wegner [45] proved the irreducibility of  $(f(x))^4 + d$  where  $m > 5, d > 0, d \not\equiv 3 \pmod{4}$ .

In 1933 Dorwart and Ore [12] generalized various of the above mentioned results. They showed that a polynomial  $P(x) \in \mathbb{Z}[x]$  of degree  $n$  taking the values  $\pm 1$  at points  $a_1, \dots, a_m \in \mathbb{Z}$  where  $4 < m \leq n$  can have factors only of the form  $h(x)f(x) \pm 1$ . The degree of a nonconstant factor of  $P(x)$  is therefore never less than  $m$ , and when  $m > n/2$ ,  $P(x)$  is irreducible over  $\mathbb{Q}$ . They derived a similar result for polynomials taking many values  $\pm p$  with  $p$  prime. They further proved that  $g(f(x))$  is irreducible if  $g(x) = b_0x^2 + b_1x + 1 \in \mathbb{Z}[x]$  is irreducible and  $m \geq 5$ , and gave all exceptions for  $m \leq 4$ . Furthermore, they obtained results for polynomials over fields  $K = \mathbb{Q}(\sqrt{-d})$  where  $d \in \mathbb{Z}_{>0}$ . For example, they proved that polynomials of the form  $af(x) \pm 1$ , with

$a \in O_K \setminus \{0\}$  and distinct  $a_1, \dots, a_m \in O_K$ , are irreducible for  $n > 8$ , where  $O_K$  denotes the ring of integers of  $K$ .

Seres ([35], [36], [37]) answered the question of Schur [34] in full generality, proving the irreducibility of the polynomials  $g(f(x))$  for all  $g(x) = x^{2^k} + 1$  with  $k \geq 3$  and, more generally for all cyclotomic polynomials  $g(x)$ , except for the case  $g(x) = x^4 - x^2 + 1$ ,  $f(x) = (x+a)(x+a+1)(x+a+2)$ ,  $a \in \mathbb{Z}$ . Further, he extended his results (cf. [38]) to every irreducible  $g(x)$  of degree  $> 5$  whose zeros are nonreal units of a cyclotomic field. Later, Győry [17, 18, 19, 20] generalized Seres' results to the even more general case when the zeros of  $f(x)$  are distinct integers from a fixed totally real number field and the splitting field of  $g(x)$  is a CM field, i.e. a totally imaginary quadratic extension of a totally real number field.

In the present paper we want to add some new results to the investigations mentioned above. We distinguish three types of results and present them in three parts. More information on the literature can be found in the appropriate section.

In Part 1 we study so-called Schur-type results which are irreducibility criteria for polynomials of the form  $f(x) + c$  where  $f(x)$  has many integer zeros from  $\mathbb{Q}$  or from  $\mathbb{Q}(\sqrt{-d})$  with  $d \in \mathbb{Z}_{>0}$ , and  $c$  is an integer from  $\mathbb{Q}$  or  $\mathbb{Q}(\sqrt{-d})$ , respectively. We also investigate of which degrees factors are possible and which shape proper divisors of  $P(x)$  should have. Theorems 2.1 and 2.2 extend the above mentioned results of Dorwart and Ore to polynomials  $P(x) \in \mathbb{Z}[x]$  taking the same value  $c$ , or dividing the same value  $c$ , respectively, for many integral values  $x$ . Theorem 3.1 presents a generalization to polynomials

$$P(x) = (x - a_1) \cdots (x - a_m)g_1(x) \cdots g_t(x) \pm 1$$

where  $a_1, \dots, a_m \in \mathbb{Z}$  are distinct and  $g_1(x), \dots, g_t(x) \in \mathbb{Z}[x]$  are of degree 2 and have negative discriminants. Such polynomials occur in relation with so-called *ABC*-fields.

In Part 2 we study so-called Pólya-type results in which we consider polynomials  $P(x) \in \mathbb{Z}[x]$  which at many integer points take small nonzero absolute values. Typical statements are the results of Pólya mentioned above. These also yield new results of Schur-type. For example, Corollary 6.1 gives an upper bound for  $|c|$  for which the polynomial  $f(x)g(x) + c$  is irreducible if  $f(x)$  is a monic polynomial with distinct integer zeros and  $f(x)g(x)$  has only simple zeros, where the upper bound depends only on the degrees of  $f(x)$  and  $g(x)$  and the minimal distance between the zeros of  $f(x)g(x)$ .

Finally, in Part 3, we study irreducibility results for polynomials of the form  $g(f(x))$ , the research of which was started by Brauer, Brauer,

and Hopf. We assume throughout that  $g(x)$  is irreducible, since otherwise  $g(f(x))$  cannot be irreducible. In Theorems 7.1 and 7.2 we deal with the case when the degree of  $g(x)$  equals 2 or 3 and  $g(f(x))$  is reducible. Furthermore Theorem 8.1 is a quantitative version of the main result of [20]. It gives an upper bound for the number of equivalence classes of monic polynomials  $f(x) \in \mathbb{Z}[x]$  of degree  $m$  with distinct zeros in a fixed totally real algebraic number field  $K$  of degree  $d$  for which  $g(f(x))$  is reducible over  $\mathbb{Q}$ , where  $g(x) \in \mathbb{Z}[x]$  is a fixed monic irreducible polynomial having splitting field of CM-type. This upper bound depends only on  $d, m, g(0)$ , the degree of  $g$  and the discriminant of  $K$ .

## PART 1. SCHUR-TYPE RESULTS

### 2. THE RATIONAL CASE

By  $\tau(c)$  we denote the number of positive divisors of a nonzero integer  $c$ . Further, for  $\alpha \in \mathbb{R}$  we define the integers  $\lfloor \alpha \rfloor$  and  $\lceil \alpha \rceil$  by  $\alpha - 1 < \lfloor \alpha \rfloor \leq \alpha \leq \lceil \alpha \rceil < \alpha + 1$ .

**Theorem 2.1.** *Let  $c$  and  $m$  be nonzero integers with*

$$m > 2\tau(c)(2 + \lfloor \log_2 |c| \rfloor).$$

*Let  $f(x)$  be a monic polynomial of degree  $m$  with distinct integer zeros,  $g(x)$  a polynomial with integral coefficients and put  $P(x) = g(x)f(x) + c$ . Then every divisor of  $P(x)$  in  $\mathbb{Z}[x]$  is of the shape  $h(x)f(x) + c_1$  where  $h(x)$  is a polynomial with integral coefficients and  $c_1$  is an integer dividing  $c$ .*

**Corollary 2.1.** *Suppose the conditions of the theorem hold. Then  $P(x)$  is reducible over  $\mathbb{Q}$  if and only if  $g(x)$  can be written as*

$$g(x) = h_1(x)h_2(x)f(x) + c_2h_1(x) + c_1h_2(x)$$

*where  $h_1(x), h_2(x)$  are nonzero polynomials with integral coefficients and  $c_1, c_2$  are integers with  $c_1c_2 = c$ .*

*Proof.* It is easy to see that with the above choice of  $g(x)$  we have

$$P(x) = (h_1(x)f(x) + c_1)(h_2(x)f(x) + c_2).$$

On the other hand, if  $P(x)$  is reducible, then by Theorem 2.1 it has a factorization of the above shape with  $c_1c_2 = c$  and it follows that  $g(x)$  is as in the statement of Corollary 2.1.  $\square$

**Corollary 2.2.** (Dorwart and Ore [12], cf. Seres [39], Pirgov [29])

*Suppose the conditions of the theorem hold.*

(i) *If  $\deg(g) < m$ , then  $P(x)$  is irreducible.*

(ii) *If  $\deg(g) = m$  and  $P(x)$  is reducible over  $\mathbb{Q}$  then  $g(x) = af(x) + b$  where  $a$  and  $b$  are nonzero integers.*

*Proof.* In case (i)  $g(x)$  cannot be written in the way displayed in Corollary 2.1. In case (ii) it follows from Corollary 2.1 that both  $h_1$  and  $h_2$  are constants.  $\square$

**Remark 2.1.** If  $c = \pm 1$ , then the condition on  $m$  in Theorem 2.1 becomes  $m > 4$ . The condition is necessary, even if  $\deg(g) = m$ , as is demonstrated by the following example. Let  $m = 4$  and

$$f(x) = x(x-1)(x-2)(x-3) \quad \text{and} \quad g(x) = x^4 - 8x^3 + 20x^2 - 14x - 3.$$

Then we have

$$g(x)f(x) + 1 = (x(x-1)(x-3)^2 - 1)^2.$$

However,  $x(x-1)(x-3)^2 - 1$  is not of the form given in the theorem.

Dorwart also classified all polynomials which take the values  $\pm 1$  at more places than their degrees, see [11] p. 378.

**Remark 2.2.** In case  $c$  is a prime  $p$ , Dorwart and Ore [12], Theorem 14, obtained an absolute lower bound for  $m$ . Let  $P(x)$  take the values  $\pm p$  at  $m > 5$  integral points  $a_1, \dots, a_m$ . They proved that if  $f(x)$  is defined by (1), then  $P(x) = h(x)f(x) \pm p$  for some  $h(x) \in \mathbb{Z}[x]$ . Consequently,  $P(x)$  can have only factors of degree  $\geq m$  if  $m > 5$ . They further showed that a polynomial  $af(x) \pm p$ , where  $m \neq 3$ , is irreducible if  $m$  is odd, and when  $m$  is even it may have only two factors of the degree  $m/2$ . The exceptions for  $m = 3$  are given by

$$P(x) = (x-1)(x+1)(x+p) + p = x(x^2 + px - 1),$$

$$P(x) = 4x(x-1)(2x+p-1) + p = (2x-1)(4x^2 + 2px - 4x - p).$$

Brauer [5] and Dorwart [10] have investigated the situation more closely, see Dorwart [11] for more information. Weisner [46] studied for general nonzero integer  $c$  the polynomials of degree  $n$  which assume the value  $c$  for  $n$  distinct integral values of  $x$ .

**Remark 2.3.** Corollary 2.2 (i) can be compared with the following result of Györy and Rimán ([24], Theorem 2). If  $c \neq 0$ ,  $m \geq 2$  are integers,  $g(x) \in \mathbb{Z}[x]$  is of degree  $< m$ , and  $f(x) = (x-a_1) \cdots (x-a_m)$

with distinct integers  $a_1, \dots, a_m$  such that

$$\max_{i,j} |a_i - a_j| > \begin{cases} |c| + 1 & \text{if } m = 2, \\ |c| + 2 & \text{if } m \geq 3, \end{cases}$$

then  $P(x) = g(x)f(x) + c$  is irreducible over  $\mathbb{Q}$ . A similar result is proved in [24], Theorem 1, with a smaller lower bound when  $g(x)$  is constant.

The argument in the proof below is an extension of proofs given by Dorwart and Ore [12].

*Proof of Theorem 2.1.* Put  $S = \tau(c)$ ,  $T = 2 + \lfloor \log_2 |c| \rfloor$  and  $f(x) = (x - a_1) \cdots (x - a_m)$ . Assume that

$$P(x) = g(x)f(x) + c = G_1(x)G_2(x)$$

for nonconstant polynomials  $G_1(x), G_2(x) \in \mathbb{Z}[x]$ . (The statement is clearly true if any of  $G_1, G_2$  is constant.) For each  $i = 1, \dots, m$  we have  $G_1(a_i) | c$ .

Suppose that there exists a  $c_1$  such that  $G_1(a_i) = c_1$  for more than  $T+1$  integers  $a_i$ , say  $a_1, \dots, a_{T+2}$ . Since  $G_1(x)$  is nonconstant, it follows that  $\deg(G_1) \geq T+2$  and

$$(2) \quad G_1(x) - c_1 = (x - a_1) \cdots (x - a_{T+2})H_1(x)$$

for some polynomial  $H_1(x)$  with integral coefficients. If now  $G_1(a_i) = c_2 \neq c_1$  for some  $i > T+2$ , then  $G_1(x) - c_2 = (x - a_i)I_1(x)$  for some  $I_1(x)$  with integral coefficients, and hence, by (2),

$$(3) \quad (a_i - a_1) \cdots (a_i - a_{T+2}) \mid c_2 - c_1.$$

At most two factors on the left-hand side are  $\pm 1$ . Each other factor contributes at least one prime factor to the product. Hence the number of prime factors of  $c_2 - c_1$  counted according to multiplicities is at least  $T$  which contradicts that  $|c_2 - c_1| \leq 2|c| < 2^T$ . Thus  $G_1(a_i) = c_1$  for each  $i = 1, \dots, m$ , i.e.

$$(4) \quad G_1(x) = h_1(x)f(x) + c_1$$

for some  $h_1(x) \in \mathbb{Z}[x]$ . Further  $G_2(a_i) = c_2$  with  $c_2 = c/c_1$  for each  $i = 1, \dots, m$  whence

$$(5) \quad G_2(x) = h_2(x)f(x) + c_2$$

where  $h_2(x) \in \mathbb{Z}[x]$ .

Suppose next that for every divisor  $c_1$  of  $c$  the number of  $a_i$  with  $G_1(a_i) = c_1$  is at most  $T+1$ . Then  $m \leq 2S(T+1)$ . Since the total number of divisors of  $c$  is  $2S$  and  $m > 2ST$ , there exists some  $c_1$  such

that  $G_1(a_i) = c_1$  for exactly  $T + 1$  integers  $a_i$ , say  $a_1, \dots, a_{T+1}$ . We distinguish between the cases  $S = 1$  and  $S > 1$ .

If  $S = 1$ , then  $c_1 \in \{-1, 1\}$ ,  $T = 2$  and  $m$  is 5 or 6. We get that the only possibility is that  $G_1(a_i) = c_1$  for  $i = 1, 2, 3$  say, and  $G_1(a_i) = -c_1$  for  $i = 4, 5$  (and also for  $i = 6$  if  $m = 6$ ). Then by a similar argument as before we deduce that

$$(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \mid 2 \quad \text{and} \quad (a_5 - a_1)(a_5 - a_2)(a_5 - a_3) \mid 2.$$

The first relation gives that two out of the numbers  $|a_4 - a_1|$ ,  $|a_4 - a_2|$  and  $|a_4 - a_3|$  equal 1 and the third one is 2, while the second one yields the same conclusion for the numbers  $|a_5 - a_1|$ ,  $|a_5 - a_2|$ ,  $|a_5 - a_3|$ . However, since  $a_i \neq a_j$  for  $i \neq j$ , this is impossible. Hence we get that (4) is valid anyhow, and the same is true for (5).

If  $S > 1$  then  $m > ST + T + 1$ . Hence either there exists some  $c_2$  such that  $c_1 c_2 > 0$  and  $G_1(a_i) = c_2$  for some  $i > T + 1$  or there exist a  $c_1^*$  and  $c_2^* \neq c_1^*$  with  $c_1 c_1^* < 0$  and  $c_1^* c_2^* > 0$  such that  $G_1(a_i) = c_1^*$  for exactly  $T + 1$  integers  $i$  and  $G_1(a_i) = c_2^*$  for another  $i$ . In the latter case, after renaming, we also have  $T + 1$  integers  $a_1, \dots, a_{T+1}$  such that  $G_1(a_j) = c_1$  for  $j = 1, \dots, T + 1$  and integers  $c_2$  and  $i > T + 1$  with  $c_1 c_2 > 0$  such that  $G_1(a_i) = c_2$ . Reasoning as for (3) we derive

$$(a_i - a_1) \cdots (a_i - a_{T+1}) \mid c_2 - c_1.$$

Using that  $|c_2 - c_1| < 2^{T-1}$  we obtain (4) and (5) in this case too.  $\square$

**Remark 2.4.** The condition on  $m$  in Theorem 2.1 and Corollary 2.1 can be improved upon when  $c$  is large. In the first place in (3) the left-hand side can be bounded from below by  $(\lfloor \frac{T+2}{2} \rfloor)! (\lfloor \frac{T+3}{2} \rfloor)!$  by using that at most two factors are  $\pm 2$ , at most two are  $\pm 3$ , and so on. The result is an improvement of order  $\log \log |c|$  for large  $|c|$ . Another improvement is obtained by replacing the upper bound  $2|c|$  in the above proof by  $|c| + 1$ : If  $|c_1| = |c|$ ,  $|c_2| > 1$  or  $|c_2| = |c|$ ,  $|c_1| > 1$ , then we consider the corresponding expression for  $G_2$ . We find instead of (3) that

$$(a_i - a_1) \cdots (a_i - a_{T+2}) \mid \left( \frac{c}{c_1} - \frac{c}{c_2} \right)$$

and use that  $|\frac{c}{c_1} - \frac{c}{c_2}| \leq |c| + 1$ . Otherwise either  $|c_1| = |c|$ ,  $|c_2| = 1$  or  $|c_2| = |c|$ ,  $|c_1| = 1$  or  $|c_1| \leq \frac{|c|}{2}$ ,  $|c_2| \leq \frac{|c|}{2}$ , and in each case  $|c_1 - c_2| \leq |c| + 1$ .

In the following variant of Theorem 2.1 we only require that  $P(a_i)|c$  for  $i = 1, \dots, m$ .

**Theorem 2.2.** *Let  $c$  and  $m$  be positive integers with*

$$(6) \quad m > 2\tau(c)(3 + \lfloor \log_2 |c| \rfloor).$$

*Let  $P(x) \in \mathbb{Z}[x]$  such that there exist integers  $a_1, \dots, a_m$  for which  $P(a_i)$  divides  $c$  for  $i = 1, \dots, m$ . Then every divisor of  $P(x)$  is of the shape  $h(x)f(x) + c_1$  where  $f(x)$  is given by (1),  $h(x) \in \mathbb{Z}[x]$  and  $c_1$  is an integer dividing  $c$ .*

As in Remark 2.4 the lower bound on  $m$  can be improved if  $|c|$  is large.

**Remark 2.5.** In case  $c$  is a prime  $p$ , Dorwart and Ore [12] proved Theorem 2.2 with  $m > 10$  instead of (6). It follows that a polynomial from  $\mathbb{Z}[x]$  taking the values dividing  $p$  at more than 10 integer points cannot have factors of degree less than  $m/2$ .

The method was used by Ore [28] to show that a polynomial  $P(x) \in \mathbb{Z}[x]$  of degree  $m$  taking values dividing a prime at  $m + 5$  integer points is irreducible. The bound  $m + 5$  is best possible in view of the example  $P(x) = ((x - 1)(x - 2) - 1)((x - 5)(x - 6) - 1)$  taking prime values or their opposites for  $x = 0, 1, 2, 3, 4, 5, 6, 7$ .

*Proof of Theorem 2.2.* As in the proof of Theorem 2.1 put  $S = \tau(c)$ ,  $T = 2 + \lfloor \log_2 |c| \rfloor$  and  $f(x) = (x - a_1) \cdots (x - a_m)$ . Assume that

$$P(x) = G_1(x)G_2(x)$$

for nonconstant polynomials  $G_1(x), G_2(x) \in \mathbb{Z}[x]$ . (If any of  $G_1, G_2$  is constant then the statement is trivial.) For each  $i = 1, \dots, m$  we have  $G_1(a_i) | c$ .

Using the box principle we know that there exists a  $c_1$  such that  $G_1(a_i) = c_1$  for more than  $T + 1$  integers  $a_i$ . Following the proof of Theorem 2.1 we conclude that

$$(7) \quad G_1(x) = h(x)f(x) + c_1$$

for some  $h(x) \in \mathbb{Z}[x]$ . Since  $G_1(x)$  is an arbitrary divisor of  $P(x)$ , the conclusion follows.  $\square$

### 3. POLYNOMIALS WITH IMAGINARY QUADRATIC ZEROS

In the formulation of Theorems 2.1 and 2.2 the condition that the zeros of  $f$  and the coefficients of  $g$  are in  $\mathbb{Z}$  can be replaced by the condition that they are in the ring of integers of  $\mathbb{Q}(\sqrt{-d})$  where  $d$  is some positive integer, not a square. However, in this case the lower bound on  $m$  will depend on  $c$  and  $d$ . We do not work this out as it is straightforward.



In this section we investigate the irreducibility of polynomials of the shape

$$(8) \quad (x - a_1) \cdots (x - a_m) g_1(x) \cdots g_t(x) \pm 1$$

where  $m, t \geq 0$ , the  $a_i$ -s are distinct integers and the distinct monic polynomials  $g_i \in \mathbb{Z}[x]$  are of degree two and have negative discriminants. Under the assumption that all the zeros of the polynomials  $g_i$  ( $i = 1, \dots, t$ ) belong to the same quadratic number field, Dorwart and Ore [12] have described all reducible polynomials of the form (8). Getting rid of the assumption, our next theorem yields a complete characterization of the reducible polynomials of the form (8). As a motivation of our work, we mention that the irreducibility of polynomials

$$(x - a_1) \cdots (x - a_m) \pm 1$$

is closely related to some important class of totally real number fields, the so-called ABC-fields; see e.g. [1], [40], [2] and the references given there. Number fields defined by polynomials of the form (8) are generalizations of such ABC-fields of arbitrary signature; see [40] again.

**Theorem 3.1.** *Put*

$$P_{\pm}(x) = (x - a_1) \cdots (x - a_m) g_1(x) \cdots g_t(x) \pm 1$$

and

$$f(x) = (x - a_1) \cdots (x - a_m) g_1(x) \cdots g_t(x),$$

where  $m + t > 0$ , the  $a_i$ -s are distinct integers and the  $g_i \in \mathbb{Z}[x]$  are distinct monic quadratic polynomials with negative discriminants. Then  $P_{\pm}(x)$  is irreducible over  $\mathbb{Q}$  except for the following cases:

- $P_+(x)$  is reducible if and only if either  $f(x)$  is equivalent to one of the polynomials

$$\begin{aligned} & [1, 5], [1, 8], [2, 8], [1, 3, 5], [1, 3, 7], [1, 3, 8], [1, 5, 7], [1, 5, 8], [1, 5, 9], [1, 7, 8], \\ & [2, 3, 10], [2, 6, 8], [1, 2, 4, 5], [1, 2, 5, 10], [1, 2, 6, 9], [1, 3, 5, 7], [1, 3, 5, 8], \\ & [1, 3, 7, 8], [1, 5, 6, 7], [1, 5, 7, 8], [2, 3, 5, 9], [2, 3, 6, 10], [2, 5, 8, 10], [5, 6, 9, 10], \\ & [1, 2, 4, 5, 9], [1, 2, 7, 9, 11], [1, 3, 5, 7, 8], [1, 2, 5, 6, 7, 9], [1, 2, 5, 7, 8, 9], \\ & [2, 3, 5, 6, 8, 10], [2, 3, 5, 7, 8, 10], \end{aligned}$$

or  $f(x)$  is of the form

$$p(x)(p(x) + 2) \text{ or } p(x)(p(x) + 1)(p(x) + 2)(p(x) + 3)$$

where  $p(x)$  is an arbitrary monic linear polynomial or a monic quadratic polynomial with negative discriminant.

•  $P_-(x)$  is reducible if and only if  $f(x)$  is equivalent to one of the polynomials

[5], [8], [3, 5], [3, 7], [3, 8], [4, 5], [5, 7], [5, 8], [7, 8], [2, 6, 7], [3, 5, 7], [3, 5, 8], [3, 5, 10], [3, 7, 8], [4, 5, 9], [5, 7, 8], [6, 7, 8], [2, 5, 7, 9], [3, 5, 6, 8], [3, 5, 7, 8].

Here a tuple  $[i_1, \dots, i_l]$  denotes the product of the corresponding polynomials from Table 1.

No.	polynomial	No.	polynomial	No.	polynomial
1	$x - 1$	5	$x^2 + 1$	9	$x^2 - x + 2$
2	$x$	6	$x^2 + 2$	10	$x^2 + x + 2$
3	$x + 1$	7	$x^2 - x + 1$	11	$x^2 - x + 3$
4	$x^2 - 2x + 2$	8	$x^2 + x + 1$		

TABLE 1. Factors of exceptional polynomials.

*Proof of Theorem 3.1.* Suppose that we have  $P_{\pm}(x) = G_1(x)G_2(x)$  with some monic polynomials  $G_1, G_2 \in \mathbb{Z}[x]$ . Then we have  $G_1(\beta_j)G_2(\beta_j) = \pm 1$  for all  $j = 1, \dots, t$ , where  $\beta_j$  is a zero of  $g_j$ . Thus, using that the  $\beta_j$ -s are quadratic imaginary algebraic integers, we deduce that

$$(9) \quad G_1(\beta_j) \in U := \{\pm 1, \pm i, \pm \varepsilon, \pm(1 - \varepsilon)\},$$

where  $\varepsilon = (1 + i\sqrt{3})/2$ . Certainly, the same holds for  $G_1(\overline{\beta_j})$ , while  $G_1(a_j)$  ( $j = 1, \dots, m$ ) may assume the values  $\pm 1$  only.

We split the proof into several parts, in accordance with (9).

**Case I.** Suppose that  $G_1(\beta_j) = \pm i$  for some  $j$ . Then  $G_1(\beta_j) \in \mathbb{Q}(\beta_j)$  yields that  $\beta_j \in \mathbb{Q}(i)$ . Since  $G_1(\beta_j) = \pm i$ , we get that

$$(10) \quad G_1(x) \mp i = (x - \beta_j)h_1(x)$$

holds with some  $h_1 \in \mathbb{Z}[i][x]$ . Taking complex conjugates we obtain that

$$G_1(x) \pm i = (x - \overline{\beta_j})h_2(x)$$

is also valid with the appropriate  $h_2 \in \mathbb{Z}[i][x]$ . The last two equalities give

$$(x - \beta_j)h_1(x) - (x - \overline{\beta_j})h_2(x) = \mp 2i,$$

whence

$$(11) \quad \beta_j - \overline{\beta_j} \mid 2 \quad \text{in the ring } \mathbb{Z}[i].$$

Write  $\beta_j = u + vi$  with some integers  $u, v$  with  $v \neq 0$ . Then  $\overline{\beta_j} = u - vi$ , which by (11) implies that  $v = \pm 1$  (and further that  $\beta_j - \overline{\beta_j} = \pm 2i$ ). Observe that this also implies  $g_j(x + u) = x^2 + 1$ . From this point on

we shall always assume that  $\beta_j = u + i$ , which we can do without loss of generality.

Now we look at all the possible other factors of  $f(x)$  in turn.

Assume first that  $f(x)$  has a linear factor  $x - a_r$ . Then by  $G_1(a_r) = \pm 1$  we have  $G_1(x) \mp 1 = (x - a_r)h_3(x)$  with some  $h_3 \in \mathbb{Z}[x]$ . Hence we deduce that  $\beta_j - a_r$  divides  $1 + i$  in  $\mathbb{Z}[i]$ , whence one of  $a_r = u - 1, u, u + 1$  must be valid. This clearly yields that

$$x + u - a_r \in \{x - 1, x, x + 1\}.$$

Assume next that for some  $\beta_r$  with  $r \neq j$  we have  $G_1(\beta_r) = \pm i$ . Then by the previous argument we already know that  $\beta_r = w \pm i$  must be valid for some  $w \in \mathbb{Z}$ . Further, we also get that either

$$(x - \beta_j)(x - \beta_r)h_4(x) - (x - \overline{\beta_j})(x - \overline{\beta_r})h_5(x) = \mp 2i$$

or

$$(x - \beta_j)(x - \overline{\beta_r})h_4(x) - (x - \overline{\beta_j})(x - \beta_r)h_5(x) = \mp 2i$$

with some  $h_4, h_5 \in \mathbb{Z}[i][x]$ . Since  $\beta_j - \overline{\beta_j} = \pm 2i$ , and without loss of generality we may assume that  $\Re(\beta_r) \geq u$ , this implies that  $w = u + 1$ . Hence we can write  $g_r(x + u) = x^2 - 2x + 2$ .

Suppose now that  $G_1(\beta_r) \in \{\pm \varepsilon, \pm(1 - \varepsilon)\}$  for some  $r$ . Then we have  $\beta_r \in \mathbb{Q}(\varepsilon)$ , and further,

$$(12) \quad G_1(x) + u_0 = (x - \beta_r)h_6(x)$$

holds with some  $u_0 \in \{\mp \varepsilon, \mp(1 - \varepsilon)\}$  and  $h_6 \in \mathbb{Z}[\varepsilon][x]$ . Taking conjugates we get that

$$(13) \quad G_1(x) + \overline{u_0} = (x - \overline{\beta_r})h_7(x)$$

is also valid with some  $h_7 \in \mathbb{Z}[\varepsilon][x]$ . Using the last two equalities we obtain

$$(x - \beta_r)h_6(x) - (x - \overline{\beta_r})h_7(x) = \pm(1 - 2\varepsilon),$$

implying  $\beta_r - \overline{\beta_r} \mid 1 - 2\varepsilon$  in  $\mathbb{Z}[\varepsilon]$ . Let  $\beta_r = w + z\varepsilon$  with  $w, z \in \mathbb{Z}$ ,  $z \neq 0$ . Then as  $\overline{\beta_r} = w + z - z\varepsilon$ , we get  $\overline{\beta_r} = w + z - z\varepsilon$ , whence  $z = \pm 1$ . Obviously, without loss of generality we may assume that  $\beta_r = w + \varepsilon$ . Now a similar argument as before yields that  $u - w + i - \varepsilon$  divides an element of the following set

$$H := \{\pm i \pm \varepsilon, \pm i \pm (1 - \varepsilon)\},$$

in the ring of integers of the number field  $L := \mathbb{Q}(i, \varepsilon)$ . A simple calculation shows that all elements of  $H$  are units in  $L$ . Hence  $u - w + i - \varepsilon$  should be a unit of this field, which after taking norm, turns out to be possible only if  $w = u$  or  $w = u - 1$ . Hence we get that either  $g_r(x + u) = x^2 - x + 1$ , or  $g_r(x + u) = x^2 + x + 1$  must hold.

Finally, let  $g_r(x)$  be a polynomial with  $G_1(\beta_r) = \pm 1$ . Observe that then we also have  $G_1(\overline{\beta_r}) = \pm 1$ , which implies that  $g_r(x)$  divides  $G_1(x) \mp 1$  in  $\mathbb{Z}[x]$ . Then using our previous arguments, we get that  $g_r(u+i) \in \{\pm 1, \pm i, \pm 1 \pm i\}$ . Hence a simple calculation gives that  $g_r(x+u)$  is one of

$$x^2 + 2, x^2 + x + 1, x^2 - x + 1, x^2 + x + 2, x^2 - x + 2.$$

Summarizing the above facts, we conclude that if  $G_1(\beta_j) = \pm i$  is valid for some  $j$ , then there exists an integer  $u$  such  $f(x+u)$  should have factors exclusively from the following set:

$$\{x-1, x, x+1, x^2+1, x^2-2x+2, x^2-x+1, \\ x^2+x+1, x^2+2, x^2+x+2, x^2-x+2\}.$$

Considering now all subsets of the above set and checking the irreducibility of the implied polynomials  $P_{\pm}(x)$ , we obtain that all the reducible cases are included in the statement. From each equivalence class we selected one representative. For example,  $[1, 2, 9]$  is not mentioned, since it is equivalent to  $[2, 3, 10]$ .

**Case II.** Suppose that  $G_1(\beta_j) \in \{\pm \varepsilon, \pm(1-\varepsilon)\}$  for some  $g_j$ . As we have already seen in Case I, we may assume that  $\beta_j = u + \varepsilon$  with some integer  $u$ . This yields  $g_j(x+u) = x^2 - x + 1$ .

As before, we look at all the possible other factors of  $f(x)$  in turn. In view of Case I, without loss of generality we may clearly assume that there is no  $\beta_r$  with  $G_1(\beta_r) = \pm i$ .

Assume first that  $f(x)$  has a linear factor  $x - a_r$ . Then using again  $G_1(a_r) = \pm 1$ , we get that  $G_1(x) \mp 1 = (x - a_r)h_8(x)$  with some  $h_8 \in \mathbb{Z}[x]$ . Similarly as above, we can deduce that  $\beta_j - a_r$  divides one of  $\pm \varepsilon, \pm 1 \pm \varepsilon, \pm(2 - \varepsilon)$  in  $\mathbb{Z}[\varepsilon]$ . Hence we obtain that one of  $a_r = u - 1, u, u + 1, u + 2$  must be valid. This clearly yields that

$$x + u - a_r \in \{x - 2, x - 1, x, x + 1\}.$$

Assume next that for some  $\beta_r$  with  $r \neq j$  we also have  $G_1(\beta_r) \in \{\pm \varepsilon, \pm(1-\varepsilon)\}$ . We already know that  $\beta_r = w \pm \varepsilon$  must be valid with some  $w \in \mathbb{Z}$ . Without loss of generality we may assume that  $w \geq u$ . Further, we also have that

$$G_1(x) + v_0 = (x - \beta_r)h_9(x)$$

and

$$G_1(x) + \overline{v_0} = (x - \overline{\beta_r})h_{10}(x)$$

hold with some  $v_0 \in \{\mp\varepsilon, \mp(1 - \varepsilon)\}$  and  $h_9, h_{10} \in \mathbb{Z}[\varepsilon][x]$ . Using (12) and (13) we get that

$$\beta_j - \beta_r | v_0 - u_0 \quad \text{and} \quad \beta_j - \overline{\beta_r} | \overline{v_0} - u_0 \quad \text{in } \mathbb{Z}[\varepsilon].$$

Checking all the possibilities one can easily verify that  $w = u + 1$  must be valid. Then we clearly have  $g_r(x + u) = x^2 - 3x + 3$ .

Finally, suppose that  $G_1(\beta_r) = \pm 1$  holds for some  $r$ . Then we also have  $G_1(\overline{\beta_r}) = \pm 1$ , i.e.  $g_r(x)$  divides  $G_1(x) \mp 1$  in  $\mathbb{Z}[x]$ . Applying our previous argument, we get that  $g_r(u + \varepsilon)$  divides one of  $\pm\varepsilon, \pm 1 \pm \varepsilon, \pm(2 - \varepsilon)$  in  $\mathbb{Z}[\varepsilon]$ . Hence a simple calculation gives that  $g_r(x + u)$  is one of

$$x^2 + 1, x^2 + 2, x^2 - x + 2, x^2 - 2x + 2, x^2 - 2x + 3.$$

Gathering all the above information, we get that in this case there is an integer  $u$  such that  $f(x + u)$  can have factors exclusively from the following set:

$$\{x - 2, x - 1, x, x + 1, x^2 - x + 1, x^2 - 3x + 3, \\ x^2 + 1, x^2 + 2, x^2 - x + 2, x^2 - 2x + 2, x^2 - 2x + 3\}.$$

Now by a similar process as for Case I we get that also in this Case II all the reducible polynomials of the form  $P_{\pm}(x)$  are listed in the statement.

**Case III.** Suppose that  $G_1(\beta_j) = \pm 1$  for each index  $j$ . Then  $g_j(x) | G_1(x) \mp 1$  for all  $j = 1, \dots, t$ . Note that since  $G_1(a_j) = \pm 1$  is also valid for all  $j = 1, \dots, m$  and certainly  $\deg(G_1(x) - 1) = \deg(G_1(x)) = \deg(G_1(x) + 1)$ ,  $2 \deg(G_1(x)) = \deg(P_{\pm}(x))$  must be valid. We distinguish two subcases.

i) Assume first that there exists a  $\beta_j$  of the form  $\beta_j = u + \alpha$  with  $u \in \mathbb{Z}$  and

$$(14) \quad \alpha \in \{i, \varepsilon, i\sqrt{2}, (1 + i\sqrt{7})/2\}.$$

In these cases we have that  $g_j(x + u)$  is given by

$$x^2 + 1, x^2 - x + 1, x^2 + 2, x^2 - x + 2,$$

respectively. Write  $G_1(\beta_j) = u_0$  with  $u_0 \in \{-1, 1\}$ . Then by similar arguments as before, using that  $G_1(x) - u_0 - (G_1(x) + u_0) = \pm 2$ , we get that for the above values of  $\alpha$  the only possible factors of  $f(x + u)$  dividing  $G_1(x + u) + u_0$  are given by

$$x - 1, x, x + 1, x^2 + 2, x^2 + 3, x^2 - x + 1, x^2 + x + 1, x^2 + x + 2, x^2 - x + 2 \quad (\alpha = i), \\ x - 1, x, x^2 - x + 2, x^2 - x + 3, x^2 + 1, x^2 - 2x + 2, x^2 + x + 1, x^2 - 3x + 3 \quad (\alpha = \varepsilon), \\ x, x^2 + 1, x^2 + 3, x^2 + 4, x^2 - x + 2, x^2 + x + 2 \quad (\alpha = i\sqrt{2}),$$

$$x - 1, x, x^2 - x + 1, x^2 - x + 3, x^2 - x + 4, x^2 + 1, x^2 + 2, \\ x^2 - 2x + 2, x^2 - 2x + 3 \ (\alpha = (1 + i\sqrt{7})/2),$$

respectively. We handle these cases in turn. We only explain our method for  $\alpha = i$ , the other cases are similar. We take a subset of the possible factors of  $G_1(x+u) + u_0$ . For example, choose  $\{x^2 - x + 1, x^2 - x + 2\}$ . Then in this hypothetical case we have  $(x^2 - x + 1)(x^2 - x + 2) \mid G_1(x+u) + u_0$ . However, these factors immediately restrict the possible factors of  $G_1(x+u) - u_0$ . Namely, in our concrete case from the above lists we get that the only possible factors of  $f(x+u)$  dividing  $G_1(x+u) - u_0$  are

$$x - 1, x, x^2 + 1, x^2 - x + 3, x^2 - 2x + 2.$$

Hence we obtain a finite (in fact rather small) set, such that all possible factors of  $f(x+u)$  belong to it. In other cases we have to produce similar lists and to compare them. Checking all possibilities, a computer calculation shows that all the cases with  $P_{\pm}(x)$  reducible are given in the statement.

ii) Finally, we are left with the case where there is no  $\beta_j$  of the form  $u + \alpha$  with  $\alpha$  satisfying (14). In this case a simple calculation shows that if  $g_j(x) \mid G_1(x) - u_0$  then there is no linear polynomial dividing  $G_1(x) + u_0$ . Let now  $g_r(x) \mid G_1(x) + u_0$  for some  $r \neq j$ . We recall the well-known fact (which can also be readily checked) that if 2 has a divisor different from  $\pm 1, \pm 2$  in the ring of integers of an imaginary quadratic number field  $K$  then we have  $K = \mathbb{Q}(\alpha)$  with  $\alpha$  satisfying (14). Hence a simple calculation yields that now  $g_r(\beta_j) \in \{-2, -1, 1, 2\}$  must be valid. However, since  $g_j(\beta_j) = 0$ , this implies that  $g_j(x) - g_r(x) \in \{-2, -1, 1, 2\}$ . Hence using that  $2 \deg(G_1(x)) = \deg(P_{\pm}(x))$ , fixing any  $p(x) := g_j(x)$ , all the possible factors of  $f(x)$  can be listed, in terms of  $p(x)$ . Hence the statement follows by a simple calculation.

**Case IV.** Finally, assume that  $t = 0$ , i.e.  $f(x)$  has only linear factors. This case has been completely treated by Flügel [16], however, for the sake of completeness we include this possibility as well. Let  $x - a_j \mid g_1(x) - u_0$  with  $u_0 = \pm 1$ . Then for any  $r \neq j$  we have that  $x - a_r \mid g_1(x) + u_0$ , which implies  $a_j - a_r \in \{-2, -1, 1, 2\}$ . Hence the statement easily follows in this case, too.  $\square$

## PART 2. PÓLYA-TYPE RESULTS

### 4. THE RATIONAL CASE. RESULTS OF LEVIT.

As was mentioned in the introduction, Pólya [30] published an irreducibility result for integral polynomials  $P(x)$  having small nonzero values in absolute value at many distinct integers. His result was based on a lemma proved by using interpolation theory (cf. the proof of Lemma 5.1). The lemma has been sharpened by several authors, see Tatzuza [42], Brauer and Ehrlich [8], and Levit [27]. We note that Tverberg [43], [44] has given asymptotic results which are asymptotically better than Proposition 4.1 below. We write  $(a)_k$  for  $a(a+1)\cdots(a+k-1)$ .

**Proposition 4.1.** (Levit [27], Theorem 1) *Let  $Q(x)$  be a monic polynomial of degree  $k$  with real coefficients, and let  $a_1 < \cdots < a_m$  be integers. If  $m > k > 0$  then, for some  $i \in \{1, \dots, m\}$ ,*

$$|Q(a_i)| \geq 2^{1-k} ((m-k)/2)_k.$$

The original lower bound of Pólya was  $2^{-k}k!$ . Pólya's argument combined with Proposition 4.1 leads immediately to the following result.

**Proposition 4.2.** *Let  $P(x) \in \mathbb{Z}[x]$  be a polynomial of degree  $m > 1$ . Let  $0 < k < m$ . Suppose there are  $k+1$  distinct integers  $a$  such that*

$$0 < |P(a)| < 2^{1-k} ((m-k)/2)_k.$$

*Then  $P(x)$  has no factor of degree  $k$  over  $\mathbb{Q}$ .*

*Proof.* If  $P(x)$  has a factor  $Q(x) \in \mathbb{Z}[x]$  of degree  $k$ , then  $0 < |Q(a)| < 2^{1-k} ((m-k)/2)_k$  for  $k+1$  distinct integers  $a$ , in contradiction with Proposition 4.1.  $\square$

Put  $N := \lceil m/2 \rceil$ . A straightforward extension of Levit's argument yields the following result in which the upper bound for  $|P(a)|$  is independent of  $k$  in contrast with Proposition 4.2.

**Theorem 4.1.** *Let  $P(x) \in \mathbb{Z}[x]$  be a polynomial of degree  $m \geq 8$ . Let  $N \leq l < m$  and  $m-l \leq k \leq l$ . Suppose there are  $l+1$  distinct integers  $a$  such that*

$$0 < |P(a)| < 2^{1-N} ((m-N)/2)_N.$$

*Then  $P(x)$  has no factor of degree  $k$  over  $\mathbb{Q}$ .*

*Proof.* If  $P(x)$  has a factor  $Q(x) \in \mathbb{Z}[x]$  of degree  $k$ , then it has a factor of degree  $m-k$ . In view of Proposition 4.2 it therefore suffices to prove that

$$2^{1-N} ((m-N)/2)_N \leq 2^{1-k} ((m-k)/2)_k$$

for  $N \leq k \leq l$ . Let  $k - N$  be even. Then it suffices to prove that

$$4^{k-N} \leq (m-k)(m-k+2) \cdots (m-N-2)(m+N)(m+N+2) \cdots (m+k-2).$$

This inequality is valid if  $(m-k)(m+k-2) \geq 16$ , thus for  $m \geq 10$ .

If  $k - N$  is odd, then it suffices to prove that

$$4^{k-N} \leq (m-k)(m-k+2) \cdots (m-N-1)(m+N+1)(m+N+3) \cdots (m+k-2).$$

This is satisfied if both  $(m-k)(m+k-2) \geq 16$  and  $m - N - 1 \geq 4$ , so if  $m \geq 10$ . The remaining cases  $(m, k)$  can be checked one by one.  $\square$

Levit, [27] Theorem 3, obtained a similar result in case  $l = m - 2$ . Besides, his Theorem 4 (and its proof) says that, for  $m - N \leq l < m - 2$ , if there are  $l + 1$  distinct integers  $a$  such that  $0 < |P(a)| < \lfloor (l^2 + 4)/8 \rfloor$ , then  $P(x)$  has no factor of degree  $k$  with  $2 \leq k \leq m - 2$ , and that this upper bound for  $|P(a)|$  cannot be improved upon.

By applying Theorem 4.1 for  $l = m - 1$  we obtain the following irreducibility result due to Levit.

**Corollary 4.1.** ([27] Theorem 2) *Let  $P(x) \in \mathbb{Z}[x]$  be a polynomial of degree  $m$ . If there are  $m$  distinct integers  $a$  such that*

$$0 < |P(a)| < 2^{1-N} ((m - N)/2)_N,$$

*then  $P(x)$  is irreducible over  $\mathbb{Q}$ .*

## 5. THE IMAGINARY QUADRATIC CASE.

We extend the results to imaginary quadratic fields. In what follows, we write, for positive integer  $m$ ,

$$T_m = \prod_{i=5}^m (\sqrt{i} - 1), \quad T_m^* = 2^{\min(9-m, 0)} \prod_{i=10}^m (\sqrt{i} - 1).$$

Here we define the empty product to be 1.

**Theorem 5.1.** *Let  $K = \mathbb{Q}(\sqrt{-d})$  where  $d$  is a positive square-free integer, and write  $O_K$  for the ring of integers of  $K$ . Let again  $N := \lceil m/2 \rceil$  and  $0 < m - l \leq k \leq l$ . Let  $P(x) \in O_K[x]$  of degree  $m$ . Suppose there are  $l + 1$  distinct integers  $a \in O_K$  such that*

$$0 < |P(a)| < (N + 1)^{-1} T_{N+1}^*.$$

*Then  $P(x)$  cannot have a factor of degree  $k$  in  $O_K[x]$ .*

**Remark 5.1.** Note that for  $m < 27$  we have

$$(N + 1)^{-1} T_{N+1}^* \leq 1.$$

Hence in this case the statement of Theorem 5.1 is empty.



As an immediate consequence of the above theorem we obtain the following statement.

**Corollary 5.1.** *Using the notation of Theorem 5.1, assume that*

$$0 < |P(a)| < (N + 1)^{-1} T_{N+1}^*$$

*holds for some  $m$  integers  $a \in O_K$ . Then  $P(x)$  is irreducible in  $O_K[x]$ .*

To prove Theorem 5.1 we need the following lemma similar to Proposition 4.1.

**Lemma 5.1.** *Using the notation of Theorem 5.1, if  $P(x) \in O_K[x]$  is of degree  $m$  and  $a_0, a_1, \dots, a_m$  are elements of  $O_K$ , then for some  $t \in \{0, 1, \dots, m\}$  we have  $|P(a_t)| \geq (m + 1)^{-1} T_{m+1}^*$ .*

The proof of this lemma is based on the following assertion which will also be used later on.

**Lemma 5.2.** *Let  $\alpha_1, \dots, \alpha_m$  ( $m \geq 2$ ) be complex numbers such that  $|\alpha_i - \alpha_j| \geq \delta$  for all  $1 \leq i < j \leq m$  with some  $\delta > 0$ . Suppose that  $z \in \mathbb{C}$  such that  $|z - \alpha_1| \leq |z - \alpha_i|$  for all  $i = 2, \dots, m$ . Then we have*

$$\prod_{i=2}^m |z - \alpha_i| \geq \left(\frac{\delta}{2}\right)^{m-1} T_m,$$

*where the right-hand side can be replaced by  $\delta^{m-1} T_m^*$  if  $z = \alpha_1$ .*

*Proof.* Let  $\alpha_1 = \gamma_1, \gamma_2, \dots, \gamma_m$  be a rearrangement of  $\alpha_1, \alpha_2, \dots, \alpha_m$  such that

$$|z - \gamma_1| \leq |z - \gamma_2| \leq \dots \leq |z - \gamma_m|$$

and let  $d_i = |z - \gamma_i|$  for  $i = 2, \dots, m$ . By the definition of  $\delta$ , the open discs with centers  $\alpha_i$  ( $i = 1, \dots, m$ ) of radius  $\delta/2$  are pairwise disjoint. Thus for  $i = 2, \dots, m$  we have

$$i \cdot \pi \left(\frac{\delta}{2}\right)^2 \leq \pi \left(d_i + \frac{\delta}{2}\right)^2.$$

Hence we immediately obtain that

$$(15) \quad d_i \geq \frac{\delta}{2} \left(\sqrt{i} - 1\right) \quad \text{for } i = 2, \dots, m.$$

Further, for every  $i > 1$  we clearly have  $d_i \geq \frac{\delta}{2}$ , and even  $d_i \geq \delta$  if  $z = \alpha_1$ . This yields

$$\prod_{i=2}^m |z - \alpha_i| \geq \left(\frac{\delta}{2}\right)^{m-1} T_m$$

where the right-hand side can be replaced by  $\delta^{m-1} T_N^*$  if  $z = \alpha_1$ .  $\square$

*Proof of Lemma 5.1.* By the interpolation formula of Lagrange we have

$$P(x) = \sum_{i=0}^m P(a_i) \prod_{\substack{j=0 \\ j \neq i}}^m \frac{x - a_j}{a_i - a_j}.$$

Since the absolute value of the leading coefficient of  $P(x)$  is at least 1, we get

$$\left| \sum_{i=0}^m P(a_i) \prod_{\substack{j=0 \\ j \neq i}}^m \frac{1}{a_i - a_j} \right| \geq 1.$$

Let  $t$  be an index such that

$$|P(a_t)| = \max_{j=0,1,\dots,m} |P(a_j)|.$$

Then we have

$$|P(a_t)| \geq \left( \sum_{i=0}^m \prod_{\substack{j=0 \\ j \neq i}}^m \frac{1}{|a_i - a_j|} \right)^{-1}.$$

Using that  $|a_i - a_j| \geq 1$  for all  $i \neq j$  and taking  $z = a_j$  in Lemma 5.2, we get

$$\prod_{\substack{j=0 \\ j \neq i}}^m |a_i - a_j| \geq T_{m+1}^*.$$

Hence the statement follows by a simple calculation.  $\square$

*Proof of Theorem 5.1.* Suppose that  $P(x)$  has a factor in  $O_K[x]$  of degree  $k$  with  $m - l \leq k \leq l$ . Then it has a factor  $Q(x)$  of degree  $k$  with  $N \leq k \leq l$ . Since  $Q(a) \mid P(a)$  in  $O_K$ , there are  $l + 1$  integers  $a \in O_K$  such that

$$0 < |Q(a)| \leq |P(a)| < (N + 1)^{-1} T_{N+1}^*.$$

One can easily check that  $m^{-1} T_m^*$  is a monotone increasing function of  $m$  for  $m \geq 10$ . Hence, as  $m \geq 27$  and  $Q(a) \neq 0$ , we get that

$$0 < |Q(a)| < (l + 1)^{-1} T_{l+1}^*$$

is valid for  $l + 1$  distinct integers  $a$ . However, this contradicts Lemma 5.1, and the statement follows.  $\square$

6. POLYNOMIALS OF THE FORM  $P(x) = f(x)g(x) + c$ 

In this section we use some lemmas from Section 5 to derive some results of the type considered in Part 1. These results depend on the minimal distance between the zeros of a polynomial  $P(x) \in \mathbb{Z}[x]$ . We define  $\text{Sep}(P)$  as the minimal distance between two zeros of the polynomial  $P(x)$ .

**Theorem 6.1.** *Let  $m$  and  $n$  be integers with  $1 \leq m \leq n$ ,  $f(x)$  a monic polynomial of degree  $m$  with integer zeros, and  $g(x) \in \mathbb{Z}[x]$  a polynomial of degree  $n - m$ . Let  $Q(x) := f(x)g(x)$  have only simple zeros and write  $\delta = \text{Sep}(fg)$ .*

(i) *Let  $k$  be an integer with  $k > n - m$  and  $c$  an integer with*

$$(16) \quad 0 < |c| < \left(\frac{\delta}{2}\right)^{n-k} T_{n-k}.$$

*Then the polynomial  $P(x) := f(x)g(x) + c$  has no factor of degree  $k$  over  $\mathbb{Z}$ . Moreover, if  $k < m$  and*

$$(17) \quad 0 < |c| < \left(\frac{\delta}{2}\right)^k T_k,$$

*then  $P(x)$  has no factor of degree  $k$  over  $\mathbb{Z}$  either.*

(ii) *Using the above notation and making the same assumptions suppose that all zeros of  $g(x)$  are real. Then both statements in (i) remain valid with  $T_{n-k}$  and  $T_k$  replaced by  $(n - k - 1)!$  and  $(k - 1)!$ , respectively.*

**Remark 6.1.** Observe that the expressions  $(n - k - 1)!$  and  $(k - 1)!$  are larger than  $T_{n-k}$  and  $T_k$ , respectively, so in the real case  $c$  can come from a larger interval.

We immediately obtain the following consequence, since every factorization of  $P(x)$  implies a factor of degree at most  $n/2$  and a factor of degree at least  $n/2$ .

**Corollary 6.1.** *Let  $2 \leq n < 2m$ . If*

$$|c| < \min_{1 \leq k \leq n/2} \left\{ \left(\frac{\delta}{2}\right)^k T_k \right\} \quad \text{or} \quad |c| < \min_{1 \leq k \leq n/2} \left\{ \left(\frac{\delta}{2}\right)^{n-k} T_{n-k} \right\},$$

*then  $P(x)$  is irreducible over  $\mathbb{Z}$ . Further, if  $g(x)$  has only real zeros, then in the above inequalities  $T_k$  and  $T_{n-k}$  can be replaced by  $(k - 1)!$  and  $(n - k - 1)!$ , respectively, and the statement remains valid.*

To prove Theorem 6.1 we need some lemmas. The first one will be used in the real case.

**Lemma 6.1.** *Let  $\delta > 0$ . Let  $Q(x) = a(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n) \in \mathbb{Z}[x]$  have real zeros such that  $\alpha_{i+1} - \alpha_i \geq \delta$  for  $i = 1, 2, \dots, n-1$ . Let  $c$  be a real number satisfying  $|c| < (n-1)!|a|(\frac{\delta}{2})^n$ . Write*

$$P(x) := Q(x) + c = a(x-\beta_1)(x-\beta_2)\dots(x-\beta_n).$$

*Then, for  $i = 1, 2, \dots, n$ , the number  $\beta_i$  is real and, if  $\beta_1 \leq \beta_2 \leq \dots \leq \beta_n$ , then*

$$(18) \quad |\alpha_i - \beta_i| \leq \frac{2^{n-1}|c|}{(n-1)!|a|\delta^{n-1}} < \frac{\delta}{2}.$$

*Proof.* Let  $\alpha_{i,i+1}$  denote the real number with  $\alpha_{i+1} - \alpha_{i,i+1} = \alpha_{i,i+1} - \alpha_i$ . We put  $\alpha_{0,1} = \alpha_1 - \delta$  and  $\alpha_{n,n+1} = \alpha_n + \delta$ . Then, for  $i = 0, \dots, n$ ,

$$\begin{aligned} |Q(\alpha_{i,i+1})| &= |a| \prod_{j=1}^n |\alpha_{i,i+1} - \alpha_j| \\ &\geq |a| \cdot \frac{\delta}{2} \cdot \frac{\delta}{2} \cdot \frac{3\delta}{2} \cdot \frac{3\delta}{2} \cdot \frac{5\delta}{2} \cdots \geq |a|(n-1)! \left(\frac{\delta}{2}\right)^n > |c|. \end{aligned}$$

Put  $P(x) = Q(x) + c$ . Observe that  $P(\alpha_{0,1}), P(\alpha_{1,2}), \dots, P(\alpha_{n,n+1})$  have alternating signs. By continuity it follows that for  $i = 1, \dots, n$  there is a zero  $\beta_i$  of  $P(x)$  between  $\alpha_{i-1,i}$  and  $\alpha_{i,i+1}$ . Hence the numbers  $\beta_i$  are all real. It further follows that, for  $i = 1, 2, \dots, n$ , the number  $\alpha_i$  is the zero of  $Q(x)$  which is the nearest to the number  $\beta_i$ . We have, for such an  $i$ ,

$$|\alpha_i - \beta_i| = \frac{|Q(\beta_i)|}{|a| \prod_{j \neq i} |\beta_i - \alpha_j|} \leq \frac{|c|}{|a| \cdot \frac{\delta}{2} \cdot \delta \cdot \frac{3\delta}{2} \cdot 2\delta \cdots \frac{(n-1)\delta}{2}} \leq \frac{2^{n-1}|c|}{(n-1)!|a|\delta^{n-1}}.$$

□

One of the basic tools in the proof of Theorem 6.1 in the complex case is the following lemma which is a straightforward consequence of Rouché's theorem.

**Lemma 6.2.** *Let  $f(z)$  be a nonzero polynomial with complex coefficients and  $c \in \mathbb{C}$ . Further, let  $\alpha \in \mathbb{C}$ ,  $r \in \mathbb{R}$ ,  $r > 0$ , and put*

$$C(\alpha, r) = \{z \in \mathbb{C} : |\alpha - z| < r\}.$$

*If for every  $z \in \mathbb{C}$  with  $|\alpha - z| = r$  we have  $|f(z)| > |c|$  then the number of zeros of the polynomials  $f(z)$  and  $f(z) + c$  in  $C(\alpha, r)$  coincide.*

The following lemma is the complex variant of Lemma 6.1.

**Lemma 6.3.** *Let  $\alpha_1, \dots, \alpha_n$  be distinct complex numbers and let  $a$  be a nonzero complex number. Put  $\delta = \min_{1 \leq i < j \leq n} |\alpha_i - \alpha_j|$  and*

$$Q(z) = a(z - \alpha_1) \dots (z - \alpha_n).$$

Let  $c \in \mathbb{C}$  with

$$|c| < |a| \cdot \left(\frac{\delta}{2}\right)^n \cdot T_n.$$

Then for each zero  $\alpha_i$  of  $Q(z)$  there exists a unique zero  $\beta_i$  of the polynomial  $P(z) = Q(z) + c$  such that

$$|\beta_i - \alpha_i| \leq \frac{|c|}{|a|T_n} \cdot \left(\frac{2}{\delta}\right)^{n-1} < \frac{\delta}{2}.$$

Further, if  $\beta_{j_1}, \dots, \beta_{j_t}$  are distinct zeros of  $P(z)$  all different from  $\beta_i$ , then we have

$$(19) \quad \prod_{s=1}^t |\beta_{j_s} - \alpha_i| \geq \left(\frac{\delta}{2}\right)^t \cdot T_{t+1}.$$

*Proof.* Let  $\alpha_i$  be any zero of  $Q(z)$ , and let  $z$  be an arbitrary complex number with  $|z - \alpha_i| = \delta/2$ . Let  $\gamma_1, \gamma_2, \dots, \gamma_n$  be a rearrangement of the zeros  $\alpha_1, \dots, \alpha_n$  such that

$$|z - \gamma_1| \leq |z - \gamma_2| \leq \dots \leq |z - \gamma_n|$$

and let  $d_i = |z - \gamma_i|$  for  $i = 1, \dots, n$ . Then we have  $d_1 = \delta/2$ . Further, following the proof of Lemma 5.2 we obtain that, for  $i > 1$ ,

$$(20) \quad d_i \geq \frac{\delta}{2} \quad \text{and also} \quad d_i \geq \frac{\delta}{2} (\sqrt{i} - 1).$$

This yields

$$|Q(z)| = |a| \cdot \prod_{i=1}^n |z - \alpha_i| = |a| \cdot \prod_{i=1}^n |z - \gamma_i| = |a| \cdot \prod_{i=1}^n d_i \geq |a| \cdot \left(\frac{\delta}{2}\right)^n \cdot T_n.$$

Since

$$|c| < |a| \cdot \left(\frac{\delta}{2}\right)^n \cdot T_n$$

we get, by Lemma 6.2, that the polynomials  $P(z)$  and  $Q(z)$  have the same number of zeros in the open disc  $C(\alpha_i, \delta/2)$  of radius  $\delta/2$  with center  $\alpha_i$ . As by the definition of  $\delta$  the only zero of  $Q(z)$  in this disc is  $\alpha_i$ , there exists a unique zero  $\beta_i$  of  $P(z)$  with  $\beta_i \in C(\alpha_i, \delta/2)$ . Then we have

$$|c| = |Q(\beta_i)| = |a| \cdot \prod_{j=1}^n |\beta_i - \alpha_j| \geq |a| \cdot |\beta_i - \alpha_i| \cdot \prod_{j=2}^n d_j,$$

whence, by (20),

$$|\beta_i - \alpha_i| \leq \frac{|c|}{|a|T_n} \cdot \left(\frac{2}{\delta}\right)^{n-1} < \frac{\delta}{2}.$$

Finally, using (20) again one can easily check that (19) is also valid, and the lemma follows.  $\square$

*Proof of Theorem 6.1.* Assume first that all the zeros of  $g(x)$  are real. Put  $P(x) = Q(x) + c$ . According to Lemma 6.1, for every  $i$  with  $1 \leq i \leq n$  there exists a zero  $\beta_i$  of  $P(x)$  such that (18) holds. Suppose  $P(x) = P_1(x)P_2(x)$  with  $P_1(x), P_2(x) \in \mathbb{Z}[x]$  and  $\deg(P_1) = k$ . Write, without loss of generality,

$$P_1(x) = a_1(x - \beta_1) \cdots (x - \beta_k), P_2(x) = a_2(x - \beta_{k+1}) \cdots (x - \beta_n).$$

Since  $m > n - k$ , there exists an  $r$  with  $1 \leq r \leq k$  such that  $P_1(\beta_r) = 0$ . Then we have for the corresponding zero  $\alpha_r$  of  $Q(x)$ ,

$$|c| = |P(\alpha_r)| = |P_1(\alpha_r)| \cdot |P_2(\alpha_r)| = |P_1(\alpha_r)| \cdot |a_2| \cdot \prod_{j=k+1}^n |\alpha_r - \beta_j|.$$

Since  $|P_1(\alpha_r)| \geq 1, |a_2| \geq 1$  and

$$\left| \prod_{j=k+1}^n (\alpha_r - \beta_j) \right| \geq \frac{\delta}{2} \cdot \frac{\delta}{2} \cdot \frac{3\delta}{2} \cdot \frac{3\delta}{2} \cdot \frac{5\delta}{2} \cdots \geq (n - k - 1)! \left(\frac{\delta}{2}\right)^{n-k},$$

we obtain  $|c| \geq (n - k - 1)! \left(\frac{\delta}{2}\right)^{n-k}$ . Thus, if (16) holds with  $T_{n-k}$  replaced by  $(n - k - 1)!$ , then  $P(x)$  cannot have a factor of degree  $k$ .

If  $m > k$ , then we can apply the above argument with  $P_1$  and  $P_2$  interchanged. This yields  $|c| \geq (k - 1)! \left(\frac{\delta}{2}\right)^k$ . Thus, if  $k < m$  and (17) holds, then  $P(x)$  cannot have a factor of degree  $k$  over  $\mathbb{Z}$ .

Suppose now that  $g(x)$  has nonreal zeros, too. Following the proof in the real case, but using Lemma 6.3 instead of Lemma 6.1, we obtain instead of (16)

$$(21) \quad |c| \geq \left(\frac{\delta}{2}\right)^{n-k} T_{n-k}.$$

If  $n - m < k$ , we find instead of (17)

$$(22) \quad |c| \geq \left(\frac{\delta}{2}\right)^k T_k.$$

$\square$

**PART 3. POLYNOMIALS OF THE FORM  $g(f(x))$** 

## 7. THE RATIONAL CASE

Let  $a_1, \dots, a_m$  be distinct integers and put  $f(x) = (x - a_1) \cdots (x - a_m)$ . Brauer, Brauer and Hopf [7] proved that if  $K = \mathbb{Q}$  or  $\mathbb{Q}(\sqrt{-d})$ , with squarefree  $d \in \mathbb{Z}_{>0}$ , and  $g(x) \in O_K[x]$  is irreducible of degree at most 3, then  $g(f(x))$  is reducible over  $K$  for only finitely many equivalence classes of polynomials  $f$ . Here  $O_K$  denotes the ring of integers of  $K$ , and  $f, f' \in O_K[x]$  are said to be equivalent if  $f'(x) = f(x + a)$  for some  $a \in O_K$ .

Dorwart and Ore [12] showed that if  $g(x) = ax^2 + bx + 1 \in \mathbb{Z}[x]$  is irreducible, then  $g(f(x))$  is also irreducible when  $m \geq 5$ . Moreover, they were able to classify all cases in which  $g(f(x))$  is reducible when  $m < 5$ . In particular,  $P(x) = a(f(x))^2 + 1$  is irreducible if  $a \neq -b^2$  and  $P(x)$  is not equivalent to

$$-8(x-1)^2x^2(x+1)^2 + 1 = (2x^2 - 1)(-4x^4 + 6x^2 - 1).$$

The following result on general polynomials  $g(x) = ax^2 + bx + c$  follows immediately from Theorem 2.1.

**Theorem 7.1.** *Let  $c$  and  $m$  be nonzero integers with  $m > 2\tau(c)(2 + \lfloor \log_2 |c| \rfloor)$ . Let  $a_1, \dots, a_m$  be distinct integers and put  $f(x) = (x - a_1) \cdots (x - a_m)$ . Let  $g(x)$  be an irreducible polynomial of degree at most 2 with integral coefficients such that  $g(0) = c$ . Then  $g(f(x))$  is irreducible over  $\mathbb{Q}$ .*

*Proof.* Assume that  $g(f(x))$  is reducible over  $\mathbb{Q}$ . The result follows immediately from Theorem 2.1 if  $g$  is linear. Let  $g(x) = ax^2 + bx + c$  with  $a, b$  integers and  $a \neq 0$ . By Corollary 2.1 we have

$$af(x) + b = h_1(x)h_2(x)f(x) + c_2h_1(x) + c_1h_2(x)$$

with  $c_1c_2 = c$ . Hence  $h_1$  and  $h_2$  are integers with  $h_1h_2 = a$  and  $c_2h_1 + c_1h_2 = b$ . It follows that  $g(x) = ax^2 + bx + c = (h_1x + c_1)(h_2x + c_2)$  is reducible.  $\square$

The next result deals with the case that the degree of  $g(x)$  is 3. We say that a set  $\{a_1, \dots, a_{2r}\}$  is a Prouhet-Tarry-Escott set if the set splits into two subsets of equal cardinality,  $A := \{a_1, \dots, a_r\}$  and  $B := \{a_{r+1}, \dots, a_{2r}\}$  say, such that  $(x - a_1) \cdots (x - a_r) - (x - a_{r+1}) \cdots (x - a_{2r})$  is a constant. We call  $(A, B)$  a PTE pair. It is a conjecture that there exist PTE pairs only for bounded  $r$ . For information on PTE pairs see [25], [3], [4]. PTE pairs in the context of this paper occur already in [10] and [32].

**Theorem 7.2.** *Suppose the conditions of Theorem 7.1 are satisfied, but  $g(x) = ax^3 + bx^2 + vx + c$  is an irreducible polynomial with integral coefficients of degree 3. If  $g(f(x))$  is reducible over  $\mathbb{Q}$ , then*

*$-ac$  and  $v^2 - 4bc$  are positive squares with  $4ac \mid (v^2 - 4bc)$ ,*

*$m$  is even,  $m \leq 2 + \log_2 \frac{v^2 - 4bc}{ac}$ ,*

*$b = 0$  or  $m \leq 2\tau(b) (\lfloor \log_2 b \rfloor + 2)$ ,*

*$(a_1, \dots, a_m)$  is a Prouhet-Tarry-Escott set, and*

*$cg(f(x))$  factorizes into two polynomials of degree  $3m/2$  each, viz.*

$$f(x) \left( \frac{1}{2}v \pm \frac{1}{2}\sqrt{v^2 - 4bc - 4acf(x)} \right) + c.$$

*Proof.* By Corollary 2.1 we have

$$a(f(x))^2 + bf(x) + v = h_1(x)h_2(x)f(x) + c_2h_1(x) + c_1h_2(x)$$

with  $h_1(x), h_2(x) \in \mathbb{Z}[x]$ . Hence  $\deg(h_1) + \deg(h_2) = \deg(f)$  and  $c_2h_1(x) + c_1h_2(x) = v + lf(x)$  for some  $l \in \mathbb{Z}$ . It follows that either  $h_1$  is a constant or  $h_2$  is a constant or  $l = 0$ .

Suppose  $h_1$  is constant. Then  $h_2(x) = \frac{l}{c_1}f(x) + \frac{v - c_2h_1}{c_1} \in \mathbb{Z}[x]$  and

$$ax^3 + bx^2 + vx + c = (h_1x + c_1)(xh_2(x) + c_2)$$

is reducible. Thus  $g(x)$  is reducible over  $\mathbb{Z}$ . The case  $h_2$  is constant is similar.

Suppose  $l = 0$ . Then  $af(x) + b = h_1(x)h_2(x)$  and  $c_2h_1(x) + c_1h_2(x) = v$ . Hence  $\deg(h_1) = \deg(h_2) = m/2$  which is possible only if  $m$  is even. The factorization of  $af(x) + b$  is possible only if  $b = 0$  or  $m \leq 2\tau(b) (2 + \lfloor \log_2 b \rfloor)$  by Theorem 2.1. Let  $x^2 - vx + bc = (x - \alpha_1)(x - \alpha_2)$ . Since  $ch_1(a_i)h_2(a_i) = bc$  and  $c_2h_1(a_i) + c_1h_2(a_i) = v$  for  $i = 1, \dots, m$ , we have  $(c_2h_1(a_i), c_1h_2(a_i)) \in \{(\alpha_1, \alpha_2), (\alpha_2, \alpha_1)\}$ . Since a nonconstant polynomial of degree  $m/2$  cannot attain the same value more than  $m/2$  times, the set  $\{a_1, \dots, a_m\}$  splits into two subsets  $A$  and  $B$  of cardinality  $m/2$  each such that  $c_2h_1(a_i) = \alpha_1$  for  $i \in A$  and  $c_2h_1(a_i) = \alpha_2$  for  $i \in B$ . Hence

$$c_2h_1(x) - \alpha_1 = c_3 \prod_{a_i \in A} (x - a_i), c_2h_1(x) - \alpha_2 = c_3 \prod_{a_i \in B} (x - a_i),$$

$$c_1h_2(x) - \alpha_2 = -c_3 \prod_{a_i \in A} (x - a_i), c_1h_2(x) - \alpha_1 = -c_3 \prod_{a_i \in B} (x - a_i)$$

for some integer  $c_3$  with  $-c_3^2 = ac_1c_2 = ac$ . Thus  $-4ac$  is the square of an integer. Furthermore, if  $b \in B$ , then  $c_3 \prod_{a_i \in A} (b - a_i) = \alpha_2 - \alpha_1$ . Hence,  $-4ac \left( \prod_{a_i \in A} (b - a_i) \right)^2 = v^2 - 4bc$ . Thus  $4ac$  divides  $v^2 - 4bc$ ,  $v^2 - 4bc$  is the square of an integer and, since the product contains at least  $m - 4$  factors with absolute value  $> 1$ ,  $v^2 - 4bc \geq |ac|2^{m-2}$ . This



yields the latter inequality for  $m$ . From  $ch_1(x)h_2(x) = acf(x) + bc$  and  $c_2h_1(x) + c_1h_2(x) = v$  we obtain

$$c_2h_1(x), c_1h_2(x) = \frac{-v \pm \sqrt{v^2 - 4bc - 4acf(x)}}{2}.$$

Hence  $\sqrt{v^2 - 4bc - 4acf(x)} \in \mathbb{Z}[x]$ . □

**Remark 7.1.** It is remarkable that even for  $m = 24$  there exist irreducible polynomials  $g(x) \in \mathbb{Z}[x]$  of degree 3 and monic polynomials  $f(x) \in \mathbb{Z}[x]$  with  $m$  distinct integer zeros such that  $g(f(x))$  is reducible. Thus the conditions on  $m$  in Theorem 7.2 are insufficient to conclude that  $g(f(x))$  is reducible. To show this suppose  $A = \{a_1, \dots, a_r\}$  and  $B = \{a_{r+1}, \dots, a_{2r}\}$  are PTE pairs with

$$\prod_{i=1}^r (x - a_i) - \prod_{i=r+1}^{2r} (x - a_i) = v \neq 2.$$

Put  $f(x) = \prod_{i=1}^{2r} (x - a_i)$ ,  $g(x) = x^3 + vx - 1$ , that is we choose  $a = 1, b = 0, c = -1$ . Then  $g(x)$  is irreducible and  $g(f(x))$  factorizes into

$$\left( f(x) \prod_{i=1}^r (x - a_i) - 1 \right) \left( f(x) \prod_{i=r+1}^{2r} (x - a_i) + 1 \right),$$

as can easily be checked. PTE pairs are known for  $r = m/2 \leq 10$  and for  $r = m/2 = 12$ . For  $m = 1, 2, 3, 4, 5, 6, 7, 8, 10$  even infinitely many essentially different PTE pairs are known. One of the two known cases for  $r = 12$  is given by

$$A = \{\pm 22, \pm 61, \pm 86, \pm 127, \pm 140, \pm 151\},$$

$$B = \{\pm 35, \pm 47, \pm 94, \pm 121, \pm 146, \pm 148\},$$

due to Chen Shuwen *et al.* [9].

## 8. POLYNOMIALS OF THE FORM $g(f(x))$ WITH $g(x)$ OF CM-TYPE

In this section we deal with the reducibility of polynomials of the form  $g(f(x))$  over  $\mathbb{Q}$ , where  $g(x)$  is a monic irreducible polynomial in  $\mathbb{Z}[x]$  and  $f(x)$  is a monic polynomial in  $\mathbb{Z}[x]$  with distinct zeros in  $\mathbb{Q}$  or, more generally, in a given algebraic number field. We assume throughout that the splitting field of  $g(x)$  over  $\mathbb{Q}$  is a CM-field, i.e., a totally imaginary quadratic extension of a totally real algebraic number field. In this case  $g(x)$  is called of *CM-type*. For example, cyclotomic polynomials and quadratic polynomials of negative discriminant are of CM-type.

It was proved in [17] that for given  $g(x)$ , there are only finitely many pairwise inequivalent monic polynomials  $f \in \mathbb{Z}[x]$  with distinct zeros in  $\mathbb{Q}$  for which  $g(f(x))$  is reducible. In [18], [19] and [20] this result was extended to polynomials  $f(x)$  having all their zeros in a given totally real algebraic number field  $K$ . It turned out that in this more general situation there can exist infinitely many pairwise inequivalent quartic exceptions  $f(x)$  for which  $g(f(x))$  is reducible for a suitable  $g(x)$ . These exceptions have been completely described in [20].

In [17] - [20] and [15] some effective and quantitative versions were also established. For example, it was shown in [15] that under the above assumptions concerning  $f$  and  $g$  there is an effectively computable positive constant  $c_1$  which depends only on the degree, class number and discriminant of  $K$  such that if

$$(23) \quad \deg(f) > c_1 |g(0)|^{2/\deg(g)}$$

then  $g(f(x))$  is irreducible over  $\mathbb{Q}$ . We now prove the following.

**Theorem 8.1.** *Let  $g \in \mathbb{Z}[x]$  be a monic irreducible polynomial of CM-type, and  $K$  a totally real algebraic number field of degree  $d$ . Further, let  $m \geq 3$  be an integer with  $m \neq 4$ . Then there are at most*

$$(24) \quad (c_2 |g(0)|^{1/\deg(g)})^{d \binom{m}{2}}$$

*equivalence classes of monic polynomials  $f \in \mathbb{Z}[x]$  of degree  $m$  with distinct zeros in  $K$  for which  $g(f(x))$  is reducible over  $\mathbb{Q}$ . Here  $c_2$  denotes an effectively computable positive constant depending only on  $d$  and the discriminant of  $K$ .*

Together with (23), this gives a quantitative version of the main result (the Theorem) of [20]. An important feature of our bound (24) is that apart from the constant term  $g(0)$ , it does not depend on the coefficients of  $g(x)$ .

As was pointed out in [20], if  $K$  has a quadratic subfield then, for a suitable  $g \in \mathbb{Z}[x]$ , there exist infinitely many pairwise inequivalent monic quartic polynomials  $f \in \mathbb{Z}[x]$  with distinct zeros in  $K$  for which  $g(f(x))$  is reducible over  $\mathbb{Q}$ . Following our proof, it is easy to see that Theorem 8.1 is true for  $m = 4$  as well, provided that  $K$  has no quadratic subfield. Finally, we note that Theorem 8.1 does not remain valid if we drop the restriction that  $g$  is of CM-type or that the zeros of  $f$  belong to a fixed number field.

**8.1. Proof of the Theorem.** We shall use some arguments from the proof of the Theorem from [20].

Let  $M$  be an arbitrary algebraic number field, and let  $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$  be a finite subset of  $O_M$ , the ring of integers of  $M$ . For given  $N \geq 1$ , let  $\mathcal{G}_M(\mathcal{A}, N)$  denote the simple graph whose vertex set is  $\mathcal{A}$  and whose edges are the unordered pairs  $[\alpha_i, \alpha_j]$  having the property

$$|N_{M/\mathbb{Q}}(\alpha_i - \alpha_j)| > N.$$

**Lemma 8.1.** *Let  $M$  be a CM-field,  $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$  a finite set of real integers in  $M$  and  $\beta$  a nonreal integer in  $M$ . If the graph  $\mathcal{G}_M(\mathcal{A}, N_{M/\mathbb{Q}}(2\beta))$  has a connected component of order  $k \geq 2$  then  $F(x) = (x - \alpha_1) \dots (x - \alpha_m) - \beta$  has no irreducible factor of degree less than  $k$  over  $M$ . In particular, if  $k > \deg(F)/2$  then  $F$  is irreducible over  $M$ .*

*Proof.* See Lemma 7 in [18] and Lemma 4 in [20].  $\square$

In general the bound given for the degrees of the irreducible factors of  $F$  is best possible. As is pointed out in [17] and [18], Lemma 8.1 is not true for arbitrary number fields  $M$ .

Let again  $M$  be an arbitrary algebraic number field, and  $\mathcal{N}$  a finite, nonempty subset of  $O_M$ . For each pair of distinct positive integers  $i, j$  we select an element of  $\mathcal{N}$ , denoted by  $\delta_{i,j}$ , such that  $\delta_{i,j} = \delta_{j,i}$ . For any finite ordered subset  $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$  of  $O_M$  with  $m \geq 3$ , we denote by  $\mathcal{H}_M(\mathcal{A}, \mathcal{D})$  or simply by  $\mathcal{H}(\mathcal{A})$  the simple graph with vertex set  $\mathcal{A}$  whose edges are the unordered pairs  $[\alpha_i, \alpha_j]$  for which

$$\alpha_i - \alpha_j \notin \delta_{i,j} O_M^*.$$

Here  $O_M^*$  is the unit group of  $O_M$ , and  $\mathcal{D}$  denotes the  $\binom{m}{2}$ -tuple  $(\delta_{i,j})_{1 \leq i, j \leq m}$ .

The ordered subsets  $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$  and  $\mathcal{A}' = \{\alpha'_1, \dots, \alpha'_m\}$  of  $O_M$  are called  $O_M^*$ -equivalent, if

$$\alpha'_i = \varepsilon \alpha_i + \gamma, \quad i = 1, \dots, m$$

for some  $\varepsilon \in O_M^*$  and  $\gamma \in O_M$ . It is clear that the graphs  $\mathcal{H}(\mathcal{A})$  and  $\mathcal{H}(\mathcal{A}')$  are then isomorphic.

The following lemma is the crucial new element in the proof of Theorem 8.1. Let  $\varrho$  denote the unit rank of  $O_M$ .

**Lemma 8.2.** *Let  $m \geq 3$  be an integer different from 4. Then for all but at most*

$$(25) \quad ((m+1)10^{78(\varrho+1)})^{4(m-2)}$$

$O_M^*$ -equivalence classes of ordered subsets  $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$  of  $O_M$ , the graph  $\mathcal{H}_M(\mathcal{A}, \mathcal{D})$  has a connected component of order at least  $m - 1$ .

This is a quantitative version of Lemma 5 of [20]. It is an important feature of our bound in (25) that it depends only on  $m$  and  $\varrho$ . For more general but much weaker quantitative versions, see Theorem 2 of [21] and Theorem 2.1 of [23].

*Proof of Lemma 8.2.* The assertion has been proved in ([21], Theorem 1), with (25) replaced by

$$(26) \quad \prod_{i=3}^m \binom{i+1}{4} (6^2 C(3, O_M^*))^{m-2}.$$

Here  $C(3, O_M^*)$  denotes an upper bound for the number of solutions of the unit equation

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 1 \text{ in } u_1, u_2, u_3 \in O_M^*$$

with  $\sum_{i \in I} a_i u_i \neq 0$  for each nonempty  $I \subseteq \{1, 2, 3\}$ , where  $a_1, a_2, a_3$  are nonzero elements of  $M$ . The existence of such a bound  $C(3, O_M^*)$  which is independent of  $a_1, a_2, a_3$  was first proved in [14]. In view of Remark 5 of [22]

$$(27) \quad 6^{2(m-2)} \prod_{i=3}^m \binom{i+1}{4} \leq (m+1)^{4(m-2)}.$$

Further, it follows from Theorem 3 of [13] that

$$(28) \quad C(3, O_M^*) \leq (2^{35} \cdot 3^2)^{3^3(\varrho+1)}.$$

Now (25) is an immediate consequence of (26), (27) and (28).  $\square$

*Proof of Theorem 8.1.* Let  $g(x) \in \mathbb{Z}[x]$  be a monic irreducible polynomial of CM-type with degree  $n$ , and let  $f \in \mathbb{Z}[x]$  be a monic polynomial of degree  $m \geq 3$  with  $m \neq 4$  and with distinct zeros in  $K$ . Suppose that  $g(f(x))$  is reducible over  $\mathbb{Q}$ . Let  $\beta$  be a fixed zero of  $g(x)$  in  $\mathbb{C}$ . Then by Capelli's theorem (cf. [31] or Lemma 3 in [20]), the polynomial  $f(x) - \beta$  is reducible over the number field  $M := K(\beta)$ . By assumption  $K$  is totally real, hence  $M$  is of CM-type. Let  $\alpha_1, \dots, \alpha_m$  be the zeros of  $f(x)$  in  $K$ , and put  $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$ . Then it follows from Lemma 8.1 that the graph  $\mathcal{G}_M(\mathcal{A}, N_{M/\mathbb{Q}}(2\beta))$  has no connected component of order greater than  $m/2$ . We note that

$$(29) \quad (N_{M/\mathbb{Q}}(2\beta))^{1/[M:K]} = (N_{\mathbb{Q}(\beta)/\mathbb{Q}}(2\beta))^{[M:\mathbb{Q}(\beta)]/[M:K]} = 2^d g(0)^{d/n},$$

where  $d$  denotes the degree of  $K$  over  $\mathbb{Q}$ .

Let  $O_K$  and  $O_K^*$  denote the ring of integers and the unit group, respectively, of  $K$ . Denote by  $\mathcal{N}$  a maximal set of pairwise nonassociate elements of  $O_K$  whose norms in absolute value do not exceed  $2^d g(0)^{d/n}$ .

As is known, the cardinality  $|\mathcal{N}|$  of  $\mathcal{N}$  is at most  $c_3 g(0)^{d/n}$  where  $c_3$  and  $c_4, c_5$  below are effectively computable positive numbers which depend only on  $d$  and the discriminant of  $K$ ; for an explicit value of  $c_3$  we refer to [41]. For each pair of distinct positive integers  $i, j$  with  $1 \leq i, j \leq m$ , we select an element of  $\mathcal{N}$ , denoted by  $\delta_{i,j}$ , for which  $\delta_{i,j} = \delta_{j,i}$ . In this way we get a set, say  $\mathcal{C}$ , of  $\binom{m}{2}$ -tuples  $(\delta_{i,j})_{1 \leq i, j \leq m}$  whose cardinality is  $|\mathcal{N}|^{\binom{m}{2}}$ . For a fixed  $\binom{m}{2}$ -tuple  $\mathcal{D} = (\delta_{i,j})_{1 \leq i, j \leq m}$  and for a subset  $\mathcal{B} = \{\beta_1, \dots, \beta_m\}$  of  $O_K$ , consider the graph  $\mathcal{H}(\mathcal{B}) = \mathcal{H}_K(\mathcal{B}, \mathcal{D})$  defined above, but with  $K$  in place of  $M$ . We recall that  $\mathcal{B}$  denotes the vertex set of  $\mathcal{H}(\mathcal{B})$ , and its edge set consists of those unordered pairs  $[\beta_i, \beta_j]$  for which  $\beta_i - \beta_j \notin \delta_{i,j} O_K^*$ .

If  $[\alpha_i, \alpha_j]$  is an edge of the complement of the graph  $\mathcal{G}_M(\mathcal{A}, N_{M/\mathbb{Q}}(2\beta))$  then, by (29),

$$|N_{K/\mathbb{Q}}(\alpha_i - \alpha_j)| \leq 2^d g(0)^{d/n}.$$

Hence  $\alpha_i - \alpha_j$  is an associate of one of the elements of  $\mathcal{N}$ . Together with the fact that  $\mathcal{G}_M(\mathcal{A}, N_{M/\mathbb{Q}}(2\beta))$  has no connected component of order  $> m/2$ , this implies that for at least one suitable  $\binom{m}{2}$ -tuple  $\mathcal{D} = (\delta_{i,j})_{1 \leq i, j \leq m}$  of  $\mathcal{C}$ , the connected components of the graph  $\mathcal{H}_K(\mathcal{A}, \mathcal{D})$  have orders at most  $m/2$ . But the number of  $\binom{m}{2}$ -tuples  $\mathcal{D}$  in question is at most  $(c_4 g(0)^{1/n})^{d \binom{m}{2}}$ . Together with Lemma 8.2 (applied with  $K$  in place of  $M$ ) and  $\text{rank}(O_K^*) + 1 \leq d$ , this gives that there are at most  $(c_5 g(0)^{1/n})^{d \binom{m}{2}}$   $m$ -tuples  $\mathcal{A}'$  in  $O_K$  such that  $\mathcal{H}_K(\mathcal{A}', \mathcal{D})$  has no connected component of order  $> m/2$  for some  $\mathcal{D}$  and that  $\mathcal{A}$  is  $O_K^*$ -equivalent to one of these  $\mathcal{A}'$ , say to  $\mathcal{A}' = \{\alpha'_1, \dots, \alpha'_m\}$ . In other words,

$$(30) \quad \alpha_i = \varepsilon \alpha'_i + \eta, \quad i = 1, \dots, m$$

with some  $\eta \in O_K$  and  $\varepsilon \in O_K^*$ . From among these  $O_K^*$ -equivalence classes consider now only those ones which contain a representative  $\mathcal{A}' = \{\alpha'_1, \dots, \alpha'_m\}$  for which  $f'(x) := (x - \alpha'_1) \dots (x - \alpha'_m) \in \mathbb{Z}[x]$ . If such a class has another representative, say  $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$  with  $f(x) := (x - \alpha_1) \dots (x - \alpha_m) \in \mathbb{Z}[x]$  then taking the discriminant of  $f$  and  $f'$  and using (30) we infer that  $\varepsilon^{m(m-1)} \in O_K^* \cap \mathbb{Q}$  whence  $\varepsilon = \pm 1$  follows. Further, summing up the relations (30) from  $i = 1$  to  $m$ , we deduce that  $\eta \in O_K \cap \mathbb{Q} = \mathbb{Z}$ . Consequently, each  $O_K^*$ -equivalence class under consideration contains at most two  $\mathbb{Z}$ -equivalence classes of  $m$ -tuples  $\mathcal{A} = \{\alpha_1, \dots, \alpha_m\}$  from  $O_K$  for which  $f(x) := (x - \alpha_1) \dots (x - \alpha_m)$  has its coefficients in  $\mathbb{Z}$ . Here two tuples  $\{\alpha_1, \dots, \alpha_m\}$  and  $\{\alpha'_1, \dots, \alpha'_m\}$  are considered to be  $\mathbb{Z}$ -equivalent if  $\alpha'_i - \alpha_i = a$  for some  $a \in \mathbb{Z}$ ,  $i = 1, \dots, m$ . This completes the proof.  $\square$

**Remark 8.1.** In the proof of the Theorem of [20] we arrived also at the relations (30). However, there we followed another argument which cannot be made quantitative.

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